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Research article

Solution of fractional boundary value problems by ψ-shifted operational matrices

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Abstract: In this paper, a numerical method is presented to solve fractional boundary value problems. In fractional calculus, the modelling of natural phenomenons is best described by fractional differential equations. So, it is important to formulate efficient and accurate numerical techniques to solve fractional differential equations. In this article, first, we introduce ψ-shifted Chebyshev polynomials then project these polynomials to formulate ψ-shifted Chebyshev operational matrices. Finally, these operational matrices are used for the solution of fractional boundary value problems. The convergence is analysed. It is observed that solution of non-integer order differential equation converges to corresponding solution of integer order differential equation. Finally, the efficiency and applicability of method is tested by comparison of the method with some other existing methods.

Keywords: fractional differential equations; boundary value problems; ψ-shifted Chebyshev polynomials; operational matrices
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1. Introduction

Fractional calculus is currently becoming one of the most studied topic. The theory dealing with derivatives and integrals of non-integer order is not new. This theory dates back to era of Liouville, Leibnitz, Gunwall-Levtinkov and Riemann. In recent years, the trend of modelling many physical problems by fractional differential equations has developed in many fields like fluid flow, viscoelasticity, porous media, mechanics and electromagnetic [1–4]. The fractional differential operator based models in science and technology provide more suitable results in many situations. The study of numerical analysis of linear and non-linear fractional differential equations is still a significant task. Different attempts have been made to formulate new methods for finding analytic and approximate solutions of variety of fractional differential equations. These methods include Wavelet method [5], Hybrid method [6], Homotopy method [7] and many more. Recently, the formulation of operational matrices of fractional derivatives and integrals using various types of polynomials, especially orthogonal polynomials is extensively progressing. A brief summary is presented here.

Ali Ahmadian in [8] solved fuzzy linear fractional differential equation using Jacobi polynomial along with Tau method. The authors in [9] presented a numerical method to solve a type of fractional spatial-temporal telegraph equation using modified fractional Legendre wavelets and modified polynomial functions. A numerical scheme to solve three-dimensional non-linear system of Voltera-Hammerstein integral equations using three-dimensional orthogonal block-pulse functions by operational matrices is presented in [10]. Muhammad Usman et al. in [11] developed an algorithm to solve time-fractional non-linear telegraph equation using shifted Gegenbauer polynomials by formulating operational matrices of fractional derivatives. B.P. Moghaddan et al. in [12] presented a numerical scheme for solution of mixed-type fractional-order functional differential equations by a collocation technique using the modified Lucas polynomials. In [13], the authors first purposed fractional collocation differentiation matrices and made the comparison with Podlubny's matrix to purpose Chebyshev collocation based matrices to solve system of linear fractional differential equations.

M. Hamid et al. in [14] devised a new spectral method to solve two-dimensional unsteady non-linear fractional partial differential equations using Chelyshkov polynomials for the development of operational matrices. The authors in [15] presented a numerical scheme to solve non-linear fractional order problems like Burgers, Schrodinger, Bloch-Torrey, Rayleigh-Stokes and Sine-Gordan by formulating the operational matrices of positive integer and non-integer order derivatives using shifted Gegenbauer wavelets. A. A. El-Sayed and P. Agarwal investigated the solution of multi-term fractional differential equations in [16] by formulating the operational matrix using shifted Legendre polynomials and then applied the collocation technique to reduce the problem into a system of algebraic equations which is then solved easily. The authors in [17] presented the solution of fractional differential equations by constructing shifted Jacobi operational matrix of fractional order derivatives.

Variety of definitions for fractional integrals and derivatives [18–20] gave rise to an idea of finding fractional derivatives of one function with respect to some other function. This class of fractional operators are based upon a kernel function ψ and unify some other definitions. In [21], Almeida introduced this concept of fractional operators where the proper choice of function ψ restore some other definitions. Many physical phenomenons can be modelled by suitable choice of trial function ψ. Researchers are developing different numerical techniques for the solution of fractional differential equations utilizing this concept of fractional operators [22–24].

In this paper, taking motivation by above cited work, we have developed a numerical method to solve fractional differential equation

$$\begin{gathered} {}^{c}D_{a}^{\alpha,\psi}u(x) + A(x)\,{}^{c}D_{a}^{\beta,\psi}u(x) + B(x)u(x) = f(x)\,, \quad x \in [a,b], \\ u(a) = u_0 \quad \text{and} \quad u(b) = u_1, \end{gathered} \tag{1.1}$$

where $\alpha, \beta \in \mathbb{R}^+, \beta \leq \alpha, 1 < \alpha \leq 2$, f is known function, A and B may be constant or variables. In our approach, we solve some specific class of fractional differential equations which involve fractional derivatives of a function with respect to some other function. Keeping the structure of these specific operators in mind, we will modify the classical Chebyshev polynomials, so that they involve same function with respect to which fractional differentiation is performed. This modification will be of great help to analyze the newly introduced polynomials from analytical and numerical point of view. We will focus on formulation of operational matrices using notified form of orthogonal polynomials, named ψ-shifted Chebyshev polynomials. By the projection of these polynomials on our problem, we get an algebraic system of equations which is then solved numerically. The paper is organized as follows.

In Section 2, we discuss some preliminaries and properties of fractional operators. In Section 3, we introduce ψ-shifted Chebyshev polynomials and discuss some properties. Section 4 presents the development of numerical scheme. In Section 5, the existence and uniqueness of the solution is discussed. Section 6 deals with the convergence analysis of the proposed technique. Finally, in Section 7, some numerical examples are presented to demonstrate the effectiveness of proposed numerical method.

2. Preliminaries

In this section, we review some definitions and properties of fractional derivatives and integrals.

