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Research article

Well-posedness of initial value problem of Hirota-Satsuma system in low regularity Sobolev space

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Abstract: In this paper, we study the initial value problem of Hirota-Satsuma system:

$$\begin{cases} u_t - \alpha(u_{xxx} + 6uu_x) = 2\beta vv_x, & x \in \mathbb{R},\ t \geq 0, \\ v_t + v_{xxx} + 3uv_x = 0, & x \in \mathbb{R},\ t \geq 0, \\ u(0,x) = \phi(x), \quad v(0,x) = \psi(x), & x \in \mathbb{R}, \end{cases}$$

where $\alpha \in \mathbb{R}, \beta \in \mathbb{R}$; $u = u(x,t)$, $v = v(x,t)$ are real functions. Aided by Fourier restrict norm method, we show that $\forall s > -\frac{1}{8}$ initial value problem (0.1) is locally well-posed in $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ which improved the results of [7].

Keywords: Hirota-Satsuma system; initial value problem; Fourier restrict norm method; local well-posed
Mathematics Subject Classification: 35E15, 35Q53

1. Introduction

Hirota-Satsuma system coupled by two KdV equations:

$$\begin{cases} u_t - \alpha(u_{xxx} + 6uu_x) = 2\beta vv_x, & x \in \mathbb{R},\ t \geq 0, \\ v_t + v_{xxx} + 3uv_x = 0, & x \in \mathbb{R},\ t \geq 0, \end{cases}$$

(where $\alpha \in \mathbb{R}, \beta \in \mathbb{R}$; $u = u(x,t)$, $v = v(x,t)$ are real functions) was introduced by Hirota, Satsuma in [1] to describe the interactions of two long waves with different dispersion relations, also was derived by Hirota R, Ohta Y in [2] as a reduction of a special hierarchy of coupled bilinear equations.

The main progress of soliton solutions of Hirota-Satsuma system is as follows: In 2000, Tam and Ma in [3] considered some particular special expansions in the direct method to derive the one- and three-cKdV soliton solutions with a profile different in form to the classical solitons. In 2003, Hu and Liu in [4] derived generalized M-solitons solutions of the Grammian type by means of a Darboux transformation. In 2020, Prado and Cisneros-Ake in [5] carried out a systematic analysis of multi-soliton solution based on the direct method to fully describe its N+M interacting multisoliton solutions holding a typical hyperbolic profile (For more detail, see [6]).

The progress of well-posedness of Hirota-Satsuma system can be summarized as: In 1994, Feng proved in [7] that Hirota-Satsuma system posed on the whole line is locally well-posed $H^s(\mathbb{R})\times H^s(\mathbb{R})$, if $s > 2$. In 2005, Angulo showed in [8] that Hirota-Satsuma system posed on periodic domain is locally well-posed in $H^s_{periodic}(0, L) \times H^s_{periodic}(0, L)$, for $s \geq 0$, when $\alpha = -1$ and globally well-posed in $H^s_{periodic}(0, L) \times H^s_{periodic}(0, L)$ for $s \geq 1$, when $\alpha \neq -1, 0$. In 2007, Panthee, Silva verified in [9] that Hirota-Satsuma system posed on periodic domain is locally well-posed in $H^s_{periodic}(0, L)\times H^{1+s}_{periodic}(0, L)$, for $s \geq -\frac{1}{2}$ and global well-posed in $H^s_{periodic}(0, L) \times H^{s+1}_{periodic}(0, L)$ for $s \geq -\frac{3}{14}$ when $\alpha = -1$.

In this paper, we will study the initial value problem of Hirota-Satsuma system:

$$\begin{cases} u_t - \alpha(u_{xxx} + 6uu_x) = 2\beta vv_x, & x \in \mathbb{R},\ t \geq 0, \\ v_t + v_{xxx} + 3uv_x = 0, & x \in \mathbb{R},\ t \geq 0, \\ u(0, x) = \phi(x), \quad v(0, x) = \psi(x), & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

As shown in [9] that asymmetrical product space $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ is more suitable to Hirota-Satsuma system than the symmetrical product space $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ since the asymmetry of nonlinear term uv_x.

