



Research article

Mathematical modeling and analysis of biological control strategy of aphid population

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Abstract: To study the biological control strategy of aphids, in this paper we propose host-parasitoid-predator models for the interactions among aphids, parasitic wasps and aphidophagous Coccinellids incorporating impulsive releases of Coccinellids, and then study the long-term control and limited time optimal control of aphids by adjusting release amount and release timing of Coccinellids. For the long-term control, the existence and stability of the aphid-eradication periodic solution are investigated and threshold conditions about the release amount and release period to ensure the ultimate extinction of the aphid population are obtained. For the limited-time control, three different optimal impulsive control problems are studied. A time rescaling technique and an optimization algorithm based on gradient are applied, and the optimal release amounts and timings of natural enemies are gained. Our simulations indicate that in the limited-time control, the optimal selection of release timing should be given higher priority compared with the release amount.

Keywords: aphid population; biological control; long-term control; limited-time control; optimal control

Mathematics Subject Classification: 92D25, 92D40

1. Introduction

The management of crop pest population has always been a complex ecological problem [1]. The traditional control technology based on chemical insecticides has caused many problems such as grain yield reduction and environmental pollution, and the development of sustainable agriculture has prompted people to accelerate the whole process of green control of various diseases and crop pests ([2–4]). Aphids are a kind of phytophagous insects, which are widely distributed in the world, and have a great impact on agricultural crop production in many countries. Therefore, biological control of aphids is an important measure to promote green development, improve the environment and enhance food security and ecosystem protection.

Biological control of aphids is to use the natural enemies of aphids to control or reduce aphid damage. Aphids have many natural enemies in nature, including insects, fungi, bacteria, viruses and birds [5]. At present, the existing observation and research show that some insects (such as aphidophagous Coccinellids) have a significant impact on the number of aphids in the field. In addition, as a parasite of aphids, parasitic wasps are also an important means to control aphids. Adult parasitic wasps deposit their eggs in aphids. After the eggs hatch, immature larvae devour the living host from the inside [1]. There is competition between predators and parasitoids for aphids. Because the offspring of parasitic wasps develop in aphids, they are also vulnerable to the predation of aphid natural enemies. The phenomenon of intraguild predation, where more than one species feed on the same prey and therefore competitors feed on each other ([6, 7]), widely exists in aphids, parasitic wasps and aphid predators. The existence of predatory natural enemies in the ecosystem makes it necessary to carefully evaluate the impact of the introduction of such natural enemies on the existing aphid parasitoid system before selecting such natural enemies to control aphids.

Experimental data of several researchers show that predation within the system can interfere with or strengthen the control effect of parasitic wasps on aphids. For example, J. Brodeur et al. found that intraguild predation leads to the fluctuation of parasitic wasp population, which greatly affected the effect of parasitic wasps on aphid control [8]. R. G. Colfer and other researchers conducted inclusion/exclusion experiments at the University of California Student Experimental Farm in 1994 [9]. They evaluated the impact of introducing a predator, the ladybird beetle, *Hippodamia convergens*, on the cotton aphid and the parasitoid *Lysiphlebus testaceipes*. W. E. Snyder et al. designed field experiments to study the interaction among the pea aphids, parasitic wasp and the predator carabid beetles. Their study also illustrated that although intraguild predation can reduce the density of aphids in a short time, the predation of beetles on parasitic wasps reduces the infection rate of parasitic wasps of aphids, which hinders the control effect of parasitic wasps on aphids [10]. E. Bilu et al. [11] analyzed the interaction among these three species and revealed that in the short term, predators in the system hinder the effect of parasitic wasps on aphids, but the superposition effect of predation and parasitism produces the best aphid control effect for a long time. L. M. Gontijo et al. investigated the effect of several types of predators on the inhibition effect of parasitic wasps on apple aphids [12], and their experiments showed that the introduction of predators gives a better aphid inhibitory effect. T. Nakazawa et al. [13] constructed an ordinary differential equation model for intraguild predation composed of aphids, parasitic wasps and Coccinellids, discussed the effects of parasitic rate and predation intensity on the stability of the system, and compared the dynamic behavior with the traditional predator-prey system in which prey is infected with diseases.

Although a large number of experimental results show that there are complex interactions among aphids, parasitic wasps and ladybugs, such as host parasitism, intraguild predation and other phenomena, there are few mathematical models to describe such phenomena, and most of them are carried out by establishing ordinary differential equations or delay differential equations. However, the human release of ladybugs is intermittent and instantaneous. In order to accurately describe the ladybug release operation, it has more advantages to constructing impulsive delay differential equation models. Therefore, in this paper we will construct an impulsive differential equation system to describe the impulsive release behavior of ladybugs, use strict mathematical theoretical analysis and numerical simulations to explore the impact of this measure on the population development of aphids, and then evaluate the effect of intraguild predation on aphid control.

In addition, the timeliness of aphid control is also a matter of concern. In the critical period of crop growth, the large-scale outbreak of aphids will seriously affect grain yield. Therefore, it is necessary to reduce the aphid density to an acceptable level within a given limited time. However, traditional aphid control research focuses more on long-term control, and there are still some theoretical and technical difficulties when dealing with limited time control problems ([14–18]). Since natural enemies can be the deciding factor in preventing pest levels of aphids developing on a particular cropped area within a growing season [1], and large-scale feeding and release of natural enemies involve high expenditure of labor and economic costs, it is of great significance to build a mathematical model based on the interaction relationship among aphids, parasitic wasps and natural enemies, and study the optimal control strategies of aphids in limited time with impulsive releases of natural enemies.

In this study, we propose host-parasitoid-predator models for the interactions among aphids, parasitic wasps and aphidophagous Coccinellids incorporating impulsive releases of Coccinellids, and then study the long-term control and limited time optimal control of aphids by adjusting release amount and release timing of Coccinellids.

The paper is structured as follows: In Section 2, a host-parasitoid-predator model for the long-term control of aphids with periodic impulsive releases of Coccinellids is constructed and its dynamical properties are studied. In Section 3, a limited-time optimal control problem about aphids is proposed. Several optimal pulse control schemes are studied, and the gradients of cost function with respect to release timing and release amount of Coccinellids are determined. Then in Section 4, the optimal release timing and release amount are gained by numerical simulations. Finally, a brief conclusion is given in Section 5.

2. Long-term control of aphids

2.1. Model formulation

According to the interaction among aphids, parasitic wasps and aphidophagous Coccinellids, we proposed the following ODE differential model incorporating intraguild predation

$$\begin{cases} \frac{dH(t)}{dt} = rH(t)\left(1 - \frac{H(t)}{K}\right) - \frac{c_1H(t)N(t)}{1+h_1H(t)} - \frac{c_2H(t)P(t)}{1+h_2H(t)}, \\ \frac{dN(t)}{dt} = \frac{e_1c_1H(t)N(t)}{1+h_1H(t)} - \frac{c_3N(t)P(t)}{1+h_3N(t)} - m_1N(t), \\ \frac{dP(t)}{dt} = \frac{e_2c_2H(t)P(t)}{1+h_2H(t)} + \frac{e_3c_3N(t)P(t)}{1+h_3N(t)} - m_2P(t), \end{cases} \quad (2.1)$$

where $K > H(0) = H_0 \geq 0$, $N(0) = N_0 \geq 0$ and $P(0) = P_0 \geq 0$. $H(t)$, $N(t)$ and $P(t)$ are the densities of aphids, parasitic wasps and aphidophagous Coccinellids at t , respectively. It is assumed that the development of aphid population conforms to the law of logistic growth, and r and K are the intrinsic growth rate and the maximum environmental capacity, respectively. The behaviors of parasitism and predation are described by Holling type II functional response function $\frac{c_i x}{1+h_i x}$, $i = 1, 2, 3$. e_i is the conversion rate, and m_1 and m_2 are the natural mortality of parasitic wasps and Coccinellids, respectively. All system parameters in (2.1) are positive constants.

For the long-term control of aphids, we consider periodic releases of Coccinellids and propose an

impulsive differential equation model as follows

$$\left\{ \begin{array}{l} \frac{dH(t)}{dt} = rH(t)\left(1 - \frac{H(t)}{K}\right) - \frac{c_1H(t)N(t)}{1+h_1H(t)} - \frac{c_2H(t)P(t)}{1+h_2H(t)}, \\ \frac{dN(t)}{dt} = \frac{e_1c_1H(t)N(t)}{1+h_1H(t)} - \frac{c_3N(t)P(t)}{1+h_3N(t)} - m_1N(t), \\ \frac{dP(t)}{dt} = \frac{e_2c_2H(t)P(t)}{1+h_2H(t)} + \frac{e_3c_3N(t)P(t)}{1+h_3N(t)} - m_2P(t), \\ H(t^+) = H(t), \\ N(t^+) = N(t), \\ P(t^+) = P(t) + \sigma, \end{array} \right. \quad \left. \begin{array}{l} t \neq k\tau, k = 1, 2, \dots, \\ t = k\tau, \end{array} \right. \quad (2.2)$$

where τ is the release period while σ is the amount released each time.

