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Research article

Global behavior of solutions to an SI epidemic model with nonlinear diffusion in heterogeneous environment

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Abstract: In this paper, a nonlinear diffusion SI epidemic model with a general incidence rate in heterogeneous environment is studied. Global behavior of classical solutions under certain restrictions on the coefficients is considered. We first establish the global existence of classical solutions of the system under heterogeneous environment by energy estimate and maximum principles. Based on such estimates, we then study the large-time behavior of the solution of system under homogeneous environment. The model and mathematical results in [M. Kirane, S. Kouachi, Global solutions to a system of strongly coupled reaction-diffusion equations, *Nonlinear Anal.*, **26** (1996), 1387–1396.] are generalized.

Keywords: SI epidemic model; general incidence rate; cross-diffusion; global solution
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1. Introduction

In this paper, we study the following strongly coupled reaction-diffusion model

$$\begin{cases} u_t = d_1\Delta u - \rho u\Delta v - \beta(x)h(u,v), & x\in\Omega, \quad t>0,\\ v_t = (d_2+\gamma u)\Delta v + \beta(x)h(u,v) - \lambda(x)v, & x\in\Omega, \quad t>0,\\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, \quad t>0,\\ u(x,0)=u_0(x)\ge 0, v(x,0)=v_0(x)\ge 0, & x\in\Omega, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ represent susceptible and infected individuals' density respectively at location x and time t; the positive constants d_1 and d_2 denote the corresponding diffusion rates for the susceptible and infected populations; and $\beta(x)$ and $\lambda(x)$ are positive Hölder continuous functions on $\Omega \subset \mathbb{R}^n$ which account for the rates of disease transmission and disease recovery at x, respectively; ρ and γ are nonnegative constant, refers to the spatial influence of infectives, ρ is called cross diffusion coefficient. The term positive cross diffusion coefficient denotes that the susceptible tends to diffuse in the direction of higher concentration of the infected. The density-dependent diffusion terms, given by γu. This form of the diffusion term was experimentally motivated [1] and can be interpreted as a collective behavior for infected populations whose activity increases significantly if they are numerous at a spot. For more details on the biological background, see [1, p.172]. The system is strongly-coupled because of the coupling in the highest derivatives in the first equation. Strongly-coupled systems occur frequently in biological and chemical models and they are notoriously difficult to analyze.

The homogeneous Neumann boundary conditions mean there is no population flux across the boundary $\partial\Omega$ and both the infected and susceptible individuals live in the self-contained environment. From the biological point of view, the incidence function $h(u,v)$ is assumed to be continuously differentiable in $\mathbb{R}_+^2$ and satisfies the following hypotheses (H):

(i) $h(u,0) = h(0,v) = 0$, for all $u, v \geq 0$;

(ii) $h(u,v) > 0$, for all $u, v > 0$;

(iii) $\frac{\partial h(u,v)}{\partial u} > 0$, for all $u \geq 0$, $v > 0$;

(iv) $\frac{\partial h(u,v)}{\partial v} \geq 0$, for all $u, v \geq 0$.

It is easy to check that class of functions $h(u,v)$ satisfying (H) include incidence functions such as

$h(u,v) = h(u,v) = u^p v, p \geq 1, h(u,v) = \frac{uv}{a+v^q}, 0 < q \leq 1$ [Holling types (1959) [2]];

$h(u,v) = \frac{uv}{av+u}$ [Ratio-dependent type (1989) [3]];

$h(u,v) = \frac{uv}{1+au+bv}$ [Beddington-DeAngelis type (1975) [4,5]];

$h(u,v) = \frac{uv}{(1+au)(1+bv)}$ [Crowley-Martin type (1989) [6]];

To the best of our knowledge, there are very few publications (see, for example, [1] and [7]) that consider a SI model with cross-diffusion and density-dependent diffusion. Their model is written by

$$\begin{cases} u_t = d_1\Delta u - \alpha\beta u\Delta v - \beta uv, & x \in \Omega, \quad t > 0, \\ v_t = (d_2 + \alpha\beta u)\Delta v + \beta uv - \lambda v, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) > 0, v(x,0) = v_0(x) > 0, & x \in \Omega. \end{cases} \tag{1.2}$$

In [7], Kirane and Kouachi showed the existence of global solutions of (1.2) when $d_1 \geq d_2$. However, the structure of the nonlinear diffusion terms ($\rho \equiv \gamma = \alpha$) and the reaction terms in the system (1.2) in those works is different than in (1.1).

