AIMS Mathematics, 7(4): 6852–6875.
DOI: 10.3934/math.2022381
Received: 13 October 2021
Revised: 31 December 2021
Accepted: 09 January 2022
Published: 26 January 2022

Mathematics

http://www.aimspress.com/journal/Math

Research article

Stochastic comparisons of extreme order statistic from dependent and heterogeneous lower-truncated Weibull variables under Archimedean copula

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Abstract: This article studies the stochastic comparisons of order statistics with dependent and heterogeneous lower-truncated Weibull samples under Archimedean copula. To begin, we obtain the usual stochastic and hazard rate orders of the largest and smallest order statistics from heterogeneous and dependent lower-truncated Weibull samples under Archimedean copula. Second, under Archimedean copula, we get the convex transform and the dispersive orders of the largest and smallest order statistics from dependent and heterogeneous lower-truncated Weibull samples. Finally, several numerical examples are given to demonstrate the theoretical conclusions.

Keywords: order statistic; Archimedean copula; majorization; lower-truncated Weibull; stochastic orders
Mathematics Subject Classification: Primary 90B25; Secondary 60E15, 60K10

1. Introduction

Order statistics are important in many probability areas, including reliability theory, auction theory, and operations research. Denote $X_{k:n}$ by the k-th order statistic of random variables $X_1, X_2, ..., X_n$, $k = 1, 2, ..., n$. In reliability theory, $X_{k:n}$ characterizes the lifetime of a $(n - k + 1)$-out-of-n system. Specifically, $X_{n:n}$ and $X_{1:n}$ can express as the lifetimes of parallel and series systems, respectively. In auction theory, $X_{n:n}$ and $X_{1:n}$ represent the first-price sealed-bid auction and final price of the first-price procurement auction, respectively. Exploring the stochastic behavior of coherent systems is one of the most important subjects in reliability. Stochastic orders, which are valuable instruments for measuring the size and variability of random variables, have been frequently employed in stochastic comparisons of reliability systems.

Many scholars have spent decades studying stochastic comparisons of order statistics from heterogeneous and independent distributions. The usual stochastic order, the hazard rate order, and the likelihood ratio order are all stochastic orders that compare the "magnitude" of random variables. For example, [1] studied the largest order statistic in terms of the hazard rate and the dispersive order. [2] obtained ordering properties of the largest order statistic with two independent heterogeneous exponential samples with respect to the likelihood ratio order and the hazard rate order. [3] investigated the likelihood ratio order and the stochastic order between the smallest order statistic of independent and dependent samples. [4] focused on stochastic orders to compare the magnitudes of two parallel systems from Weibull distributions when one set of scale parameters majorizes the other. There have been lots of extensions works to the case of stochastic comparisons with independent heterogeneous Weibull distribution, for example, [5–7]. Besides, we study stochastic orders that compare the "variability" or the "dispersion" of random variables. The most important and common orders are the convex and dispersive orders. The variability of order statistics from heterogeneous and independent random variables has nice applications in the reliability theory and actuarial science, and has been studied by many researchers. For instance, the convex transform, star, and dispersive orders are used for comparing variability in probability distributions. [8] proved the star order between the largest order statistic from heterogeneous(homogeneous) and independent proportional hazard rates(PHR) models. [9] investigated the stochastic comparisons of the largest and the smallest order statistics with independent heterogeneous generalized exponential samples in terms of various stochastic orders. [10] gave some sufficient conditions for stochastic comparisons between the largest order statistic with exponentiated Weibull samples with respect to the usual stochastic, the likelihood ratio order, and the dispersive order. [11] compared two of the largest order statistics having heterogeneous exponentiated Weibull samples in terms of the reversed hazard rate and likelihood ratio orders. [12] considered stochastic comparisons of the smallest order statistic from the location-scale family of distributions with respect to different stochastic orders. [13] focused on some stochastic comparisons between the corresponding order statistics based on modified proportional hazard rates and modified proportional reversed hazard rates models. [14] presented some new ordering properties between two parallel systems comprising general independent heterogeneous samples in the sense of the usual stochastic and reversed hazard rate orders. [15] dealt with some stochastic comparisons of both the largest and the smallest order statistics comprising dependent Burr Type XII samples under the Archimedean copula with respect to the star order and the convex transform order. [16] considered the largest order statistic with Pareto samples and studied the effect of heterogeneity on the skewness of such systems. [17] established some properties of the new measures for various classes of symmetric and asymmetric distributions, and characterized the generalized Pareto distribution in terms of the convex transform order. For more investigations, one may refer to [18–31].

All of the above studies are restricted to the independent case. However, in most practical scenarios, due to the common environment or share the same workload, which results in the dependence among the samples. In recent years, the dependent samples of order statistics have attracted widespread attention in the academic world. [32] investigated the ordering properties of order statistics from random variables of Archimedean copulas. [33] studied order statistics from random variables following the scale model with respect to the usual stochastic order of the sample extremes and the second smallest order statistic, the dispersive order, and the star order of the sample extremes. [34] further obtained the stochastic properties of the largest and the smallest order statistics with heterogeneous and dependent PHR samples in the sense of the usual stochastic order. [35] discussed stochastic comparisons of the largest and the smallest order statistics with heterogeneous and dependent resilience-scaled samples. [36] provided sufficient conditions for the hazard rate order on the smallest and proportional hazard rates or scales, and the reversed hazard rate order on the largest of the sample with Archimedean copula. [37] obtained the variability of both the largest and the smallest order statistics of heterogeneous samples, sufficient conditions are established for the dispersive and the star orders between the smallest order statistic consisting of dependent samples having multiple-outlier proportional hazard rates and Archimedean copulas. [38] focused on the usual stochastic, star, and convex transform orders of extreme order statistic comparing heterogeneous and dependent extended exponential samples under Archimedean copulas. [39] carried out stochastic comparisons of the largest and the smallest order statistics with dependent heterogeneous Topp-Leone generated samples in terms of the usual stochastic order and the reversed hazard rate order. [40] investigated stochastic comparisons on extreme order statistic from heterogeneous and dependent samples following modified proportional reversed hazard rated and modified proportional hazard rates models. [41] provided distribution-free results to compare, in the usual stochastic order under some majorization conditions, coherent systems with heterogeneous and dependent components where the dependency structure can be defined by any copula. The stochastic comparisons with statistically dependent samples has attracted widespread attention among a lot of scholars. The interested readers may refer to [42–54].

Truncated Weibull distributions are comprehensively applied to product reliability, modeling product failures, quality control and life testing, so it is very meaningful that we research the ordering statistics results of truncated Weibull models. As discussed in [55, 56], in many sampling settings, observations are limited to a subset of the population's potential values. As a result of the limiting process, only n of N prospective observations remain visible, while (N-n) are deleted. When N(or N-n) is known, the resulting collection of partial data is referred to as censored; otherwise, it is referred to as truncated. A defined number of objects are generally evaluated for a fixed amount of time in life testing, which is aimed to estimate the average life span of, for example, transistors from a certain production line. Because the lives of things surviving the life test are unknown, this results in a censored sample of lifetimes. In certain real-life testing circumstances, the total number of items on the exam is unknown. This happens, for example, when a certain unknown number of products are put through a life test and one of them has a specific fault that is only discovered after the item fails. If the lifespan of an item with this specific defect is the variable of interest, the sample is truncated in the sense that the number of missing observations with lifetime larger than the burn-in or testing period is unknown. Therefore, conducting sample research with Lower-truncated Weibull distribution has very important theoretical needs and practical significance. [57] discussed the stochastic comparisons of extreme order statistics with heterogeneous and independent lower-truncated Weibull samples. A random variable X is said to have doubly truncated Weibull distribution if its cumulative distribution function is

$$G(x) = \frac{F(x) - F(T)}{F(L) - F(T)}, \qquad 0 \le T \le x \le L < \infty,$$

where $F(x) = 1 - e^{(-\lambda x)^{\alpha}} (x \ge 0, \lambda > 0, \alpha > 0)$. When $T = 0$ and $L \to \infty$, it becomes two-parameter Weibull distribution. When $T = 0$ and $L < \infty$, it is upper-truncated Weibull distribution, and when $T > 0$ and $L \to \infty$, it is lower-truncated Weibull distribution. For more details on Weibull distribution and its truncated forms, one may refer to [58, 59].

In this paper, we will focus attention on the lower-truncated Weibull with cumulative distribution function and probability density function

$$G(x) = 1 - e^{1-(\lambda x)^{\alpha}}, \qquad x \geq \frac{1}{\lambda},$$

$$g(x) = \alpha\lambda^{\alpha} x^{\alpha-1} e^{1-(\lambda x)^{\alpha}}, \qquad x \geq \frac{1}{\lambda},$$

respectively. We denote $X \sim LTW(\alpha, \lambda)$ if X has the distribution functions $G(x)$. Motivated by the work of [57], this paper focus on studying the usual stochastic, the hazard rate, the convex transform and the dispersive orders of the extreme order statistic from dependent and heterogeneous lower-truncated Weibull samples. We derive the magnitude and variability with the heterogeneity considered in the shape and scale parameter. The difference from [57] is that we extend the independent situation to the dependent situation.

The rest of this paper is organized as follows. In Section 2, we review some definitions of stochastic orders, majorizations, and some lemmas. In Section 3, we discuss the usual stochastic order and the hazard rate order of the smallest and the largest order statistics with dependent and heterogeneous lower-truncated Weibull samples. In Section 4, we get the convex transform and the dispersive orders of the largest and the smallest order statistics with dependent and heterogeneous lower-truncated Weibull samples. Finally, Section 5 concludes the paper with some remarks.

2. Preliminaries

Before proceeding to the main results, we briefly introduce some basic concepts about stochastic orders, majorization, and Archimedean copula, which are very useful tools to compare random variables arising from reliability theory, operations research, actuarial science, and so on. Furthermore, the notion '$a \overset{sgn}{=} b$' means that a and b have the same sign.

2.1. Stochastic orders

Let X be a non-negative random variable with distribution function $F_X(t)$, survival function $\bar{F}_X(t) = 1 - F_X(t)$, probability density function $f_X(t)$, the hazard rate function $h_X(t) = f_X(t)/\bar{F}_X(t)$, and the right-continuous inverse F_X^{-1}.

Definition 1. *A non-negative random variable X is said to be smaller than Y in the sense of*

(i) *Usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}_X(t) \leq \bar{F}_Y(t)$ for all $t \in [0, \infty)$;*
(ii) *Hazard rate order (denoted by $X \leq_{hr} Y$) if $\bar{F}_Y(t)/\bar{F}_X(t)$ is increasing in $t \in [0, \infty)$ or $h_X(t) \geq h_Y(t)$ in $t \in [0, \infty)$;*
(iii) *Dispersive order (denoted by $X \leq_{disp} Y$) if $F_X^{-1}(v) - F_X^{-1}(u) \leq F_Y^{-1}(v) - F_Y^{-1}(u)$ for all $0 \leq u \leq v \leq 1$. When X and Y have densities f_X and f_Y, respectively, then $X \leq_{disp} Y$ if and only if $f_Y(F_Y^{-1}(t)) \leq f_X(F_X^{-1}(t))$ for all $t \in (0, 1)$;*
(iv) *Convex transform order (denoted by $X \leq_c Y$) if $F_Y^{-1}F_X(t)$ is convex in $t \in [0, \infty)$, or equivalently, $X \leq_c Y$ if and only if $F_X^{-1}F_Y(t)$ is concave in $t \in [0, \infty)$;*
(v) *Star order (denoted by $X \leq_* Y$) if $F_Y^{-1}F_X(t)/t$ is increasing in $t \in [0, \infty)$;*
(vi) *Lorenz order (denoted by $X \leq_{Lorenz} Y$) if $L_X(t) \geq L_Y(t)$ for all $t \in [0, 1]$, where the Lorenz curve L_X is defined as $L_X(t) = \int_0^t F_X^{-1}(u)du/\mu_X$, and $\mu_X = \mathbb{E}[X]$.*

It is known that

$$X \le_{hr} Y \Rightarrow X \le_{st} Y \qquad \textit{and} \qquad X \le_{c} Y \Rightarrow X \le_{*} Y \Rightarrow X \le_{Lorenz} Y,$$

but the reversed statement is not true in general. The convex transform is usually used to describe skewness. Skewness is an important quantity used for measuring the asymmetry in a distribution. For a unimodal distribution, nagative skewness indicates that the tail on the left side of the density function is longer than the right side, while positive skewness indicates that the tail on the right side of the density function is longer than the left side. The convex transform order in this regard implies that one distribution is more skewed to the right than the other distribution. It should be noticed that the convex transform suitable for a non-negative random variable, hence the implication $X \le_c Y \Rightarrow X \le_* Y$ holds when X and Y are two non-negative random variables (see [60]). For more comprehensive discussions on various stochastic orders and their applications, one may refer to [61, 62].