For the long-term control of aphids, we will focus on how to eliminate aphids by adjusting the release strategy of Coccinellids. To this end, we will first analysis the dynamical behaviors of the system (2.2).

2.2. Dynamics analysis

Consider an arbitrary solution $X(t) = (H(t), N(t), P(t))^T$ of system (2.2), we can know easily that it is continuous between any two adjacent pulse moments and the right limit of it at the pulse point $X(k\tau^+) = \lim_{\varepsilon \rightarrow 0^+} X(k\tau + \varepsilon)$ exists. According to the smooth properties of $f = (f_1, f_2, f_3)^T$, which is the function at the right hand of the first three equations in (2.2), the existence and uniqueness of solutions of (2.2) under given initial values are guaranteed ([19, 20]).

Firstly, the positivity and boundedness of the solution of system (2.2) are investigated. Clearly, $dH(t)/dt|_{H=0} = 0$, $dN(t)/dt|_{N=0} = 0$ and $dP(t)/dt|_{P=0} = 0$. Therefore we have

Proposition 1. *Suppose that $X(t) = (H(t), N(t), P(t))^T$ is a solution of the system (2.2) with $X(0) \geq 0$, then $X(t) \geq 0$ for all $t \geq 0$.*

Let $V_0 = \{V : R_+ \times R_+^3 \rightarrow R_+, \text{ continuous on } (k\tau, (k+1)\tau) \times R_+^3, \text{ and } \lim_{(t,Y) \rightarrow (k\tau,X), t > k\tau} V(t, Y) = V(k\tau^+, X) \text{ exists}\}$.

Definition 1. *Let $V \in V_0$, then for $V(t, X) \in (k\tau, (k+1)\tau) \times R_+^3$, the upper right derivative of $V(t, X)$ with respect to the impulsive differential system (2.2) is defined as*

$$D^+V(t, X) = \limsup_{h \rightarrow 0} \frac{1}{h} [V(t+h, X+hf(t, X)) - V(t, X)].$$

Proposition 2. *If $e_2 > e_1e_3$, then there exists a constant $L > 0$ satisfying $H(t) \leq L, N(t) \leq L$ and $P(t) \leq L$ with all t large enough.*

Proof. Suppose $X(t) = (H(t), N(t), P(t))^T$ is a solution of system (2.2) with non-negative initial conditions. Let $V = e_2H(t) + e_3N(t) + P(t)$, then $V \in V_0$ and we have

$$\left\{ \begin{array}{l} D^+V(t) = e_2rH(t) - \frac{e_2rH^2(t)}{K} - \frac{(e_2 - e_1e_3)c_1H(t)N(t)}{1+h_1H(t)} \\ \quad \quad \quad - e_3m_1N(t) - m_2P(t), \quad t \neq k\tau, \\ V(t^+) = V(t) + \sigma, \quad t = k\tau. \end{array} \right. \quad (2.3)$$

Since $e_2 > e_1 e_3$, for any $\eta > 0$ and $t \neq k\tau$, there is

$$D^+V(t) + \eta V(t) \leq e_2(\eta + r)H(t) - \frac{e_2 r H^2(t)}{K} - e_3(m_1 - \eta)N(t) - (m_2 - \eta)P(t).$$

Select a $\eta = \eta_0 \in (0, \min\{m_1, m_2\})$, then there must exist $K_1 > 0$ such that

$$(\eta_0 + e_2 r)H(t) - \frac{e_2 r H^2(t)}{K} - (e_3 m_1 - \eta_0)N(t) - (m_2 - \eta_0)P(t) \leq K_1,$$

and we derive that

$$\begin{cases} D^+V(t) \leq -\eta_0 V(t) + K_1, \\ V(t^+) = V(t) + \sigma, \quad t = k\tau, \\ V(0^+) = V(0). \end{cases} \quad (2.4)$$

Now we consider the impulsive differential equation

$$\begin{cases} \frac{dx(t)}{dt} = K_1 - \eta_0 x(t), \quad t \neq k\tau, \\ x(t^+) = x(t) + \sigma, \quad t = k\tau, \\ x(0^+) = V(0). \end{cases} \quad (2.5)$$

The system (2.5) is a linear impulsive differential equation, According to the basic properties of linear impulsive differential equation, we give some results of system (2.5) without proof. Interested readers can refer to [21, 22]. The system (2.5) has a globally asymptotically stable positive periodic solution

$$\begin{cases} \tilde{x}(t) = \frac{K_1}{\eta_0} + \frac{\sigma \exp(-\eta_0(t - k\tau))}{1 - \exp(-\eta_0\tau)}, \quad t \in (k\tau, (k+1)\tau], \\ \tilde{x}(0^+) = \frac{K_1}{\eta_0} + \frac{\sigma}{1 - \exp(-\eta_0\tau)}, \end{cases} \quad (2.6)$$

and the solution of the system (2.5) has the form

$$x(t) = (x(0^+) - \tilde{x}(0^+)) \exp(-\eta_0 t) + \tilde{x}(t),$$

which satisfies $\lim_{t \rightarrow \infty} x(t) = \tilde{x}(t)$.

By comparison theorem, we have that

$$V(t) \leq x(t) \leq \frac{K_1}{\eta_0} + \frac{\sigma}{1 - \exp(-\eta_0\tau)} + |x(0^+) - \tilde{x}(0^+)|, \quad \text{for } t \geq 0,$$

then $V(t)$ is ultimately bounded by a constant and there exists a positive number L such that $H(t) \leq L$, $N(t) \leq L$ and $P(t) \leq L$ with all t large enough. This completes the proof. \square

According to (2.2), if aphids are eventually eliminated in the system, the population of parasitic wasps will also go extinct. When both species disappear and only Coccinellids survive in the system, that is, $H(t) = 0$, $N(t) = 0$, then system (2.2) is changed as follows

$$\begin{cases} \frac{d(P)}{dt} = -m_2 P(t), \quad t \neq k\tau, k = 1, 2, \dots, \\ P(t^+) = P(t) + \sigma, \quad t = k\tau. \end{cases} \quad (2.7)$$

By system (2.7), similar to the discussion of system (2.5), we have that

Theorem 1. System (2.2) has a unique aphid-eradication periodic solution $(0, 0, \tilde{P}(t))$, where

$$\tilde{P}(t) = \frac{\sigma \exp(-m_2(t - k\tau))}{1 - \exp(-m_2\tau)}, \quad t \in (k\tau, (k+1)\tau], k = 0, 1, \dots$$

and $\tilde{P}(0^+) = \sigma / (1 - \exp(-m_2\tau))$.

Now we study the stability of the τ -period solution $(0, 0, \tilde{P}(t))$.

Theorem 2. If $r < \frac{c_2\sigma}{\exp(m_2\tau) - 1}$, then the aphid-eradication periodic solution $(0, 0, \tilde{P}(t))$ of system (2.2) is locally asymptotically stable.

Proof. To discuss the local stability of the aphid-eradication periodic solution $(0, 0, \tilde{P}(t))$, we use the classical small perturbation method. Let

$$H(t) = \omega_1(t), \quad N(t) = \omega_2(t), \quad P(t) = \tilde{P}(t) + \omega_3(t).$$

By linearization, (2.2) is transformed into

$$\begin{cases} \frac{d\omega_1(t)}{dt} = (r - c_2\tilde{P}(t))\omega_1(t), \\ \frac{d\omega_2(t)}{dt} = -(c_3\tilde{P}(t) + m_1)\omega_2(t), \\ \frac{d\omega_3(t)}{dt} = e_2c_2\tilde{P}(t)\omega_1(t) + e_3c_3\tilde{P}(t)\omega_2(t) - m_2\omega_3(t). \end{cases}$$

Rewrite it in matrix form, then we have

$$(\omega_1(t), \omega_2(t), \omega_3(t))^T = \Phi(t)(\omega_1(0), \omega_2(0), \omega_3(0))^T,$$

and $\Phi(t)$ is the solution of

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} r - c_2\tilde{P}(t) & 0 & 0 \\ 0 & -(c_3\tilde{P}(t) + m_1) & 0 \\ e_2c_2\tilde{P}(t) & e_3c_3\tilde{P}(t) & -m_2 \end{pmatrix} \Phi(t),$$

where $\Phi(0) = I$. The fourth to sixth equations of system (2.2) are transformed into

$$(\omega_1(k\tau^+), \omega_2(k\tau^+), \omega_3(k\tau^+))^T = (\omega_1(k\tau), \omega_2(k\tau), \omega_3(k\tau))^T.$$

Consider the matrix $M = \Phi(\tau)$ and we investigate its three eigenvalues. Obviously,

$$\Phi(\tau) = \begin{pmatrix} \exp\left(\int_0^\tau (r - c_2\tilde{P}(t)) dt\right) & 0 & 0 \\ 0 & \exp\left(-\int_0^\tau (c_3\tilde{P}(t) + m_1) dt\right) & 0 \\ \cdot & \cdot & \exp(-m_2\tau) \end{pmatrix}.$$