The coefficient in the model (1.2) are all spatially-independent. However, it has been shown that environmental heterogeneity can make a great difference to infections disease. There has been considerable an SIS epidemic model with heterogeneous environment [21,22]. On the hand, pattern formation, anomalous diffusion, nonlocal dispersal and chemotaxis effect of the epidemic models are paid more and more attention [23–27]. In recent years, the fractional order epidemic models has attracted great interests(see, for example, [36–43]).

Here we mention that global existence and boundedness of classical solutions to SKT competition systems with cross-diffusion

$$\begin{cases} u_t = \Delta\left[(d_1 + a_{11}u + a_{12}v)\, u\right] + \mu_1 u\,(1 - u - a_1 v)\,, & x \in \Omega, t > 0 \\ v_t = \Delta\left[(d_2 + a_{22}v)\, v\right] + \mu_2 v\,(1 - v - a_2 u)\,, & x \in \Omega, t > 0 \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0 \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega \end{cases} \tag{1.3}$$

For the system (1.3), it is straightforward to find out that maximum principles can be applied to the second equation of (1.3) to obtain the boundedness of v . Then the key issue is to establish the boundedness for u. However, for the first equation of (1.3), the boundedness of u cannot be obtained directly by using the maximum principle. This is the biggest obstacle in studying the global existence of system (1.3).

Global existence (in time) of (1.3) has recently received great attention [8, 9, 28–35], The global existence is proved for $n = 2$ by Lou-Ni-Wu [32], thereafter for $n \le 5$ by Le-Nguyen-Nguyen [28] and Choi-Liu-Yamada [8], for $n \le 9$ by phan [9], and the uniform boundedness was asserted when $n \le 9$ and Ω is convex by Tao-Winkler [34]. In these papers, to get the boundedness of the solution of (1.3), authors first obtained L^p -estimates of the solution and then used the Sobolev embeddings. Therefore, they have a restriction on the dimension n of Ω. Recently, the global existence is proved for arbitrary $n \ge 1$ by Hoang-Nguyen-Pan [29]. Their first obtained L^p -estimates for ∇v for large p, and then obtained L^p -estimates of u for large p. In a different approach, Phan [30] who proved the existence of global solutions of (1.3) without any restrictions on space dimension, but with some restrictions on the amplitude of cross-diffusion coefficient. the authors introduce a new function w of the form $w = G(u, v)$ and then use maximum principles to obtain the boundedness of the solution u (1.3). Using test function techniques, Le-Nguyen [31] obtained some global existence results for $n \ge 1$.

Here we should stress that the assumption $a_{11} > 0$ plays a crucial role in the analysis in the aforementioned works. When $a_{11} = 0$, whether the solution of the system (1.3) exists globally in time for $n \ge 1$ is still a well-known open problem made by Y. Yamada in [33]. Liu and Tao [35] recently established the existence of global classical solutions for a simplified parabolic-elliptic system (1.3) when $a_{11} = 0$. However, parabolic-parabolic system (1.3) is still a open problem $a_{11} > 0$.

We also remark that while there have been many results on global solutions to cross-diffusion systems, such as [8–12]. However, in [8–12], the authors utilize the fact that one component is ‘trivially’ uniformly bounded and use it to bound the other component(s).

To understand the global dynamics of the system (1.1), an crucial step is to establish the existence of classical solutions of (1.1). Since the dispersal includes cross-diffusion and density-dependent diffusion, the global well-posedness of system (1.1) is nontrivial.

