AIMS Mathematics

AIMS Mathematics, 7(4): 6915–6935.
DOI: 10.3934/math.2022384
Received: 17 November 2021
Revised: 07 January 2022
Accepted: 18 January 2022
Published: 28 January 2022

http://www.aimspress.com/journal/Math

Research article

Influence of weight function for similarity measures

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Abstract: The mainstream for dealing with pattern recognition problems is to develop new similarity measures, and then to compare outcomes among different measures. Along with a study trend focusing on developing new similarity measures for pattern recognition problems, this study tackles the issue of tuning weight functions of the existing measures. In this study, a detailed examination is executed to point out that a chosen weight function decides the pattern for a given example. The main contribution of the paper is to provide analytic derivations to explain the influence of weights for both discrete and continuous cases which supports our claims with mathematical foundations. With findings from this study, we expect a sensitivity analysis of the weights and exploring procedures in deciding a reasonable weight function for applications that can be set for future studies.

Keywords: similarity measure; intuitionistic fuzzy sets; pattern recognition
Mathematics Subject Classification: 03E72, 90B50

1. Introduction

1.1. Background, related studies

To apply distance or similarity measures for real-world problems, researchers developed many new distances and their corresponding similarity measures. For example, Yang [1] is a generalization of Hsieh [2] to extend the Graded Mean Integration Representation method to a weighted average operation. Following Szmidt and Kacprzyk [3], Hung [4] developed an enhanced entropy method to help physicians execute the examination of the preliminary diagnosis. Chao and Chu [5] is a generalized result for the following four papers, Zhang and Fu [6], Yusoff et al. [7], and Lin and Julian [8]. The new measure of Chao and Chu [5] is examined to hold the monotonic property. Tuan [9] studied a new aggregation method with uncertain conditions to simplify the previously tedious algorithm proposed by Li [10] and Lin et al. [11]. Lin et al. [12] applied the both-side attainment index to extend Hop [13]. Chao et al. [14] provided two extended findings for interval-valued intuitionistic fuzzy sets. However, after their papers had been published, several authors pointed out their similarity measures containing severe questionable dilemmas that sometimes the proposed measure cannot help decision-makers select the pattern for a given sample in a pattern recognition problem or decide the optimal location for a hydroelectric dam. Hence, many new similarity measures continuously had been constructed and then criticized by further studies by providing counter-examples to reveal their incapability. For example, Chou et al. [15] improved the solution method of Hop [16] for linear programming problems with randomness and fuzziness by attainment values. Chou et al. [17] showed that the solution procedure of Roy and Maiti [18] for fuzzy inventory models with storage space and budget constraints contained questionable results and then Chou et al. [17] presented the revised solution approach. Mitchell [19] developed a new similarity measure to improve Li and Cheng [20]. Julian et al. [21] tried to revise Mitchell [19] to verify the similarity values deriving by one-norm should be greater than two-norm and to construct a new similarity measure with a scattered property. Deng and Chao [22] pointed out the questionable proof of Julian et al. [21] and then Deng and Chao [22] provided the improved proof to compare one-norm and two-norm with weight functions. Hung and Tuan [23] challenged De et al. [24] to show that they arbitrarily applied the max-min operator to handle medical diagnosis issues that contained questionable findings. Chao [25] examined Park et al. [26] and Liang and Shi [27] to show that sometimes the pattern recognition problems cannot be solved by their similarity measures. Chuang et al. [28] showed that the similarity measures proposed by Xu [29] did not satisfy his assertion of the fourth axiom. Szmidt and Kacprzyk [30] constructed a new similarity measure to deal with the intuitionistic fuzzy set and its complement, and then Tung and Hopscotch [31] found a counterexample to indicate that their new similarity measure is unreasonable. Chuang et al. [32] demonstrated that the linguistic hybrid geometric averaging operator proposed by Xu [33] is influent by the relative weights. Yusoff et al. [7] criticized that the new similarity measure developed by Zhang and Fu [6] cannot solve a pattern recognition problem and then constructed a new similarity. Chao and Butler [34] proved that the new measure created by Yusoff et al. [7] did not satisfy the fourth axiom proposed by Mitchell [19]. Xu [35] amended Xu [36], however, Lin [37] further revised Xu [35] for his proof of the transitivity property. Tuan and Chao [38] examined theorem 2 of Gerstenkorn and Mańko [39] with correlation coefficient similarity measures and then provided a revision. Chu et al. [40] studied the similarity measure proposed by Xu [36] to simplify his redundant proof and then the criticism of Xu [36] concerning Szmidt and Kacprzyk [3] and Yang and Chiclana [41] are revised. Feng [42] showed that the inexact optimal solution approach with the fuzzy criterion proposed by Wang [43] contained questionable results. Yen [44] showed that the fuzzy operations proposed by Wang et al. [45] contained questionable results.

1.2. Motivation

On the other hand, Yen et al. [46] is the first paper to develop a comparison algorithm in which the steps are related to the number of elements in the universe of disclose. Yen et al. [46] had proven that they can solve any discrete pattern recognition problems without an unsolvable dilemma. Chu et al. [47] derived a new comparison algorithm for two similarity measures in which the size of their algorithm is proportional to the cardinal number of the universe of disclose. Chu et al. [47] also verified that their lengthy algorithm can handle any discrete universe set of disclose. Hung et al. [48] constructed a new algorithm to solve the pattern recognition problems and then applied their algorithm to solve a radar target issue. However, Hung et al. [48] did not prove that their algorithm can solve every discrete pattern recognition problem. Chou [49] constructed a new comparison algorithm for pattern recognition problems to prove that his algorithm can solve problems under the discrete case of the universe of disclose. We present a discussion for Yen et al. [46] since their article is the first paper to create an algorithm in which steps are proportional to the size of the universe of disclose. If $\mathrm{X} = \{\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_\mathrm{n}\}$ is the universe of disclose, then the pattern recognition algorithm in Yen et al. [46] will contain $6\mathrm{n} + 6$steps. Following Yen et al. [46], there are two consequent papers: Chu et al. [47] and Chou [49] in which algorithms contained $6\mathrm{n} + 6$ and $4\mathrm{n} + 2$steps in their pattern recognition algorithms, respectively. To construct similarity measures, Yen et al. [46] only referred to membership function and hesitation function. Chu et al. [47] considered membership, non-membership, and hesitation function. On the other hand, Chou [49] used the membership and non-membership functions. In the past, researchers paid attention to developing new similarity measures and constructing new algorithms to apply their similarity measures such that just a few papers focused on the weighted function or the relative weights for elements in the universe of disclose. The purpose of this paper is to provide a theoretical explanation to demonstrate that the outcome of pattern recognition problems is influent by the relative weight for elements in the universe of disclose. We will select a famous paper, Li and Cheng [20] to develop our explanation. After Mitchell [19] presented a counter-example to illustrate that the similarity measures constructed by Li and Cheng [20] contained questionable results, there are almost four hundred papers still cited Li and Cheng [20] in their references to show that Li and Cheng [20] is a very important paper in the academic history of intuitionistic fuzzy sets. Hence, we will use the numerical examples in Li and Cheng [20] as our foundation to derive our results.