(\cdot) represents the term which has no need to be calculated for the exact form. For every $k = 0, 1, \dots$ and $t \in (k\tau, (k+1)\tau]$, there is

$$\frac{\sigma}{\exp(m_2\tau) - 1} = \tilde{P}(\tau) \leq \tilde{P}(t) = \frac{\sigma \exp(-m_2(t - k\tau))}{1 - \exp(-m_2\tau)} \leq \tilde{P}(0^+) = \frac{\sigma \exp(m_2\tau)}{\exp(m_2\tau) - 1}.$$

From the expression of $\Phi(\tau)$, we know that if $r < \frac{c_2\sigma}{\exp(m_2\tau) - 1}$, then we get $r < c_2\tilde{P}(\tau) \leq c_2\tilde{P}(t)$ and three eigenvalues of $\Phi(\tau)$ are positive and less than one. According to the Floquet theory, the aphid-eradication periodic solution $(0, 0, \tilde{P}(t))$ of system (2.2) is locally asymptotically stable. The proof is completed. \square

Remark 1. *The suppressing intensity of aphid population is mainly reflected in the release of Coccinellids, which is determined by the amount of release σ and the length of release interval τ . In Theorem 2, $\frac{c_2\sigma}{\exp(m_2\tau) - 1}$ is used as a critical value to measure whether the release intensity of Coccinellids is large enough. If the intrinsic growth rate of aphids r is less than this critical value, then the aphid-eradication periodic solution $(0, 0, \tilde{P}(t))$ is locally stable.*

In the following, we further prove that $(0, 0, \tilde{P}(t))$ is globally attractive.

Theorem 3. *Assume $r < \frac{c_2\sigma}{\exp(m_2\tau) - 1}$, then the aphid-eradication periodic solution $(0, 0, \tilde{P}(t))$ of system (2.2) is globally asymptotically stable if one of the following two conditions holds.*

- (i) $h_2K \leq 1$;
- (ii) $h_2K > 1$ and $r(1 + h_2K)^2 < \frac{4h_2Kc_2\sigma}{\exp(m_2\tau) - 1}$.

Proof. From the third and sixth equation of (2.2), we have

$$\begin{cases} \frac{dP(t)}{dt} \geq -m_2P(t), & t \neq k\tau, \\ P(t^+) = P(t) + \sigma, & t = k\tau, \\ P(0) = P_0 > 0. \end{cases} \quad (2.8)$$

Consider the following corresponding comparison system

$$\begin{cases} \frac{dx_1(t)}{dt} = -m_2x_1(t), & t \neq k\tau, \\ x_1(t^+) = x_1(t) + \sigma, & t = k\tau, \\ x_1(0) = P_0 > 0, \end{cases} \quad (2.9)$$

and from the relevant conclusions of systems (2.5) and (2.7), we can obtain that

$$\lim_{t \rightarrow \infty} x_1(t) = \tilde{P}(t) = \frac{\sigma \exp(-m_2(t - k\tau))}{1 - \exp(-m_2\tau)}, \quad t \in (k\tau, (k+1)\tau], k = 0, 1, \dots$$

Then for $\varepsilon > 0$ small enough, there exists $t_1 > 0$ such that $x_1(t) > \tilde{P}(t) - \varepsilon$, $t > t_1$. Thus, it follows from the comparison theorem that

$$P(t) \geq x_1(t) > \tilde{P}(t) - \varepsilon, \quad t > t_1.$$

Since $\frac{\sigma \exp(-m_2\tau)}{1 - \exp(-m_2\tau)} = \tilde{P}(\tau) \leq \tilde{P}(t) = \frac{\sigma \exp(-m_2(t - k\tau))}{1 - \exp(-m_2\tau)}$, $t \in (k\tau, (k+1)\tau]$, according to the first equation of the system (2.2), we obtain that for $t > t_1$,

$$\begin{aligned} \frac{dH(t)}{dt} &\leq rH(t)\left(1 - \frac{H(t)}{K}\right) - \frac{c_2(\tilde{P}(\tau) - \varepsilon)}{1 + h_2H(t)}H(t) \\ &= \frac{H(t)[(r - c_2(\tilde{P}(\tau) - \varepsilon)) + r(h_2 - 1 \setminus K)H(t) - rh_2H^2(t) \setminus K]}{1 + h_2H(t)}. \end{aligned} \quad (2.10)$$

If $r < \frac{c_2\sigma}{\exp(m_2\tau) - 1}$, $\varepsilon > 0$ is small enough and (i) or (ii) holds, then we get $r < c_2(\tilde{P}(\tau) - \varepsilon)$ and $\frac{dH(t)}{dt} < 0$, thus there must exist $t_2 > t_1$ such that $0 \leq H(t) < \varepsilon$, $t > t_2$.

Combined with the conclusions we have obtained about $H(t)$ and $N(t)$ and the second equation of system (2.2), we derive that for $t > t_2$, there is

$$\frac{dN(t)}{dt} \leq \left(\frac{e_1c_1\varepsilon}{1+h_1\varepsilon} - m_2 \right) N(t).$$

Because $\varepsilon > 0$ is small enough, so $\frac{e_1c_1\varepsilon}{1+h_1\varepsilon} - m_2 < 0$ can be guaranteed. There must be $t_3 > t_2$ such that $0 \leq N(t) < \varepsilon$, $t > t_3$.

Finally, we consider the variable $P(t)$ again for $t > t_3$. From the third and sixth equations of system (2.2), we derive

$$\begin{cases} \frac{dP(t)}{dt} \leq \left(\frac{e_2c_2\varepsilon}{1+h_2\varepsilon} + \frac{e_3c_3\varepsilon}{1+h_3\varepsilon} - m_2 \right) P(t), & t \neq k\tau, \\ P(t^+) = P(t) + \sigma, & t = k\tau, \\ P(0) = P_0 > 0. \end{cases} \quad (2.11)$$

Similarly, we construct a comparison system

$$\begin{cases} \frac{dx_2(t)}{dt} = \left(\frac{e_2c_2\varepsilon}{1+h_2\varepsilon} + \frac{e_3c_3\varepsilon}{1+h_3\varepsilon} - m_2 \right) x_2(t), & t \neq k\tau, \\ x_2(t^+) = x_2(t) + \sigma, & t = k\tau, \\ x_2(0) = P_0 > 0, \end{cases} \quad (2.12)$$

and by the conclusions of systems (2.5), we can obtain that

$$\lim_{t \rightarrow \infty} x_2(t) = \tilde{x}_2(t) = \frac{\sigma \exp\left(\left(\frac{e_2c_2\varepsilon}{1+h_2\varepsilon} + \frac{e_3c_3\varepsilon}{1+h_3\varepsilon} - m_2\right)(t - k\tau)\right)}{1 - \exp\left(\left(\frac{e_2c_2\varepsilon}{1+h_2\varepsilon} + \frac{e_3c_3\varepsilon}{1+h_3\varepsilon} - m_2\right)\tau\right)}, \quad t \in (k\tau, (k+1)\tau].$$

Then for $\varepsilon > 0$ small enough, there exists $t_4 > t_3$ such that $x_2(t) < \tilde{x}_2(t) + \varepsilon$, $t > t_4$. Thus, it follows from the comparison theorem that

$$P(t) \leq x_2(t) < \tilde{x}_2(t) + \varepsilon, \quad t > t_4.$$

Letting $\varepsilon \rightarrow 0$ then yields $H(t) \rightarrow 0$, $N(t) \rightarrow 0$ and $P(t) \rightarrow \tilde{P}(t)$ as $t \rightarrow \infty$, which means that $(0, 0, \tilde{P}(t))$ is a global attractor. The proof is completed. \square

Remark 2. To ensure the global stability of the periodic solution of aphid-extinction, only the condition about the intrinsic growth rate of aphids and release intensity of Coccinellids in Theorem 2 is far from enough, and the environmental capacity of aphids should also be considered. From Theorem 3, we can see that if the environmental capacity is small enough (condition (i) in Theorem 3), or the environmental capacity is large but the release strength of ladybug increases to a certain extent (condition (ii) in Theorem 3), then the aphid-eradication periodic solution $(0, 0, \tilde{P}(t))$ is globally stable.

2.3. Long-term control strategies

In this subsection, we study control strategies by exploring the impacts of the release amount σ and the release period τ of Coccinellids on the population development trend of aphids.

To eliminate aphids in the long term, we can adjust the release parameters σ and τ such that the conditions in Theorems 2 and 3 are satisfied. To this end, we consider the following function of σ and τ :

$$\mathcal{M}(\sigma, \tau) = r - \frac{c_2\sigma}{\exp(m_2\tau) - 1}. \quad (2.13)$$

For any fixed release period τ^* , $\mathcal{M}(\sigma, \tau^*)$ is monotonically decreasing with respect to σ . Obviously, $\mathcal{M}(0, \tau^*) = r$ and $\mathcal{M}(+\infty, \tau^*) = -\infty$. So there must be a threshold of release amount $\sigma_{crit}(\tau^*) > 0$ such that $\mathcal{M}(\sigma_{crit}(\tau^*), \tau^*) = 0$. Therefore the aphid-eradication periodic solution $(0, 0, \tilde{P}(t))$ of (2.2) is asymptotically stable provided $\sigma > \sigma_{crit}(\tau^*)$ according to Theorem 2.