We would like to mention that (1.1) is very similar to SKT system with cross-diffusion (1.3) when $a_{11} = 0$. Nevertheless, maximum principles can be not applied to the second equation of (1.1) to obtain the boundedness of v We would like to stress that their approachs for SKT systems (1.3) cannot be applied to system (1.1). Our approach first obtain L^2 -estimates for u, v, ∇u and ∇v, then introduce a new function $L(u, v)$, and this function allows us to use maximum principles to get the boundedness of the solution u and v.

Main results. The purpose of this paper is to establish the global existence of classical solutions to (1.1) under heterogeneous environment and the large time behaviour of solution to (1.1) under homogeneous environment. Precisely, we prove the following results:

Theorem 1.1. *Assume that* $u_0, v_0 > 0$ *satisfy the zero Neumann boundary condition and belong to* $C^{2+\delta}(\overline{\Omega})$*, and suppose* $\beta(x), \lambda(x) \in C^{2+\delta}(\overline{\Omega})$ *for some* $0 < \delta < 1$*. Then* (1.1) *possesses a unique non-negative solution* $u, v \in C^{2+\delta, 1+\frac{\delta}{2}}(\overline{\Omega} \times [0, \infty))$ *if* $d_1 \geq d_2$ *and* $\rho \leq \gamma$*.*

Theorem 1.2. *Assume that* $d_1 \geq d_2$*,* $\rho \leq \gamma$ *and* β*,* λ *are positive constants. Then, the problem* (1.1) *possesses a unique non-negative global classical solutions* (u, v) *which satisfies*

$$\|u(\cdot, t) - \overline{u}\|_{L^2(\Omega)} + \|v(\cdot, t)\|_{L^2(\Omega)} \to 0 \quad as \quad t \to \infty. \tag{1.4}$$

Remark 1.3. *Theorem 1.1 also valid for* (1.1) *but with homogeneous Dirichlet boundary condition.*

Remark 1.4. *From Theorem 1.1, it is not difficult to see that the conditions* $d_1 \geq d_2$ *and* $\rho \leq \gamma$ *play crucial roles in the study of global boundedness of solutions to problem* (1.1)*. It is an open question whether solutions of system* (1.1) *with bounded non-negative initial data exist globally for* $d_1 < d_2$ *or* $\rho > \gamma$ *[7]. We believe that the conditions* $d_1 \geq d_2$ *and* $\rho \leq \gamma$ *of Theorem 1.1 are just the technical conditions. To drop these conditions, more new ideas and techniques must be developed, and we expect to completely solve it in the future.*

Remark 1.5. *The model presented by Kirane and Kouachi in [7] is a particular case of our model* (1.1) *if we choose* $\rho = \gamma = \alpha\beta, h(u, v) = \beta uv$*.*

The paper is organized as follows. In Section 2, we introduce some known results as priminaries. In Section 3, we prove Theorem 1.1. In Section 4, the large time behaviour of solution to (1.1) are studied. In Section 5, we give an example to illustrate our theoretical results. Conclusions are drawn in Section 6.

2. Local existence and a priori estimate

2.1. Local existence

For the time-dependent solutions of (1.1), the local existence of non-negative solutions is established by Amann in the seminal papers [15, 16]. The results can be summarized as follows:

Theorem 2.1. *Suppose that* u_0*,* v_0 *are in* $W_p^1(\Omega)$ *for some* $p > n$*. Then* (1.1) *has a unique non-negative smooth solution* u, v *in*

$$C([0, T), W_p^1(\Omega)) \bigcap C^\infty((0, T), C^\infty(\Omega))$$

with maximal existence time T*. Moreover, if the solution* (u, v) *satisfies the estimate*

$$\sup\{\|u(\cdot, t)\|_{W_p^1(\Omega)}, \|v(\cdot, t)\|_{W_p^1(\Omega)} : t \in (0, T)\} < \infty,$$