2.2. *Majorization order*

Majorization order is an important tool in establishing various inequaties arising from many research areas. Denote $\mathbb{I}_n = 1, 2, ..., n$.

Definition 2. *For two real vectors* $\mathbf{a} = (a_1, a_2, ..., a_n)$ *and* $\mathbf{b} = (b_1, b_2, ..., b_n) \in \mathbb{R}^n$, $a_{(1)} \le a_{(2)} \le \cdots \le a_{(n)}$ *and* $b_{(1)} \le b_{(2)} \le \cdots \le b_{(n)}$ *denote the increasing arrangement of the components of* $\mathbf{a}$ *and* $\mathbf{b}$*, respectively. Then*

(i) *Vector* $\mathbf{a}$ *is said to be majorized by vector* $\mathbf{b}$ *(denoted by* $\mathbf{a} \overset{m}{\preceq} \mathbf{b}$*) if* $\sum_{j=1}^{i} a_{(j)} \ge \sum_{j=1}^{i} b_{(j)}$ *for* $i = 1, 2, ..., n-1$, *and* $\sum_{j=1}^{n} a_{(j)} = \sum_{j=1}^{n} b_{(j)}$*;*

(ii) *Vector* $\mathbf{a}$ *is said to be weakly supermajorized by vector* $\mathbf{b}$ *(denoted by* $\mathbf{a} \overset{w}{\preceq} \mathbf{b}$*) if* $\sum_{j=1}^{i} a_{(j)} \ge \sum_{j=1}^{i} b_{(j)}$ *for* $i = 1, 2, ..., n$.

Definition 3. ([63]) A real-valued function φ, defined on a set $\mathbb{A} \subseteq \mathbb{R}^n$, is said to be Schur-convex (Schur-concave) on $\mathbb{A}$ if $\boldsymbol{a} \overset{\mathrm{m}}{\succeq} \boldsymbol{b}$ implies $\varphi(\boldsymbol{a}) \ge (\le)\varphi(\boldsymbol{b})$ for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{A}$.

It is clear that $\boldsymbol{a} \overset{\mathrm{m}}{\succeq} \boldsymbol{b}$ implies $\boldsymbol{a} \overset{\mathrm{w}}{\succeq} \boldsymbol{b}$. For more details on majorization, weak majorization and Schur-convex (Schur-convave) functions, one may refer to [63]. The sufficient and necessary conditions for the characterization of Schur-convex (Schur-concave) function is presented in the following lemma.

Lemma 1. ([63]) Let $\mathcal{I} \subseteq \mathbb{R}$ be an open interval and let $\psi : \mathcal{I}^n \to \mathbb{R}$ be continuously differentiable. Then, ψ is Schur-convex (Schur-concave) on $\mathcal{I}$ if and only if ψ is symmetric on $\mathcal{I}^n$ and for all $i \ne j$,

$$(x_i - x_j)\left[\frac{\partial\psi(\boldsymbol{x})}{\partial x_i} - \frac{\partial\psi(\boldsymbol{x})}{\partial x_j}\right] \ge (\le) 0, \text{ for all } \boldsymbol{x} \in \mathcal{I}^n,$$

where $\partial\psi(\boldsymbol{x})/\partial x_i$ denotes the partial derivative of ψ with respect to its i-th argument.

Lemma 2. ([63]) For a real-valued function ψ defined on a set $\mathcal{A} \subseteq \mathbb{R}^n$, $\boldsymbol{a} \overset{\mathrm{w}}{\preceq} \boldsymbol{b}$ implies $\psi(\boldsymbol{a}) \le \psi(\boldsymbol{b})$ if and only if ψ is decreasing and Schur-convex on $\mathcal{A}$.

2.3. Archimedean copula

Archimedean copula has been used in reliability theory and many other areas due to its capability of capturing wide ranges of dependence and mathematical tractability. By definition, for a continuous and decreasing function $\phi : \mathbb{R}^+ \mapsto [0,1]$ such that $\phi(0) = 1$, $\phi(+\infty) = 0$, let $\psi = \phi^{-1}$ be the pseudo-inverse,

$$C_\psi(u_1, ..., u_n) = \psi\left(\sum_{i=1}^{n} \phi(u_i)\right), \text{ for all } u_i \in [0,1], \qquad i = 1, 2, ..., n,$$

is called an Archimedean copula with the generator ψ if $(-1)^{n-2}\psi^{(n-2)}(x)$ is decreasing and convex and $(-1)^k\psi^{(k)}(x) \geq 0$ for all $x \geq 0$, $k = 0, 1, ..., n-2$. For more discussions on copulas and their properties, one may refer to [64, 65].

Lemma 3. *([32]) For two n-dimensional Archimedean copulas C_{ψ_1} and C_{ψ_2}, if $\phi_2 \circ \psi_1$ is super-additive, then $C_{\psi_1}(\boldsymbol{u}) \leq C_{\psi_2}(\boldsymbol{u})$ for all $\boldsymbol{u} \in [0,1]^n$.*

Lemma 4. *([34]) For generators ψ_1 and ψ_2 of Archimedean copulas, if $\psi_2(\phi_2(t)/n)/\psi_1(\phi_1(t)/n)$ increases in t, then, for all $t \in [0,1]$, $\psi_2(n\phi_2(t)) \geq \psi_1(n\phi_1(t))$ and*

$$\frac{\psi_2'(\phi_2(t)/n)}{\psi_2'(\phi_2(t))\psi_2(\phi_2(t)/n)} \geq \frac{\psi_1'(\phi_1(t)/n)}{\psi_1'(\phi_1(t))\psi_1(\phi_1(t)/n)}.$$

3. Stochastic comparisons of the magnitude

In this section, we provide the ordering properties of the largest and the smallest order statistics with dependent and heterogeneous lower-truncated Weibull samples in terms of the usual stochastic order and the hazard rate order. We denote $\boldsymbol{X} \sim LTW(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi)$ as the sample arising from non-negative random variables $X_1, X_2, ..., X_n$ assembled with an Archimedean copula having generator ψ, where $X_i \sim LTW(\alpha_i, \lambda_i)$ for $i = 1, 2, ..., n$.

3.1. Orders of the largest order statistic

First, Theorem 1 gives sufficient conditions for the largest order statistic from dependent and heterogeneous lower-truncated Weibull samples in terms of the usual stochastic order.

Theorem 1. *Let $\boldsymbol{X} \sim LTW(\boldsymbol{\alpha}, \lambda, \psi_1)$ and $\boldsymbol{Y} \sim LTW(\boldsymbol{\beta}, \lambda, \psi_2)$. If $\phi_2 \circ \psi_1$ be super-additive, then, for $\lambda > 0$, $x \geq 1/\lambda$, we have*

$$\boldsymbol{\alpha} \overset{w}{\geq} \boldsymbol{\beta} \Rightarrow Y_{n:n} \leq_{st} X_{n:n}.$$

Proof. First, the distribution functions of $X_{n:n}$ and $Y_{n:n}$ are given by

$$F_{X_{n:n}}(x) = \psi_1\left(\sum_{i=1}^{n} \phi_1\left(1 - e^{1-(\lambda x)^{\alpha_i}}\right)\right), \qquad x \geq 1/\lambda$$

and

$$F_{Y_{n:n}}(x) = \psi_2\left(\sum_{i=1}^{n} \phi_2\left(1 - e^{1-(\lambda x)^{\beta_i}}\right)\right), \qquad x \geq 1/\lambda,$$

respectively. Note that super-additivity of $\phi_2 \circ \psi_1$ implies

$$\psi_1\left(\sum_{i=1}^{n} \phi_1\left(1 - e^{1-(\lambda x)^{\beta_i}}\right)\right) \leq \psi_2\left(\sum_{i=1}^{n} \phi_2\left(1 - e^{1-(\lambda x)^{\beta_i}}\right)\right).$$

Hence, to obtain the required result, it suffices to show that

$$\psi_1\left(\sum_{i=1}^{n} \phi_1\left(1 - e^{1-(\lambda x)^{\alpha_i}}\right)\right) \leq \psi_1\left(\sum_{i=1}^{n} \phi_1\left(1 - e^{1-(\lambda x)^{\beta_i}}\right)\right).$$

Denote $\Pi(\alpha_1, \alpha_2, ..., \alpha_n) = \psi_1\left(\sum_{i=1}^{n} \phi_1\left(1 - e^{1-(\lambda x)^{\alpha_i}}\right)\right)$. According to Lemma 2, we need to show that $\Pi(\alpha_1, \alpha_2, ..., \alpha_n)$ is Schur-concave in $(\alpha_1, \alpha_2, ..., \alpha_n)$ and increasing in α_s. Taking the partial derivative of $\Pi(\alpha_1, \alpha_2, ..., \alpha_n)$ with respect to α_s, we have

$$\begin{aligned}
\frac{\partial \Pi(\alpha_1, \alpha_2, ..., \alpha_n)}{\partial \alpha_s} &= \psi_1^{'}\left(\sum_{i=1}^{n} \phi_1\left(1 - e^{1-(\lambda x)^{\alpha_i}}\right)\right) \frac{e^{1-(\lambda x)^{\alpha_s}}}{\psi_1^{'}(\phi_1(1 - e^{1-(\lambda x)^{\alpha_s}}))}(\lambda x)^{\alpha_s} \ln(\lambda x) \\
&= \psi_1^{'}\left(\sum_{i=1}^{n} \phi_1\left(e^{1-(\lambda x)^{\alpha_i}}\right)\right) G(\alpha_s, x) F(\alpha_s, x) \ln(\lambda x) \geq 0,
\end{aligned}$$

where $G(\alpha_s, x) = (\psi_1^{'}(\phi_1(1 - e^{1-(\lambda x)^{\alpha_s}})))^{-1}$, $F(\alpha_s, x) = (\lambda x)^{\alpha_s} e^{1-(\lambda x)^{\alpha_s}}$. Because $G(\alpha_s, x)$ is non-positive and increasing in α_s for $\lambda > 0, x \geq 1/\lambda$, and $F(\alpha_s, x)$ is non-negative and decreasing in α_s, for $\lambda > 0$, $x \geq 1/\lambda$. Hence, $\Pi(\alpha_1, \alpha_2, ..., \alpha_n)$ and $G(\alpha_s, x)F(\alpha_s, x)$ are increasing in α_s, for $\lambda > 0$, $x \geq 1/\lambda$. Therefore, for any $s \neq t$, we have

$$\begin{aligned}
&(\alpha_s - \alpha_t)\left(\frac{\partial \Pi(\alpha_1, \alpha_2, ..., \alpha_n)}{\partial \alpha_s} - \frac{\partial \Pi(\alpha_1, \alpha_2, ..., \alpha_n)}{\partial \alpha_t}\right) \\
= \; &(\alpha_s - \alpha_t)\, \psi_1^{'}\left(\sum_{i=1}^{n} \phi_1\left(1 - e^{1-(\lambda x)^{\alpha_i}}\right)\right) [G(\alpha_s, x)F(\alpha_s, x) - G(\alpha_t, x)F(\alpha_t, x)] \ln(\lambda x) \leq 0.
\end{aligned}$$