Similarly, for any fixed release amount σ^* , $\mathcal{M}(\sigma^*, \tau)$ is monotonically increasing with respect to τ . It can be obtained by simple calculation that $\mathcal{M}(\sigma^*, 0) = -\infty$ and $\mathcal{M}(\sigma^*, +\infty) = r$. Similarly, there must be a threshold of release period $\tau_{crit}(\sigma^*) > 0$ such that $\mathcal{M}(\sigma^*, \tau_{crit}(\sigma^*)) = 0$. Therefore the aphid-eradication periodic solution $(0, 0, \tilde{P}(t))$ of (2.2) is asymptotically stable provided $\tau < \tau_{crit}(\sigma^*)$ according to Theorem 2.

Now we provide examples to demonstrate these results. Choose parameters as follows

$$\begin{aligned} r = 2.5, \quad K = 100, \quad h_1 = 0.03, \quad c_1 = 0.03, \\ h_2 = 0.03, \quad c_2 = 0.03, \quad h_3 = 0.03, \quad c_3 = 0.03, \quad e_1 = 3, \\ e_2 = 0.8, \quad e_3 = 0.2, \quad m_1 = 0.1, \quad m_2 = 0.2. \end{aligned} \quad (2.14)$$

Firstly, we fix the release period $\tau^* = 4$. Through simple calculation, we obtain that the threshold of release amount $\sigma_{crit}(4) \approx 102.1284$. Consider the release amount $\sigma = 280$, through direct calculation we get $r = 2.5 < \frac{c_2\sigma}{\exp(m_2\tau)-1} \approx 6.8543$, $h_2K = 3 > 1$ and $r(1 + h_2K)^2 = 40 < \frac{4h_2Kc_2\sigma}{\exp(m_2\tau)-1} \approx 82.2516$, which verify conditions listed in Theorem 3, and the global stability of the aphid-eradication periodic solution $(0, 0, \tilde{P}(t))$ of system (2.2) is guaranteed. From Figure 1, we can see that solutions from different initial values all tend to the aphid-eradication periodic solution $(0, 0, \tilde{P}(t))$. We reduce the release amount to $\sigma = 100 < \sigma_{crit}(4)$, then conditions in Theorems 2 and 3 are not satisfied, and we can see that there exists positive periodic solutions for the coexistence of these species (see Figure 2).

We next keep the same parameters but fix $\sigma^* = 300$, then we can obtain the threshold of release period $\tau_{crit}(5) \approx 7.6303$. If we extend the release period to $\tau = 8 > \tau_{crit}(5)$, we can see that there also exists positive periodic solutions for the coexistence of these species (see Figure 3). While if we shorten the release period to $\tau = 4.5 < \tau_{crit}(5)$, through direct calculation we get $r = 2.5 < \frac{c_2\sigma}{\exp(m_2\tau)-1} \approx 6.1661$, $h_2K = 3 > 1$ and $r(1 + h_2K)^2 = 40 < \frac{4h_2Kc_2\sigma}{\exp(m_2\tau)-1} \approx 73.9929$, then conditions listed in Theorem 3 hold and the aphid-eradication periodic solution is globally stable (see Figure 4).

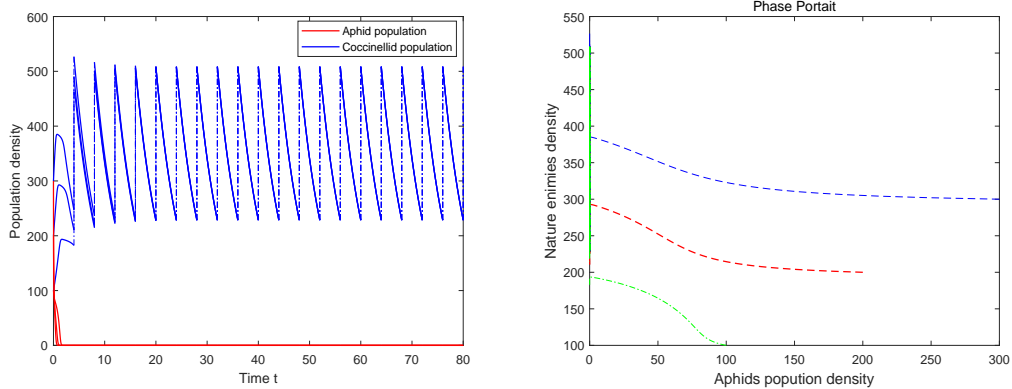


Figure 1. The global stability of the aphid-eradication periodic solution $(0, 0, \tilde{P}(t))$ of (2.2) with $\sigma = 280$ and $\tau = 4$.

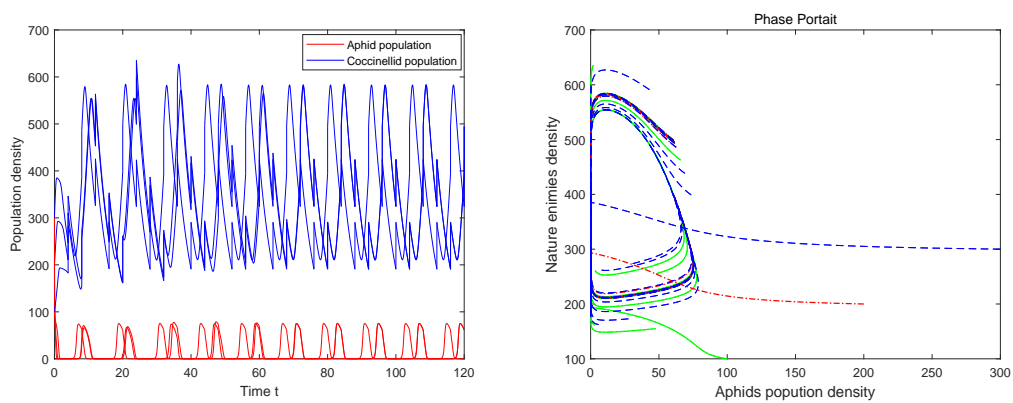


Figure 2. Population coexistence of system (2.2) with $\sigma = 100$ and $\tau = 4$.

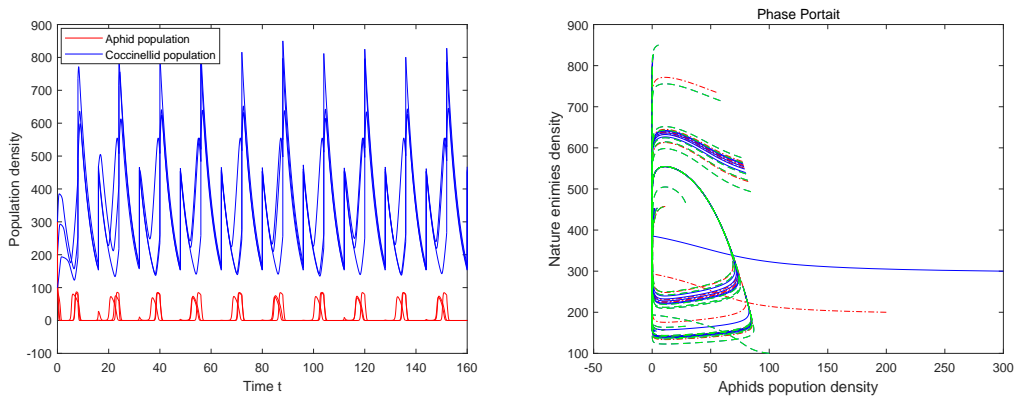


Figure 3. Population coexistence of system (2.2) with $\sigma = 300$ and $\tau = 8$.

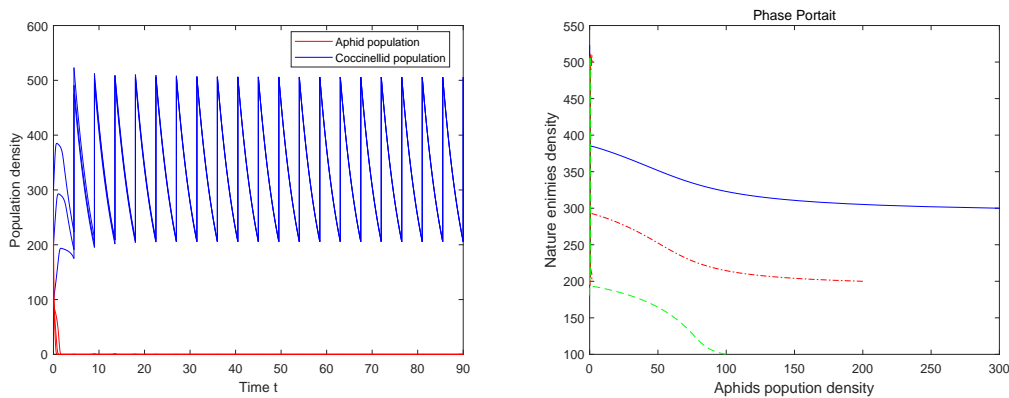


Figure 4. The global stability of the aphid-eradication periodic solution $(0, 0, \tilde{P}(t))$ of (2.2) with $\sigma = 300$ and $\tau = 4.5$.