then $T = \infty$*.*

We denote

$$
\begin{aligned}
&Q_T = \Omega \times [0, T),\\
&\|u\|_{L^{p,q}(Q_T)} = \Big(\int_0^T (\int_\Omega |u(x,t)|^p dx)^{\frac{q}{p}} dt \Big)^{1/q}, L^p(Q_T) := L^{p,p}(Q_T),\\
&\|u\|_{W_p^{2,1}(Q_T)} := \|u\|_{L^p(Q_T)} + \|u_t\|_{L^p(Q_T)} + \|\nabla u\|_{L^p(Q_T)} + \|\nabla^2 u\|_{L^p(Q_T)},
\end{aligned}
$$

T be the maximal existence time for the solution (u, v) of (1.1).

Let Z be a Banach space and $a \in R^+$, $C_B([a, +\infty), Z)$ denote the space of continuous functions such that remains bounded in Z for $t > a$. In order to prove Theorem 1.1, we need the following some preliminary Lemmas.

2.2. Apriori estimates

Lemma 2.2. *Let* $3 < p < \infty$. *Suppose* w *is a solution to the following equation:*

$$
\begin{aligned}
&\frac{\partial w}{\partial t} = a^{ij}(x,t) D_{ij} w + h(x,t) && \text{in } \Omega \times [0, T),\\
&\frac{\partial w}{\partial \nu} = 0 && \text{on } \partial\Omega \times [0, T),\\
&w(x, 0) = w_0(x) && \text{in } \Omega,
\end{aligned}
\tag{2.1}
$$

where $T < \infty$ *and* $\{a^{ij}(x,t)\}_{i,j=1,\dots,N}$ *are bounded continuous functions on* $\overline{Q}_T$ *satisfying*

$$
\lambda |\xi|^2 \le a^{ij}(x,t)\xi_i \xi_j \le \Lambda |\xi|^2, \forall \xi \in \mathbb{R}^N,
$$

where λ, Λ *are positive constants. Suppose* $h \in L^p(\overline{Q}_T)$. *Then there exists a constant* C_p *depending on the bounds of* $\{a^{ij}(x,t)\}_{i,j=1,\dots,N}$, $\lambda, \Lambda, \Omega, T$ *and* p *such that*

$$
\|w\|_{W_p^{2,1}(\overline{Q}_T)} \le C_p \left(\|h\|_{L^p(\overline{Q}_T)} + \|w_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} \right),
\tag{2.2}
$$

where the constant C_p *remains bounded for finite values of* T *and* $w_0(x)$ *satisfies the compatibility condition* $\frac{\partial w_0}{\partial \nu} = 0$ *on* $\partial\Omega$.

This lemma can be found in [17, Theorem 9.1 p.341 and Remark on p.351].

Lemma 2.3. *Let* $\beta, \lambda \in C^{2+\delta}(\overline{\Omega})$, $\gamma \ge \rho$. *Then there exists a positive constant* C *such that*

$$
\begin{aligned}
&\|\nabla u\|_{L^2(Q_T)} \le C, \quad \|\nabla v\|_{L^2(Q_T)} \le C,\\
&\|u\|_{L^2(\Omega)} \le C, \quad \|v\|_{L^2(\Omega)} \le C
\end{aligned}
\tag{2.3}
$$

for any $T > 0$.

Proof. By the first two equations in (1.1) we derive

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\left\{\frac{1}{2}\delta_1u^2+uv+\frac{1}{2}\delta_2v^2+\frac{d_1+d_2}{\rho}u\right\}dx\\
=&\int_\Omega\left\{\delta_1uu_t+vu_t+uv_t+\delta_2vv_t+\frac{d_1+d_2}{\rho}u_t\right\}dx\\
=&-\int_\Omega\left[d_1\delta_1|\nabla u|^2+d_2\delta_2|\nabla v|^2+(d_1+d_2)\nabla u\nabla v\right]dx+\int_\Omega(d_1+d_2)\nabla u\nabla vdx \qquad (2.4)\\
&+\int_\Omega(\gamma-\delta_1\rho)u^2\Delta vdx+\int_\Omega(\delta_2\gamma-\rho)uv\Delta vdx\\
&+\int_\Omega\beta(x)h(u,v)\left\{(1-\delta_1)u+(\delta_2-1)v-\frac{d_1+d_2}{\rho}\right\}dx-\int_\Omega\lambda(x)v(u+\delta_2v)dx.
\end{aligned}
$$