Definition 1.1. *Let $s \in \mathbb{R}$, $b \in \mathbb{R}$, Bourgain space $X_{s,b}$ associated with $\partial_t \pm \alpha\partial_x^3$ is defined to be the closure of the Schwartz space $S(R^2)$ under the norm:*

$$\|u\|_{X_{s,b}} = \|(1 + |\xi|)^s(1 + |\tau \mp \alpha\xi^3|)^b\mathcal{F}u(\xi, \tau)\|_{L^2_\xi L^2_\tau},$$

where $\langle\cdot\rangle = (1 + |\cdot|)$, $\mathcal{F}u = \widehat{u}(\xi, \tau)$ denote as Fourier transformation of u with respect to t and x.

Obviously, when $s_1 \leq s_2$, $b_1 \leq b_2$, $\|u\|_{X_{s_1,b_1}} \leq \|u\|_{X_{s_2,b_2}}$.

The main result is:

Theorem 1.2. *Let $s > -\frac{1}{8}$. Then for any initial data $(\phi, \psi) \in H^s(\mathbb{R}) \times H^{1+s}(\mathbb{R})$, there exists $T = T(\|(\phi, \psi)\|_{H^s\times H^{1+s}})$, such that there is unique solution of initial value problem (1.1) on [0,T).*

Conservative of mass

$$\frac{1}{2}\int \Big[u^2 + \frac{2}{3}\beta v^2\Big]dx,$$

conservative of energy

$$\int \Big[\frac{1 + \alpha}{2}u_x^2 + \beta v_x^2 - (1 + \alpha)u^3 - \beta uv^2\Big]dx$$

and local well-posedness (Theorem 1.2) imply that: For $\alpha = -1$ and $\beta > 0$, initial value problem (1.1) is globally well-posed in $H^s(\mathbb{R}) \times H^{1+s}(\mathbb{R})$ if $s \geq 0$.

The following sections are arranged as follows: Bilinear estimate will be established in Section 2 which is the core of the Fourier restriction norm method; Locally well-posedness will be proved in Section 3 by Banch's fixed point theorem; We give some remarks in Section 4 to point out some simple facts about the Hirota-Satsuma system.

In the following, without lose of generalization, we assume that $\alpha = -1, \beta = 1$.

2. Bilinear estimates

2.1. Some lemmas

D^s denote the s-order derivative defined by:

$$\mathcal{F}(D^s f)(\xi) = |\xi|^s \mathcal{F} f(\xi), \quad \forall f \in S(\mathbb{R}).$$

Lemma 2.1. *Denote* $\widehat{F_\rho}(\xi,\tau) = \frac{f(\xi,\tau)}{(1+|\tau-\xi^3|)^\rho}$, *then*

(1) If $\rho > \frac{1}{2}$, *then*

$$\|\chi(\xi) F_\rho\|_{L_x^2 L_t^\infty} \le C\|f\|_{L_\xi^2 L_\tau^2}, \tag{2.1}$$

where $\chi \in C_0^\infty$ *satisfying: When* $|\xi| \le 1$, $\chi(\xi) = 1$*; when* $|\xi| > 2$ *then* $\chi(\xi) = 0$.

(2) If $\rho > \frac{3}{8}$, $0 \le \theta \le \frac{1}{8}$, *then*

$$\|D_x^\theta F_\rho\|_{L_x^4 L_t^4} \le C\|f\|_{L_\xi^2 L_\tau^2}. \tag{2.2}$$

(3) If $\rho > \frac{5}{12}$, *then*

$$\|F_\rho\|_{L_x^4 L_t^6} \le C\|f\|_{L_\xi^2 L_\tau^2}. \tag{2.3}$$

(4) If $\rho > \frac{\theta}{2}$, *where* $\theta \in [0,1]$,*then*

$$\|D_x^\theta F_\rho\|_{L_x^{\frac{2}{1-\theta}} L_t^2} \le C\|f\|_{L_\xi^2 L_\tau^2}. \tag{2.4}$$

(5) If $\rho > \frac{1}{3}$, *then*

$$\|D_x^{\frac{1}{4}} F_\rho\|_{L_x^4 L_t^3} \le C\|f\|_{L_\xi^2 L_\tau^2}. \tag{2.5}$$

Proof. (2.1)–(2.5) are Lemmas 2.3–2.7 in [10]. □

Lemma 2.2. *Assume* f, f_1, f_2 *are Schwartz functions, then*

$$\int_* \overline{\widehat{f}}(\xi,\tau)\widehat{f_1}(\xi_1,\tau_1)\widehat{f_2}(\xi_2,\tau_2)d\delta = \int_{R\times R} \overline{f} f_1 f_2(x,t)dxdt,$$

where $\int_* d\delta = \int_{\xi=\xi_1+\xi_2,\ \tau=\tau_1+\tau_2} d\xi_1 d\xi_2 d\tau_1 d\tau_2.$