1.3. Main contributions

This paper is organized as follows. In Section 2, two numerical examples are discussed for a discrete case and a continuous case for the universe of disclose, respectively. In Section 3, we develop different relative weights for the discrete case to show that sometimes, the sample can be assigned to patterns A_1 or A_2, besides the assertion of Li and Cheng [20], that is assigned to the pattern A_3. In Section 4, for a continuous universe of disclose, we extend the weighted function to a general expression to reveal that the outcome of the comparison is determined by the weighted function. In Section 5, we present our revisions for Wang [52] and Yang et al. [70] and direction for future research. In section 6, we conclude our study. We analytically show the findings of pattern recognition problems with weighted similarity measures under intuitionistic fuzzy sets environment that is dominated by relative weights of elements in the universe of discourse for the discrete case and the weighted function for the continuous case. In the past, researchers focus on constructing new similarity measures or developing new algorithms applying their similarity measures. Hence, previous results depended on a special weight to decide the pattern of the sample that may be required further considerations. How to select a proper weight will be an important issue for researchers in the future when dealing with pattern recognition problems. Our consideration will offer a patchwork to enhance the operational development of similarity measures for pattern recognition under IFSs.

2. Review of previous results

Let X be a fixed set that is usually denoted as the universe of disclose. An Intuitionistic Fuzzy Sets (IFS) A in X is an object having the form $A = \{\langle x, u_A(x), v(x)\rangle : x \in X\}$ where the function $u_A: X \to [0,1]$ and $v_A: X \to [0,1]$ define the degree of membership and degree of non-membership, respectively, and for every element x in X, $0 \le u(x) + v(x) \le 1$. The set of all IFSs in X is assumed to be $IFS_S(X)$. The order relation between two IFSs, A and B with $A, B \in IFS_S(X)$, expressed as $A \subseteq B$ that is assumed as $u_A(x) \le u_B(x)$ and $v_A(x) \ge v_B(x)$ for every element x in X. For the pattern recognition problem, with a given sample B and a set of patterns, $\{A_1, A_2, \ldots, A_m\}$, the decision-maker wants to decide the pattern of the given sample through a similarity measure, denoted as *Sim*. If

$$Sim\left(A_{i_0}, B\right) = \max_{1 \le i \le m} Sim(A_i, B), \tag{2.1}$$

based on the principle of the maximum degree of similarity among IFSs, researchers can decide that sample B should be assigned belonging to the pattern A_{i_0}.

We will focus on the numerical examples discussed by Li and Cheng [20] to illustrate that the outcome comparison results is influent by the relative weights for the elements in the universe of disclose, where the weight of an element x_i is denoted as w_i, for $i = 1,2, \ldots, n$ that satisfies $w_i \ge 0$, for $i = 1,2, \ldots, n$, and $\sum_{i=1}^{n} w_i = 1$.

The universe of discourse is expressed as $X = \{x_1, x_2, \ldots, x_n\}$, we recall the similarity measure proposed by Li and Cheng [20] for two IFSs *A* and *B*, *Sim* as follows

$$Sim(A, B) = 1 - \sqrt[p]{\sum_{i=1}^{n} w_i |\rho_A(x_i) - \rho_B(x_i)|^p} \tag{2.2}$$

with an auxiliary function, $\rho(x_i)$ or $\rho(x)$ as follows

$$\rho_A(x_i) = \frac{u_A(x_i) + 1 - v_A(x_i)}{2}, \tag{2.3}$$

for a discrete universe of disclose $X = \{x_1, x_2, \ldots, x_n\}$, and

$$\rho_A(x) = \frac{u_A(x) + 1 - v_A(x)}{2}, \tag{2.4}$$

for a continuous universe of disclose, $X = \{x: a \le x \le b\}$, where u_A and v_A are membership and non-membership function of IFS *A*, and u_B and v_B are membership and non-membership function of IFS *B*.

The first example 1 of Li and Cheng [20] is examined in the following. The universe of disclose is $X = \{x_1, x_2, x_3\}$, and three patters $\{A_1, A_2, A_3\}$ and one sample B are provided with $A_1 = \{\langle 1, 0\rangle, \langle 0.8, 0\rangle, \langle 0.7, 0.1\rangle\}$,$A_2 = \{\langle 0.8, 0.1\rangle, \langle 1,0\rangle, \langle 0.9, 0\rangle\}$, and $A_3 = \{\langle 0.6, 0.2\rangle, \langle 0.8, 0\rangle, \langle 1, 0\rangle\}$, and $B = \{\langle 0.5, 0.3\rangle, \langle 0.6, 0.2\rangle, \langle 0.8, 0.1\rangle\}$ such that $u_B(x_1) = 0.5$ and $v_B(x_1) = 0.3$. The decision-maker will decide which pattern should be assigned for sample *B*. First, we recall the results presented in Li and Cheng [20]. They assumed three possible combinations for weights and then applied the similarity measure of Eq (2.2). Their findings are listed in the following Table 1.

Table 1. Their results in Li and Cheng [20].