Choosing $\delta_1=\frac{\gamma}{\rho}$, $\delta_2=\frac{\rho}{\gamma}$, we see from condition $\gamma\geq\rho$ that

$$1-\delta_1\leq0,\ \ \delta_2-1\leq0.$$

Here from (2.4) and $\beta(x),\lambda(x)>0$ to gain the estimate

$$\frac{d}{dt}\int_\Omega\left\{\frac{1}{2}\delta_1u^2+uv+\frac{1}{2}\delta_2v^2+\frac{d_1+d_2}{\rho}u\right\}dx\leq-\int_\Omega\left[d_1\delta_1|\nabla u|^2+d_2\delta_2|\nabla v|^2\right]dx. \qquad (2.5)$$

Integrating the above inequality from 0 to t $(t<T)$, we have

$$\int_\Omega\left\{\frac{1}{2}\delta_1u^2+uv+\frac{1}{2}\delta_2v^2+\frac{d_1+d_2}{\rho}u\right\}dx+\int_{Q_t}\left[d_1\delta_1|\nabla u|^2+d_2\delta_2|\nabla v|^2\right]dxdt\leq C, \qquad (2.6)$$

where the constant C depends only on $d_1,\ d_2,\ \gamma,\ \rho,\ \|u_0\|_{L^2(\Omega)},\ \|u_0\|_{L^1(\Omega)}$ and $\|v_0\|_{L^2(\Omega)}$. □

Lemma 2.4. *Let* $\beta,\lambda\in C^{2+\delta}(\overline{\Omega})$. *For any* $0<t<T$, *we have*

$$u,\ v\in C_B(R^+;C(\overline{\Omega})) \qquad (2.7)$$

and

$$\|u\|_{L^p(Q_T)}\leq C,\quad \|v\|_{L^p(Q_T)}\leq C \qquad (2.8)$$

whenever $d_1>d_2$ *and* $\rho\leq\gamma$.

Proof. Define the function

$$L(u,v)=u+\frac{\rho}{\gamma}v+d+d\log(-u/d),$$

where $d=\frac{d_2-d_1}{\gamma}<0$.

Notice that $L(u,v)>0$ for $u,v\in\mathbb{R}^+$, $u\neq-d, v\neq0$ and $L(-d,0)=0$. Now define $E(x,t):=L(u(x,t),v(x,t))$, we have

$$\frac{dE}{dt}\ =\ \left(u+\frac{\rho}{\lambda}v\right)_t+du_t/u$$

$$= \quad d_1(1 + d/u)\Delta u + \frac{d_1\rho}{\lambda}\Delta v + \left(\frac{\rho}{\gamma} - 1 - \frac{d}{u}\right)\beta h(u, v) - \frac{\rho\lambda}{\gamma}v,$$

and

$$\Delta E = (1 + d/u)\Delta u + \frac{\rho}{\gamma}\Delta v - d|\nabla \log u|^2.$$

Therefore

$$\begin{aligned} &E_t - d_1\Delta\Sigma = d_1 d|\nabla \log u|^2 + + \left(\tfrac{\rho}{\gamma} - 1 - \tfrac{d}{u}\right)\beta h(u, v) - \tfrac{\rho\lambda}{\gamma}v, \quad x \in \Omega, \quad t > 0, \\ &\tfrac{\partial E}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t > 0, \\ &E_\delta(x) = u_\delta(x) + \tfrac{\rho}{\gamma}v_\delta(x) + d + d\log(-u_\delta(x)/d) > 0, \quad x \in \Omega, \end{aligned} \tag{2.9}$$