Let Z be Abelian addition group with invariable measure $d\xi$. For integer $k \ge 2$, we denote $\Gamma_k(Z)$ as the hyperplane:

$$\Gamma_k(Z) = \{(\xi_1,\xi_2,\cdots,\xi_k) \in Z^k,\ \xi_1+\xi_2+\cdots+\xi_k = 0\}.$$

Define $[k,Z]$-multiplier as function $m : \Gamma_k(Z) \mapsto C$. If m is $[k,Z]$-multiplier, define $\|m\|_{[k,Z]}$ the norm of $[k,Z]$-multiplier as the infimum of C such that

$$\Big|\int_{\Gamma_k(Z)} m(\xi)\prod_{j=1}^k f_j\Big| \le C\prod_{j=1}^k \|f_j\|_{L^2(Z)}.$$

Lemma 2.3. *If* $m(\xi)$ *and* $M(\xi)$ *both are* $[k,Z]$*-multipliers, and* $\forall\ \xi \in \Gamma_k(Z)$, $|m(\xi)| \le |M(\xi)|$, *then*

$$\|m\|_{[k,Z]} \le \|M\|_{[k,Z]}.$$

Proof. See [11] for the detail. □

2.2. Bilinear estimates

Proposition 2.4. *If* $s \geq -\frac{1}{8}$, $\frac{1}{2} < b < \frac{9}{16}$, *then* $\forall\, b' > \frac{1}{2}$, *we have*

$$\|(\partial_x u_1)u_2\|_{X_{1+s,b-1}} \leq C\|u_1\|_{X_{1+s,b'}}\|u_2\|_{X_{s,b'}} \tag{2.6}$$

and

$$\|\partial_x(u_1u_2)\|_{X_{s,b-1}} \leq C\|u_1\|_{X_{s,b'}}\|u_2\|_{X_{s,b'}}. \tag{2.7}$$

Proof. It is enough to prove (2.6). Since the proof of (2.7) is just a minor modification of that of (2.6). Besides, it is enough to show the case of $s \leq 0$. Since when $s > 0$, we have:

$$\langle\xi\rangle^s \leq \langle\xi_1\rangle^s\langle\xi_2\rangle^s.$$

This inequality and the results of $s = 0$ implies the result of $s > 0$.

By Plancherel Theorem, in order to prove (2.6), it is enough to prove

$$\begin{aligned}
I &= \int_{\Gamma_3(\mathbb{R}\times\mathbb{R})} \frac{\langle\xi\rangle^{1+s}\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}\frac{|\xi_1|f_1(\xi_1,\tau_1)}{\langle\xi_1\rangle^{1+s}\langle\sigma_1\rangle^{b'}}\frac{f_2(\xi_2,\tau_2)}{\langle\xi_2\rangle^s\langle\sigma_2\rangle^{b'}}d\delta\\
&= \int_{\Gamma_3(\mathbb{R}\times\mathbb{R})} \frac{\langle\xi\rangle^{1+s}|\xi_1|}{\langle\sigma\rangle^{1-b}\langle\xi_1\rangle^{1+s}\langle\sigma_1\rangle^{b'}\langle\xi_2\rangle^s\langle\sigma_2\rangle^{b'}}\overline{f}(\xi,\tau)f_1(\xi_1,\tau_1)f_2(\xi_2,\tau_2)d\delta\\
&\leq C\left\|\frac{\langle\xi\rangle^{1+s}|\xi_1|}{\langle\sigma\rangle^{1-b}\langle\xi_1\rangle^{1+s}\langle\sigma_1\rangle^{b'}\langle\xi_2\rangle^s\langle\sigma_2\rangle^{b'}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}\|f\|_{L^2_\xi L^2_\tau}\Pi_{j=1}^2\|f_j\|_{L^2_\xi L^2_\tau},
\end{aligned}$$

where $\bar{f} \in L^2(\mathbb{R}^2)$ and $\bar{f} \geq 0$;

$$\begin{aligned}
&f_1 = \langle\xi_1\rangle^{1+s}\langle\sigma_1\rangle^{b'}\widehat{u_1}(\xi_1,\tau_1); \quad f_2 = \langle\xi_2\rangle^s\langle\sigma_2\rangle^{b'}\widehat{u_2}(\xi_2,\tau_2);\\
&\xi = \xi_1 + \xi_2,\ \tau = \tau_1 + \tau_2; \quad \sigma = \tau - \xi^3,\ \sigma_1 = \tau_1 - \xi_1^3;\\
&\sigma_2 = \tau_2 - \xi_2^3.
\end{aligned}$$