				$Sim(A_1, B)$	$Sim(A_2, B)$	$Sim(A_3, B)$
x_1	x_2	x_3	p			
$1/3$	$1/3$	$1/3$	1	0.78	0.80	0.85
$1/3$	$1/3$	$1/3$	2	0.74	0.78	0.84
0.5	0.3	0.2	2	0.696	0.779	0.853

Owing to the findings in the above Table 1, $Sim(A_3, B)$ always bigger than $Sim(A_1, B)$ and $Sim(A_2, B)$, Li and Cheng [20] implied that sample B should be assigned to the pattern A_3.

For the continuous example, under the universe of disclose, $X = \{x: a \leq x \leq b\}$, Li and Cheng [20] constructed the similarity measure for two IFSs A and B as follows

$$\mathrm{Sim}(A, B) = 1 - \int_a^b w(x)|\rho_A(x) - \rho_B(x)|dx, \tag{2.5}$$

with the weight function, $w(x)$. Li and Cheng [20] adopted the uniform distribution for the weight function $w(t): [a, b] \to [0,1]$, that satisfies $\int_a^b w(t)\, dt = 1$. Hence, they applied the weight function as $w(t) = 1/(b - a)$, for $a \leq t \leq b$.

We recall the second example in Li and Cheng [20] for two patters $\{A_1, A_2\}$ and one sample B such that their membership and non-membership functions are expressed in the following,

$$u_{A_1}(x) = \begin{cases} \frac{4}{5(x-1)}, & 1 \leq x < 2, \\ \frac{4(5-x)}{15}, & 2 \leq x \leq 5, \end{cases} \tag{2.6}$$

$$v_{A_1}(x) = \begin{cases} \frac{19-9x}{19}, & 1 \leq x < 2, \\ \frac{3x-5}{10}, & 2 \leq x \leq 5, \end{cases} \tag{2.7}$$

$$u_{A_2}(x) = \begin{cases} \frac{x-1}{5}, & 1 \leq x < 4, \\ \frac{3(5-x)}{5}, & 4 \leq x \leq 5, \end{cases} \tag{2.8}$$

$$v_{A_2}(x) = \begin{cases} \frac{13-3x}{10}, & 1 \leq x < 4, \\ \frac{9x-35}{10}, & 4 \leq x \leq 5, \end{cases} \tag{2.9}$$

$$u_B(x) = \begin{cases} \frac{3(x-1)}{10}, & 1 \leq x < 3, \\ \frac{3(5-x)}{10}, & 3 \leq x \leq 5, \end{cases} \tag{2.10}$$

and

$$v_B(x) = \begin{cases} \frac{14-4x}{10}, & 1 \leq x < 3, \\ \frac{4x-10}{10}, & 3 \leq x \leq 5. \end{cases} \tag{2.11}$$

Li and Cheng [20] mentioned that $\mathrm{Sim}(A_1, B) = 0.85$ and $\mathrm{Sim}(A_2, B) = 0.86$. Therefore, Li and Cheng [20] decided that sample *B* should be assigned as the pattern A_2.

3. Our examination for example 1 in [20]

Applying Eq (2.3), we find that $\rho_{A_1} = (1.0, 0.9, 0.8)$, $\rho_{A_2} = (0.85, 1.0, 0.95)$, $\rho_{A_3} = (0.7, 0.9, 1.0)$ and $\rho_B = (0.6, 0.7, 0.85)$ to indicate that $\rho_B(x_1) = 0.6$, $\rho_B(x_2) = 0.7$ and $\rho_B(x_3) = 0.8$. We will compute the similarity measures in advance and then compare those $\mathrm{Sim}(A_1, B)$, $\mathrm{Sim}(A_2, B)$, and $\mathrm{Sim}(A_3, B)$ to select a proper combination of the relative weights w_1, w_2, and w_3 for elements x_1, x_2, and x_3 in the universe of disclose, $X = \{x_1, x_2, x_3\}$. With the assumption $p = 1$, we obtain that

$$\mathrm{Sim}(A_1, B) = 1 - 0.4w_1 - 0.2w_2 - 0.05w_3. \tag{3.1}$$

$$\mathrm{Sim}(A_2, B) = 1 - 0.25w_1 - 0.3w_2 - 0.1w_3. \tag{3.2}$$

and

$$\mathrm{Sim}(A_3, B) = 1 - 0.1w_1 - 0.2w_2 - 0.15w_3. \tag{3.3}$$

In Li and Cheng [20], they claimed that

$$\mathrm{Sim}(A_3, B) = \max\{\mathrm{Sim}(A_1, B), \mathrm{Sim}(A_2, B), \mathrm{Sim}(A_3, B)\} \tag{3.4}$$

and then they concluded that sample B should be assigned to the pattern A_3. Our purpose is to construct different relatives such that sample B will be assigned to other patterns.
If we try to select relative weights w_1, w_2, and w_3 such that

$$\mathrm{Sim}(A_2, B) < Sim(A_1, B), \tag{3.5}$$

and

$$\mathrm{Sim}(A_3, B) < Sim(A_1, B). \tag{3.6}$$

We refer to our findings of Eqs (3.1–3.3) to derive that

$$3w_1 < 2w_2 + w_3, \tag{3.7}$$

and

$$3w_1 < w_3. \tag{3.8}$$

Based on our derivations of Eqs (3.7–3.8), we select $w_1 = 0.2$, $w_2 = 0.1$, and $w_3 = 0.7$, then we obtain the wanted result,

$$\mathrm{Sim}(A_2, B) = 0.85 < Sim(A_3, B) = 0.855 < Sim(A_1, B) = 0.865. \tag{3.9}$$

Hence, we show that there is a combination of relative weights such that sample B should be assigned to the pattern A_1.
Next, we examine the conditions to obtain that

$$\mathrm{Sim}(A_2, B) \geq \max\{\mathrm{Sim}(A_1, B), \mathrm{Sim}(A_3, B)\}. \tag{3.10}$$