Definition 2.1. [25] Let $\beta > 0$ and u be the integrable function which is defined on J where $J = [a, b]$ be a finite or infinite interval, $n \in \mathbb{N}$, and $\psi \in C^n(J; \mathbb{R})$ is an increasing function with $\psi'(x) \neq 0$ for all $x \in J$. Then fractional integral of function u with respect to another function ψ is defined as

$$I_a^{\beta,\psi} u(x) := \frac{1}{\Gamma(\beta)} \int_a^x \psi'(s)(\psi(x) - \psi(s))^{\beta-1} u(s) ds, \tag{2.1}$$

where Γ is the Gamma function.

Definition 2.2. [21] Let $\beta > 0$, $n \in \mathbb{N}$, $J = [a, b]$ and $u, \psi \in C^n(J; \mathbb{R})$ where ψ is increasing and $\psi'(x) \neq 0$ for all $x \in J$, then ψ-Caputo fractional derivative is defined as

$${}^c D_a^{\beta,\psi} u(x) := \Big(\frac{1}{\psi'(x)} \frac{d}{dx}\Big)^n I_a^{n-\beta,\psi} u(x), \tag{2.2}$$

where $n = [\beta] + 1$ for $\beta \notin \mathbb{N}$, $[\beta]$ denotes the integer part of β and $n = \beta$ for $\beta \in \mathbb{N}$. Thus

$${}^c D_a^{\beta,\psi} u(x) = \Big(\frac{1}{\psi'(x)} \frac{d}{dx}\Big)^m u(x), \quad if \;\; \beta = m \in \mathbb{N},$$

$${}^c D_a^{\beta,\psi} u(x) = \frac{1}{\Gamma(n-\beta)} \Big(\frac{1}{\psi'(x)} \frac{d}{dx}\Big)^n \int_a^x \psi'(s)(\psi(x) - \psi(s))^{n-\beta-1} u(s) ds, \;\; if \;\; \beta \notin \mathbb{N}.$$

Theorem 2.3. *[25] Let $u : [a, b] \to \mathbb{R}$, $n - 1 < \beta < n$, $n \in \mathbb{N}$, then*

- *for $u \in C[a, b]$,* $\quad {}^c D_a^{\beta,\psi} I_a^{\beta,\psi} u(x) = u(x)$,
- *for $u \in C^{n-1}[a, b]$,* $\quad I_a^{\beta,\psi}\, {}^c D_a^{\beta,\psi} u(x) = u(x) - \sum_{k=0}^{n-1} \frac{u_\psi^{[k]}(a)}{k!} (\psi(x) - \psi(a))^k$,

where

$$u_\psi^{[k]}(x) = \Big(\frac{1}{\psi'(x)} \frac{d}{dx}\Big)^k u(x).$$

Lemma 2.4. *[21] Let $\alpha, \gamma \in \mathbb{R}$ and $u \in C[a,b]$, the integral operator $I_a^{\alpha,\psi}$ has the property*

$$I_a^{\alpha,\psi} I_a^{\gamma,\psi} u(x) = I_a^{\alpha+\gamma,\psi} u(x).$$

Lemma 2.5. *[26] Let $\alpha > 0,\ \gamma > -1$, then for the power function $u(x) = (\psi(x) - \psi(a))^{\gamma}$,*

$$I_a^{\alpha,\psi}(\psi(x) - \psi(a))^{\gamma} = \frac{\Gamma((\gamma+1)}{\Gamma(\alpha+\gamma+1)}(\psi(x) - \psi(a))^{\alpha+\gamma}.$$

Definition 2.6. [26] Let $J = [a,b]$, $a, b \in \mathbb{R}$, $a < b$, and $\psi \in C^1(J;[0,1])$ be an increasing function such that $\psi^{'}(x) \neq 0$, $\psi(J) = [0,1]$and $\sigma(x)$ is the weight function then the space

$$H_{\psi}^2(J;\mathbb{R}) = \Big\{ g : J \to \mathbb{R}\ : g \text{ is measurable and } \int_J |g(x)|^2 \sigma(x) \psi^{'}(x) dx < \infty \Big\},$$

endowed with the inner product

$$(g,h)_{H_{\psi}^2(J;\mathbb{R})} = \int_J g(x)h(x)\sigma(x)\psi'(x)dx, \qquad g,h \in H_{\psi}^2(J;\mathbb{R}),$$

and the induced norm

$$\|g\|_{H_{\psi}^2(J;\mathbb{R})} = \sqrt{(g,g)_{H_{\psi}^2(J;\mathbb{R})}} = \Big(\int_J |g(x)|^2 \sigma(x) \psi^{'}(x) dx \Big)^{\frac{1}{2}}, \quad g \in H_{\psi}^2(J;\mathbb{R}),$$

is a Hilbert space. Also, for a function $g : J \to \mathbb{R}$, we define

$$\widehat{g} : [0,1] \to \mathbb{R} \quad by \quad \widehat{g(t)} = g(\psi^{-1}(t)), \qquad 0 \leq t \leq 1. \tag{2.3}$$

Definition 2.7. [26] Let $(e_k)_{k \in \mathbb{N}_o}$ be a Hilbertian basis of Hilbert space H where $\mathbb{N}_o = \{0, 1, 2...\}$ then for $m \in \mathbb{N}$, the orthogonal operator $\mathfrak{T}_m$ is defined as

$$\mathfrak{T}_m : H \to span\{e_k : k = 0, 1, ..., m-1\},$$
$$\mathfrak{T}_m(x) = \sum_{k=0}^{m-1} (x, e_k)_H\ e_k, \qquad x \in H.$$