By the definition of $[k, Z]$-multiplier, if

$$\left\|\frac{\langle\xi\rangle^{1+s}|\xi_1|}{\langle\sigma\rangle^{1-b}\langle\xi_1\rangle^{1+s}\langle\sigma_1\rangle^{b'}\langle\xi_2\rangle^s\langle\sigma_2\rangle^{b'}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]} \leq C,$$

then (2.6) holds.

By symmetry, it is enough to consider $|\xi_1| \leq |\xi_2|$. Let $r = -s$, then $\frac{1}{8} > r \geq 0$.

Denote $\widehat{F_\rho}(\xi,\tau) = \frac{\bar{f}(\xi,\tau)}{(1+|\tau-\xi^3|)^\rho}$, $\widehat{F_\rho^j}(\xi,\tau) = \frac{f_j(\xi,\tau)}{(1+|\tau-\xi^3|)^\rho}$, $j = 1, 2$.

Case 1. $|\xi| \leq 2$.

Subcase 1.1. $|\xi_1| \leq 1$. We have $|\xi_2| = |\xi - \xi_1| \leq |\xi| + |\xi_1| \leq 3$, thus,

$$\begin{aligned}
I &= \int_{\Gamma_3(\mathbb{R}\times\mathbb{R})} \frac{\chi_{|\xi|\leq 2}\overline{f}(\xi,\tau)}{\langle\xi\rangle^{r-1}\langle\sigma\rangle^{1-b}}\frac{\chi_{|\xi_1|\leq 1}|\xi_1|\langle\xi_1\rangle^{r-1}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\leq 3}\langle\xi_2\rangle^r f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&\leq C\int_{\Gamma_3(\mathbb{R}\times\mathbb{R})} \frac{\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}\frac{f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta
\end{aligned}$$

$$
\begin{aligned}
&\leq C\int \overline{F}_{1-b}\cdot F^1_{b'}\cdot F^2_{b'}(x,t)dxdt\\
&\leq C\|F_{1-b}\|_{L^2_xL^2_t}\|F^1_{b'}\|_{L^4_xL^4_t}\|F^2_{b'}\|_{L^4_xL^4_t}\\
&\leq C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}.
\end{aligned}
$$

We applied (2.2) of Lemma 2.1 and Lemma 2.2 here.

Subcase 1.2. $|\xi_1|\geq 1$. By symmetrical assumption, $|\xi_2|\geq 1$. For $r\leq \frac{1}{8}$, we have

$$
\begin{aligned}
I &= \int_{\Gamma_3(\mathbb{R}\times\mathbb{R})}\frac{\chi_{|\xi|\leq 2}\overline{f}(\xi,\tau)}{\langle\xi\rangle^{r-1}\langle\sigma\rangle^{1-b}}\frac{\chi_{|\xi_1|\geq 1}|\xi_1|\langle\xi_1\rangle^{r-1}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\geq 1}\langle\xi_2\rangle^{r}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&\leq C\int_{\Gamma_3(\mathbb{R}\times\mathbb{R})}\frac{\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}\frac{\chi_{|\xi_1|\geq 1}|\xi_1|^{r}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\geq 1}|\xi_2|^{r}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&\leq C\int_{\Gamma_3(\mathbb{R}\times\mathbb{R})}\frac{\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}\frac{|\xi_1|^{\frac{1}{8}}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{|\xi_2|^{\frac{1}{8}}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&= C\int \overline{F}_{1-b}\cdot D_x^{\frac{1}{8}}F^1_{b'}\cdot D_x^{\frac{1}{8}}F^2_{b'}(x,t)dxdt\\
&\leq C\|F_{1-b}\|_{L^2_xL^2_t}\|D_x^{\frac{1}{8}}F^1_{b'}\|_{L^4_xL^4_t}\|D_x^{\frac{1}{8}}F^2_{b'}\|_{L^4_xL^4_t}\\
&\leq C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}.
\end{aligned}
$$

We applied (2.2) of 2.1 and Lemma 2.2 here.

Case 2. $|\xi|\geq 2$.