We recall Eqs (3.1–3.3) to derive that

$$3w_1 \geq 2w_2 + w_3, \tag{3.11}$$

and

$$3w_1 + 2w_2 \leq w_3. \tag{3.12}$$

We combine Eqs (3.11–3.12) to imply that

$$2w_2 + w_3 \leq 3w_1 \leq w_3 - 2w_2, \tag{3.13}$$

such that we derive that

$$w_2 = 0, \tag{3.14}$$

and

$$w_3 = 3w_1. \tag{3.15}$$

Owing to $\sum_{i=1}^{3} w_i = 1$, we find a combination as

$$w_1 = 0.25,\ w_2 = 0, \text{ and } w_3 = 0.75, \tag{3.16}$$

such that

$$\mathrm{Sim}(A_2, B) = \max\{\mathrm{Sim}(A_1, B), \mathrm{Sim}(A_2, B), \mathrm{Sim}(A_3, B)\} \tag{3.17}$$

and then, we claim that sample *B* can be assigned to the pattern A_2.
Li and Cheng [20] mentioned that sample *B* can be assigned to the pattern A_3 that is Eq (3.4).
In the above demonstration, by Eq (3.9), sample *B* can be assigned to the pattern A_1.
Moreover, referring to Eq (3.17), sample *B* can be assigned to the pattern A_2.

Hence, we provide different combinations of relative weights to illustrate that the outcome of comparisons is influent by the choice of relative weights.

4. Our discussion for example 2 of [20]

Based on Eq (2.4), we evaluate the auxiliary function $\rho(x)$ to yield that

$$\rho_{A_1}(x) = \begin{cases} \frac{17(x-1)}{20}, & 1 \leq x < 2, \\ \frac{17(5-x)}{60}, & 2 \leq x \leq 5, \end{cases} \tag{4.1}$$

$$\rho_{A_2}(x) = \begin{cases} \frac{x-1}{4}, & 1 \leq x < 4, \\ \frac{3(5-x)}{4}, & 4 \leq x \leq 5, \end{cases} \tag{4.2}$$

and

$$\rho_B(x) = \begin{cases} \frac{7(x-1)}{20}, & 1 \leq x < 3, \\ \frac{7(5-x)}{20}, & 3 \leq x \leq 5. \end{cases} \tag{4.3}$$

In Li and Cheng [20], they used the following weighted function,

$$w(x) = \frac{1}{4}, \tag{4.4}$$

for$1 \leq x \leq 5$ that satisfies $w(x) \geq 0$ for $1 \leq x \leq 5$ and $\int_1^5 w(x)dx = 1$.
We will abstractly treat their weighted function in a generalized expression,

$$w(x) = \begin{cases} a, & 1 \leq x \leq 3, \\ b, & 3 \leq x \leq 5, \end{cases} \tag{4.5}$$

that satisfies $a \geq 0$, $b \geq 0$, and $2a + 2b = 1$. Therefore, the weighted function proposed by Li and Cheng [20] is a special case for our generalized expression, with $a = b = \frac{1}{4}$.
Next, we begin to evaluate $\mathrm{Sim}(A_1, B)$ and $\mathrm{Sim}(A_2, B)$ and then we obtain that

$$\begin{aligned}
\mathrm{Sim}(A_1, B) &= 1 - \int_1^5 w(x)\left|\rho_{A_1}(x) - \rho_B(x)\right|dx, \\
&= 1 - a\int_1^2 \left|\frac{17(x-1)}{20} - \frac{7(x-1)}{20}\right| dx - a\int_2^3 \left|\frac{17(5-x)}{60} - \frac{7(x-1)}{20}\right| dx, \\
&\quad -b\int_3^5 \left|\frac{17(5-x)}{60} - \frac{7(5-x)}{20}\right| dx, \\
&= 1 - \frac{a}{2}\int_1^2 (x-1)dx - \frac{a}{30}\int_2^3 |53 - 19x|dx - \frac{b}{15}\int_3^5 (5-x)dx, \\
&= 1 - \frac{a}{2}\int_1^2 (x-1)dx - \frac{a}{30}\int_2^{53/19} (53 - 19x)dx, \\
&\quad - \frac{a}{30}\int_{53/19}^3 (19x - 53)dx - \frac{b}{15}\int_3^5 (5-x)dx, \\
&= 1 - \frac{263a}{570} - \frac{2b}{15},
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
\mathrm{Sim}(A_2, B) &= 1 - \int_1^5 w(x)\left|\rho_{A_2}(x) - \rho_B(x)\right|dx, \\
&= 1 - a\int_1^3 \left|\frac{x-1}{4} - \frac{7(x-1)}{20}\right| dx - b\int_3^4 \left|\frac{x-1}{4} - \frac{7(5-x)}{20}\right| dx - b\int_4^5 \left|\frac{3(5-x)}{4} - \frac{7(5-x)}{20}\right| dx, \\
&= 1 - \frac{a}{10}\int_1^3 (x-1)dx - \frac{b}{5}\int_3^4 |3x - 10|dx - \frac{2b}{5}\int_4^5 (5-x)dx, \\
&= 1 - \frac{a}{10}\int_1^3 (x-1)dx - \frac{b}{5}\int_3^{10/3} (10 - 3x)dx - \frac{b}{5}\int_{10/3}^4 (3x - 10)dx - \frac{2b}{5}\int_4^5 (5-x)dx, \\
&= 1 - \frac{a}{5} - \frac{11b}{30}.
\end{aligned} \tag{4.7}$$

Next, we compare our findings with that of Li and Cheng [20] with the condition $a = b = \frac{1}{4}$, and then we obtain that

$$\mathrm{Sim}(A_1, B) = 0.851, \tag{4.8}$$

and

$$\mathrm{Sim}(A_2, B) = 0.858. \tag{4.9}$$

Our results are consistent with that of Li and Cheng [20] in which $\mathrm{Sim}(A_1, B) = 0.85$ and $\mathrm{Sim}(A_2, B) = 0.86$.

We begin to examine the influence of the selection of a and b for the ordering of $\mathrm{Sim}(A_1, B)$ and $\mathrm{Sim}(A_2, B)$.