Lemma 2.8. *Let $g, h \in H_{\psi}^2(J;\mathbb{R})$ and $\widehat{g}, \widehat{h} \in H_{\psi}^2([0,1];\mathbb{R})$, then*

$$(g,h)_{H_{\psi}^2(J;\mathbb{R})} = (\widehat{g}, \widehat{h})_{H^2([0,1];\mathbb{R})}.$$

Lemma 2.9. *For $m \in \mathbb{N}$, $\mathfrak{T}_m$ is an operator on H which is linear and continuous so that for all $x \in H$, $\|\mathfrak{T}_m(x)\|_H \leq \|x\|$ and $\lim_{m\to\infty} \|x - \mathfrak{T}_m(x)\|_H = 0$.*

Lemma 2.10. *For the sequences $\{x_m\} \subset H$ and $\{a_m\} \subset \mathbb{R}$ where*

$$x_m = \sum_{k=0}^{m-1} a_k e_k, \qquad m \in \mathbb{N},$$

there exists $x \in H$ such that $\lim_{m\to\infty} \|x_m - x\|_H = 0$, then $x_m = \mathfrak{T}_m(x)$.

The proof of Lemmata 2.8–2.10 can be made on the same lines as the proof of Lemmata 2.1, 2.4 and 2.5 respectively in [26].

3. ψ-shifted Chebyshev polynomials

Although approximation by orthogonal polynomials is old and well developed in theory and applications, however approximation of solutions of fractional differential equations by orthogonal polynomials is an art in which a moderate number of researchers are investigating. There are different treatments of the subject of approximation by orthogonal polynomials. One of the common approach is to use orthogonal polynomials in their original form. Alternatively, one can modify the orthogonal polynomials of interest to fit in the framework of differential and integral operators which appear in mathematical model under consideration. In this work, we adopt the later approach.

Chebyshev polynomials are special type of polynomials that are very suitable for approximation of other functions. In recent decades, theses are widely used in numerical analysis, implementation of spectral methods, polynomial approximation and so on. Most recently, for numerical purposes, shifted Chebyshev polynomials are used in which the range of independent variable is $[0, 1]$ instead of $[-1, 1]$. Several researchers are using these polynomials for the approximation of the solution of different types of differential equations [27–30]. Chebyshev polynomials of the first kind on the interval $[-1, 1]$ are defined as

$$T_n(x) = \cos(n \arccos(x)), \qquad \text{when} \qquad x = \cos\theta.$$

Chebyshev polynomials are orthogonal with weight function $\sigma(x) = \frac{1}{\sqrt{1-x^2}}$ and the orthogonality relation is defined as [31]

$$\int_{-1}^{1} T_m(x)\, T_n(x)\, \sigma(x) dx = \kappa_n \delta_{nm},$$

with $\kappa_0 = \pi$ and $\kappa_n = \frac{1}{2}\pi$ if $n \neq 0$ and δ_{nm} is Kronecker delta. Further, we have $(T_n, T_m) = 0$ for $n \neq m$. So, that $\{T_n(x) : n = 0, 1, 2, ...\}$ is orthogonal but not orthonormal. The proper scaling leads to an orthonormal system [32]. Shifted Chebyshev polynomials of the first kind on the interval $[0, 1]$ are defined as

$$T_n^*(x) = T_n(2x - 1).$$

In shifted Chebyshev polynomials, the orthogonality relation is

$$\int_{0}^{1} T_m^*(x)\, T_n^*(x)\, \sigma^*(x) dx = \kappa_n \delta_{nm}, \tag{3.1}$$

with weight function $\sigma^*(x) = \frac{1}{\sqrt{x-x^2}}$. The power series representation of shifted Chebyshev polynomials is [33]

$$T_n^*(x) = n \sum_{k=0}^{n} \frac{(-1)^{(n-k)}(2)^{2k}(n+k-1)!}{(2k)!(n-k)!}(x)^k, \qquad n > 0. \tag{3.2}$$

We already have presented some definitions and properties of fractional operators depending on the function ψ. Now, we introduce ψ-shifted Chebyshev polynomials as

$$\mathcal{G}_n^{*\psi}(x) = n \sum_{k=0}^{n} \frac{(-1)^{(n-k)}(2)^{2k}(n+k-1)!}{(2k)!(n-k)!}(\psi(x))^k, \qquad n > 0, \tag{3.3}$$

where $\mathcal{G}_n^{*\psi}(x) = T_n^*(\psi(x))$ for all x.

Lemma 3.1. *The set $\{\mathcal{G}_n^{*\psi} : n \in \mathbb{N}_o\}$ is an orthonormal basis of the Hilbert space $H_\psi^2(J;\mathbb{R})$.*

Proof. The proof of lemma involves the orthonormality of shifted Chebyshev polynomials [34]. Let $i, j \in \mathbb{N}_o$, then by using Lemma 2.8, we have

$$(\mathcal{G}_i^{*\psi}, \mathcal{G}_j^{*\psi})_{H_\psi^2(J;\mathbb{R})} = (\widehat{\mathcal{G}_i^{*\psi}}, \widehat{\mathcal{G}_j^{*\psi}})_{H^2([0,1];\mathbb{R})},$$

$$(\widehat{\mathcal{G}_i^{*\psi}}, \widehat{\mathcal{G}_j^{*\psi}})_{H^2([0,1];\mathbb{R})} = \int_0^1 \widehat{\mathcal{G}_i^{*\psi}(s)}\ \widehat{\mathcal{G}_j^{*\psi}(s)}\ \widehat{\sigma^{*\psi}(s)} ds.$$

Using Eq 2.3, we get

$$(\mathcal{G}_i^{*\psi}, \mathcal{G}_j^{*\psi})_{H_\psi^2(J;\mathbb{R})} = \int_0^1 \mathcal{G}_i^{*\psi}(\psi^{-1}(s))\ \mathcal{G}_j^{*\psi}(\psi^{-1}(s))\ \sigma^{*\psi}(\psi^{-1}(s)) ds.$$