Case 2.1. $|\xi_1|\leq 1$. We have $|\xi_2|=|\xi-\xi_1|\geq|\xi|-|\xi_1|\geq 1$, thus

$$
\begin{aligned}
I &= \int_{\Gamma_3(\mathbb{R}\times\mathbb{R})}\frac{\chi_{|\xi|\geq 2}\overline{f}(\xi,\tau)}{\langle\xi\rangle^{r-1}\langle\sigma\rangle^{1-b}}\frac{\chi_{|\xi_1|\leq 1}|\xi_1|\langle\xi_1\rangle^{r-1}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\geq 1}\langle\xi_2\rangle^{r}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&\leq C\int_{\Gamma_3(\mathbb{R}\times\mathbb{R})}\frac{\chi_{|\xi|\geq 2}|\xi|^{1-r}\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}\frac{\chi_{|\xi_1|\leq 1}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\geq 1}|\xi_2|^{r}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&\leq C\int_{\Gamma_3(\mathbb{R}\times\mathbb{R})}\frac{\chi_{|\xi|\geq 2}|\xi|^{1-r}\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}\frac{\chi_{|\xi_1|\leq 1}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\geq 1}|\xi_2|^{r}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&\leq C\int D_x^{1-r}\overline{F}_{1-b}\cdot\chi_{|\xi_1|\leq 1}F^1_{b'}\cdot D_x^{r}F^2_{b'}(x,t)dxdt\\
&\leq C\|D_x^{1-r}F_{1-b}\|_{L_x^{\frac{2}{r}}L^2_t}\|\chi_{|\xi_1|\leq 1}F^1_{b'}\|_{L^2_xL^\infty_t}\|D_x^{r}F^2_{b'}\|_{L_x^{\frac{1}{1-r}}L^2_t}\\
&\leq C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}.
\end{aligned}
$$

Here (2.1) and (2.4) of Lemma 2.1 and Lemma 2.2 are used. Besides, $b<\frac{9}{16}$ is also required.

Case 2.2. $|\xi_1|\geq 1$. By symmetrical assumption, $1\leq|\xi_1|\leq|\xi_2|$.

Since $(\tau_1-\xi_1^3)+(\tau_2-\xi_2^3)-(\tau-\xi^3)=3\xi\xi_1\xi_2$, at lease one of the following 3 cases will occur:

$$
\begin{aligned}
&(a)\quad |\tau-\xi^3|\geq|\xi||\xi_1||\xi_2|,\\
&(b)\quad |\tau_1-\xi_1^3|\geq|\xi||\xi_1||\xi_2|,\\
&(c)\quad |\tau_2-\xi_2^3|\geq|\xi||\xi_1||\xi_2|.
\end{aligned}
$$

By this fact, we divide Case 2.2 into 3 different subcases as follows:

Case 2.2.1. When (a) occurs. If $r+b-1\le\frac{1}{8}$ and $r\ge b>\frac{1}{2}$, then

$$\begin{aligned}
I &= \int_{\Gamma_3(\mathbb{R}\times\mathbb{R})}\frac{\chi_{|\xi|\ge 2}\overline{f}(\xi,\tau)}{\langle\xi\rangle^{r-1}\langle\sigma\rangle^{1-b}}\frac{\chi_{|\xi_1|\ge 1}|\xi_1|\langle\xi_1\rangle^{r-1}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\ge 1}\langle\xi_2\rangle^{r}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&\le C\int_{\Gamma_3(\mathbb{R}\times\mathbb{R})}\frac{\chi_{|\xi|\ge 2}|\xi|^{1-r}\overline{f}(\xi,\tau)}{(|\xi||\xi_1||\xi_2|)^{1-b}}\frac{\chi_{|\xi_1|\ge 1}|\xi_1|^{r}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\ge 1}|\xi_2|^{r}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&\le C\int_{\Gamma_3(\mathbb{R}\times\mathbb{R})}\chi_{|\xi|\ge 2}|\xi|^{b-r}\overline{f}(\xi,\tau)\frac{\chi_{|\xi_1|\ge 1}|\xi_1|^{r+b-1}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\ge 1}|\xi_2|^{r+b-1}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&\le C\int_{\Gamma_3(\mathbb{R}\times\mathbb{R})}\overline{f}(\xi,\tau)\frac{|\xi_1|^{\frac{1}{8}}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{|\xi_2|^{\frac{1}{8}}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&= C\int\overline{F}_0\cdot D_x^{\frac{1}{8}}F^1_{b'}\cdot D_x^{\frac{1}{8}}F^2_{b'}(x,t)dxdt\\
&\le C\|F_0\|_{L^2_xL^2_t}\|D_x^{\frac{1}{8}}F^1_{b'}\|_{L^4_xL^4_t}\|D_x^{\frac{1}{8}}F^2_{b'}\|_{L^4_xL^4_t}\\
&\le C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}.
\end{aligned}$$

Here (2.2) of Lemma 2.1 and Lemma 2.2 are used.