We try to find the relation of a and b such that the following goal is achieved

$$\mathrm{Sim}(A_1, B) > Sim(A_2, B), \tag{4.10}$$

that is equivalent to

$$1 - \frac{263a}{570} - \frac{2b}{15} > 1 - \frac{a}{5} - \frac{11b}{30}. \tag{4.11}$$

Under the condition, $2a + 2b = 1$, we can simplify the inequality of Eq (4.11) as follows

$$b > 0.264. \tag{4.12}$$

Therefore, if we select $a = 0.2$ and $b = 0.3$, then we derive that

$$\mathrm{Sim}(A_1, B) = 0.868, \tag{4.13}$$

and

$$\mathrm{Sim}(A_2, B) = 0.850. \tag{4.14}$$

Hence, we construct a weighted function $w(x) = 0.2$, for $1 \le x \le 3$, and $w(x) = 0.3$, for $3 \le x \le 5$ then sample B should be assigned to the pattern A_1. We demonstrate that the assertion of Li and Cheng [20] mentioned sample B belonging to the pattern A_2 is questionable that is influent by the weight function.

5. The recent development for the relative weights and our comments

The goal of multiple attribute group decision-making problems is to select the best alternative or provide an order for alternatives. How to decide weights for elements in the universe of discourse is an important issue for multiple attribute decision-making problems. In the traditional multiple attribute group decision-making, decision-makers adopted predefined weights for elements in the discourse of universe that has several drawbacks: (a) Weights are independent of data of alternatives, (b) A group of experts may have different universes of discourses, (c) The alternatives for experts may be heterogeneous. Wallenius et al. [50], and Durbach and Stewart [51] tried to solve multiple attribute decision-making problems with weights are related to information of alternatives. If an element in the universe of discourse has a smaller deviation value that indicates all alternatives have almost identical values with respect to this element such that this element will not help researchers decide which alternative is the best alternative or the ranking for alternatives. Hence, this element should be given a smaller weight. On the contrary, if an element has a larger deviation, then values corresponding to this element are varied to help researchers decide the best alternative or the ranking for alternatives, and then this element should be given a larger weight. Hence, many researchers applied the maximizing deviation method proposed by Wang [52] and then this approach has extended to heterogeneous settings. In Wang [52], for a group decision-making problem with the discourse of universe $X = \{x_1, x_2, \dots, x_n\}$, alternatives $\{A_1, A_2, \dots, A_m\}$ and $A_i(x_j)$ is the evaluation of alternative A_i for element x_j. Wang [52] constructed the following maximum problem to decide weights for $X = \{x_1, x_2, \dots, x_n\}$:

$$\text{Max } \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} w_j \, d\left(A_i(x_j), A_k(x_j)\right), \tag{5.1}$$

s.t.$w_j \geq 0$, $j = 1,2,\dots,n$, $\sum_{j=1}^{n} w_j^2 = 1$,
where $d\left(A_i(x_j), A_k(x_j)\right)$ is the deviation of $A_i(x_j)$ with $A_k(x_j)$.
We claim that there are many ways to evaluate $d\left(A_i(x_j), A_k(x_j)\right)$. For example, if $A_i(x_j)$ is a crisp number, then we can apply one norm to define that

$$d\left(A_i(x_j), A_k(x_j)\right) = \left|A_i(x_j) - A_k(x_j)\right|. \tag{5.2}$$

If $A_i(x_j)$ is an intuitionistic fuzzy number, with $A_i(x_j) = \langle \mu_i(x_j), v_i(x_j) \rangle$, then we can apply two-norm to assume that

$$d\left(A_i(x_j), A_k(x_j)\right) = \sqrt{\left(\mu_i(x_j) - \mu_k(x_j)\right)^2 + \left(v_i(x_j) - v_k(x_j)\right)^2}. \tag{5.3}$$

Wang [52] did not offer us a detailed description of how to derive his findings. Instead, he directly claimed that

$$w_j = \frac{\sum_{i=1}^{m} \sum_{k=1}^{m} d\left(A_i(x_j), A_k(x_j)\right)}{\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m} \sum_{k=1}^{m} d\left(A_i(x_j), A_k(x_j)\right)\right)^2}}. \tag{5.4}$$

The above solution did not satisfy the condition that $\sum_{j=1}^{n} w_j = 1$ such that Wang [52] then normalized the results to find that

$$w_j = \frac{\sum_{i=1}^{m} \sum_{k=1}^{m} d\left(A_i(x_j), A_k(x_j)\right)}{\sum_{j=1}^{n}\left(\sum_{i=1}^{m} \sum_{k=1}^{m} d\left(A_i(x_j), A_k(x_j)\right)\right)}. \tag{5.5}$$

Herrera-Viedmaet al. [53], and Hartmann et al. [54] divided the decision procedure into two stages: (i) The selection procedure; (ii) The consensus procedure. For each expert, he operated his selection procedure to obtain collective solutions based on his weights for the universe of discourse. The group of experts runs the consensus procedure to achieve maximum content among them over collective solutions. Wang et al. [55] established a set of discounted belief degrees by the basic probability assignment function to merge with analytical and evidential reasoning rules. Yang and Xu [56] developed a novel discounting approach to construct discount evidence concerning reliability and weight. Li and Wan [57,58] and Wan and Li [59,60] applied the linear programming technique to minimize the inconsistency measure for multidimensional analysis of preference to decide the weights for elements in the discourse of the universe. In Zhang et al. [61], they pointed out that for multiple criteria group decision-making problems, weights of several experts and elements in the discourse of universe occupied a significant character to decide the optimal alternative such that Zhang et al. [61] invited researchers to pay attention to this issue. Owing to experts with their different special backgrounds, it is very difficult for them to reach a consensus agreement for the weights for elements in the universe of discourse. Zhang et al. [61] tried to construct algorithms with different criteria for diverse experts. They developed that every expert provided his weight to replace the traditional approach to find a global weight for all experts.

We recall that based on the expression of Zhang et al. [61], if there are k elements in the universe of discourse, then the weighting vector for elements is expressed as

$$\mathrm{v} = (\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_\mathrm{k}) \tag{5.6}$$

that satisfies $\mathrm{v_j} \geq 0$, for j=1,2,...,k, and $\sum_{\mathrm{j}=1}^{\mathrm{k}} \mathrm{v_j} = 1$.