3. Limited-time optimal control problems

In recent years, the use of natural enemies to protect crops from aphids has been widely promoted, especially in the critical period of crop growth season. However, there are still some obstacles to the biological control of agricultural pests by natural enemies, such as the large-scale artificial feeding of natural enemies and the difficulty of field release technology. Therefore, when releasing natural enemies (such as Coccinellids), it is inevitable to consider the cost. In addition, from the perspective of ecosystem stability, it is not necessary to completely eliminate pests, but only to control their quantity within a reasonable range. Therefore in this section, we consider optimal control problems for aphids in a limited time (for example, the critical period of crop growth season). With the optimization method, on the one hand, the number of aphid population is restrained, on the other hand, the control cost is reduced as much as possible. The release moments and release amount of Coccinellids are the main control parameters.

3.1. Mixed control

Suppose we need to control the number of aphids in the field within a limited time of T . We plan to achieve this goal through $q - 1$ releases of natural enemies within this limited time. Suppose that an amount σ_k of Coccinellids is released into the field at moments $t_k \in [0, T]$, $k = 1, 2, \dots, q - 1$, where $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_{q-1} \leq t_q = T$, then we propose the following limited-time control system

$$\left\{ \begin{array}{l} \frac{dH(t)}{dt} = rH(t)\left(1 - \frac{H(t)}{K}\right) - \frac{c_1H(t)N(t)}{1 + h_1H(t)} - \frac{c_2H(t)P(t)}{1 + h_2H(t)}, \\ \frac{dN(t)}{dt} = \frac{e_1c_1H(t)N(t)}{1 + h_1H(t)} - \frac{c_3N(t)P(t)}{1 + h_3N(t)} - m_1N(t), \\ \frac{dP(t)}{dt} = \frac{e_2c_2H(t)P(t)}{1 + h_2H(t)} + \frac{e_3c_3N(t)P(t)}{1 + h_3N(t)} - m_2P(t), \\ H(t^+) = H(t), N(t^+) = N(t), \\ P(t^+) = P(t) + \sigma_k, \end{array} \right. \quad \left. \begin{array}{l} t \neq t_k, t \in [0, T], \\ t = t_k, k = 1, 2, \dots, q - 1 \end{array} \right\} \quad (3.1)$$

with initial conditions

$$H(0) = H_0, N(0) = N_0, P(0) = P_0. \quad (3.2)$$

Let $\tau_k = t_k - t_{k-1}$ and

$$0 \leq \tau_k^L \leq \tau_k \leq \tau_k^U, k = 1, 2, \dots, q, \quad (3.3)$$

where τ_k^L and τ_k^U are given values used to constrain the interval length between the $(k-1)$ th and k th releases. Besides, we also give constrain condition for the amount of the k th release

$$0 \leq \sigma_k^L \leq \sigma_k \leq \sigma_k^U, k = 1, 2, \dots, q-1, \quad (3.4)$$

where σ_k^L and σ_k^U are predefined constants.

Let $\Theta = (\tau_1, \tau_2, \dots, \tau_q)^T$ and $\Upsilon = (\sigma_1, \sigma_2, \dots, \sigma_{q-1})^T$, where τ_k and σ_k satisfy (3.3) and (3.4) respectively, and denote the sets of all $\Theta \in R^q, \Upsilon \in R^{q-1}$ satisfying (3.3) and (3.4) by Ω_1 and Ω_2 , respectively.

Since the functions on the right hand of the first three equations of system (3.1) are differentiable, for each pair $(\Theta, \Upsilon) \in (\Omega_1, \Omega_2)$, the initial value problem of (3.1) and (3.2) has a unique solution ([19,20]).

Considering the periodic change of aphid population, simple terminal control is not enough to reflect the control effect. So in this paper, we consider the process control of aphid and define a cost function as

$$J(\Theta, \Upsilon) = \int_0^T H(t)dt + r_0 \sum_{k=1}^{q-1} \sigma_k, \quad (3.5)$$

where r_0 is a balance factor between the feeding and release cost of Coccinellids and the aphid population level. Then we can describe the optimal control problem of aphids as follows:

(P1) For the host-parasitoid-predator system (3.1) with initial condition (3.2), determine a parameter vector pair $(\Theta, \Upsilon) \in (\Omega_1, \Omega_2)$ such that the cost function $J(\Theta, \Upsilon)$ is minimized.

There is a technical difficulty for this kind of optimal impulsive control problem, which is mainly caused by uncertain pulse effects. In ([14–18]), the authors introduced a time rescaling method, and changed these uncertain time points into fixed ones. In the following, we will use the same method and transform (P1) into an equivalent optimal parameter selection problem, then solve the problem by using gradient-based optimization techniques.

Let $t = \sum_{i=1}^{k-1} \tau_i + \tau_k s$ for $t \in (\sum_{i=1}^{k-1} \tau_i, \sum_{i=1}^k \tau_i]$, and let

$$H_k(s) = H\left(\sum_{i=1}^{k-1} \tau_i + \tau_k s\right), N_k(s) = N\left(\sum_{i=1}^{k-1} \tau_i + \tau_k s\right), P_k(s) = P\left(\sum_{i=1}^{k-1} \tau_i + \tau_k s\right). \quad (3.6)$$

Then system (3.1) with initial condition (3.2) is transformed into q subsystems

$$\left\{ \begin{array}{l} \frac{dH_k(s)}{ds} = F_1^k(s) = \tau_k \left[rH_k(s) \left(1 - \frac{H_k(s)}{K} \right) - \frac{c_1 H_k(s) N_k(s)}{1 + h_1 H_k(s)} - \frac{c_2 H_k(s) P_k(s)}{1 + h_2 H_k(s)} \right], \\ \frac{dN_k(s)}{ds} = F_2^k(s) = \tau_k \left[\frac{e_1 c_1 H_k(s) N_k(s)}{1 + h_1 H_k(s)} - \frac{c_3 N_k(s) P_k(s)}{1 + h_3 N_k(s)} - m_1 N_k(s) \right], \\ \frac{dP_k(s)}{ds} = F_3^k(s) = \tau_k \left[\frac{e_2 c_2 H_k(s) P_k(s)}{1 + h_2 H_k(s)} + \frac{e_3 c_3 N_k(s) P_k(s)}{1 + h_3 N_k(s)} - m_2 P_k(s) \right], \\ H_k(0) = H_{k-1}(1), N_k(0) = A_{k-1}(1), P_k(0) = P_{k-1}(1) + \sigma_k, \end{array} \right\} s \in (0, 1], k = 1, 2, \dots, q, \quad (3.7)$$

with

$$H_1(0) = H(0) = H_0, N_1(0) = N(0) = N_0, P_1(0) = P(0) = P_0. \quad (3.8)$$

According to (3.7) and (3.8), the cost function (3.5) can be changed as

$$J_1(\Theta, \Upsilon) = \int_0^1 \sum_{k=1}^q H_k(s) ds + r_0 \sum_{k=1}^{q-1} \sigma_k. \quad (3.9)$$

Then the optimal control problem (P1) is converted into

(P2) For the host-parasitoid-predator system (3.7) with initial condition (3.8), find a parameter vector pair $(\Theta, \Upsilon) \in (\Omega_1, \Omega_2)$ such that the cost function (3.9) is minimized.

To solve problem, we define the following Hamiltonian functions $\mathcal{H}_k, k = 1, 2, \dots, q$ like [23],

$$\mathcal{H}_k(H_k(s), N_k(s), P_k(s), \Theta, \Upsilon) = H_k(s) + (\lambda_1^k(s), \lambda_2^k(s), \lambda_3^k(s))(F_1^k(s), F_2^k(s), F_3^k(s))^T, \quad (3.10)$$

where $\lambda^k(s) = (\lambda_1^k(s), \lambda_2^k(s), \lambda_3^k(s))$ is the corresponding costate which is governed by

$$\begin{cases} \dot{\lambda}_1^k(s) = -\frac{\partial \mathcal{H}}{\partial H_k} = -1 - \tau_k [\lambda_1^k(s)r(1 - \frac{2H_k(s)}{K}) - (\lambda_1^k(s) - \lambda_2^k(s)e_1) \\ \quad \times \frac{c_1 N_k(s)}{(1 + h_1 H_k(s))^2} - (\lambda_1^k(s) - \lambda_3^k(s)e_2) \frac{c_2 P_k(s)}{(1 + h_2 H_k(s))^2}], \\ \dot{\lambda}_2^k(s) = -\frac{\partial \mathcal{H}}{\partial N_k} = -\tau_k [(\lambda_2^k(s)e_1 - \lambda_1^k(s)) \frac{c_1 H_k(s)}{1 + h_1 H_k(s)} \\ \quad - (\lambda_2^k(s) - \lambda_3^k(s)e_3) \frac{c_3 P_k(s)}{(1 + h_3 N_k(s))^2} - m_1 \lambda_2^k(s)], \\ \dot{\lambda}_3^k(s) = -\frac{\partial \mathcal{H}}{\partial P_k} = -\tau_k [(\lambda_3^k(s)e_2 - \lambda_1^k(s)) \frac{c_2 H_k(s)}{1 + h_2 H_k(s)} \\ \quad + (\lambda_3^k(s)e_3 - \lambda_2^k(s)) \frac{c_3 N_k(s)}{1 + h_3 N_k(s)} - m_2 \lambda_3^k(s)] \end{cases} \quad (3.11)$$

with

$$\begin{cases} \lambda_1^q(1) = 0, \lambda_2^q(1) = 0, \lambda_3^q(1) = 0, \\ \lambda_1^k(1) = \lambda_1^{k+1}(0), \lambda_2^k(1) = \lambda_2^{k+1}(0), \lambda_3^k(1) = \lambda_3^{k+1}(0), \quad k = 1, 2, \dots, q-1. \end{cases} \quad (3.12)$$