The above results implies that if $r+b-1\le\frac{1}{8}$ and $r\ge b>\frac{1}{2}$, then

$$\left\|\frac{\langle\xi_1\rangle^{r-1}|\xi_1|\langle\xi_2\rangle^{r}}{\langle\sigma\rangle^{1-b}\langle\xi\rangle^{r-1}\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b'}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}\le C. \tag{2.8}$$

By Lemma 2.3, when $r\le\frac{1}{8}$, (2.8) still holds. Indeed, since $\xi=\xi_1+\xi_2$, we have $\langle\xi\rangle\le\langle\xi_1\rangle\langle\xi_2\rangle$. If $r_1\le r_2$, then

$$\begin{aligned}
m &= \frac{\langle\xi_1\rangle^{r_1-1}|\xi_1|\langle\xi_2\rangle^{r_1}}{\langle\sigma\rangle^{1-b}\langle\xi\rangle^{r_1-1}\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b'}}=\frac{\langle\xi_1\rangle^{r_1}\langle\xi_2\rangle^{r_1}}{\langle\xi\rangle^{r_1}}\frac{\langle\xi_1\rangle^{-1}|\xi_1|}{\langle\sigma\rangle^{1-b}\langle\xi\rangle^{-1}\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b'}}\\
&\le\frac{\langle\xi_2\rangle^{r_2}\langle\xi_2\rangle^{r_2}}{\langle\xi\rangle^{r_2}}\frac{\langle\xi_1\rangle^{-1}|\xi_1|}{\langle\sigma\rangle^{1-b}\langle\xi\rangle^{-1}\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b'}}=\frac{\langle\xi_1\rangle^{r_2-1}|\xi_1|\langle\xi_2\rangle^{r_2}}{\langle\sigma\rangle^{1-b}\langle\xi\rangle^{r_2-1}\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b'}}=M.
\end{aligned}$$

Case 2.2.2. When (b) occurs. If $r+b'\ge 1$, $0<r-b'\le\frac{1}{16}$, we have

$$\begin{aligned}
I &= \int_{\Gamma_3(\mathbb{R}\times\mathbb{R})}\frac{\chi_{|\xi|\ge 2}\overline{f}(\xi,\tau)}{\langle\xi\rangle^{r-1}\langle\sigma\rangle^{1-b}}\frac{\chi_{|\xi_1|\ge 1}|\xi_1|\langle\xi_1\rangle^{r-1}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\ge 1}\langle\xi_2\rangle^{r}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&\le C\int_{\Gamma_3(\mathbb{R}\times\mathbb{R})}\frac{\chi_{|\xi|\ge 2}|\xi|^{1-r}\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}\frac{\chi_{|\xi_1|\ge 1}|\xi_1|^{r}f_1(\xi_1,\tau_1)}{(|\xi||\xi_1||\xi_2|)^{b'}}\frac{\chi_{|\xi_2|\ge 1}|\xi_2|^{r}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&\le C\int_{\Gamma_3(\mathbb{R}\times\mathbb{R})}\frac{\chi_{|\xi|\ge 2}|\xi|^{1-b'-r}\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}\cdot\chi_{|\xi_1|\ge 1}|\xi_1|^{r-b'}f_1(\xi_1,\tau_1)\cdot\frac{\chi_{|\xi_2|\ge 1}|\xi_2|^{r-b'}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&\le C\int_{\Gamma_3(\mathbb{R}\times\mathbb{R})}\frac{\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}\cdot f_1(\xi_1,\tau_1)\cdot\frac{|\xi_2|^{2(r-b')}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&= C\int\overline{F}_{1-b}\cdot F^1_0\cdot D_x^{\frac{1}{8}}F^2_{b'}(x,t)dxdt
\end{aligned}$$

$$\leq \quad C\|F_{1-b}\|_{L_x^4L_t^4}\|F_0^1\|_{L_x^2L_t^2}\|D_x^{\frac{1}{8}}F_{b'}^2\|_{L_x^4L_t^4}$$
$$\leq \quad C\|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2},$$

where (2.2) of Lemma 2.1 and Lemma 2.2 is used here. Besides, it is required that $b < \frac{5}{8}$.