Zhang et al. [61] claimed that Kim and Ahn [62], Wang et al. [63], and Chen [64] developed five basic forms for weights:

(1) A weak ranking: $\mathrm{v_i} \geq \mathrm{v_j}$, for $\mathrm{i} \in \Delta_1$ and $\mathrm{j} \in \Delta_2$, where Δ_1 and Δ_2 are two disjoint subsets, with $\Delta_1 \cup \Delta_2 = \{1,2,\dots,\mathrm{k}\}$.

(2) A strict ranking: $\mathrm{L} \leq \mathrm{v_s} - \mathrm{v_t} \leq \mathrm{U}$, for $\mathrm{s} \in \Delta_3$ and $\mathrm{t} \in \Delta_4$, where Δ_3 and Δ_4 are two disjoint subsets, with $\Delta_3 \cup \Delta_4 = \{1,2,\dots,\mathrm{k}\}$, where L and U are constraints to satisfy $0 < L < U$.

(3) A ranking of differences: $\mathrm{v_a} - \mathrm{v_b} \geq \mathrm{v_c} - \mathrm{v_d}$, for $\mathrm{a} \in \Delta_5, \mathrm{b} \in \Delta_6$, $\mathrm{c} \in \Delta_7$ and $\mathrm{d} \in \Delta_8$, where Δ_5, Δ_6, Δ_7 and Δ_8 are four disjoint subsets, with $\Delta_5 \cup \Delta_6 \cup \Delta_7 \cup \Delta_8 = \{1,2,\dots,\mathrm{k}\}$.

(4) A ranking with multiples: $\mathrm{v_i} \geq \delta \mathrm{v_j}$, $\mathrm{i} \in \Delta_9$ and $\mathrm{j} \in \Delta_{10}$, where Δ_9 and Δ_{10} are two disjoint subsets, with $\Delta_9 \cup \Delta_{10} = \{1,2,\dots,\mathrm{k}\}$ and δ is a constant with $\delta > 0$.

(5) An interval form: $\mathrm{S} \leq \mathrm{v_i} \leq \mathrm{T}$, for $\mathrm{i} \in \Delta_{11}$, where S and T are two constants that satisfy $0 < S < T$ and Δ_{11} is a subset of $\{1,2,\dots,\mathrm{k}\}$.

Zhang et al. [61] claimed that the expressions of weights usually contain several forms of above mentioned five basic forms that are varied by the conditions of problem settings and oriented applications.

Zhang et al. [61] firstly developed a deviation modeling method to derive the optimal weights of elements in the universe of discourse. Secondly, Zhang et al. [61] constructed the following minimizing deviation model to decide the weights of $\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_\mathrm{k}$,

$$\min \quad \mathrm{Z} = \sum_{\mathrm{i}=1}^{\mathrm{m}} \sum_{\mathrm{j}=1}^{\mathrm{k}} \mathrm{v_j} \left| \mathrm{CI}_\mathrm{i}^* - \mathrm{CI}_\mathrm{i}^\mathrm{j} \right| \tag{5.7}$$

s.t.$\sum_{\mathrm{j}=1}^{\mathrm{k}} \mathrm{v_j} = 1$, $\mathrm{v_j} \geq 0$, for $\mathrm{j} \in \{1,2,\dots,\mathrm{k}\}$.

We can simplify the expression to rewrite the above problem as follows,

$$\min \ \mathrm{Z} = \sum_{\mathrm{j}=1}^{\mathrm{k}} \mathrm{v_j} \alpha_\mathrm{j} \tag{5.8}$$

such that $\sum_{\mathrm{j}=1}^{\mathrm{k}} \mathrm{v_j} = 1$, $\mathrm{v_j} \geq 0$, for $\mathrm{j} \in \{1,2,\dots,\mathrm{k}\}$, and $\alpha_\mathrm{j} = \sum_{\mathrm{i}=1}^{\mathrm{m}} \left| \mathrm{CI}_\mathrm{i}^* - \mathrm{CI}_\mathrm{i}^\mathrm{j} \right|$.

Without loss of generality, we assume that

$$\alpha_1 = \min\{\alpha_1, \alpha_2, \dots, \alpha_\mathrm{k}\}, \tag{5.9}$$

then we derive that

$$\sum_{\mathrm{j}=1}^{\mathrm{k}} \mathrm{v_j} \alpha_\mathrm{j} \geq \sum_{\mathrm{j}=1}^{\mathrm{k}} \mathrm{v_j} \alpha_1 = \alpha_1. \tag{5.10}$$

Hence, if we select $\mathrm{v}_1 = 1$ and $\mathrm{v_j} = 0$, for $\mathrm{j} \in \{2,3,\dots,\mathrm{k}\}$, then we attain the minimum. However, only focusing on one element in the universe of discourse is not a proper approach to derive the optimal alternative such that we can claim that the minimizing deviation model proposed by Zhang et al. [61] contained severe questionable results.

Dong et al. [65] mentioned that experts may come from different backgrounds and have various interests such that they may have diverse universes of discourses. In the recent development, researchers considered that (a) Different discourse of universe for individual expert; (b) Experts may net reach a uniform conclusion for the optimal alternative, but they can find a compromise alternative that is accepted by the majority of experts; (c) Discourses of universe and alternatives may change during the decision process. Dong et al. [65] applied the maximizing deviation method proposed by Wang [52] to derive weights for elements in the universe of discourse under heterogeneous environments. Xu et al. [66] pointed out that only considering the positive ideal solution to minimize the inconsistency that cannot guarantee to attain the maximum of the consistency measure concerning the negative ideal solution. Xu et al. [66] constructed a new approach not only to handle crisp values, intervals, fuzzy sets, and intuitionistic fuzzy sets but also to deal with hesitant fuzzy sets to solve multiple attribute decision-making problems in the real-world environment. Wan et al. [67] mentioned that if researchers adopted the uniform weights for elements in the universe of disclose, then it may imply contradicted results in the selection process. Hence, how to decide weights becomes a critical matter for decision-makers. Lin and Wang [68] adopted score functions to rank intuitionistic fuzzy sets.