Thus, $\Pi(\alpha_1, \alpha_2, ..., \alpha_n)$ is Schur-concave in $(\alpha_1, \alpha_2, ..., \alpha_n)$, the desired result follows from Lemma 2. □

Remark 1. *We have to mention that the condition '$\phi_2 \circ \psi_1$' is super-additive in Theorem 1 is general and easy to be satisfied for many Archimedean copulas. For example,*

(i) *The Clayton copula with generator $\psi(t) = (\theta t + 1)^{-1/\theta}$ for $\theta \geq 0$. Let us set $\psi_1(t) = (\theta_1 t + 1)^{-1/\theta_1}$ and $\psi_2(t) = (\theta_2 t + 1)^{-1/\theta_2}$. It can be observed that $\phi_2 \circ \psi_1(t) = (\theta_1 t + 1)^{\frac{\theta_2}{\theta_1}} /\theta_2 - 1/\theta_2$. Taking the derivative of $\phi_2 \circ \psi_1$ with respect to t twice, we can see that $[\phi_2 \circ \psi_1]'' \geq 0$ for $\theta_2 \geq \theta_1 \geq 0$. Thus, $\phi_2 \circ \psi_1$ is super-additivity (cf. Table 4.2.3 on Page 116 of [64]);*

(ii) *The Gumbel-Hougaard copula with generator $\psi(t) = e^{1-(1+t)^{\theta}}$ for $\theta \in [1, \infty)$. Let us set $\psi_1(t) = e^{1-(1+t)^{\alpha}}$ and $\psi_2(t) = e^{1-(1+t)^{\beta}}$. It can be seed that $\phi_2 \circ \psi_1(t) = (1 + t)^{\alpha/\beta - 1}$. Taking the derivative of $\phi_2 \circ \psi_1$ with respect to t twice, we can see that $[\phi_2 \circ \psi_1]'' \geq 0$ for $\alpha \geq \beta \geq 1$ which implies the superadditivity of $\phi_2 \circ \psi_1$ (cf. Table 4.2.3 on Page 116 of [64]).*

The following Example 1 illustrates the result of Theorem 1.

Example 1. *Consider the case of* $n = 3$. *Let generators* $\psi_1(x) = (\theta_1 x + 1)^{-1/\theta_1}$, $\psi_2(x) = (\theta_2 x + 1)^{-1/\theta_2}$, $\theta_i \geq 0, i = 1, 2$. *Set* $\theta_1 = 0.1$, $\theta_2 = 1$, $\lambda = 0.5$, $\boldsymbol{\alpha} = (0.2, 0.4, 0.6) \overset{w}{\succeq} (0.4, 0.6, 0.8) = \boldsymbol{\beta}$. *One can check all conditions of Theorem 1 are statisfied. Plot the whole of distribution function curves of* $X_{3:3}$ *and* $Y_{3:3}$ *on* $(2, \infty)$. *As is seen in Figure 1, the distribution function curve of* $X_{3:3}$ *is always beneath that of* $Y_{3:3}$, *that is,* $X_{3:3} \geq_{st} Y_{3:3}$, *which coincided with the result of Theorem 1.*

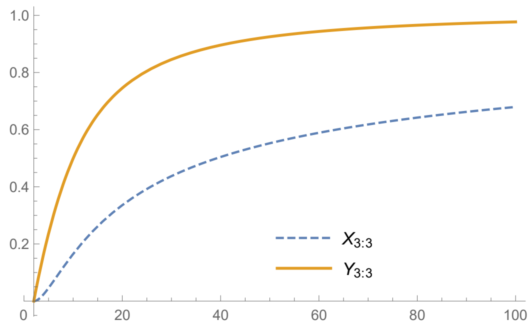

Figure 1. The distribution function curves of $X_{3:3}$ and $Y_{3:3}$.

The proof of Theorem 2 can be obtained similarly that of Theorem 1, thus we omit the proof process of Theorem 2.

Theorem 2. *Let* $\boldsymbol{X} \sim LTW(\alpha, \boldsymbol{\lambda}, \psi_1)$ *and* $\boldsymbol{Y} \sim LTW(\alpha, \boldsymbol{\mu}, \psi_2)$. *If* $\phi_2 \circ \psi_1$ *be superadditive. Then, for* $x \geq max(1/\lambda_1, 1/\lambda_2, ..., 1/\lambda_n, 1/\mu_1, 1/\mu_2, .., 1/\mu_n)$ *and* $0 < \alpha \leq 1$, *we have*

$$\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\mu} \Rightarrow Y_{n:n} \leq_{st} X_{n:n}.$$

Remark 2. *It should be noted that for independent samples,* $\psi(x) = e^{-x}$, *thus Theorem 1 and Theorem 2 generalizes Theorem 3.2 of [57] from independent samples to the case of statistically dependent components with Archimedean copulas.*

Remark 3. *Proposition 3.7 in [41] is considerably more generic, because the random variables studied in such proposition have arbitrary distribution functions, however, in this paper, we explore a specific sort of distribution function, namely the lower-truncated Weibull distribution. In comparison to Proposition 3.7 in [41], the Theorem 1 and the Theorem 2 do not need the requirement* ψ_1 *is log-convex, hence our conditions are weaker for lower-truncated Weibull distribution in the largest order statistic.*

We present the following Example 2 to illustrate the result of Theorem 2.

Example 2. *Take* $\psi_1(x) = (\theta_1 x + 1)^{-1/\theta_1}$, $\psi_2(x) = (\theta_2 x + 1)^{-1/\theta_2}$. *Set* $\theta_1 = 0.2$, $\theta_2 = 0.5$, $\alpha = 0.2$, $\boldsymbol{\lambda} = (0.2, 0.4, 0.6) \overset{w}{\succeq} (0.4, 0.6, 0.8) = \boldsymbol{\mu}$. *These satisfy all conditions of Theorem 2, and the whole of distribution function curves of* $X_{3:3}$ *and* $Y_{3:3}$ *on* $(5, \infty)$ *are plotted in Figure 2, which asserts* $X_{3:3} \leq_{st} Y_{3:3}$.

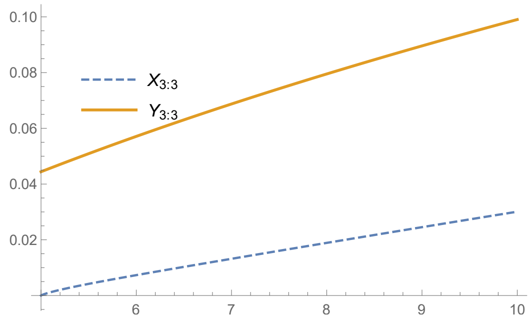

Figure 2. The distribution function curves of $X_{3:3}$ and $Y_{3:3}$.

3.2. Orders of the smallest order statistic

First, we will give a stochastic comparison of the smallest order statistic from dependent and heterogeneous lower-truncated Weibull samples in terms of the usual stochastic order.

Corollary 1. *Let* $\boldsymbol{X} \sim LTW(\boldsymbol{\alpha}, \lambda, \psi_1)$ *and* $\boldsymbol{Y} \sim LTW(\boldsymbol{\beta}, \lambda, \psi_2)$*. If* $\phi_2 \circ \psi_1$ *be super-additive and* ψ_1 *is log-convex. Then, for* $\lambda > 0$*,* $x \geq 1/\lambda$*. We have*

$$\boldsymbol{\alpha} \stackrel{m}{\succeq} \boldsymbol{\beta} \Rightarrow Y_{1:n} \geq_{st} X_{1:n}.$$

Proof. The distribution function of X is increasing in α, and the survival function of X is log-concave in α, which satisfies conditions in Proposition 3.16 of [41]. Hence, according to Proposition 3.16 of [41], we can easily establish the result of Corollary 1. □

The next Example 3 is provided to explain the result of Corollary 1.

Example 3. *Let generators* $n = 3$*,* $\psi_1(x) = (\theta_1 x + 1)^{-1/\theta_1}$*,* $\psi_2(x) = (\theta_2 x + 1)^{-1/\theta_2}$*,* $\theta_i \geq 0, i = 1, 2$*. Set* $\theta_1 = 0.2$*,* $\theta_2 = 2$*,* $\lambda = 0.5$*,* $\boldsymbol{\alpha} = (0.3, 0.4, 0.5) \stackrel{m}{\succeq} (0.5, 0.6, 0.1) = \boldsymbol{\beta}$*. It is apparent to the* $\phi_2 \circ \psi_1$ *be super-additive and* ψ_1 *is log-convex. The whole of survival function curves of* $X_{1:3}$ *and* $Y_{1:3}$ *are display in Figure 3, which confirmed that* $X_{1:3} \leq_{st} Y_{1:3}$*.*

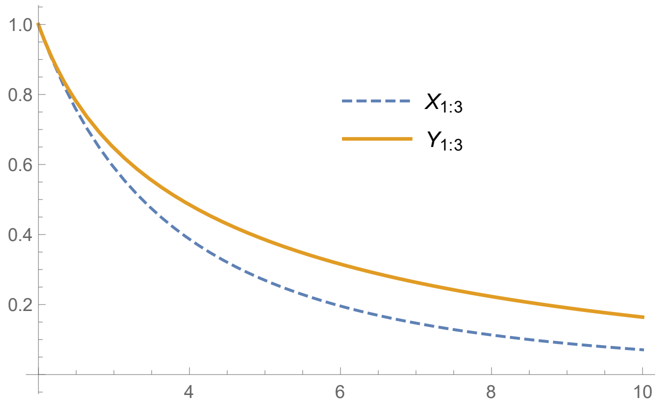

Figure 3. The survival function curves of $X_{1:3}$ and $Y_{1:3}$.

Corollary 2. *Let* $\boldsymbol{X} \sim LTW(\alpha, \boldsymbol{\lambda}, \psi_1)$ *and* $\boldsymbol{Y} \sim LTW(\alpha, \boldsymbol{\mu}, \psi_2)$*. Suppose* $\phi_2 \circ \psi_1$ *be super-additive.*

(i) *If* ψ_1 *is log-concave and* $\boldsymbol{\lambda} \overset{w}{\leq} \boldsymbol{\mu}$, *then* $Y_{1:n} \geq_{st} X_{1:n}$ *for all* $0 < \alpha \leq 1$ *and* $x \geq max(1/\lambda_1, ..., 1/\lambda_n, 1/\mu_1, ..., 1/\mu_n)$.

(ii) *If* ψ_1 *is log-convex and* $\boldsymbol{\lambda} \overset{m}{\geq} \boldsymbol{\mu}$, *then* $Y_{1:n} \geq_{st} X_{1:n}$ *for all* $\alpha \geq 1$ *and* $x \geq max(1/\lambda_1, ..., 1/\lambda_n, 1/\mu_1, ..., 1/\mu_n)$.

Proof. (i) The distribution function of X is increasing in λ, and the survival function of X is log-concave in λ for $0 < \alpha \leq 1$, which satisfies conditions in Proposition 3.16 of [41]. As a result, Corollary 2(i) can be obtained immediately from proposition 3.7 in [41].

(ii) The proof of Corollary 2(ii) can be obtained similarly that of Corollary 2(i), thus we omit the proof process of Corollary 2(ii).