Now using Eq 3.3, the above equation reduces to

$$\begin{aligned}(\mathcal{G}_i^{*\psi}, \mathcal{G}_j^{*\psi})_{H_\psi^2(J;\mathbb{R})} &= \int_0^1 T_i^*(\psi(\psi^{-1}(s)))\ T_j^{*\psi}(\psi(\psi^{-1}(s)))\ \sigma^*(\psi(\psi^{-1}(s))) ds,\\ &= \int_0^1 T_i^*(s)\ T_j^*(s)\ \sigma^*(s) ds,\\ &= (T_i^*, T_j^*)_{H^2([0,1];\mathbb{R})}.\end{aligned} \tag{3.4}$$

Thus, the result follows. □

Let $f \in H_\psi^2(J;\mathbb{R})$, $m \in \mathbb{N}$ and $\mathfrak{T}_m$ is the orthogonal projection, then, we have

- $\mathfrak{T}_m(f)(x) = \sum_{k=0}^{m-1} (f, \mathcal{G}_k^{*\psi})_{H_\psi^2(J;\mathbb{R})}\ \mathcal{G}_k^{*\psi}(x), \quad x \in J.$
- $\lim_{m\to\infty} \|f - \mathfrak{T}_m(f)\|_{H_\psi^2(J;\mathbb{R})} = 0.$

A function $f(x)$ for $x \in [a, b]$ may be expanded as

$$f(x) = \sum_{n=0}^{\infty} b_n\ \mathcal{G}_n^{*\psi}(x).$$

The expansion coefficients b_n are associated with family $\{\mathcal{G}_n^{*\psi}\}$ and can be calculated by

$$b_n = \frac{(f, \mathcal{G}_n^{*\psi})}{\|\mathcal{G}_n^{*\psi}\|}.$$

Practically, we use first m-terms of ψ-shifted Chebyshev polynomials and approximate $f(x)$ as

$$f(x) \simeq f_m(x) = \sum_{n=0}^{m-1} b_n\ \mathcal{G}_n^{*\psi}(x) = \mathfrak{B}\Omega_m^{*\psi}(x),$$

where

$$\mathfrak{B} = [b_0, b_1, ..., b_{m-1}],$$

$$\Omega_m^{*\psi}(x) = [\mathcal{G}_0^{*\psi}(x), \mathcal{G}_1^{*\psi}(x), ..., \mathcal{G}_{m-1}^{*\psi}(x)]^T, \qquad x \in J. \tag{3.5}$$

Next, we define generalized fractional Taylor's formula in term of ψ.

Theorem 3.2. *[21] Assume $\beta \in (0,1)$, $m \in \mathbb{N}$ and f is such that ${}^{c}D_{a}^{k\beta,\psi} f$ exists and is continuous for all k=0,1,2,...,m+1, and $x \in [a,b]$, then*

$$f(x) = \sum_{k=0}^{m} \frac{{}^{c}D_{a}^{k\beta,\psi} f(a)}{\Gamma(k\beta+1)} (\psi(x)-\psi(a))^{k\beta} + \frac{{}^{c}D_{a}^{(m+1)\beta,\psi} f(\zeta)}{\Gamma((m+1)\beta+1)} (\psi(x)-\psi(a))^{(m+1)\beta},$$

for some $\zeta \in (a,x)$ and

$$\left| f(x) - \sum_{k=0}^{m} \frac{{}^{c}D_{a}^{k\beta,\psi} f(a)}{\Gamma(k\beta+1)} (\psi(x)-\psi(a))^{k\beta} \right| \leq \frac{\mathbf{C}}{\Gamma((m+1)\beta+1)} (\psi(x)-\psi(a))^{(m+1)\beta},$$

where

$$\left| {}^{c}D_{a}^{(m+1)\beta,\psi} f(\zeta) \right| \leq \mathbf{C}. \tag{3.6}$$

The following Theorem 3.3 is the generalization of Theorem 1 in [35].

Theorem 3.3. *Let ${}^{c}D_{a}^{k\beta,\psi} f(x) \in C(0,1)$ for $k = 0,1,2,...,m$ and $\mathcal{H}_{m}^{*\psi}$ is generated by $\{\mathcal{G}_{0}^{*\psi}(x), \mathcal{G}_{1}^{*\psi}(x), ..., \mathcal{G}_{m-1}^{*\psi}(x)\}$. Further, if $f_m(x) = \mathfrak{B}\Omega_{m}^{*\psi}(x)$ is the best approximation to $f(x)$ from $\mathcal{H}_{m}^{*\psi}$, then the error bound is,*

$$\|f(x) - f_m(x)\| \leq \frac{\mathbf{C}' \, \mathbf{\Psi}}{\Gamma((m+1)\beta+1)\sqrt{2(m+1)\beta+\frac{1}{2}}},$$

where $\mathbf{\Psi} = \left((\psi(1)-\psi(a))^{2(m+1)\beta+\frac{1}{2}} - (\psi(0)-\psi(a))^{2(m+1)\beta+\frac{1}{2}} \right)^{\frac{1}{2}}$ and $\left| {}^{c}D_{a}^{(m+1)\beta,\psi} f(\zeta) \right| \leq \mathbf{C}$.