In Van et al. [69], for interval-valued intuitionistic fuzzy sets, they used $A_j^+ = \langle[1,1],[0,0],[0,0]\rangle$ as the positive ideal solution and $A_j^- = \langle[0,0],[1,1],[1,1]\rangle$ as the positive ideal solution for the alternative A_j. Hence, their positive and negative ideal solutions are predesigned that are independent of data information about alternatives.

Van et al. [69] defined the closeness coefficient for alternative A_j as

$$CC_j = \frac{d_j^+}{d_j^+ + d_j^-} \tag{5.11}$$

where d_j^+ is the shortest distance for alternative A_j, and d_j^- is the farthest distance for alternative A_j. We cite the results of Table 2 of Van et al. [69] in the following.

Table 2. Reproduction of Table 7 for distance measurement of Van et al. [69].

Suppliers	d^+	d^-
A_1	0.768	0.251
A_2	0.767	0.252
A_3	0.751	0.268
A_4	0.768	0.251

For the closeness coefficient of suppliers, we cite their results in Table 3 of Van et al. [69] in the following table.

Table 3. Reproduction of Table 8 for Closeness coefficient of Van et al. [69].

Suppliers	Closeness coefficient	Ranking
A_1	0.247	3
A_2	0.248	2
A_3	0.263	1
A_4	0.246	4

In the following, we present our comments for Van et al. [69].

Based on Table 2, the values of d^+ and d^- for A_1 and A_4 are identical such that findings in Table 3 of Van et al. [69] for A_1 and A_4 are different which is questionable. We can say that Van et al. [69] should express their results to the fourth decimal place to show the difference between A_1 and A_4.

Moreover, we can claim that Van et al. [69] adopted a pair of predesigned positive and negative ideal solutions that will result in computing findings for d^+ and d^- with respect to A_1, A_2, A_3 and A_4 are very close to each other. If they constructed their positive and negative ideal solutions depending on the data information of alternatives that will imply varies for d^+ and d^- and then the values of the "closeness coefficient" will be expressed separately. It will help researchers decide which alternative is the optimal choice.

Yang et al. [70] claimed that if weights of elements in the universe of discourse are completely predesigned, then they can decide weights by applying the maximizing deviation method proposed by Wang [52]. We cite the third multiple-objective programming model of Yang et al. [70], A reasonable weight vector should make the deviation value as large as possible. Then we set up the following model

$$\text{(M-3)Max } \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} w_j\, d\left(\breve{h}_{i\,j}, \breve{h}_{k\,j}\right) \tag{5.12}$$

s.t. $w_j \geq 0,\ j = 1,2,\dots,n,\ \sum_{j=1}^{n} w_j^2 = 1$.”

The whole derivation of Yang et al. [70] for their M-3 model is lengthy and contains questionable results such that we only mention their goal in Eq (5.12). We will provide our revisions in the next section.

We remark that $d\left(\breve{h}_{i\,j}, \breve{h}_{k\,j}\right)$ is the deviation of $\breve{h}_{i\,j}$ with the evaluation of $\breve{h}_{k\,j}$. $\breve{h}_{i\,j}$ maybe expressed in a crisp value, a fuzzy number, an intuitionistic fuzzy set, or interval-valued intuitionistic fuzzy set of the ith alternative for the jth attributive (the jth element in the universe of disclose).

In the following, we briefly introduce the derivation procedure of Yang et al. [70].

Yang et al. [70] tried to solve the model (M-3) to define the Lagrange function $L(w, \xi)$ as

$$L(w,\xi) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} w_j\, d\left(\breve{h}_{i\,j}, \breve{h}_{k\,j}\right) + \frac{\xi}{2}\left(\left(\sum_{j=1}^{n} w_j^2\right) - 1\right), \tag{5.13}$$

where ξ is the Lagrange multiplier variable. Yang et al. [70] took the partial derivatives with respect towandξ and then they solved the simultaneous system of $\frac{\partial L(w,\xi)}{\partial w} = 0$ and $\frac{\partial L(w,\xi)}{\partial \xi} = 0$ to derive that

$$w_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} d\left(\breve{h}_{i\,j},\breve{h}_{k\,j}\right)}{\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m} d\left(\breve{h}_{i\,j},\breve{h}_{k\,j}\right)\right)^2}}, \tag{5.14}$$

for $j = 1,2,\dots,n$.

The above solution did not satisfy the condition that $\sum_{j=1}^{n} w_j = 1$ such that Yang et al. [70] normalized the results to find that

$$w_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} d\left(\breve{h}_{i\,j},\breve{h}_{k\,j}\right)}{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m} d\left(\breve{h}_{i\,j},\breve{h}_{k\,j}\right)\right)}, \tag{5.15}$$

for $j = 1,2,\dots,n$.

6. Our improvement for Wang [52] and Yang et al. [70] and direction for future research

We point out that the Lagrange multiplier method only derives possible candidates for local minimum or local maximum points.

Based on the solution approach of Yang et al. [70], they should try to solve the following different maximum problems by applying the Lagrange multiplier method,

$$\text{Max}\ \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} w_j\, d\left(\check{h}_{i\,j}, \check{h}_{k\,j}\right) \tag{6.1}$$

such that $\sum_{j=1}^{n} w_j^2 = 1$.

We must point out that the maximum problem of Eq (6.1) is different from the maximum problem proposed by Yang et al. [70] of Eq (5.12) because the inequality constraints $w_j \geq 0,\ j = 1,2,\dots,n$, had been removed.

We write the results for $\frac{\partial L(w,\xi)}{\partial w} = 0$ and $\frac{\partial L(w,\xi)}{\partial \xi} = 0$ in detail,

$$\sum_{i=1}^{m}\sum_{k=1}^{m} d\left(\check{h}_{i\,j}, \check{h}_{k\,j}\right) + \xi w_j = 0, \tag{6.2}$$

and

$$\frac{1}{2}\left(\left(\sum_{j=1}^{n} w_j^2\right) - 1\right) = 0. \tag{6.3}$$

Form Eq (6.2), we derive that

$$w_j = \left(\frac{-1}{\xi}\right)\sum_{i=1}^{m}\sum_{k=1}^{m} d\left(\check{h}_{i\,j}, \check{h}_{k\,j}\right), \tag{6.4}$$

for $j = 1,2,\dots,n$.