□

In the following, we will give a numerical example to illustrate the Corollary 2.

Example 4. (i) *Consider the case of* $n = 3$. *Let generators* $\psi_1(x) = e^{(1-e^x)/\theta_1}, 0 \leq \theta_1 \leq 1$, $\psi_2(x) = (\theta_2 x + 1)^{-1/\theta_2}$, $\theta_2 \geq 0$. *Set* $\theta_1 = 0.1$, $\theta_2 = 1.2$, $\alpha = 0.5$, $\boldsymbol{\mu} = (0.2, 0.4, 0.6) \overset{w}{\geq} (0.4, 0.6, 0.8) = \boldsymbol{\lambda}$. *One can check all conditions of Corollary 2(i) are statisfied. The survival function curves of* $X_{1:3}$ *and* $Y_{1:3}$ *are plotted in Figure 4(i), which coincided with the result of Corollary 2(i).*

(ii) *Let* $\psi_1(x) = (\theta_1 x + 1)^{-1/\theta_1}$, $\psi_2(x) = (\theta_2 x + 1)^{-1/\theta_2}$, $\theta_i \geq 0, i = 1, 2$. *Set* $n = 3$, $\theta_1 = 0.1$, $\theta_2 = 1$, $\alpha = 2$, $\boldsymbol{\lambda} = (0.2, 0.3, 0.4) \overset{m}{\geq} (0.3, 0.4, 0.2) = \boldsymbol{\mu}$. *As is seen in Figure 4(ii), the survival function curve of* $X_{1:3}$ *is always beneath that of* $Y_{1:3}$, *that is,* $X_{1:3} \leq_{st} Y_{1:3}$, *which verified Corollary 2 (ii).*

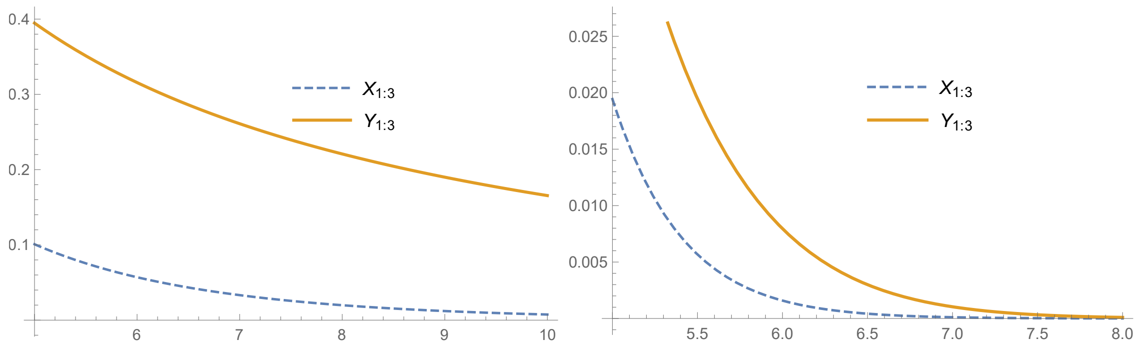

Figure 4. The graphs of the survival function of $X_{1:3}$ and $Y_{1:3}$, for the lower-truncated Weibull ($\alpha = 0.5$) (left) and ($\alpha = 2$) (right).

The following Theorem 3 carries out stochastic comparison of the smallest order statistics from dependent and heterogeneous lower-truncated Weibull samples in the sense of the hazard rate order.

Theorem 3. *Let* $\boldsymbol{X} \sim LTW(\alpha, \boldsymbol{\lambda}, \psi)$ *and* $\boldsymbol{Y} \sim LTW(\alpha, \boldsymbol{\mu}, \psi)$. *If* ψ *is log-concave, and* $-\psi'/\psi$ *is log-convex. Then, for* $0 < \alpha \leq 1/2$ *and* $x \geq max(1/\lambda_1, ..., 1/\lambda_n, 1/\mu_1, .., 1/\mu_n)$, *then*

$$\boldsymbol{\mu} \overset{w}{\geq} \boldsymbol{\lambda} \Rightarrow Y_{1:n} \geq_{hr} X_{1:n}.$$

Proof. For ease of reference, let us list the following facts.

ϱ_1: $\psi(x) \geq 0$ and $\psi'(x) \leq 0$.

ϱ_2: The ψ is log-concave implies that ψ'/ψ decreasing and hence $\psi''(x)\psi(x) \leq \left[\psi'(x)\right]^2$.

ϱ_3: The $-\psi'/\psi$ is log-convex implies that $\left\{\psi''(x)\psi(x) - \left[\psi'(x)\right]^2\right\}/\psi(x)\psi'(x) \geq 0$ increases in $x \geq 0$.

The survival function of $X_{1:n}$ is given by

$$\bar{F}_{X_{1:n}}(x) = \psi\left(\sum_{i=1}^{n}\phi\left(e^{1-(\lambda_i x)^\alpha}\right)\right), \qquad x \geq max(1/\lambda_1, ..., 1/\lambda_n, 1/\mu_1, .., 1/\mu_n),$$

the hazard rate function of $X_{1:n}$ can be expressed as

$$h_{X_{1:n}}(x) = \frac{\psi'\left(\sum_{i=1}^{n}\phi(e^{1-(\lambda_i x)^\alpha})\right)}{\psi\left(\sum_{i=1}^{n}\phi(e^{1-(\lambda_i x)^\alpha})\right)}\sum_{i=1}^{n}\frac{\psi(\phi(e^{1-(\lambda_i x)^\alpha}))}{\psi'(\phi(e^{1-(\lambda_i x)^\alpha}))}\left(\alpha\lambda_i^\alpha x^{\alpha-1}\right) = L(x, \boldsymbol{\lambda}, \alpha, \psi),$$

$x \geq max(1/\lambda_1, ..., 1/\lambda_n, 1/\mu_1, .., 1/\mu_n)$.

Likewise, $Y_{1:n}$ gets the hazard rate function $h_{Y_{1:n}} = L(x, \boldsymbol{\mu}, \alpha, \psi)$. Further denote

$$A(\boldsymbol{\lambda}_{(s,t)}, x) = \sum_{i\neq s,t}\frac{\psi(\phi(e^{1-(\lambda_i x)^\alpha}))}{\psi'(\phi(e^{1-(\lambda_i x)^\alpha}))}\left(\alpha\lambda_i^\alpha x^{\alpha-1}\right), \qquad B(\boldsymbol{\lambda}, x) = \frac{\psi(\sum_{i=1}^{n}\phi(e^{1-(\lambda_i x)^\alpha}))}{\psi'(\sum_{i=1}^{n}\phi(e^{1-(\lambda_i x)^\alpha}))},$$

$$J_\psi(x) = \frac{\psi''(x)\psi(x) - (\psi'(x))^2}{(\psi(x))^2}, \qquad C(\boldsymbol{\lambda}, x) = J_\psi\left(\sum_{i=1}^{n}\phi\left(e^{1-(\lambda_i x)^\alpha}\right)\right)B(\boldsymbol{\lambda}, x).$$

Then, for any $s, t \in \mathbb{I}_n$ with $s \neq t$, we have

$$\begin{aligned}
&\frac{\partial L(x, \boldsymbol{\lambda}, \alpha, \psi)}{\partial\lambda_s} \\
=\;& J_\psi\left(\sum_{i=1}^{n}\phi\left(e^{1-(\lambda_i x)^\alpha}\right)\right)B(\lambda_s, x)(-\alpha\lambda_s^{\alpha-1}x^\alpha)\sum_{i=1}^{n}\frac{\psi(\phi(e^{1-(\lambda_i x)^\alpha}))}{\psi'(\phi(e^{1-(\lambda_i x)^\alpha}))}\left(\alpha\lambda_i^\alpha x^{\alpha-1}\right) \\
&+\frac{\psi'(\sum_{i=1}^{n}\phi(e^{1-(\lambda_i x)^\alpha}))}{\psi(\sum_{i=1}^{n}\phi(e^{1-(\lambda_i x)^\alpha}))}J_\psi\left(\phi(e^{1-(\lambda_s x)^\alpha})\right)B^3(\lambda_s, x)(\alpha\lambda_s^{2\alpha-1}x^{2\alpha-1}) \\
&+\frac{\psi'(\sum_{i=1}^{n}\phi(e^{1-(\lambda_i x)^\alpha}))}{\psi(\sum_{i=1}^{n}\phi(e^{1-(\lambda_i x)^\alpha}))}B(\lambda_s, x)(\alpha^2\lambda_s^{\alpha-1}x^{\alpha-1}) \\
=\;& J_\psi\left(\sum_{i=1}^{n}\phi\left(e^{1-(\lambda_i x)^\alpha}\right)\right)B(\lambda_s, x)(-\alpha\lambda_s^{\alpha-1}x^\alpha) \\
&\times\left(A(\boldsymbol{\lambda}_{(s,t)}, x) + B(\lambda_s, x)(\alpha\lambda_s^\alpha x^{\alpha-1}) + B(\lambda_t, x)(\alpha\lambda_t^\alpha x^{\alpha-1})\right) \\
&+\frac{B^2(\lambda_s, x)}{B(\boldsymbol{\lambda}, x)}C(\boldsymbol{\lambda}_s e_s, x)(\alpha^2\lambda_s^{2\alpha-1}x^{2\alpha-1}) + \frac{B(\lambda_s, x)}{B(\boldsymbol{\lambda}, x)}(\alpha^2\lambda_s^{\alpha-1}x^{\alpha-1}).
\end{aligned}$$

Owing to ϱ_3, it holds that $C(\boldsymbol{\lambda}, x) \geq C(\lambda_s, x)$. Combining ϱ_1, ϱ_2 with $C(\boldsymbol{\lambda}, x) \geq C(\lambda_s, x)$ we have

$$\begin{aligned}
&\frac{\partial L(x, \boldsymbol{\lambda}, \alpha, \phi)}{\partial\lambda_s} \\
=\;& -J_\psi\left(\sum_{i=1}^{n}\phi\left(e^{1-(\lambda_i x)^\alpha}\right)\right)A(\boldsymbol{\lambda}_{(s,t)}, x)B(\lambda_s, x)(\alpha\lambda_s^{\alpha-1}x^\alpha) \\
&-J_\psi\left(\sum_{i=1}^{n}\phi\left(e^{1-(\lambda_i x)^\alpha}\right)\right)B(\lambda_s, x)B(\lambda_t, x)(\alpha^2 x^{2\alpha-1}\lambda_t^\alpha\lambda_s^{\alpha-1})
\end{aligned}$$

$$
\begin{aligned}
&-\frac{B^2(\lambda_s,x)}{B(\boldsymbol{\lambda},x)}\left(C(\boldsymbol{\lambda},x)-C(\lambda_s,x)\right)(\alpha^2\lambda_s^{2\alpha-1}x^{2\alpha-1})\\
&+\frac{B(\lambda_s,x)}{B(\boldsymbol{\lambda},x)}(\alpha^2\lambda_s^{\alpha-1}x^{\alpha-1})\geq 0.
\end{aligned}
$$