$E_\delta(x)$ is bounded, where $0 < \delta < T$. Since $v \le E$ and $v \in L^\infty((0, +\infty); L^2(\Omega))$. By the maximum principle [19] and the proposition 3.3 of [18], we have

$$E \in C_B(R^+; C(\overline{\Omega})).$$

As $u + H + H\log(-u/H) > 0$, we have

$$0 < v(x, t) < M,$$

and

$$0 < C_0(M) \le u(x, t) \le C_1(M) < +\infty,$$

where M depends only on $\|u_\delta\|_{L^\infty}$ and $\|v_\delta\|_{L^\infty}$, and $C_0(M)$ and $C_1(M)$ are the solutions of

$$M = v + d + d\log(-v/d).$$

By (1.1), we have

$$(u + v)_t = \Delta(d_1 u + d_2 v) + (\rho - \lambda)u\Delta v - \lambda(x)v. \tag{2.10}$$

Multiplying the Eq (2.10) by $\frac{1}{p}(u + v)^{p-1}$ and integrating by parts, using the Young's inequality, $\lambda(x) \in C^{2+\delta}(\overline{\Omega})$ and (2.7), we have

$$\|u + v\|^p_{L^p(\Omega)} \le C\left(\|\nabla u\|^2_{L^2(Q_T)} + \|\nabla v\|^2_{L^2(Q_T)}\right) + \|u_0 + v_0\|^p_{L^p(\Omega)}, \tag{2.11}$$

which implies that (2.8) holds by the Lemma 2.3. □

3. Global existence

Proof of Theorem 1.1. Now, We will divide the proof of Theorem 1.1 into two cases according to $d_1 > d_2$ and $d_1 = d_2$.

Case (a). $d_1 > d_2$.

The second equation of (1.1) can be written as the following form

$$v_t = (d_2 + \gamma u)\Delta v + \beta(x)h(u, v) - \lambda(x)v, \tag{3.1}$$

where $d_2 + \gamma u$ and $\beta(x)h(u,v) - \lambda(x)v$ are bounded in $\overline{Q}_T$ by Lemma 2.4, $\beta, \lambda \in C^{2+\delta}(\overline{\Omega})$ and the assumption (H). Applying the Lemma 2.2 to the Eq (3.1) ensures that $\|v\|_{W_p^{2,1}(Q_T)}$ is bounded, which implies

$$\|v_t\|_{L^p(Q_T)} \le C_{3,1}, \quad \|\Delta v\|_{L^p(Q_T)} \le C_{3,1}, \tag{3.2}$$

where $C_{3,1}$ is a positive constant independent t. It follows from the first equation of (1.1), Lemma 2.4, $\beta, \lambda \in C^{2+\delta}(\overline{\Omega})$, the assumption (H) and (3.2) that

$$\|u_t\|_{L^p(Q_T)} \le C_{3,2}, \quad \|\Delta u\|_{L^p(Q_T)} \le C_{3,2}, \tag{3.3}$$

where $C_{3,2}$ is a positive constant independent t. Therefore, $u, v \in W^{2,1}(Q_T) \hookrightarrow C^{\frac{\sigma}{2},\sigma}(\overline{Q}_T)$. By the Schauder theory for parabolic equations and the bootstrap argument, we have

$$u, v \in C^{2+\delta, 1+\frac{\delta}{2}}(\overline{\Omega} \times [0, \infty)).$$

Case (b). $d_1 = d_2$.