Proof. In fractional Taylor's Theorem 3.2, take

$$u = \sum_{k=0}^{m} \frac{{}^{c}D_{a}^{k\beta,\psi} f(a)}{\Gamma(k\beta+1)} (\psi(x)-\psi(a))^{k\beta},$$

so that, we have

$$|f(x) - u(x)| \leq \frac{\mathbf{C}}{\Gamma((m+1)\beta+1)} (\psi(x)-\psi(a))^{(m+1)\beta}.$$

As $f_m(x) = \mathfrak{B}\Omega_{m}^{*\psi}(x)$ is the best approximation to f(x) from $\mathcal{H}_{m}^{*\psi}$ and $u \in \mathcal{H}_{m}^{*\psi}$. Thus, we have

$$\begin{aligned}
\|f(x) - f_m(x)\| &\leq \|f(x) - u(x)\| \\
&\leq \left(\frac{\mathbf{C}^2}{\Gamma((m+1)\beta+1)^2} \int_{0}^{1} \frac{(\psi(x)-\psi(a))^{2(m+1)\beta}}{\sqrt{(\psi(x)-\psi(a)) - (\psi(x)-\psi(a))^2}} \, \psi'(x) dx \right)^{\frac{1}{2}} \\
&\leq \frac{\mathbf{C}'}{\Gamma((m+1)\beta+1)} \left(\int_{0}^{1} (\psi(x)-\psi(a))^{2(m+1)\beta-\frac{1}{2}} \, \psi'(x) dx \right)^{\frac{1}{2}} \\
&= \frac{\mathbf{C}' \, \mathbf{\Psi}}{\Gamma((m+1)\beta+1)\sqrt{2(m+1)\beta+\frac{1}{2}}}.
\end{aligned}$$

where $\mathbf{C}' = \mathbf{C}\widehat{\mathbf{C}}$ and $\widehat{\mathbf{C}}$ is a positive number. Thus, by increasing m, the approximation solution $f_m(x)$ converges to $f(x)$. $\square$

4. Formulation of numerical method

While dealing the fractional type operators, the analytic solution of the problem becomes difficult to determine, so, we use numerical methods to find the approximate solution of the problem. For different numerical techniques, approximation of continuous functions is vital in scientific computing. Interpolating polynomial can be represented by using different basis functions. Suitable transformations can be used to relate these representations. Polynomials can be characterized in several ways with different basis and each form of basis has its advantage [36]. Let us consider the polynomial

$$P_n(x) = a_0 + a_1\psi(x) + a_2(\psi(x))^2 + \cdots + a_n(\psi(x))^n, \tag{4.1}$$

where the monomial basis are defined as

$$M(x) = \{1, \psi(x), (\psi(x))^2, ..., (\psi(x))^n\}.$$

Consider a set of grid points x_j, where x_j are shifted Chebyshev points defined in [33] as

$$x_j = \frac{1}{2}\left(\cos\left(\frac{\pi j}{n}\right) + 1\right), \qquad 0 \leq j \leq n, \;\; n = 1, 2, \ldots. \tag{4.2}$$

Let $u_1, u_2, ..., u_n$ represent function values at $x_j \in [0, 1]$ for $j = 1, 2, 3, ..., n$ and $x_j \neq x_i$ for $i \neq j$. Suppose $P_n(x)$ be the polynomial that interpolates the function values, then the interpolating condition is [37]

$$P_n(x_j) = u_j, \qquad j = 1, 2, \ldots.$$

In order to calculate the coefficients in (4.1), we solve the system

$$\Re C = U, \tag{4.3}$$

where $\Re = [(\psi(x_i))^j]$, $0 \leq i, j \leq n$ and $C = [a_0, a_1, a_2, \cdots, a_n]^T$ and $U = [u_0, u_1, u_2, \cdots, u_n]^T$. ψ-shifted Chebyshev polynomials defined in (3.3) can be written as

$$\mathcal{G}_n^{*\psi}(x) = \sum_{k=0}^{n} n\widetilde{a}_{kn}(\psi(x))^k, \qquad n > 0, \tag{4.4}$$

where

$$\widetilde{a}_{kn} = \frac{(-1)^{(n-k)}(2)^{2k}(n + k - 1)!}{(2k)!(n - k)!}.$$

Now, the interpolating polynomial in terms of ψ-shifted Chebyshev polynomial basis can be written as

$$P_n(x) = \sum_{i=0}^{n} \eta_i \, \mathcal{G}_i^{*\psi}(x).$$

In the next step, we relate ψ-shifted Chebyshev polynomial basis with the Lagrange basis. Lagrangian polynomials are useful in polynomial interpolaton. Interpolating polynomials $P_n(x)$ are represented in terms of Lagrangian polynomials as

$$P_n(x) = \sum_{i=0}^{n} P_n(x_i) l_i(x), \tag{4.5}$$

where

$$l_i(x) = \prod_{i=0, i\neq j}^{n} \frac{x - x_j}{x_i - x_j}, \qquad i = 0, 1, 2, \ldots.$$

Using the interpolating conditions, Eq 4.5 becomes

$$P_n(x) = \sum_{i=0}^{n} u(x_i) l_i(x).$$

Thus

$$\sum_{i=0}^{n} \eta_i \, \mathcal{G}_i^{*\psi}(x) = \sum_{i=0}^{n} u(x_i) l_i(x).$$

In matrix form,

$$\mathcal{G}^{*T} \, C = L^T U. \tag{4.6}$$

The coefficients η_i can be calculated by solving $\Re C = U$.
Combining (4.4) and (4.6), $L = \omega \mathcal{G}^*$ where $\omega = (\Re^T)^{-1}$.
So, we get

$$u(x) \approx \sum_{i=0}^{n} \sum_{j=0}^{n} u(x_i) \omega_{ij} \mathcal{G}_j^{*\psi}(x). \tag{4.7}$$

Fractional order derivative of shifted Chebyshev polynomial is defined in [33] as

$${}^{c}D_a^{\alpha} \, (T_n^*(x)) = \begin{cases} \sum\limits_{k \geq \lceil \alpha \rceil}^{n} \frac{n \widetilde{a}_{kn} k!}{(k-\alpha)!} (x)^{k-\alpha}, & n \geq \lceil \alpha \rceil, \\ 0, & n < \lceil \alpha \rceil. \end{cases}$$