Therefore, $L(x,\boldsymbol{\lambda},\alpha,\psi)$ is increasing in λ_s. Furthermore, for $s,t\in\mathbb{I}_n$ with $s\neq t$, we obtain

$$
\begin{aligned}
&(\lambda_s-\lambda_t)\left(\frac{\partial L(x,\boldsymbol{\lambda},\alpha,\psi)}{\partial\lambda_s}-\frac{\partial L(x,\boldsymbol{\lambda},\alpha,\psi)}{\partial\lambda_t}\right)\\
=\;&(\lambda_s-\lambda_t)\,J_\psi\left(\sum_{i=1}^n\phi\left(e^{1-(\lambda_i x)^\alpha}\right)\right)A(\boldsymbol{\lambda}_{(s,t)},x)\,(-\alpha x^\alpha)\left(B(\lambda_s,x)\lambda_s^{\alpha-1}-B(\lambda_t,x)\lambda_t^{\alpha-1}\right)\\
&+(\lambda_s-\lambda_t)^2\,J_\psi\left(\sum_{i=1}^n\phi\left(e^{1-(\lambda_i x)^\alpha}\right)\right)\alpha^2x^{2\alpha-1}B(\lambda_s,x)B(\lambda_t,x)\lambda_t^{\alpha-1}\lambda_s^{\alpha-1}\\
&-(\lambda_s-\lambda_t)\,\frac{\alpha^2x^{2\alpha-1}}{B(\boldsymbol{\lambda},x)}\\
&\times\left[B^2(\lambda_s,x)\left(C(\boldsymbol{\lambda},x)-C(\lambda_s,x)\right)\lambda_s^{2\alpha-1}-B^2(\lambda_t,x)\left(C(\boldsymbol{\lambda},x)-C(\lambda_t,x)\right)\lambda_t^{2\alpha-1}\right]\\
&+(\lambda_s-\lambda_t)\,\frac{\alpha^2x^{\alpha-1}}{B(\boldsymbol{\lambda},x)}(B(\lambda_s,x)\lambda_s^{\alpha-1}-B(\lambda_t,x)\lambda_t^{\alpha-1}).
\end{aligned}
$$

It is easy to verify that $B(\lambda_s,x)\lambda_s^{\alpha-1}$ is increasing in λ_s for $0<\alpha\leq 1$, therefore

$$
(\lambda_s-\lambda_t)\,J_\psi\left(\sum_{i=1}^n\phi\left(e^{1-(\lambda_i x)^\alpha}\right)\right)A(\boldsymbol{\lambda}_{(s,t)},x)\,(-\alpha x^\alpha)\left(B(\lambda_s,x)\lambda_s^{\alpha-1}-B(\lambda_t,x)\lambda_t^{\alpha-1}\right)\leq 0. \tag{3.1}
$$

By ϱ_1, we have

$$
(\lambda_s-\lambda_t)^2\,J_\psi\left(\sum_{i=1}^n\phi(e^{1-(\lambda_i x)^\alpha})\right)\alpha^2x^{2\alpha-1}B(\lambda_s,x)B(\lambda_t,x)\lambda_t^{\alpha-1}\lambda_s^{\alpha-1}\leq 0. \tag{3.2}
$$

$B^2(\lambda_s,x)\lambda_s^{2\alpha-1}$ is decreasing in λ_s, for $0<\alpha\leq 1/2$, which implies that $0\leq B^2(\lambda_s,x)\lambda_s^{2\alpha-1}\leq B^2(\lambda_t,x)\lambda_t^{2\alpha-1}$. By ϱ_3, for $\lambda_s\geq\lambda_t$, we have $C(\boldsymbol{\lambda},x)-C(\lambda_t,x)\geq C(\boldsymbol{\lambda},x)-C(\lambda_s,x)\geq 0$. It holds that

$$
\begin{aligned}
&-(\lambda_s-\lambda_t)\,\frac{\alpha^2x^{2\alpha-1}}{B(\boldsymbol{\lambda},x)}\\
&\times\left\{B^2(\lambda_s,x)(C(\boldsymbol{\lambda},x)-C(\lambda_s,x))\lambda_s^{2\alpha-1}-B^2(\lambda_t,x)\left(C(\boldsymbol{\lambda},x)-C(\lambda_t,x)\right)\lambda_t^{2\alpha-1}\right\}\leq 0.
\end{aligned} \tag{3.3}
$$

Likewise, for $\lambda_s\geq\lambda_t$, then

$$
(\lambda_s-\lambda_t)\,\frac{\alpha^2x^{\alpha-1}}{B(\boldsymbol{\lambda},x)}(B(\lambda_s,x)\lambda_s^{\alpha-1}-B(\lambda_t,x)\lambda_t^{\alpha-1})\leq 0. \tag{3.4}
$$

Combing (3.1)–(3.3) with (3.4), we conclude that $L(x,\boldsymbol{\lambda},\alpha,\psi)$ is Schur-concave in $(\lambda_1,...,\lambda_n)$. Thus, the desired result follows immediately from Lemma 2. □

Remark 4. *It should be noted that for independent samples, $\psi(x) = e^{-x}$, thus Theorem 1 generalizes Theorem 3.1 of [57] from independent components to the case of dependent samples.*

We present the following Example 5 to illustrate the result of Theorem 3.

Example 5. *Take $\psi(x) = e^{(1-e^x)/\theta}, 0 \le \theta \le 0.5(3-\sqrt{5})$. Set $n = 3$, $\theta = 0.1$, $\alpha = 0.25$, $\boldsymbol{\mu} = (0.2, 0.4, 0.6) \overset{w}{\succeq} (0.4, 0.6, 0.8) = \boldsymbol{\lambda}$. As is seen in Figure 5, the hazard rate function curve of $Y_{1:3}$ is always beneath that of $X_{1:3}$, that is, $X_{1:3} \le_{hr} Y_{1:3}$, which coincides Theorem 3.*

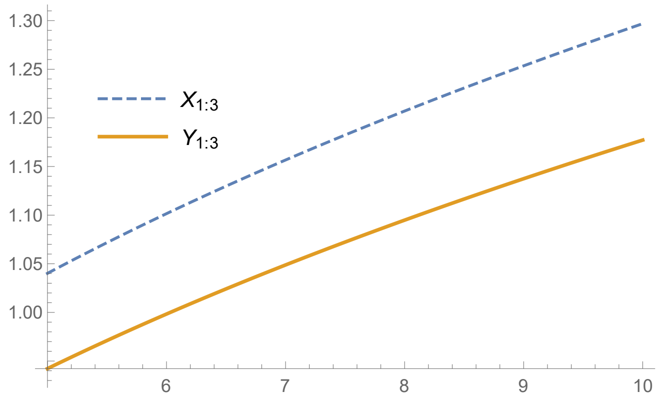

Figure 5. Curves of the hazard rate function $h_{X_{1:3}}(X)$ and $h_{Y_{1:3}}(X)$.

Next counterexample 1 explains the result in Theorem 3 couldn't hold if $\alpha \ge 1/2$.

Counterexample 1. *Let $\psi(x) = e^{(1-e^x)/\theta}, 0 \le \theta \le 0.5(3-\sqrt{5})$. Set $n = 3$, $\theta = 0.382$, $\alpha = 0.55$, $\boldsymbol{\mu} = (0.2, 0.4, 0.6) \overset{w}{\succeq} (0.4, 0.6, 0.8) = \boldsymbol{\lambda}$. As is seen in Figure 6, the difference between the hazard rate functions $h_{X_{1:n}}$ and $h_{Y_{1:n}}$ is not always non-negative and this means that $Y_{1:n} \nleq_{hr} X_{1:n}$. So the result in Theorem 3 couldn't hold if $\alpha \ge 1/2$.*

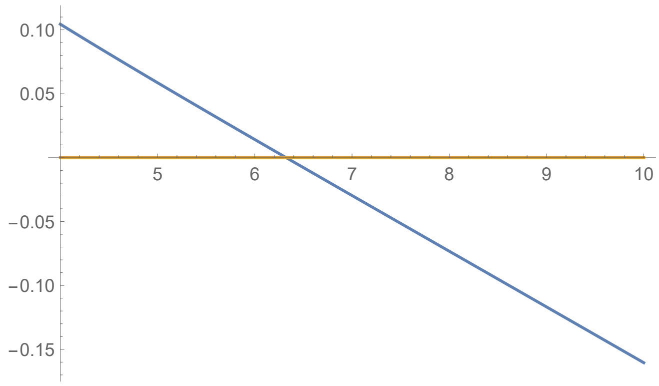

Figure 6. Curve of difference functions of $h_{X_{1:n}}(X) - h_{Y_{1:n}}(X)$.

4. Stochastic comparisons based on variability orders

In this section, we study the convex transform order and the dispersive order of the smallest and the largest order statistics from dependent and heterogeneous lower-truncated Weibull samples.

4.1. Stochastic comparisons based on the convex transform order

First, we compare the smallest and the largest order statistics between dependent and heterogeneous-homogeneous samples with respect to the convex transform order.

Theorem 4. *Let $\boldsymbol{X} \sim LTW(\alpha, \boldsymbol{\lambda}, \psi)$ and $\boldsymbol{Y} \sim LTW(\alpha, \lambda, \psi)$.*

(i) *If ψ is log-concave, then $Y_{1:n} \geq_c X_{1:n}$ for all $0 < \alpha \leq 1$.*
(ii) *$Y_{n:n} \geq_c X_{n:n}$ for all $0 < \alpha \leq 1$.*

Proof. (i) The survival functions of $X_{1:n}$ and $Y_{1:n}$ are given by

$$\bar{F}_{X_{1:n}}(x) = \psi\left(\sum_{i=1}^{n} \phi\left(e^{1-(\lambda_i x)^\alpha}\right)\right), \qquad x \geq max(1/\lambda_1, ..., 1/\lambda_n, 1/\lambda)$$

and

$$\bar{F}_{Y_{1:n}}(x) = \psi\left(n\phi\left(e^{1-(\lambda x)^\alpha}\right)\right), \qquad x \geq max(1/\lambda_1, ..., 1/\lambda_n, 1/\lambda),$$

respectively. It holds that

$$F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(x)) \;=\; \frac{1}{\lambda}\left[1 - \ln\psi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(e^{1-(\lambda_i x)^\alpha}\right)\right)\right]^{\frac{1}{\alpha}} = \frac{1}{\lambda}\left[1 - \ln L(x)\right]^{\frac{1}{\alpha}},$$

where $L(x) = \psi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(e^{1-(\lambda_i x)^\alpha}\right)\right)$. The first and the second partial derivatives of $F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(x))$ with respect to x are

$$\frac{\partial}{\partial x}\left[F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(x))\right] = \frac{1}{\lambda\alpha}\left[1 - \ln L(x)\right]^{\frac{1-\alpha}{\alpha}}\left(\frac{-L'(x)}{L(x)}\right)$$

and

$$\begin{aligned}&\frac{\partial^2}{\partial x^2}\left[F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(x))\right]\\ &= \frac{1}{\lambda\alpha}\left\{\frac{1-\alpha}{\alpha}\left[1 - \ln L(x)\right]^{\frac{1-2\alpha}{\alpha}}\left(\frac{-L'(x)}{L(x)}\right)^2 + \left[1 - \ln L(x)\right]^{\frac{1-\alpha}{\alpha}}\frac{\left(L'(x)\right)^2 - L(x)L''(x)}{L^2(x)}\right\},\end{aligned}$$

respectively. Thus, $F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(x))$ is convex if $0 < \alpha \leq 1$ and $\left(L'(x)\right)^2 - L(x)L''(x) \geq 0$, for which it is sufficient to have $0 < \alpha \leq 1$ and ψ is log-concave.