When $r \leq \frac{1}{8}$, the results is implied by Lemma 2.3.

Case 2.2.3. When (c) occurs. The proof is similar to Case 2.2.2, we omit the detail. □

3. Proof of the main theorem

Take $\theta \in C_0^\infty(\mathbb{R})$ such that: When $t \in [-\frac{1}{2}, \frac{1}{2}]$, $\theta \equiv 1$ and supp $\theta \subseteq (-1,\ 1)$. Denote $\theta_\delta(t) = \theta(\frac{t}{\delta})$. Let $U(t)$ $(t \in \mathbb{R})$ denote fundamental solution operator of the Airy equation: $v_t \pm v_{xxx} = 0$:

$$U(t)\varphi = \int_{-\infty}^{\infty} e^{i(x\xi \mp t\xi^3)}\widehat{\varphi}(\xi)d\xi,\ \forall \varphi \in H^s(\mathbb{R}),\ s \in \mathbb{R}.$$

Lemma 3.1. *Let $s \in R$, $\frac{1}{2} < b < b' \leq 1$, $0 < \delta \leq 1$, then*

$$\|\theta_\delta(t)U(t)u_0\|_{X_{s,b}} \leq C\delta^{\frac{(1-2b)}{2}}\|u_0\|_{H^s}, \tag{3.1}$$

$$\|\theta_\delta(t)\int_0^t U(t-s)F(s)ds\|_{X_{s,b}} \leq C\delta^{\frac{(1-2b)}{2}}\|F\|_{X_{s,b-1}}, \tag{3.2}$$

$$\|\theta_\delta(t)F\|_{X_{s,b-1}} \leq C\delta^{b'-b}\|F\|_{X_{s,b'-1}}. \tag{3.3}$$

Proof. See [10]. □

In the following, we will give the

Proof of Theorem 1.1:

Proof. For $s \geq -\frac{1}{8}$, let $(\phi, \psi) \in H^s \times H^{1+s}$ and $\|(\phi, \psi)\|_{H^s \times H^{1+s}} \equiv \|\phi\|_{H^s} + \|\psi\|_{H^{1+s}} = r$. Define

$$B_r = \{(u, v) \in X_{s,b} \times X_{1+s,b} :\ \|(u, v)\|_{X_{s,b}\times X_{1+s,b}} \leq 2Cr\},$$

then B_r is Banach space, whose norm is

$$\|(u, v)\|_{X_{s,b}\times X_{1+s,b}} \equiv \|u\|_{X_{s,b}} + \|v\|_{X_{1+s,b}}.$$

For $(u, v) \in B_r$, define the mapping

$$\begin{cases} \Phi_\phi[u, v] = \theta_1(t)U(t)\phi - \theta_1(t)\int_0^t U(t-s)\theta_\delta(t)[6uu_x - 2\beta vv_x](s)ds, \\ \Psi_\psi[u, v] = \theta_1(t)U(t)\psi - \theta_1(t)\int_0^t U(t-s)\theta_\delta(t)[3uv_x](s)ds. \end{cases}$$

We will prove that $\Phi \times \Psi_{(\phi,\psi)}[u, v]$ map B_r into B_r.

By (3.1)–(3.3) in Lemma 3.1 and bi-linear estimate (2.7), there exists b, b' satisfying $\frac{1}{2} < b < b' \leq \frac{9}{16}$ such that

$$\begin{aligned} \|\Phi_\phi[u, v]\|_{X_{s,b}} &\leq \|\theta_1(t)U(t)\phi\|_{X_{s,b}} + \left\|\theta_1(t)\int_0^t U(t-s)\theta_\delta(t)[6uu_x - 2\beta vv_x](s)ds\right\|_{X_{s,b}} \\ &\leq C\|\phi\|_{H^s} + C\|\theta_\delta(t)uu_x\|_{X_{s,b-1}} + C\|\theta_\delta(t)vv_x\|_{X_{s,b-1}} \end{aligned}$$

$$\begin{aligned}
&\le C\|\phi\|_{H^s} + C\delta^{b'-b}\|uu_x\|_{X_{s,b'-1}} + C\delta^{b'-b}\|vv_x\|_{X_{s,b'-1}} \\
&\le C\|\phi\|_{H^s} + C\delta^{b'-b}\|u\|^2_{X_{s,b}} + C\delta^{b'-b}\|v\|^2_{X_{s,b}} \\
&\le C\|\phi\|_{H^s} + C\delta^{b'-b}\|u\|^2_{X_{s,b}} + C\delta^{b'-b}\|v\|^2_{X_{1+s,b}}. \qquad (3.4)
\end{aligned}$$