The term $\lceil \alpha \rceil$ is ceiling function that means the smallest integer greater than or equal to α. The ψ-Caputo fractional derivative of the function $h(x) = (\psi(x))^k$ is [26]

$${}^{c}D_a^{\alpha,\psi} \, h(x) = \frac{k!}{(k-\alpha)!} (\psi(x))^{k-\alpha}, \quad \alpha > 0, \; k > n - 1,$$

so, ψ-Caputo fractional derivative of ψ-shifted Chebyshev polynomials will be

$$\mathcal{G}_n^{*\psi,\alpha}(x) = \begin{cases} \sum\limits_{k \geq \lceil \alpha \rceil}^{n} \frac{n \widetilde{a}_{kn} k!}{(k-\alpha)!} (\psi(x))^{k-\alpha}, & n \geq \lceil \alpha \rceil, \\ 0, & n < \lceil \alpha \rceil, \end{cases} \tag{4.8}$$

where

$$\mathcal{G}_n^{*\psi,\alpha} = {}^{c}D_a^{\alpha,\psi} \, (\mathcal{G}_n^{*\psi}).$$

Now, taking fractional derivative of (4.7)

$$^{c}D_{a}^{\alpha,\psi}\, u(x) \approx \sum_{i=0}^{n}\ \sum_{j=0}^{n} u(x_i)\omega_{ij}\mathcal{G}_{j}^{*\psi,\alpha}(x). \tag{4.9}$$

Using Eqs (4.7) and (4.9) in (1.1), we get following algebraic system of equations.

$$[G^{*\psi,\alpha} + diag(\tilde{A})G^{*\psi,\beta} + diag(\tilde{B})]\ U^{T} = F^{T}, \tag{4.10}$$

where

$$G^{*\psi,\alpha} = \begin{bmatrix} \sum\limits_{j=0}^{n}\omega_{0j}\mathcal{G}_{j0}^{*\psi,\alpha} & \sum\limits_{j=0}^{n}\omega_{1j}\mathcal{G}_{j0}^{*\psi,\alpha} & \dots & \sum\limits_{j=0}^{n}\omega_{nj}\mathcal{G}_{j0}^{*\psi,\alpha} \\ \sum\limits_{j=0}^{n}\omega_{0j}\mathcal{G}_{j1}^{*\psi,\alpha} & \sum\limits_{j=0}^{n}\omega_{1j}\mathcal{G}_{j1}^{*\psi,\alpha} & \dots & \sum\limits_{j=0}^{n}\omega_{nj}\mathcal{G}_{j1}^{*\psi,\alpha} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \sum\limits_{j=0}^{n}\omega_{0j}\mathcal{G}_{jn}^{*\psi,\alpha} & \sum\limits_{j=0}^{n}\omega_{1j}\mathcal{G}_{jn}^{*\psi,\alpha} & \dots & \sum\limits_{j=0}^{n}\omega_{nj}\mathcal{G}_{jn}^{*\psi,\alpha} \end{bmatrix},$$

is the fractional differentiation matrix.
$diag(\tilde{A}) = [A(x_0), A(x_1), ..., A(x_n)]$, $diag(\tilde{B}) = [B(x_0), B(x_1), ..., B(x_n)]$,
$F^{T} = [f(x_0), f(x_1), ..., f(x_n)]$.
The system in Eq (4.10) is an algebraic system which is then solved to get the solution vector u. All numerical calculations are performed using Matlab.

5. Existence and uniqueness

This section deals with the detailed investigation of the existence and uniqueness of fractional boundary value problems. The author in [38] has investigated the existence and uniqueness of fractional boundary value problem by converting it into equivalent integral equation and used the properties of Greens function and Guo-Kransnoselskii fixed point theorem. In [39] the authors investigated the existence and uniqueness of the fractional differential equation by establishing the equivalence result between the fractional problem and Voltera integral equation. Here, we will use the similar method to establish the existence and uniqueness.

Theorem 5.1. *Let us suppose that $f \in L_1[a,b]$, then u is the solution of problem (1.1) if and only if u is the solution of integral equation*

$$u(x) - \int_{a}^{b} \omega(x,s)\psi'(s)u(s)ds = F(x),$$

with

$$\omega(x,s) = A\omega_{\alpha-\beta}(x,s) + B(s)\omega_{\alpha}(x,s),$$

where A is constant and B is variable and

$$\omega_\alpha(x,s) := \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(\psi(x)-\psi(a))(\psi(b)-\psi(s))^{\alpha-1}}{(\psi(b)-\psi(a))} - (\psi(x)-\psi(s))^{\alpha-1}, & s \le x, \\ \frac{(\psi(x)-\psi(a))(\psi(b)-\psi(s))^{\alpha-1}}{(\psi(b)-\psi(a))}, & x \le s, \end{cases}$$

$$\begin{aligned} F(x) :=& u_0 + I_a^{\alpha,\psi} f(x) + \frac{1}{(\psi(b)-\psi(a))}[u_1 - u_0 - I_a^{\alpha,\psi} f(b)](\psi(x)-\psi(a)) \\ &+ \frac{Au_0}{\Gamma(\alpha-\beta+1)}\Big[(\psi(x)-\psi(a))^{\alpha-\beta} - (\psi(x)-\psi(a))(\psi(b)-\psi(a))^{\alpha-\beta-1}\Big]. \end{aligned}$$