(ii) The survival functions of $X_{n:n}$ and $Y_{n:n}$ are given by

$$F_{X_{n:n}}(x) = \psi\left(\sum_{i=1}^{n}\phi\left(1 - e^{1-(\lambda_i x)^\alpha}\right)\right), \qquad x \geq max(1/\lambda_1, ..., 1/\lambda_n, 1/\lambda)$$

and

$$F_{Y_{n:n}}(x) = \psi\left(n\phi\left(1 - e^{1-(\lambda x)^\alpha}\right)\right), \qquad x \geq max(1/\lambda_1, ..., 1/\lambda_n, 1/\lambda),$$

respectively. Note that

$$F^{-1}_{Y_{n:n}}\left(F_{X_{n:n}}(x)\right) = \frac{1}{\lambda}\left[1-\ln\left(1-\psi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(1-e^{1-(\lambda_i x)^{\alpha}}\right)\right)\right)\right]^{\frac{1}{\alpha}} = \frac{1}{\lambda}\left[1-\ln\left(1-L(x)\right)\right]^{\frac{1}{\alpha}},$$

where $L(x) = \psi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(1-e^{1-(\lambda_i x)^{\alpha}}\right)\right)$. The first and second partial derivatives of $F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(x))$ with respect to x are

$$\frac{\partial}{\partial x}\left[F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(x))\right] = \frac{1}{\lambda\alpha}\left[1-\ln\left(1-L(x)\right)\right]^{\frac{1-\alpha}{\alpha}}\left(\frac{L^{'}(x)}{1-L(x)}\right)$$

and

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\left[F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(x))\right] &= \frac{1}{\lambda\alpha}\left\{\frac{1-\alpha}{\alpha}\left[1-\ln\left(1-L(x)\right)\right]^{\frac{1-2\alpha}{\alpha}}\left(\frac{L^{'}(x)}{1-L(x)}\right)^2\right.\\ &\left.\quad+\left[1-\ln(1-L(x))\right]^{\frac{1-\alpha}{\alpha}}\frac{(1-L(x))L^{''}(x)+\left(L^{'}(x)\right)^2}{(1-L(x))^2}\right\}.\end{aligned}$$

respectively. Thus, for $0<\alpha\le 1$, $F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(x))$ is convex in $x\in\mathbb{R}^+$.

□

The following Corollary 3 is an obvious consequence of Theorem 4.

Corollary 3. *Let* $\boldsymbol{X}\sim LTW(\alpha,\boldsymbol{\lambda},\psi)$ *and* $\boldsymbol{Y}\sim LTW(\alpha,\lambda,\psi)$.

(i) *If* ψ *is log-concave, then* $Y_{1:n}\ge_{Lorenz(*)}X_{1:n}$ *for all* $0<\alpha\le 1$.
(ii) $Y_{n:n}\ge_{Lorenz(*)}X_{n:n}$ *for all* $0<\alpha\le 1$.

Next counterexample 2 explains the result in Theorem 4 (i) couldn't hold if $\alpha\ge 1$.

Counterexample 2. *Take* $\psi(x) = e^{(1-e^x)/\theta}, 0\le\theta\le 0.5(3-\sqrt{5})$. *Set* $n=2$, $\theta=0.382$, $\alpha=1.51$, $\lambda=0.2$, $\boldsymbol{\lambda}=(0.6,0.8)$. *As is seen in Figure 7, the curve of* $\partial^2F^{-1}_{Y_{1:2}}(F_{X_{1:2}}(x))/\partial x^2$ *is not positive and this means that* $Y_{1:2}\not\ge_c X_{1:2}$. *Hence the result in Theorem 4 couldn't hold if* $\alpha\ge 1$.

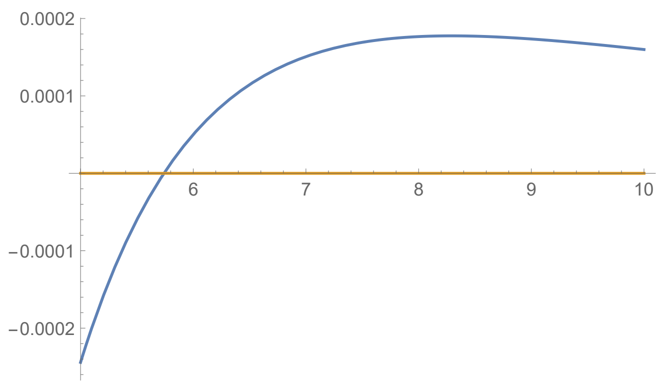

Figure 7. The curve of $\partial^2F^{-1}_{Y_{1:2}}(F_{X_{1:2}}(x))/\partial x^2$.

Theorem 4 indicates that the larger and the smaller order statistics, from dependent homogeneous lower-truncated Weibull samples, are more skewed to the right than those from dependent heterogeneous lower-truncated Weibull samples.

4.2. Stochastic comparisons based on the dispersive order

Aside from the convex transform order, which commonly measures skewness, the dispersive order also measures skewness from a different standpoint. Theorem 5 looks at the dispersive order of the smallest order statistic from dependent heterogeneous lower-truncated Weibull samples and dependent homogeneous lower-truncated Weibull samples with the identical Archimedean copulas.

Theorem 5. *Let $\boldsymbol{X} \sim LTW(\boldsymbol{\alpha}, \lambda, \psi)$ and $\boldsymbol{Y} \sim LTW(\alpha, \lambda, \psi)$. If ψ is log-convex, ψ/ψ' is concave and $0 < \alpha \leq (1/n)\sum_{i=1}^{n} \alpha_i = \bar{\alpha} \leq 1$, $\lambda \geq 0$, then we have $Y_{1:n} \geq_{disp} X_{1:n}$.*

Proof. The survival functions of $X_{1:n}$ and $Y_{1:n}$ are given by

$$\bar{F}_{X_{1:n}}(x) = \psi\left(\sum_{i=1}^{n} \phi\left(e^{1-(\lambda x)^{\alpha_i}}\right)\right), \qquad x \geq 1/\lambda$$

and

$$\bar{F}_{Y_{1:n}}(x) = \psi\left(n\phi\left(e^{1-(\lambda x)^{\alpha}}\right)\right), \qquad x \geq 1/\lambda,$$

respectively. The corresponding density functions of $X_{1:n}$ and $Y_{1:n}$ can be express

$$f_{X_{1:n}} = \psi'\left(\sum_{i=1}^{n} \phi\left(e^{1-(\lambda x)^{\alpha_i}}\right)\right)\sum_{i=1}^{n} \frac{\psi(\phi(e^{1-(\lambda x)^{\alpha_i}}))}{\psi'(\phi(e^{1-(\lambda x)^{\alpha_i}}))}\left(\alpha_i \lambda^{\alpha_i} x^{\alpha_i - 1}\right)$$

and

$$f_{Y_{1:n}} = \psi'\left(n\phi(e^{1-(\lambda x)^{\alpha}})\right) n \frac{\psi(\phi(e^{1-(\lambda x)^{\alpha}}))}{\psi'(\phi(e^{1-(\lambda x)^{\alpha}}))}\left(\alpha \lambda^{\alpha} x^{\alpha-1}\right),$$

respectively. Note that

$$F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(x)) = \frac{1}{\lambda}\left[1 - \ln\psi\left(\frac{1}{n}\sum_{i=1}^{n} \phi\left(e^{1-(\lambda x)^{\alpha_i}}\right)\right)\right]^{\frac{1}{\alpha}},$$

therefore,

$$\begin{aligned} & f_{Y_{1:n}}(F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(x))) \\ = \; & \psi'\left(\sum_{i=1}^{n} \phi\left(e^{1-(\lambda x)^{\alpha_i}}\right)\right) n \frac{\psi(\frac{1}{n}\sum_{i=1}^{n} \phi(e^{1-(\lambda x)^{\alpha_i}}))}{\psi'(\frac{1}{n}\sum_{i=1}^{n} \phi(e^{1-(\lambda x)^{\alpha_i}}))}\alpha\lambda\left[1 - \ln\psi\left(\frac{1}{n}\sum_{i=1}^{n} \phi\left(e^{1-(\lambda x)^{\alpha_i}}\right)\right)\right]^{\frac{\alpha-1}{\alpha}}. \end{aligned}$$

Since ψ is n-monotone, $\phi(e^{1-(\lambda x)^{\alpha}})$ is increasing and convex in α when ψ is log-convex and when $\alpha \leq (1/n)\sum_{i=1}^{n} \alpha_i = \bar{\alpha}$, we have

$$\phi\left(e^{1-(\lambda x)^{\alpha}}\right) \leq \phi\left(e^{1-(\lambda x)^{\bar{\alpha}}}\right) \leq \frac{1}{n}\sum_{i=1}^{n} \phi\left(e^{1-(\lambda x)^{\alpha_i}}\right),$$

which implies

$$x \leq \frac{1}{\lambda}\left[1 - \ln\psi\left(\frac{1}{n}\sum_{i=1}^{n} \phi\left(e^{1-(\lambda x)^{\alpha_i}}\right)\right)\right]^{\frac{1}{\alpha}}.$$

Notice $0 < \alpha \le 1$, then

$$\alpha\lambda^{\alpha}x^{\alpha-1} \ge \alpha\lambda\left[1-\ln\psi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(e^{1-(\lambda x)^{\alpha_i}}\right)\right)\right]^{\frac{\alpha-1}{\alpha}}.$$

It is easy to see that $\alpha\lambda^{\alpha}x^{\alpha-1}$ is increasing and convex in α. When $\alpha \le \frac{1}{n}\sum_{i=1}^{n}\alpha_i = \bar{\alpha}$, we have

$$\alpha\lambda^{\alpha}x^{\alpha-1} \le \bar{\alpha}\lambda^{\bar{\alpha}}x^{\bar{\alpha}-1} \le \frac{1}{n}\sum_{i=1}^{n}\alpha_i\lambda^{\alpha_i}x^{\alpha_i-1}.$$

Thus

$$\alpha\lambda\left[1-\ln\psi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(e^{1-(\lambda x)^{\alpha_i}}\right)\right)\right]^{\frac{\alpha-1}{\alpha}} \le \frac{1}{n}\sum_{i=1}^{n}\alpha_i\lambda^{\alpha_i}x^{\alpha_i-1}. \tag{4.1}$$

Observe ψ/ψ' is concave, then we have

$$-\frac{\psi(\frac{1}{n}\sum_{i=1}^{n}\phi(e^{1-(\lambda x)^{\alpha_i}}))}{\psi'(\frac{1}{n}\sum_{i=1}^{n}\phi(e^{1-(\lambda x)^{\alpha_i}}))} \le -\frac{1}{n}\sum_{i=1}^{n}\frac{\psi(\phi(e^{1-(\lambda x)^{\alpha_i}}))}{\psi'(\phi(e^{1-(\lambda x)^{\alpha_i}}))}. \tag{4.2}$$

Thus

$$\begin{aligned}&\alpha\lambda\left[1-\ln\psi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(e^{1-(\lambda x)^{\alpha_i}}\right)\right)\right]^{\frac{\alpha-1}{\alpha}}\left(-\frac{\psi(\frac{1}{n}\sum_{i=1}^{n}\phi(e^{1-(\lambda x)^{\alpha_i}}))}{\psi'(\frac{1}{n}\sum_{i=1}^{n}\phi(e^{1-(\lambda x)^{\alpha_i}}))}\right)\\ &\le \frac{1}{n}\sum_{i=1}^{n}\alpha_i\lambda^{\alpha_i}x^{\alpha_i-1}\left(-\frac{1}{n}\sum_{i=1}^{n}\frac{\psi(\phi(e^{1-(\lambda x)^{\alpha_i}}))}{\psi'(\phi(e^{1-(\lambda x)^{\alpha_i}}))}\right).\end{aligned} \tag{4.3}$$

By Chebyshev's inequality, we have

$$\frac{1}{n}\sum_{i=1}^{n}\alpha_i\lambda^{\alpha_i}x^{\alpha_i-1}\left(-\frac{1}{n}\sum_{i=1}^{n}\frac{\psi(\phi(e^{1-(\lambda x)^{\alpha_i}}))}{\psi'(\phi(e^{1-(\lambda x)^{\alpha_i}}))}\right) \le \frac{1}{n}\sum_{i=1}^{n}\alpha_i\lambda^{\alpha_i}x^{\alpha_i-1}\left(-\frac{\psi(\phi(e^{1-(\lambda x)^{\alpha_i}}))}{\psi'(\phi(e^{1-(\lambda x)^{\alpha_i}}))}\right). \tag{4.4}$$

From (4.1)–(4.4) and $\psi'\left(\sum_{i=1}^{n}\phi(e^{1-(\lambda x)^{\alpha_i}})\right)$ is non-positive, we have

$$f_{Y_{1:n}}(F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(x))) \le f_{X_{1:n}}(x).$$

Hence, the theorem follows. □

Theorem 6 studies the dispersive order of the smallest order statistic from dependent homogeneous lower-truncated Weibull samples with different Archimedean copulas.