Let $x_k(s) = (H_k(s), N_k(s), P_k(s))^T$, and obviously

$$x_k(0) = x_{k-1}(1) + (0, 0, \sigma_{k-1})^T, \quad i = 2, 3, \dots, q.$$

According to Theorems 4.1 and 4.2 in [23], we have

Proposition 3. *The gradients of the cost functional J_1 with respect to Θ and Υ are given by*

$$\frac{\partial J_1}{\partial \Upsilon} = \frac{\partial(r_0 \sum_{k=1}^{q-1} \sigma_k)}{\partial \Upsilon} + \sum_{k=1}^q (\lambda^k(0))^T \left(\frac{\partial x_k(0)}{\partial \Upsilon} \right) + \int_0^1 \sum_{k=1}^q \frac{\partial \mathcal{H}_k(x_k(s), \Upsilon, \Theta, \lambda^k(s))}{\partial \Upsilon}$$

and

$$\frac{\partial J_1}{\partial \Theta} = \frac{\partial(r_0 \sum_{k=1}^{q-1} \tau_k)}{\partial \Theta} + \sum_{k=1}^q (\lambda^k(0))^T \left(\frac{\partial x_k(0)}{\partial \Theta} \right) + \int_0^1 \sum_{k=1}^q \frac{\partial \mathcal{H}_k(x_k(s), \Upsilon, \Theta, \lambda^k(s))}{\partial \Theta},$$

respectively.

Applying these formulas, we obtain the following results.

Theorem 4. *The gradients of the cost function $J_1(\Theta, \Upsilon)$ with respect to the release parameters τ_k and σ_i are determined as follows:*

$$\begin{aligned} \nabla_{\tau_k} J_1(\Theta, \Upsilon) &= \int_0^1 \sum_i^q \frac{\partial \mathcal{H}_i(s, H_i(s), N_i(s), P_i(s), \Theta, \Upsilon)}{\partial \tau_k} ds \\ &= \int_0^1 [\lambda_1^k(s)(rH_k(s)(1 - \frac{H_k(s)}{K}) - \frac{c_1 H_k(s)N_k(s)}{1 + h_1 H_k(s)} - \frac{c_2 H_k(s)P_k(s)}{1 + h_2 H_k(s)}) \\ &\quad + \lambda_2^k(s)(\frac{e_1 c_1 H_k(s)N_k(s)}{1 + h_1 H_k(s)} - \frac{c_3 N_k(s)P_k(s)}{1 + h_3 N_k(s)} - m_1 N_k(s)) \\ &\quad + \lambda_3^k(s)(\frac{e_2 c_2 H_k(s)P_k(s)}{1 + h_2 H_k(s)} + \frac{e_3 c_3 N_k(s)P_k(s)}{1 + h_3 N_k(s)} - m_2 P_k(s))] ds \end{aligned} \quad (3.13)$$

for $k = 1, 2, \dots, q$, and

$$\begin{aligned} \nabla_{\sigma_i} J_1(\Theta, \Upsilon) &= r_0 + \sum_k^{q-1} (\lambda^{k+1}(0))^T \frac{\partial x_{k+1}(0)}{\partial \sigma_i} ds \\ &= r_0 + (\lambda_1^{i+1}(0), \lambda_2^{i+1}(0), \lambda_3^{i+1}(0))(0, 0, 1)^T \\ &= r_0 + \lambda_3^{i+1}(0) \end{aligned} \quad (3.14)$$

for $i = 1, 2, \dots, q - 1$.

3.2. Release amount control

A common case of field operations is considered in this subsection. Suppose that Coccinellids are released periodically in the limited time $[0, T]$ and the release amount is the same each time, which is denoted by σ_{fix} . Similarly, totally $q - 1$ times of releases are planned, so the release period is $\tau = \frac{T}{q}$. Then the system (3.1) is transformed into

$$\left\{ \begin{array}{l} \frac{dH(t)}{dt} = rH(t)(1 - \frac{H(t)}{K}) - \frac{c_1 H(t)N(t)}{1 + h_1 H(t)} - \frac{c_2 H(t)P(t)}{1 + h_2 H(t)}, \\ \frac{dN(t)}{dt} = \frac{e_1 c_1 H(t)N(t)}{1 + h_1 H(t)} - \frac{c_3 N(t)P(t)}{1 + h_3 N(t)} - m_1 N(t), \\ \frac{dP(t)}{dt} = \frac{e_2 c_2 H(t)P(t)}{1 + h_2 H(t)} + \frac{e_3 c_3 N(t)P(t)}{1 + h_3 N(t)} - m_2 P(t), \\ H(t^+) = H(t), N(t^+) = N(t), \\ P(t^+) = P(t) + \sigma_{fix}, \end{array} \right. \quad \left. \begin{array}{l} t \neq k\tau, t \in [0, T], \\ t = k\tau, k = 1, 2, \dots, q - 1 \end{array} \right\} \quad (3.15)$$

with initial conditions (3.2).

We also give constraint condition

$$0 \leq \sigma_L \leq \sigma_{fix} \leq \sigma_U, \quad (3.16)$$

where σ_L and σ_U are given constants.

The cost function of control problem (P1) is converted as follows

$$\bar{J}(\sigma_{fix}) = \int_0^T H(t)dt + r_0(q-1)\sigma_{fix}. \quad (3.17)$$

Obviously, σ_{fix} is the only control parameter, and what we need to do is determining a $\sigma_{fix} \in [\sigma_L, \sigma_U]$ to minimize $\bar{J}(\sigma_{fix})$.

Similarly, let $t = (k-1)\tau + s\tau$ for $k = 1, 2, \dots, q$, and then the initial-value problem (3.15) and (3.2) is equivalent to

$$\left\{ \begin{array}{l} \frac{dH_k(s)}{ds} = F_1^k(s) = \tau_k \left[rH_k(s) \left(1 - \frac{H_k(s)}{K} \right) - \frac{c_1 H_k(s) N_k(s)}{1 + h_1 H_k(s)} - \frac{c_2 H_k(s) P_k(s)}{1 + h_2 H_k(s)} \right], \\ \frac{dN_k(s)}{ds} = F_2^k(s) = \tau_k \left[\frac{e_1 c_1 H_k(s) N_k(s)}{1 + h_1 H_k(s)} - \frac{c_3 N_k(s) P_k(s)}{1 + h_3 N_k(s)} - m_1 N_k(s) \right], \\ \frac{dP_k(s)}{ds} = F_3^k(s) = \tau_k \left[\frac{e_2 c_2 H_k(s) P_k(s)}{1 + h_2 H_k(s)} + \frac{e_3 c_3 N_k(s) P_k(s)}{1 + h_3 N_k(s)} - m_2 P_k(s) \right], \\ H_k(0) = H_{k-1}(1), N_k(0) = A_{k-1}(1), P_k(0) = P_{k-1}(1) + \sigma_{fix}, \end{array} \right\} \quad s \in (0, 1], k = 1, 2, \dots, q, \quad (3.18)$$

with same initial conditions (3.8).

The cost function (3.17) can be expressed as

$$\bar{J}_1(\sigma_{fix}) = \int_0^1 \sum_{k=1}^q H_k(s) ds + r_0(q-1)\sigma_{fix} \quad (3.19)$$

and we only need to find a $\sigma_{fix} \in [\sigma_L, \sigma_U]$ to minimize $\bar{J}_1(\sigma_{fix})$.