Proof. Suppose that u is a solution of (1.1). We apply fractional integral $I_a^{\alpha,\psi}$ to both sides of Eq (1.1). Using Theorem 2.3, semigroup property of fractional integrals and first boundary condition, above equation reduces to

$$u(x) - u_0 - c_2(\psi(x)-\psi(a)) + AI_a^{\alpha-\beta,\psi} I_a^{\beta,\psi}\, {}^cD_a^{\beta,\psi} u(x) + I_a^{\alpha,\psi} B(x)u(x) = I_a^{\alpha,\psi} f(x).$$

Applying Theorem 2.3 again

$$u(x) - u_0 - c_2(\psi(x)-\psi(a)) - AI_a^{\alpha-\beta,\psi} u_0 + AI_a^{\alpha-\beta,\psi} u(x) + I_a^{\alpha,\psi} B(x)u(x) = I_a^{\alpha,\psi} f(x).$$

Now, applying Lemma 2.5

$$\begin{aligned} & u(x) - u_0 - c_2(\psi(x)-\psi(a)) - \frac{Au_0}{\Gamma(\alpha-\beta+1)}(\psi(x)-\psi(a))^{\alpha-\beta} + AI_a^{\alpha-\beta,\psi} u(x) \\ & + I_a^{\alpha,\psi} B(x)u(x) = I_a^{\alpha,\psi} f(x). \end{aligned}$$

Applying second boundary condition

$$\begin{aligned} u(x) =& u_0 + \frac{1}{(\psi(b)-\psi(a))}\Big[u_1 - u_0 + AI_a^{\alpha-\beta,\psi} u(b) - \frac{Au_0}{\Gamma(\alpha-\beta+1)}(\psi(b)-\psi(a))^{\alpha-\beta} \\ & + I_a^{\alpha,\psi} B(b)u(b) - I_a^{\alpha,\psi} f(b)\Big](\psi(x)-\psi(a)) - AI_a^{\alpha-\beta} u(x) - I_a^{\alpha,\psi} B(x)u(x) + I_a^{\alpha,\psi} f(x) \\ & + \frac{Au_0}{\Gamma(\alpha-\beta+1)}(\psi(x)-\psi(a))^{\alpha-\beta}. \end{aligned}$$

Thus, we get

$$\begin{aligned} u(x) &= \frac{1}{(\psi(b)-\psi(a))}\Big[AI_a^{\alpha-\beta,\psi} u(b) + I_a^{\alpha,\psi} B(b)u(b)\Big](\psi(x)-\psi(a)) \\ &- AI_a^{\alpha-\beta,\psi} u(x) - I_a^{\alpha,\psi} B(x)u(x) + F(x). \end{aligned} \tag{5.1}$$

Using the definition of fractional integral, the above system reduces to

$$u(x) = \int_a^b \omega(x,s)\psi'(s)u(s)ds + F(x).$$

Conversely, let u is the solution of integral equation. Rewriting (5.1) as

$$u(x) \quad + AI_a^{\alpha-\beta,\psi} u(x) - \frac{Au_0}{\Gamma(\alpha-\beta+1)}(\psi(x)-\psi(a))^{\alpha-\beta} + I_a^{\alpha,\psi} B(x)u(x)$$

$$= \quad I_a^{\alpha,\psi} f(x) + u_0 + \frac{1}{(\psi(b)-\psi(a))}\Big[u_1 - u_0 - I_a^{\alpha,\psi} f(b) + A I_a^{\alpha-\beta,\psi} u(b) + I_a^{\alpha,\psi} B(b)u(b)$$
$$- \quad \frac{Au_0}{\Gamma(\alpha-\beta+1)}(\psi(b)-\psi(a))^{\alpha-\beta}\Big](\psi(x)-\psi(a)).$$

Using Lemma 2.5, Theorem 2.3 and then applying the operator ${}^cD_a^{\alpha,\psi}$ to both sides, we see that u solves the differential equation in (1.1). The boundary conditions are also obtained immediately. Thus, u solves boundary value problem (1.1). □

Theorem 5.2. *Suppose $f \in L_1[a,b]$ and f is bounded, then for $\alpha > 0$, (1.1) has unique continuous solution.*

Proof. The proof of the theorem involves following steps.
Step 1. In this step, we will construct a sequence of functions in such a way that it become closer and closer to exact solution. For the purpose, let us consider the iterations

$$u_n(x) = F(x) + \int_a^b \omega(x,s)\psi'(s)u_{n-1}(s)ds, \qquad n = 1,2,3...,$$

with $u_0(x) = F(x)$. We define

$$\xi_n(x) := u_n(x) - u_{n-1}(x), \quad \text{with} \quad \xi_0(x) := F(x). \tag{5.2}$$

This is a telescopic sum which is easy to analyze. These type of telescopic sums play a vital role in Calculus theory. So, we get

$$u_n(x) := \sum_{j=0}^{n} \xi_j(x).$$

Step 2. Now, we have to show that sequence of functions $\{u_n\}_{n=0}^{\infty}$ converges to u uniformly by Weistrass M-test. By Eq (5.2), we have

$$\xi_n(x) = \int_a^b \omega(x,s)\psi'(s)\xi_{n-1}(s)ds, \qquad n = 1,2,3,\ldots. \tag{5.3}$$