Theorem 6. *Let $\boldsymbol{X} \sim LTW(\alpha, \lambda, \psi_1)$ and $\boldsymbol{Y} \sim LTW(\alpha, \lambda, \psi_2)$. If $0 < \alpha \le 1$, $\lambda \ge 0$, $\psi_2(\phi_2(x)/n)/\psi_1(\phi_1(x)/n)$ is increasing in x, we have $Y_{1:n} \ge_{disp} X_{1:n}$.*

Proof. The survival functions of $X_{1:n}$ and $Y_{1:n}$ are

$$\bar{F}_{X_{1:n}}(x) = \psi_1(n\phi_1(e^{1-(\lambda x)^{\alpha}})), \qquad x \ge 1/\lambda$$

and

$$\bar{F}_{Y_{1:n}}(x) = \psi_2(n\phi_2(e^{1-(\lambda x)^\alpha})), \qquad x \geq 1/\lambda,$$

respectively. The $X_{1:n}$ and $Y_{1:n}$ corresponding density functions are

$$f_{X_{1:n}} = \psi_1' \left(n\phi_1(e^{1-(\lambda x)^\alpha})\right) n \frac{\psi_1(\phi_1(e^{1-(\lambda x)^\alpha}))}{\psi_1'(\phi_1(e^{1-(\lambda x)^\alpha}))} \left(\alpha\lambda^\alpha x^{\alpha-1}\right)$$

and

$$f_{Y_{1:n}} = \psi_2' \left(n\phi_2(e^{1-(\lambda x)^\alpha})\right) n \frac{\psi_2(\phi_2(e^{1-(\lambda x)^\alpha}))}{\psi_2'(\phi_2(e^{1-(\lambda x)^\alpha}))} \left(\alpha\lambda^\alpha x^{\alpha-1}\right),$$

respectively. Note that

$$F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(x)) = \frac{1}{\lambda}\left[1 - \ln\psi_2\left(\frac{1}{n}\phi_2(\psi_1(n\phi_1(e^{1-(\lambda x)^\alpha})))\right)\right]^{\frac{1}{\alpha}},$$

$$\begin{aligned} & f_{Y_{1:n}}\left(F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(x))\right) \\ = \; & \psi_2'\left(\phi_2\left(\psi_1\left(n\phi_1\left(e^{1-(\lambda x)^\alpha}\right)\right)\right)\right) n \\ & \times \frac{\psi_2(\frac{1}{n}\phi_2(\psi_1(n\phi_1(e^{1-(\lambda x)^\alpha}))))}{\psi_2'(\frac{1}{n}\phi_2(\psi_1(n\phi_1(e^{1-(\lambda x)^\alpha}))))} \alpha\lambda \left[1 - \ln\psi_2\left(\frac{1}{n}\phi_2\left(\psi_1(n\phi_1(e^{1-(\lambda x)^\alpha}))\right)\right)\right]^{\frac{\alpha-1}{\alpha}}. \end{aligned}$$

By Lemma 4, observe that $\psi_2(\phi_2(x)/n)/\psi_1(\phi_1(x)/n)$ is increasing in x, we have

$$\psi_2\left(n\phi_2\left(e^{1-(\lambda x)^\alpha}\right)\right) \geq \psi_1\left(n\phi_1\left(e^{1-(\lambda x)^\alpha}\right)\right),$$

which implies

$$e^{1-(\lambda x)^\alpha} \geq \psi_2\left(\frac{1}{n}\phi_2\left(\psi_1\left(n\phi_1\left(e^{1-(\lambda x)^\alpha}\right)\right)\right)\right).$$

Notice $0 < \alpha \leq 1$, we obtain

$$\alpha\lambda\left[1 - \ln\psi_2\left(\frac{1}{n}\phi_2\left(\psi_1\left(n\phi_1\left(e^{1-(\lambda x)^\alpha}\right)\right)\right)\right)\right]^{\frac{\alpha-1}{\alpha}} \leq \alpha\lambda^\alpha x^{\alpha-1}. \tag{4.5}$$

From Lemma 4, by substituting $t = \psi_1\left(n\phi_1\left(e^{1-(\lambda x)^\alpha}\right)\right)$, we have

$$\begin{aligned} & \psi_2'\left(\phi_2\left(\psi_1\left(n\phi_1\left(e^{1-(\lambda x)^\alpha}\right)\right)\right)\right) \frac{\psi_2(\frac{1}{n}\phi_2(\psi_1(n\phi_1(e^{1-(\lambda x)^\alpha}))))}{\psi_2'(\frac{1}{n}\phi_2(\psi_1(n\phi_1(e^{1-(\lambda x)^\alpha}))))} \\ \leq \; & \psi_1'\left(n\phi_1\left(e^{1-(\lambda x)^\alpha}\right)\right) \frac{\psi_1(\phi_1(e^{1-(\lambda x)^\alpha}))}{\psi_1'(\phi_1(e^{1-(\lambda x)^\alpha}))}. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we have $f_{Y_{1:n}}\left(F^{-1}_{Y_{1:n}}\left(F_{X_{1:n}}(x)\right)\right) \leq f_{X_{1:n}}(x)$. Hence the theorem follows.

□

The following Corollary 4 derives from Theorem 5 and Theorem 6. Corollary 4 compares the smallest of two samples, one group from n dependent and heterogeneous lower-truncated Weibull samples and another group from n dependent and homogeneous lower-truncated Weibull samples with different Archimedean copulas.

Corollary 4. *Let $\boldsymbol{X} \sim LTW(\boldsymbol{\alpha}, \lambda, \psi_1)$ and $\boldsymbol{Y} \sim LTW(\alpha, \lambda, \psi_2)$. If ψ_1 is log-convex, $\psi_1/\psi_1^{'}$ is concave and $\psi_2(\phi_2(t)/n)/\psi_1(\phi_1(t)/n)$ is increasing in t, for $0 < \alpha \leq (1/n)\sum_{i=1}^{n}\alpha_i = \bar{\alpha} \leq 1$, $\lambda \geq 0$, then we have $Y_{1:n} \geq_{disp} X_{1:n}$.*

Proof. Let $Z_i \sim LTW(\alpha, \lambda)(i = 1, 2, ..., n)$ and the associated Archimedean copula is with generator ψ_1. Then from Theorem 5, we have $Z_{1:n} \geq_{disp} X_{1:n}$, and from Theorem 6, we have $Y_{1:n} \geq_{disp} Z_{1:n}$.

□

In reliability, the smallest order statistics represents a series system. Corollary 4 indicates that the aging rate of dependent and homogeneous lower-truncated Weibull samples is usually faster than dependent and heterogeneous lower-truncated Weibull samples.

5. Applications

The study of ordering findings of order statistics is critical in many practical domains, including actuarial science, reliability theory, auction theory and multivariate statistics. In this part, we will look at how the established theoretical conclusions may be put to use. In industrial engineering areas. These are referred to as the parallel and series systems, respectively. Consider two series systems with n dependent components that are modeled using lower-truncated Weibull models. More heterogeneous scale parameters in the weakly supermajority order yield a series system with a stochastically longer lifetime, according to Theorem 1. Similar findings may be drawn from Theorems 2 for more heterogeneous reciprocals of location parameters in terms of weak supermajority orders.

6. Conclusions

We studied the usual stochastic order, the hazard rate order, the dispersive order, and the convex transform order in this paper considering dependent and heterogeneous lower-truncated Weibull samples of both the biggest and smallest order statistics. It is highly significant to compare distinct extremes order statistics using the convex transform order, which may be used to examine the relative aging properties of systems from diverse perspectives. These findings generalize several previous findings in the literature. The fundamental restriction of the researched topic may be the absence of treatment of the kth order statistic, which has a significant practical impact. However, due to the complexities of modeling for statistically dependent order statistics, these intriguing problems remain unsolved and demand more investigation.

Acknowledgments

All authors thank the insightful and constructive comments and suggestions from the Editor and three anonymous reviewers, which have greatly improved the presentation of this manuscript. This research was supported by the National Natural Science Foundation of China (No. 11861058 and No. 71471148).

Conflict of interest

The authors declare no conflicts of interest.

References

1. R. Dykstra, S. Kochar, J. Rojo, Stochastic comparisons of parallel systems of heterogeneous exponential components, *J. Stat. Plan. Infer.*, **65** (1997), 203–211. https://doi.org/10.1016/S0378-3758(97)00058-X
2. P. Zhao, N. Balakrishnan, Some characterization results for parallel systems with two heterogeneous exponential components, *Statistics*, **45** (2011), 593–604. https://doi.org/10.1080/02331888.2010.485276
3. C. Li, X. Li, Likelihood ratio order of sample minimum from heterogeneous Weibull random variables, *Stat. Probabil. Lett.*, **97** (2015), 46–53. https://doi.org/10.1016/j.spl.2014.10.019
4. N. Torrado, S. C. Kochar, Stochastic order relations among parallel systems from Weibull distributions, *J. Appl. Probab.*, **52** (2015), 102–116. https://doi.org/10.1239/jap/1429282609
5. B. E. Khaledi, S. Kochar, Weibull distribution: Some stochastic comparisons results, *J. Stat. Plan. Infer.*, **136** (2006), 3121–3129. https://doi.org/10.1016/j.jspi.2004.12.013
6. S. Kochar, M. Xu, Stochastic comparisons of parallel systems when components have proportional hazard rates, *Probab. Eng. Inform. Sci.*, **21** (2007), 597–609. https://doi.org/10.1017/S0269964807000344
7. N. Torrado, Comparisons of smallest order statistics from Weibull distributions with different scale and shape parameters, *J. Korean Stat. Soc.*, **44** (2015), 68–76. https://doi.org/10.1016/j.jkss.2014.05.004
8. S. Kochar, M. Xu, On the skewness of order statistics with applications, *Ann. Oper. Res.*, **212** (2012), 127–138. https://doi.org/10.1007/s10479-012-1212-4
9. N. Balakrishnan, A. Haidari, K. Masoumifard, Stochastic comparisons of series and parallel systems with generalized exponential components, *IEEE T. Reliab.*, **64** (2015), 333–348. https://doi.org/10.1109/TR.2014.2354192
10. L. Fang, X. Zhang, Stochastic comparisons of parallel systems with exponentiated Weibull components, *Stat. Probabil. Lett.*, **97** (2015), 25–31. https://doi.org/10.1016/j.spl.2014.10.017
11. A. Kundu, S. Chowdhury, Ordering properties of order statistics from heterogeneous exponentiated Weibull models, *Stat. Probabil. Lett.*, **114** (2016), 119–127. https://doi.org/10.1016/j.spl.2016.03.017
12. N. K. Hazra, M. R. Kuiti, M. Finkelstein, A. K. Nanda, On stochastic comparisons of minimum order statistics from the location-scale family of distributions, *Metrika*, **81** (2017), 105–123. https://doi.org/10.1007/s00184-017-0636-x

13. N. Balakrishnan, G. Barmalzan, A. Haidari, Modified proportional hazard rates and proportional reversed hazard rates models via marshall olkin distribution and some stochastic comparisons, *J. Korean Stat. Soc.*, **47** (2018), 127–138. https://doi.org/10.1016/j.jkss.2017.10.003