We next consider the case $d_1 = d_2$. By (1.1), we have

$$\left(\frac{\gamma}{\rho}u + v\right)_t = d_1 \Delta\left(\frac{\gamma}{\rho}u + v\right) + \left(1 - \frac{\gamma}{\rho}\right)\beta h(u,v) - \lambda(x)v.$$

Since $\gamma \ge \rho, \beta, \lambda > 0$ and $\beta, \lambda \in C^{2+\delta}(\overline{\Omega})$, using the maximum principle [19] yields

$$\left\|\frac{\gamma}{\rho}u + v\right\|_{L^\infty(\Omega)} \le C_{3,3},$$

where $C_{3,3} > 0$ only dependent $\|u_0\|_{L^\infty(\Omega)}$, $\|v_0\|_{L^\infty(\Omega)}$, d_1, γ and ρ. The rest of the proof is same as in the case $d_1 > d_2$. Finally, by Theorem 2.1 we have (u, v) exists globally in time. The proof of Theorem 1.1 is now complete. □

4. Large time behaviour

Proof of Theorem 1.2. Define the Lyapunov functional

$$V(t) = \int_\Omega \left\{\frac{1}{2}\left(\sqrt{\frac{\gamma}{\rho}}\, u + \sqrt{\frac{\rho}{\gamma}}\, v\right)^2 + \frac{d_1 + d_2}{\rho} u\right\} dx$$

Then

$$\begin{aligned}
\frac{dV}{dt} &= \int_\Omega \left\{\left(\sqrt{\frac{\gamma}{\rho}}\, u + \sqrt{\frac{\rho}{\gamma}}\, v\right)\left(\sqrt{\frac{\gamma}{\rho}}\, u + \sqrt{\frac{\rho}{\gamma}}\, v\right)_t + \frac{d_1 + d_2}{\rho} u_t\right\} dx \\
&= -\int_\Omega \left[d_1 \frac{\gamma}{\rho}|\nabla u|^2 + d_2 \frac{\rho}{\gamma}|\nabla v|^2\right] dx \\
&\quad + \int_\Omega \beta h(u,v)\left\{\left(1 - \frac{\gamma}{\rho}\right)u + \left(\frac{\rho}{\gamma} - 1\right)v - \frac{d_1 + d_2}{\rho}\right\} dx - \int_\Omega \lambda v\left(u + \frac{\rho}{\gamma}v\right) dx
\end{aligned}$$

$$\begin{aligned} &\leq -\int_\Omega \left[d_1 \frac{\gamma}{\rho}|\nabla u|^2 + d_2 \frac{\rho}{\gamma}|\nabla v|^2 \right] dx \\ &:= -\psi(t). \end{aligned} \tag{4.1}$$

Here we use the condition $\gamma \geq \rho$ and the assumption (H). $\psi(t)$ is bounded by Theorem 1.1. Applying [20, Lemma 1] to (4.1), we have

$$\lim_{t\to\infty} \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) dx = 0. \tag{4.2}$$

From (4.2) and the Poincaré inequality, we deduce that

$$\lim_{t\to\infty} \int_\Omega \left\{(u-\overline{u})^2 + (v-\overline{v})^2\right\} dx = 0, \tag{4.3}$$

where $\overline{g} = \frac{1}{|\Omega|}\int_\Omega g dx$ for a function $g \in L^1(\Omega)$.

On the other hand, we claim that

$$\|v(\cdot,t)\|_{L^2(\Omega)} = 0, \quad as \quad t \to \infty. \tag{4.4}$$

To achieve this, suppose that $\|v(t)\|_{L^2(\Omega)}$ does not converge to 0. Then, there would exist a number $K > 0$ and a time sequence $\{t_m\}_{m=1,2,3,\cdots}$ tending to ∞ such that $\|v(t_m)\|_{L^2(\Omega)} \geq K$.