Similarly, by (3.1)–(3.3) of Lemma 3.1 and bilinear estimate (2.6), we have

$$\begin{aligned}
\|\Psi_\psi[u,v]\|_{X_{1+s,b}} &\le \|\theta_1(t)U(t)\psi\|_{X_{1+s,b}} + \left\|\theta_1(t)\int_0^t U(t-s)\theta_\delta(t)[3uv_x](s)ds\right\|_{X_{1+s,b}} \\
&\le C\|\psi\|_{H^{1+s}} + C\|\theta_\delta(t)uv_x\|_{X_{1+s,b-1}} \\
&\le C\|\psi\|_{H^{1+s}} + C\delta^{b'-b}\|uv_x\|_{X_{1+s,b'-1}} \\
&\le C\|\psi\|_{H^{1+s}} + C\delta^{b'-b}\|u\|_{X_{s,b}}\|v\|_{X_{1+s,b}} \\
&\le C\|\psi\|_{H^{1+s}} + C\delta^{b'-b}\|u\|^2_{X_{s,b}} + \delta^{b'-b}\|v\|^2_{X_{1+s,b}}. \qquad (3.5)
\end{aligned}$$

Thus, by the estimates (3.4) and (3.5), we have

$$\begin{aligned}
\|\Phi\times\Psi_{(\phi,\psi)}[u,v]\|_{X_{s,b}\times X_{1+s,b}} &\le C\|\phi\|_{H^s} + C\|\psi\|_{H^{1+s}} + C\delta^{b'-b}\|u\|^2_{X_{s,b}} + C\delta^{b'-b}\|v\|^2_{X_{1+s,b}} \\
&\le C\|(\phi,\psi)\|_{H^s\times H^{1+s}} + C\delta^{b'-b}[\|u\|^2_{X_{s,b}} + \|v\|^2_{X_{1+s,b}}] \\
&\le C\|(\phi,\psi)\|_{H^s\times H^{1+s}} + C\delta^{b'-b}\|(u,v)\|^2_{X_{s,b}\times X_{1+s,b}}.
\end{aligned}$$

Thus, when taking $\delta < [(2C)^2 r]^{\frac{1}{b-b'}}$, $\Phi\times\Psi_{(\phi,\psi)}[u,v]$ mapping B_r into B_r.

Similar to (3.4) and (3.5), for δ determined above, we have

$$\|\Phi\times\Psi_{(\phi,\psi)}[u_1,v_1] - \Phi\times\Psi_{(\phi,\psi)}[u_2,v_2]\|_{X_{s,b}\times X_{1+s,b}} < \frac{1}{2}\|(u,v)\|_{X_{s,b}\times X_{1+s,b}}.$$

Thus,$\Phi\times\Psi_{(\phi,\psi)}[u,v]$ is contract mapping.

Finally, by Banach theorem, $\forall\ t\ (0 < t \le 1)$, in the ball B_r, the mapping $\Phi\times\Psi_{(\phi,\psi)}[u,v]$ have unique fixed point (u,v) satisfying

$$\begin{cases} u = U(t)\phi - \int_0^t U(t-s)[6uu_x - 2\beta vv_x](s)ds, \\ v = U(t)\psi - \int_0^t U(t-s)[3uv_x](s)ds. \end{cases}$$

□

4. Conclusions

Remark 4.1. *Although, the main result in this paper covered the results of [7], it must be not the sharp results when compare it with [9].*

Remark 4.2. *When compare it with [9], we conjecture that the initial value problem of Hirota-Satsuma system maybe locally well-posed in $H^s(\mathbb{R})\times H^{s+1}(\mathbb{R})$, for any $s > -\frac{3}{4}$. We'll investigate this question in the future.*

Remark 4.3. *We are interested in well-posedness of initial boundary value problem of the Hirota-Satsum system, especially well-posedness with low regularity datum. We'll show the results in elsewhere.*

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Conflict of interest

The authors declare that they have no conflicts of interest.

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