14. G. Barmalzan, S. Kosari, N. Balakrishnan, Usual stochastic and reversed hazard orders of parallel systems with independent heterogeneous components, *Commun. Stat.-Theory Methods*, 2020, 1–26. https://doi.org/10.1080/03610926.2020.1823415

15. G. Barmalzan, S. Ayat, N. Balakrishnan, Stochastic comparisons of series and parallel systems with dependent burr type XII components, *Commun. Stat.-Theory Methods*, 2020, 1–22. https://doi.org/10.1080/03610926.2020.1772307

16. S. Naqvi, W. Ding, P. Zhao, Stochastic comparison of parallel systems with pareto components, *Probab. Eng. Inform. Sci.*, 2021, 1–13. https://doi.org/10.1017/S0269964821000176

17. A. Arriaza, A. Di Crescenzo, M. A. Sordo, A. Suárez-Llorens, Shape measures based on the convex transform order, *Metrika*, **82** (2019), 99–124. https://doi.org/10.1007/s00184-018-0667-y

18. J. Zhang, R. Yan, Stochastic comparison at component level and system level series system with two proportional hazards rate components, *J. Quant. Econ.*, **35** (2018), 91–95. https://doi.org/10.16339/j.cnki.hdjjsx.2018.04.035

19. G. S. Mudholkar, D. K. Srivastava, Exponentiated Weibull family for analyzing bathtub failure-rate data, *IEEE T. Reliab.*, **42** (1993), 299–302. https://doi.org/10.1109/24.229504

20. P. Zhao, N. Balakrishnan, New results on comparisons of parallel systems with heterogeneous gamma components, *Stat. Probabil. Lett.*, **81** (2011), 36–44. https://doi.org/10.1016/j.spl.2010.09.016

21. B. E. Khaledi, S. Farsinezhad, S. C. Kochar, Stochastic comparisons of order statistics in the scale model, *J. Stat. Plan. Infer.*, **141** (2011), 276–286. https://doi.org/10.1016/j.jspi.2010.06.006

22. N. Misra, A. K. Misra, New results on stochastic comparisons of two-component series and parallel systems, *Stat. Probabil. Lett.*, **82** (2012), 283–290. https://doi.org/10.1016/j.spl.2011.10.010

23. N. Balakrishnan, P. Zhao, Hazard rate comparison of parallel systems with heterogeneous gamma components, *J. Multivariate Anal.*, **113** (2013), 153–160. https://doi.org/10.1016/j.jmva.2011.05.001

24. P. Zhao, N. Balakrishnan, Comparisons of largest order statistics from multiple-outlier gamma models, *Methodol. Comput. Appl. Probab.*, **17** (2015), 617–645. https://doi.org/10.1007/s11009-013-9377-0

25. W. Ding, Y. Zhang, P. Zhao, Comparisons of k-out-of-n systems with heterogenous components, *Stat. Probabil. Lett.*, **83** (2013), 493–502. https://doi.org/10.1016/j.spl.2012.10.012

26. Z. Guo, J. Zhang, R. Yan, The residual lifetime of surviving components of coherent system under periodical inspections, *Mathematics*, **8** (2020), 2181. https://doi.org/10.3390/math8122181

27. S. C. Kochar, N. Torrado, On stochastic comparisons of largest order statistics in the scale model, *Commun. Stat.-Theory Methods*, **44** (2015), 4132–4143. https://doi.org/10.1080/03610926.2014.985839

28. A. Panja, P. Kundu, B. Pradhan, Variability and skewness ordering of sample extremes from dependent random variables following the proportional odds model, *arXiv*, 2020. Available from: `https://arxiv.org/abs/2006.04454`.

29. Z. Guo, J. Zhang, R. Yan, On inactivity times of failed components of coherent system under double monitoring, *Probab. Eng. Inform. Sci.*, 2021, 1–18. https://doi.org/10.1017/S0269964821000152

30. A. Kundu, S. Chowdhury, A. K. Nanda, N. K. Hazra, Some results on majorization and their applications, *J. Comput. Appl. Math.*, **301** (2016), 161–177. https://doi.org/10.1016/j.cam.2016.01.015

31. R. Yan, B. Lu, X. Li, On redundancy allocation to series and parallel systems of two components, *Commun. Stat.-Theory Methods*, **48** (2019), 4690–4701. https://doi.org/10.1080/03610926.2018.1500603

32. X. Li, R. Fang, Ordering properties of order statistics from random variables of Archimedean copulas with applications, *J. Multivariate Anal.*, **133** (2015), 304–320. https://doi.org/10.1016/j.jmva.2014.09.016

33. C. Li, R. Fang, X. Li, Stochastic somparisons of order statistics from scaled and interdependent random variables, *Metrika*, **79** (2016), 553–578. https://doi.org/10.1007/s00184-015-0567-3

34. R. Fang, C. Li, X. Li, Stochastic comparisons on sample extremes of dependent and heterogenous observations, *Statistics*, **50** (2016), 930–955. https://doi.org/10.1080/02331888.2015.1119151

35. Y. Zhang, X. Cai, P. Zhao, H. Wang, Stochastic comparisons of parallel and series systems with heterogeneous resilience-scaled components, *Statistics*, **53** (2019), 126–147. https://doi.org/10.1080/02331888.2018.1546705

36. C. Li, X. Li, Hazard rate and reversed hazard rate orders on extremes of heterogeneous and dependent random variables, *Stat. Probabil. Lett.*, **146** (2019), 104–111. https://doi.org/10.1016/j.spl.2018.11.005

37. Y. Zhang, X. Cai, P. Zhao, Ordering properties of extreme claim amounts from heterogeneous portfolios, *ASTIN Bull.*, **49** (2019), 525–554. https://doi.org/10.1017/asb.2019.7

38. G. Barmalzan, S. M. Ayat, N. Balakrishnan, R. Roozegar, Stochastic comparisons of series and parallel systems with dependent heterogeneous extended exponential components under archimedean copula, *J. Comput. Appl. Math.*, **380** (2020), 112965. https://doi.org/10.1016/j.cam.2020.112965

39. L. Zhang, R. Yan, Stochastic comparisons of series and parallel systems with dependent and heterogeneous topp leone generated components, *AIMS Math.*, **6** (2021), 2031–2047. https://doi.org/10.3934/math.2021124

40. M. Zhang, B. Lu, R. Yan, Ordering results of extreme order statistics from dependent and heterogeneous modified proportional (reversed) hazard variables, *AIMS Math.*, **6** (2021), 584–606. https://doi.org/10.3934/math.2021036

41. N. Torrado, Comparing the reliability of coherent systems with heterogeneous, dependent and distribution-free components, *Qual. Technol. Quant. M.*, **18** (2021), 740–770. https://doi.org/10.1080/16843703.2021.1963033

42. E. Amini-Seresht, J. Qiao, Y. Zhang, P. Zhao, On the skewness of order statistics in multiple-outlier PHR models, *Metrika*, **79** (2016), 817–836. https://doi.org/10.1007/s00184-016-0579-7

43. B. E. Khaledi, S. Kochar, Dispersive ordering among linear combinations of uniform random variables, *J. Stat. Plan. Infer.*, **100** (2002), 13–21. https://doi.org/10.1016/S0378-3758(01)00091-X

44. J. Jeon, S. Kochar, C. G. Park, Dispersive ordering some applications and examples, *Stat. Pap.*, **47** (2006), 227–247. https://doi.org/10.1007/s00362-005-0285-4

45. S. Kochar, Stochastic comparisons of order statistics and spacings: A review, *Probab. Stat.*, **2012** (2012), 839473. https://doi.org/10.5402/2012/839473

46. W. Ding, J. Yang, X. Ling, On the skewness of extreme order statistics from heterogeneous samples, *Commun. Stat.-Theory Methods*, **46** (2016), 2315–2331. https://doi.org/10.1080/03610926.2015.1041984

47. J. Wu, M. Wang, X. Li, Convex transform order of the maximum of independent Weibull random variables, *Stat. Probabil. Lett.*, **156** (2020), 108597. https://doi.org/10.1016/j.spl.2019.108597

48. R. Yan, J. Wang, Component level versus system level at active redundancies for coherent systems with dependent heterogeneous components, *Commun. Stat.-Theory Methods*, 2020, 1–21. https://doi.org/10.1080/03610926.2020.1767140

49. R. Yan, J. Zhang, Y. Zhang, Optimal allocation of relevations in coherent systems, *J. Appl. Probab.*, **58** (2021), 1152–1169. https://doi.org/10.1017/jpr.2021.23

50. T. Lando, I. Arab, P. E. Oliveira, Second-order stochastic comparisons of order statistics, *Statistics*, **55** (2021), 561–579. https://doi.org/10.1080/02331888.2021.1960527

51. T. Lando, L. Bertoli-Barsotti, Second-order stochastic dominance for decomposable multiparametric families with applications to order statistics, *Stat. Probabil. Lett.*, **159** (2020), 108691. https://doi.org/10.1016/j.spl.2019.108691

52. J. Zhang, R. Yan, J. Wang, Reliability optimization of parallel-series and series-parallel systems with statistically dependent components, *Appl. Math. Model.*, **102** (2022), 618–639. https://doi.org/10.1016/j.apm.2021.10.003

53. L. Jiao, R. Yan, Stochastic comparisons of lifetimes of series and parallel systems with dependent heterogeneous MOTL-G components under random shocks, *Symmetry*, **13** (2021), 2248. https://doi.org/10.3390/sym13122248

54. B. Lu, J. Zhang, R. Yan, Optimal allocation of a coherent system with statistical dependent subsystems, *Probab. Eng. Inform. Sci.*, 2021, 1–20. https://doi.org/10.1017/S0269964821000437

55. S. Blumenthal, A survey of estimating distributional parameters and sample sizes from truncated samples, In: *Statistical distributions in scientific work*, Springer Netherlands: Dordrecht, **79** (1981), 75–86. https://doi.org/10.1007/978-94-009-8552-0_6

56. S. Blumenthal, R. Marcus, Estimating population size with exponential failure, *J. Am. Stat. Assoc.*, **70** (1975), 913–922. https://doi.org/10.2307/2285457

57. L. Fang, J. Ling, N. Balakrishnan, Stochastic comparisons of series and parallel systems with independent heterogeneous lower-truncated Weibull components, *Commun. Stat.-Theory Methods*, **45** (2015), 540–551. https://doi.org/10.1080/03610926.2015.1099671

58. F. Famoye, Continuous univariate distributions, *Technometrics*, **37** (1995), 466. https://doi.org/10.1080/00401706.1995.10484392
59. D. N. P. Murthy, M. Xie, R. Jiang, *Weibull models*, Wiley, 2004.
60. W. R. Van Zwet, *Convex transformations of random variables*, Mathematisch Centrum Amsterdam, 1964.
61. M. Shaked, J. G. Shanthikumar, *Stochastic orders*, New York: Springer, 2007. https://doi.org/10.1007/978-0-387-34675-5
62. H. Li, X. Li, *Stochastic orders in reliability and risk*, New York: Springer, 2013. https://doi.org/10.1007/978-1-4614-6892-9
63. A. W. Marshall, I. Olkin, B. C. Arnold, *Inequalities: Theory of majorization and its applications*, Springer, 1979.
64. R. B. Nelsen, *An introduction to copulas*, New York: Springer, 2007. https://doi.org/10.1007/0-387-28678-0
65. A. J. McNeil, J. Neslehova, Multivariate Archimedean copulas, d-monotone functions and ℓ_1-norm symmetric distributions, *Ann. Stat.*, **37** (2009), 3059–3097. https://doi.org/10.1214/07-AOS556

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