In the meantime, from (1.1) and Theorem 1.1, we have

$$\left|\frac{d}{dt}\|v(t)\|^2_{L^2(\Omega)}\right| \leq C \int_\Omega (|\nabla u|^2 + |\nabla v|^2) dx + C \leq M_0, \quad 0 < t < \infty. \tag{4.5}$$

So, consider for each m, a continuous function $\varphi_m(t)$ for $0 < t < \infty$ such that $\varphi(t) \equiv 0$ for $|t - t_m| \geq \frac{K}{M_0}$, $\varphi_m(t_m) = K$ for $t = t_m$, and $\varphi_m(t)$ is linear for $t_m - \frac{K}{M_0} \leq t \leq t_m$ and for $t_m \leq t \leq t_m + \frac{K}{M_0}$. Then by the mean value theorem, it must hold that $\|v(t)\|^2_{L^2(\Omega)} \geq \varphi_m(t)$ for all $-\infty < t < \infty$. Furthermore, $|v(t)\|^2_{L^2(\Omega)} \geq \sup_m \varphi_m(t)$. But this contradicts $\int_0^\infty \|v(t)\|^2_{L^2(\Omega)} dt < \infty$ by (4.5) .

It follows from (4.3) and (4.4) that

$$\|u(\cdot,t) - \overline{u}\|_{L^2(\Omega)} \to 0, \quad \|v(\cdot,t)\|_{L^2(\Omega)} \to 0, \quad as \quad t \to \infty. \tag{4.6}$$

Thus the proof of Theorem 1.2 is completed. □

5. Examples

5.1. Example 1 (SI model with Beddington-DeAngelis type incidence rate)

Choose $h(u,v) = \frac{uv}{1+au+bv}$, then hypotheses (H) hold. System (1.1) reduces to

$$\begin{cases} u_t = d_1\Delta u - \rho u\Delta v - \dfrac{\beta(x)uv}{1+au+bv}, & x \in \Omega, \quad t > 0, \\ v_t = (d_2 + \gamma u)\Delta v + \dfrac{\beta(x)uv}{1+au+bv} - \lambda(x)v, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{5.1}$$

According to Theorem 1.1 and Theorem 1.2, one can obtain the following.

Theorem 5.1. *Assume that $u_0, v_0 > 0$ satisfy the zero Neumann boundary condition and belong to $C^{2+\delta}(\overline{\Omega})$, and suppose $\beta(x), \lambda(x) \in C^{2+\delta}(\overline{\Omega})$ for some $0 < \delta < 1$. Then (5.1) possesses a unique non-negative solution $u, v \in C^{2+\delta, 1+\frac{\delta}{2}}(\overline{\Omega} \times [0, \infty))$ if $d_1 \geq d_2$ and $\rho \leq \gamma$.*

Theorem 5.2. *Assume that $d_1 \geq d_2$, $\rho \leq \gamma$ and β, λ are positive constants. Then, the problem (5.1) possesses a unique non-negative global classical solutions (u, v) which satisfies*

$$\|u(\cdot, t) - \overline{u}\|_{L^2(\Omega)} + \|v(\cdot, t)\|_{L^2(\Omega)} \to 0 \quad as \quad t \to \infty. \tag{5.2}$$

6. Conclusions

This paper presents a mathematical study on the dynamical behavior of a nonlinear diffusion SI epidemic model with general nonlinear incidence rate of the form $h(u, v)$. The functions $h(u, v)$ includes a number of especial incidence rates. For instance, $h(u, v) = u^p v, p \geq 1$, $h(u, v) = \frac{uv}{a+v^q}, 0 < q \leq 1$, $h(u, v) = \frac{uv}{av+u}$,$h(u, v) = \frac{uv}{1+au+bv}$ and $h(u, v) = \frac{uv}{(1+au)(1+bv)}$. The well-posedness of the model, including local existence, nonnegativity, global existence of solutions under heterogeneous environment and the large time behaviour of solution to (1.1) under homogeneous environment have been established if $d_1 \geq d_2$ and $\rho \leq \gamma$. Our results cover and improve some known results. However, it is an open question whether solutions of system (1.1) with bounded non-negative initial data exist globally for $d_1 < d_2$ or $\rho > \gamma$. We believe that the conditions $d_1 \geq d_2$ and $\rho \leq \gamma$ of Theorem 1.1 are just the technical conditions.

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Conflict of interest

The authors declare no conflicts of interest.

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