Through similar discussion, we get

Theorem 5. *The gradient of $\bar{J}_1(\sigma_{fix})$ with respect to σ_{fix} is*

$$\nabla \bar{J}_1(\sigma_{fix}) = r_0(q-1) + \sum_{k=1}^{q-1} \lambda_3^{k+1}(0). \quad (3.20)$$

3.3. Release timing control

Another common case in field operations is considered in this subsection. Suppose that Coccinellids are released at irregular moments $0 \leq t_1 \leq \dots \leq t_{q-1} \leq T$ with a same release amount σ_{fix} . Then the system (3.1) is changed as

$$\left\{ \begin{array}{l} \frac{dH(t)}{dt} = rH(t) \left(1 - \frac{H(t)}{K} \right) - \frac{c_1 H(t) N(t)}{1 + h_1 H(t)} - \frac{c_2 H(t) P(t)}{1 + h_2 H(t)}, \\ \frac{dN(t)}{dt} = \frac{e_1 c_1 H(t) N(t)}{1 + h_1 H(t)} - \frac{c_3 N(t) P(t)}{1 + h_3 N(t)} - m_1 N(t), \\ \frac{dP(t)}{dt} = \frac{e_2 c_2 H(t) P(t)}{1 + h_2 H(t)} + \frac{e_3 c_3 N(t) P(t)}{1 + h_3 N(t)} - m_2 P(t), \\ H(t^+) = H(t), N(t^+) = N(t), \\ P(t^+) = P(t) + \sigma_{fix}, \end{array} \right\} \quad t \neq t_k, t \in [0, T], \quad (3.21)$$

with initial conditions (3.2), and $t_k, k = 1, 2, \dots, q$ and σ_{fix} satisfy constraint conditions (3.3) and (3.16), respectively.

Obviously, the cost function of control problem (P1) can be expressed as

$$\hat{J}(\Theta, \sigma_{fix}) = \int_0^T H(t)dt + r_0(q-1)\sigma_{fix}, \quad (3.22)$$

where $\Theta = (\tau_1, \tau_2, \dots, \tau_q)^T$, $\tau_k = t_k - t_{k-1}$, and σ_{fix} and $\tau_1, \tau_2, \dots, \tau_q$ are all decision variables.

Let $t = \sum_{i=1}^{k-1} \tau_i + \tau_k s$ for $k = 1, 2, \dots, q$, and the system (3.21) is reduced to

$$\left\{ \begin{array}{l} \frac{dH_k(s)}{ds} = F_1^k(s) = \tau_k \left[rH_k(s) \left(1 - \frac{H_k(s)}{K} \right) - \frac{c_1 H_k(s) N_k(s)}{1 + h_1 H_k(s)} - \frac{c_2 H_k(s) P_k(s)}{1 + h_2 H_k(s)} \right], \\ \frac{dN_k(s)}{ds} = F_2^k(s) = \tau_k \left[\frac{e_1 c_1 H_k(s) N_k(s)}{1 + h_1 H_k(s)} - \frac{c_3 N_k(s) P_k(s)}{1 + h_3 N_k(s)} - m_1 N_k(s) \right], \\ \frac{dP_k(s)}{ds} = F_3^k(s) = \tau_k \left[\frac{e_2 c_2 H_k(s) P_k(s)}{1 + h_2 H_k(s)} + \frac{e_3 c_3 N_k(s) P_k(s)}{1 + h_3 N_k(s)} - m_2 P_k(s) \right], \\ H_k(0) = H_{k-1}(1), N_k(0) = A_{k-1}(1), P_k(0) = P_{k-1}(1) + \sigma_{fix}, \end{array} \right\} s \in (0, 1], k = 1, 2, \dots, q, \quad (3.23)$$

with initial conditions (3.8).

Then cost function (3.22) can be rewritten as

$$\hat{J}_1(\Theta, \sigma_{fix}) = \int_0^1 \sum_{k=1}^q H_k(s) ds + r_0(q-1)\sigma_{fix}. \quad (3.24)$$

Then, we can obtain

Theorem 6. *The gradients of $\hat{J}_1(\Theta, \sigma_{fix})$ with respect to the release timing τ_k and release amount σ_{fix} are*

$$\nabla_{\tau_k} \hat{J}_1(\Theta, \sigma_{fix}) = \nabla_{\tau_k} \hat{J}_1(\Theta, \sigma_{fix}), \quad k = 1, 2, \dots, q \quad (3.25)$$

and

$$\nabla_{\sigma_{fix}} \hat{J}_1(\Theta, \sigma_{fix}) = r_0(q-1) + \sum_k^{q-1} \lambda_3^{k+1}(0). \quad (3.26)$$

Here $\nabla_{\tau_k} J_1(\Theta, \sigma_{fix})$ is defined in (3.13).

4. Numerical simulations for the optimal parameter selection

A series of numerical simulations for the optimal selection of control parameters for systems (3.1), (3.15) and (3.21) are performed in this section.

For the computation of the cost function and its gradients with respect to the control parameters, we adopt the approach given in ([14–17]). Taking (3.1) as an example, we list the following computation steps.

- (1) Using the initial value condition (3.8), the solution of (3.7) is calculated, and $H_k(s), N_k(s), P_k(s), s \in [0, 1], k = 1, 2, \dots, q$ are obtained.
- (2) Applying $H_k(s), N_k(s), P_k(s)$, the solution of (3.11) can be obtained by inverse calculation with boundary conditions (3.12), then $\lambda_1^k(s), \lambda_2^k(s)$ and $\lambda_3^k(s), k = 1, 2, \dots, q$ are gained.
- (3) By (3.9), the cost function $J_1(\Theta, \Upsilon)$ can be calculated.
- (4) Applying $H_k(s), N_k(s), P_k(s), \lambda_1^k(s), \lambda_2^k(s)$ and $\lambda_3^k(s)$, we calculate $\nabla_{\tau_k} J_1(\Theta, \Upsilon)$ for $k = 1, 2, \dots, q$ and $\nabla_{\sigma_i} J_1(\Theta, \Upsilon)$ for $i = 1, 2, \dots, q - 1$.

In our numerical simulations, time is measured in days and we take 30 days as the total control time, that is, $T = 30$. These 30 days will be divided into $q = 5$ parts, and natural enemies will be released $q - 1 = 4$ times in different modes. The balance factor parameter is selected as $r_0 = 0.01$.

In the following, by using parameters given in (2.14), we will study three different optimal strategies in impulsive control by numerical simulations. Surely there is no guarantee that the optimal solution we find numerically is unique and global, and we just give a set of optimal ones under some initial release periods and amounts by the above steps.

Example 1. *Release amount control.*

For a fixed release period $\tau = 6$, starting with a initial release amount $\sigma_{fix} = 450$, we can obtain that the cost value $\bar{J}_0 = 38.4752$.

Under constraint $0 \leq \sigma_{fix} \leq 500$, the optimal problem is solved according to the above algorithm by using Matlab program, and we obtain an optimal release amount $\sigma_{fix}^* = 360.87$ and the corresponding cost value $\bar{J}^* = 36.0234$. In the left figure of Figure 5, the optimal amount control, non-control and simple impulsive control are compared, and we find that the optimal amount control and simple impulsive control have almost the same significant control effect, however, the former needs to release fewer natural enemies.

Besides, the impact of the intensity of each release on the objective function are exploited, see the right figure in Figure 5. With the release amount σ_{fix} varies in the interval $[0, 500]$, we find that the objective function has a minimum value which is consistent with the optimal value we get.

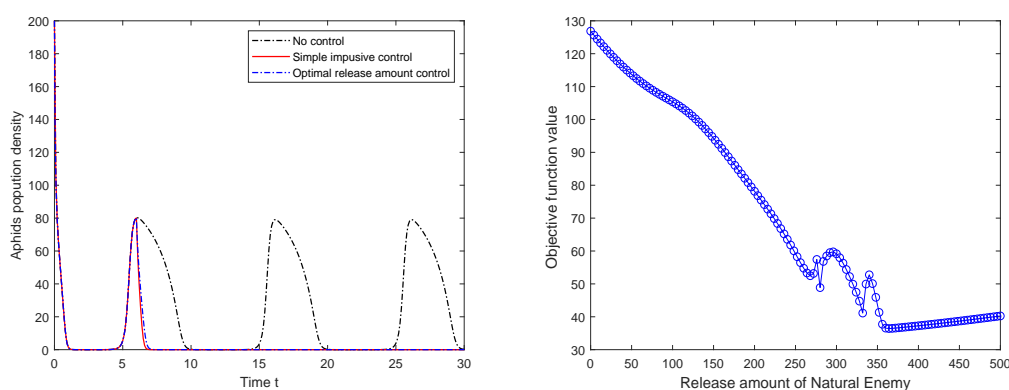


Figure 5. Release amount control: (a) Comparisons of total aphid population under different biological controls; (b) Impact of the intensity of each release on the objective function.

Example 2. Release timing control.

Using the same initial release amount of natural enemies $\sigma_{fix} = 450$ and initial release intervals $\tau_1 = \tau_2 = \dots = \tau_5 = 6$. In order to determine optimal parameter value of τ_k and σ_{fix} by the algorithm given above, we give constraints as follows

$$0 \leq \tau_k \leq 10, k = 1, 2, \dots, 5, \quad \sum_1^5 T_k = 30 \quad (4.1)$$

and $0 \leq \sigma_{fix} \leq 500$.