We plug the findings of Eq (6.4) back to Eq (6.3) to find that

$$\xi^2 = \sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m} d\left(\check{h}_{i\,j}, \check{h}_{k\,j}\right)\right)^2, \tag{6.5}$$

such that ξ has two solutions:

$$\xi = \sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m} d\left(\check{h}_{i\,j}, \check{h}_{k\,j}\right)\right)^2}, \tag{6.6}$$

and

$$\xi = -\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m} d\left(\check{h}_{i\,j}, \check{h}_{k\,j}\right)\right)^2}. \tag{6.7}$$

Based on Eq (6.7), we derive results as that of Eq (5.14).

On the other hand, based on Eq (6.6), we imply that

$$w_j = \frac{-\sum_{i=1}^{m}\sum_{k=1}^{m} d\left(\check{h}_{i\,j}, \check{h}_{k\,j}\right)}{\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m} d\left(\check{h}_{i\,j}, \check{h}_{k\,j}\right)\right)^2}}, \tag{6.8}$$

for $j = 1,2,\dots,n$.

Therefore, by the Lagrange multiplier method, we obtain the maximum solution of Eq (5.14) and the minimum solution of Eq (6.8) for our proposed problem of Eq (6.1) to support our claim that researchers can not directly say the findings is a maximum solution or a minimum solution.

Therefore, we claim that the patchwork proposed by Yang et al. [70] for the maximum problem constructed by Wang [52] contained questionable results.

Next, we provide our patchwork for Wang [52] and Yang et al. [70].

We can abstractly handle maximum problems of Eq (5.1) of Wang [52] and Eq (5.12) of Yang et al. [70] simultaneously,

$$\mathrm{Max}\ \sum_{j=1}^{n} w_j\,\beta_j \tag{6.9}$$

such that $w_j \geq 0,\ j = 1,2,\ldots,n$, and $\sum_{j=1}^{n} w_j^2 = 1$.

When $\beta_j = \sum_{i=1}^{m}\sum_{k=1}^{m} d\left(A_i(x_j), A_k(x_j)\right)$, our proposed problem of Eq (6.9) converts to Eq (5.1). When $\beta_j = \sum_{i=1}^{m}\sum_{k=1}^{m} d(\breve{h}_{i\,j}, \breve{h}_{k\,j})$, our proposed problem of Eq (6.9) converts to Eq (5.12) such that we can claim that our maximum problem of Eq (6.9) stands for Eq (5.1) of Wang [52] and Eq (5.12) of Yang et al. [70] at the same time.

For $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$, we recall the Cauchy-Schwarz inequality to imply that

$$-\sqrt{\sum_{j=1}^{n} a_j^2}\sqrt{\sum_{j=1}^{n} b_j^2} \leq \sum_{j=1}^{n} a_j b_j \leq \sqrt{\sum_{j=1}^{n} a_j^2}\sqrt{\sum_{j=1}^{n} b_j^2}, \tag{6.10}$$

and inequalities change to equalities when there is a number, k, satisfying

$$a_j = k\,b_j, \tag{6.11}$$

for $j = 1,2,\ldots,n$.

Based on our discussion of Eqs (6.10) and (6.11), owing to $\sum_{j=1}^{n} w_j^2 = 1$, we solve the problem of Eq (6.9) as

$$-\sqrt{\sum_{j=1}^{n} \beta_j^2} \leq \sum_{j=1}^{n} w_j\,\beta_j \leq \sqrt{\sum_{j=1}^{n} \beta_j^2}, \tag{6.12}$$

and the maximum value attained when

$$\beta_j = k\,w_j, \tag{6.13}$$

for $j = 1,2,\ldots,n$.

Using Eq (6.13), we derive that

$$\sum_{j=1}^{n} \beta_j^2 = \sum_{j=1}^{n} k^2 w_j^2. \tag{6.14}$$

We recall $\sum_{j=1}^{n} w_j^2 = 1$ obtaining

$$k = \sqrt{\sum_{j=1}^{n} \beta_j^2}, \tag{6.15}$$

and

$$w_j = \frac{\beta_j}{k} = \frac{\beta_j}{\sqrt{\sum_{j=1}^{n} \beta_j^2}}. \tag{6.16}$$

For the final step, we normalize the findings of Eq (6.16) to find that

$$w_j = \frac{\beta_j}{\sum_{j=1}^{n} \beta_j}. \tag{6.17}$$

The present version of our paper was based on a definition of similarity of intuitionistic fuzzy sets in the literature to draw some conclusions through a specific example such that it does not have universal significance. In the future, we should observe the real world, for example, earth and cosmos, to find significant issues and then develop our results with real applications. Moreover, we are motivated by Hung et al. [71], to incorporate with the computer-based interface in the diagnosis system can accelerate the estimating process to save the precious time of physicians.

7. Conclusions

A similarity measure is a useful tool for determining the similarity of two objects. Based on the same numerical examples of Li and Cheng [20], we demonstrated that their proposed similarity measures are dominated by the relative weight of the domain for an IFS in pattern recognition problems. In the past, researchers focus on developing new similarities to replace previously established similarity measures, moreover, Yen et al. [46], Chu et al. [47], and Chou [49] constructed algorithms that are related to the size of the universe of discourse for the discrete cases to repeatedly applied their proposed similarity measures. However, Yen et al. [46], Chu et al. [47], and Chou [49] did not pay attention to how to decide relative weights for elements in the universe of discourse. Based on our discussion, we show that applying the same similarity measure with different relative weights will result in different findings for pattern recognition problems. Consequently, we point out their proposed measures to analyze the behavior of decision making that should be put more attention to the relative weight of the universe of disclose for an IFS. The study can be extended to picture fuzzy sets. For example, Ganie and Singh [72] is a very important lineage to future work. We may extend the paper to consider spherical fuzzy sets and their applications in selecting mega projects as Khan et al. [73].

Conflict of interest

The author declares that there is no conflict of interest in this manuscript.

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