Then we solve the optimal problem and obtain a set of optimal release intervals

$$\tau_1^* = 4.9754, \tau_2^* = 6.3439, \tau_3^* = 6.2288, \tau_4^* = 6.3275, \tau_5^* = 6.1244 \quad (4.2)$$

and an optimal release amount

$$\sigma_{fix}^* = 385.28. \quad (4.3)$$

The minimum cost value we gained is $\hat{J}^* = 30.0030$.

The optimal release timing control is compared with non-control and optimal release amount control (see the left figure of Figure 6), and we find that the former has obvious advantages and it can suppress the number of aphids at a lower level in the whole control process.

Furthermore, for every $\sigma_{fix} \in [0, 500]$, we also determine the corresponding optimal time intervals under constraint (4.1) and then calculate the value of cost function. The results are displayed in the right figure of Figure 6. With the release amount σ_{fix} varies in the interval $[0, 500]$, we see that the objective function also has a minimum value which is consistent with the optimal value we get.

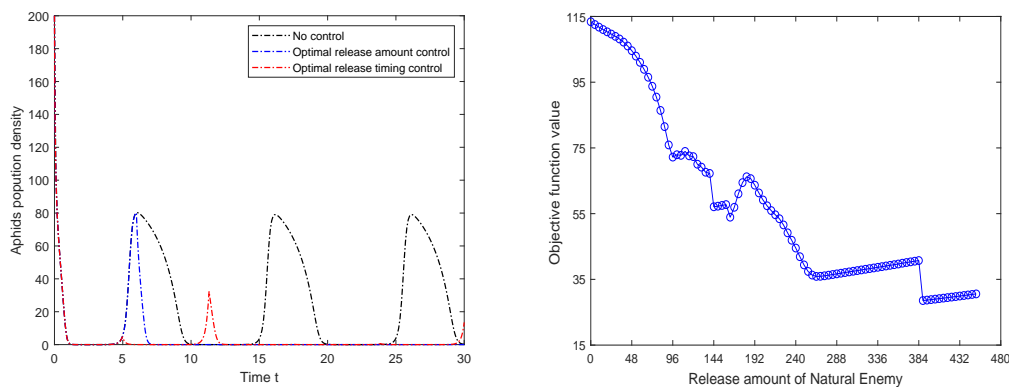


Figure 6. Release timing control: (a) Comparisons of aphid population under different biological controls; (b) Impact of the intensity of each release on the optimal cost value.

Example 3. Mixed control.

Still use the initial release intervals $\tau_1 = \tau_2 = \dots = \tau_5 = 6$ and initial release amounts $\sigma_1 = \sigma_2 = \dots = \sigma_4 = 450$, and under the constraint (4.1) and $0 \leq \sigma_k \leq 500, k = 1, 2, 3, 4$, we solve the optimal problem by the same method, then obtain a set of optimal release amounts

$$\sigma_1^* = 439.65, \sigma_2^* = 439.65, \sigma_3^* = 338.15, \sigma_4^* = 338.15, \quad (4.4)$$

and a set of optimal release intervals

$$\tau_1^* = 5.1809, \tau_2^* = 6.4559, \tau_3^* = 6.4541, \tau_4^* = 5.9545, \tau_5^* = 5.9545. \quad (4.5)$$

This control strategy is showed in the right figure of Figure 7. Besides, the minimum cost value is $J^* = 27.1708$.

The time series diagrams of aphid population under four types of control modes are showed in the left figure of Figure 7. We can see that although the optimal mixed control corresponds to a smaller value of the cost function, and it does show a good control effect in the first stage of the control process, however we also clearly see that the number of aphid population rebounded significantly in the last stage of the control process. From this point of view, the comprehensive effect of optimal timing control is better.

Finally, we compare these three optimal release strategies together (refer to Table 1 and the left figure in Figure 8). Combining with the right figure in Figure 8, we know that the optimal amount control releases the least natural enemies in the whole control process, while optimal mixed control releases the most. Further more, we can see that for process control, compared with the release amount, the optimal selection of release timing should be given higher priority. In addition, the rebound of the number of aphids in the last stage of optimal mixed control will bring trouble to the subsequent population management of aphids.

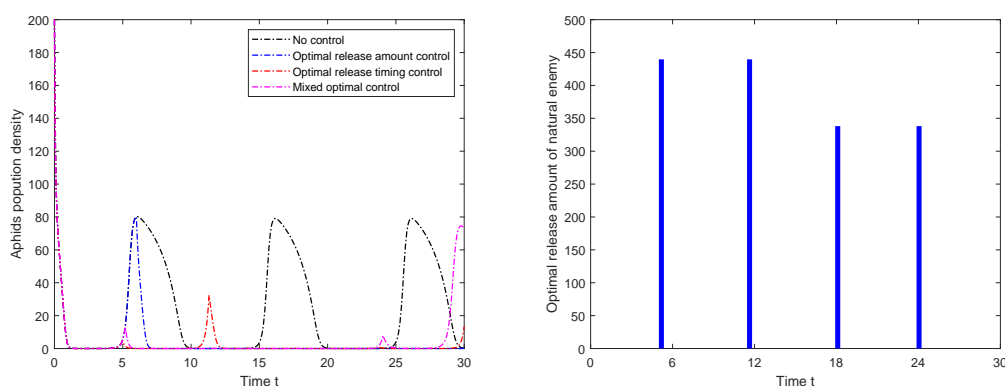


Figure 7. Mixed control: (a) Comparisons of aphid population under different biological controls; (b) Release strategy of the mixed optimal control.

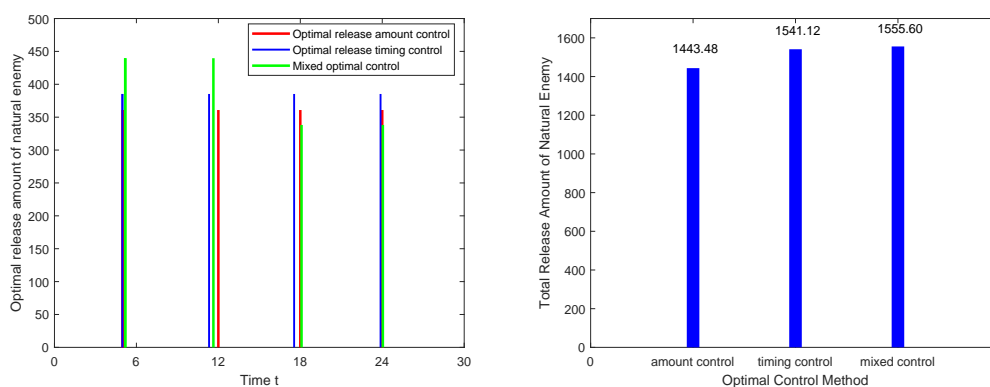


Figure 8. (a) Release strategies of three kinds of optimal control; (b) Comparisons of total release amounts of natural enemies for three optimal control methods.

Table 1. Comparison of different release strategies.

	Optimal control parameters	\mathcal{J}^*
Release amount control	$\sigma_{fix}^* = 360.87$	36.0234
Release timing control	$\tau_1^* = 4.9754, \tau_2^* = 6.3439,$ $\tau_3^* = 6.2288, \tau_4^* = 6.3275,$ $\tau_5^* = 6.1244, \sigma_{fix}^* = 385.28$	30.0030
Mixed control	$\tau_1^* = 5.1809, \tau_2^* = 6.4559,$ $\tau_3^* = 6.4541, \tau_4^* = 5.9545,$ $\tau_5^* = 5.9545, \sigma_1^* = 439.65$ $\sigma_2^* = 439.65, \sigma_3^* = 338.15$ $\sigma_4^* = 338.15$	27.1708

5. Conclusions

In this paper, impulsive dynamical systems for the interaction among aphids, parasitic wasps and aphidophagous Coccinellids incorporating impulsive releases of Coccinellids are proposed, and the release strategies of Coccinellids in the long-term and limited-time control of aphids are explored.

We first investigate the long-term control system. The existence and stability of the aphid-eradication periodic solution are studied. In addition, we discuss the threshold conditions for the release amount and release period to ensure that the aphid population is eliminated eventually.

For the limited-time control, three different optimal strategies in impulsive control are studied. Time rescaling technique and an optimization algorithm based on gradient are applied to overcome the technical difficulty of the impulsive optimal control problem which is caused by uncertain pulse effects. Gradients of objective function with respect to each control parameter are gained, and optimal values of control parameters are obtained by numerical simulations. Our simulations indicate that for process control, the optimal selection of release timing should be given higher priority compared with the release amount.

In the limited-time control, we focus on process control but ignore terminal control. By comparing the three control results, we notice that there are some disadvantages in the construction of cost function

by only emphasizing process control. For example, in the mixed control mode, although the cost function is reduced to the lowest value, it can be seen that the number of aphids has rebounded greatly at the end. This can not be said to have achieved a good control effect. Therefore in our future work, we will consider both terminal control and process control in the limited time control, so as to ensure the best comprehensive control effect on aphid population. In addition, the selection of model parameters in this work is more from the perspective of theoretical research. Therefore, in further research in the future, we will focus on the specific experimental data for numerical fitting, so as to better apply the research results to practice.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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