$F(x)$ and $B(x)$ are continuous on $[a,b]$, so for some $K_1, K_2 > 0$, $|F(x)| \le K_1$ and $|B(x)| \le K_2$ holds for all $x \in [a,b]$.
For $n = 1$, Eq (5.3) gives

$$|\xi_1(x)| = \left|\int_a^b \omega(x,s)\psi'(s)\xi_o(s)ds\right| \le K_1 \left|\int_a^b \omega(x,s)\psi'(s)ds\right|$$
$$\le \widetilde{K_1}\left|\frac{1}{\Gamma(\alpha-\beta)}\Big[\int_a^x \frac{(\psi(x)-\psi(a))(\psi(b)-\psi(s))^{\alpha-\beta-1}}{(\psi(b)-\psi(a))} - (\psi(x)-\psi(s))^{\alpha-\beta-1}\right.$$
$$\left. + \int_x^b \frac{(\psi(x)-\psi(a))(\psi(b)-\psi(s))^{\alpha-\beta-1}}{(\psi(b)-\psi(a))}\Big]\psi'(s)ds\right|$$
$$+ \widetilde{K_2}\left|\frac{1}{\Gamma(\alpha)}\Big[\int_a^x \frac{(\psi(x)-\psi(a))(\psi(b)-\psi(s))^{\alpha-1}}{(\psi(b)-\psi(a))} - (\psi(x)-\psi(s))^{\alpha-1}\right.$$

$$+\int_x^b \frac{(\psi(x)-\psi(a))(\psi(b)-\psi(s))^{\alpha-1}}{(\psi(b)-\psi(a))}\Big]\psi'(s)ds\Big|.$$

where $\widetilde{K_1} = K_1\ |A|$ and $\widetilde{K_2} = K_1\ K_2$. So,

$$\begin{aligned}
|\xi_1(x)| \leq &\frac{\widetilde{K_1}}{\Gamma(\alpha-\beta+1)}\Big|(\psi(x)-\psi(a))(\psi(b)-\psi(a))^{\alpha-\beta-1}-(\psi(x)-\psi(a))^{\alpha-\beta}\Big|\\
&+\frac{\widetilde{K_2}}{\Gamma(\alpha+1)}\Big|(\psi(x)-\psi(a))(\psi(b)-\psi(a))^{\alpha-1}-(\psi(x)-\psi(a))^{\alpha}\Big|\\
\leq &\frac{\widetilde{K_1}}{\Gamma(\alpha-\beta+1)}(\psi(b)-\psi(a))^{\alpha}+\frac{\widetilde{K_2}}{\Gamma(\alpha+1)}(\psi(b)-\psi(a))^{\alpha}\\
= &\frac{K\ (\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)},
\end{aligned}$$

where

$$K=\left[\widetilde{K_2}+\frac{\widetilde{K_1}\ (\Gamma(\alpha+1))}{\Gamma(\alpha-\beta+1)}\right].$$

Using induction process, we get

$$|\xi_n(x)| \leq \frac{K}{(\Gamma(\alpha+1))^n}(\psi(b)-\psi(a))^{n\alpha}\ ,$$

which leads to the fact that

$$u(x)=\lim_{n\to\infty}u_n(x)=\sum_{j=0}^{\infty}\xi_j(x).$$

Step 3. Here, we have to prove that $u(x)$ satisfies the given differential equation. For this purpose, we proceed as

$$\sum_{j=0}^{\infty}\xi_j(x)=\xi_0(x)+\sum_{j=0}^{\infty}\xi_{j+1}(x),$$

$$u(x)=F(x)+\sum_{j=0}^{\infty}\xi_{j+1}(x).$$

In case of uniform convergence, the interchange of sum and integrals is possible. So, applying this property of convergence in Eq (5.3), we get

$$u(x)=F(x)+\int_a^b \omega(x,s)\psi'(s)u(s)ds. \tag{5.4}$$

Step 4. Finally, we have to show that the solution $u(x)$ is unique. On the contrary, let us suppose that another solution $w(x)$ exists which is continuous too, then

$$|u(x) - w(x)| \leq \int_a^b \omega(x, s)|u(s) - w(s)|\psi'(s)ds.$$

Imposing the condition of continuity for $|u(x) - w(x)|$ on $[a, b]$, let $L > 0$ be a constant, such that for all $x \in [a, b]$, $|u(x) - w(x)| \leq L$, then above equation reduces to,

$$|u(x) - w(x)| \leq \frac{L}{\Gamma(\alpha + 1)}(\psi(b) - \psi(a))^{\alpha}.$$

By repeated application, we get

$$|u(x) - w(x)| \leq \frac{L}{(\Gamma(\alpha + 1))^n}(\psi(b) - \psi(a))^{n\alpha}.$$

Thus, for $n \to \infty$, $u(x) = w(x)$ for $x \in [a, b]$. □

6. Convergence analysis

In this section, we are going to present the convergence analysis of our proposed technique. Assume $u \in H^2_{\psi}(J; \mathbb{R})$ then for ${}^cD_a^{\alpha,\psi}u$, ${}^cD_a^{\beta,\psi}u \in H^2_{\psi}(J; \mathbb{R})$

$$\mathfrak{T}_m({}^cD_a^{\alpha,\psi}u)(x) = \sum_{k=0}^{m-1}({}^cD_a^{\alpha,\psi}u, \mathcal{G}_k^{*\psi})_{H^2_{\psi}(J;\mathbb{R})}\mathcal{G}_k^{*\psi}(x),$$

$$\mathfrak{T}_m({}^cD_a^{\beta,\psi}u)(x) = \sum_{k=0}^{m-1}({}^cD_a^{\beta,\psi}u, \mathcal{G}_k^{*\psi})_{H^2_{\psi}(J;\mathbb{R})}\mathcal{G}_k^{*\psi}(x).$$

Define two vectors $W_m^{\alpha,\psi}, W_m^{\beta,\psi} \in \mathbb{R}^m$ for $m \in \mathbb{N}$ as

$$W_m^{\alpha,\psi} = [({}^cD_a^{\alpha,\psi}u, \mathcal{G}_o^{*\psi})_{H^2_{\psi}(J;\mathbb{R})}, ({}^cD_a^{\alpha,\psi}u, \mathcal{G}_1^{*\psi})_{H^2_{\psi}(J;\mathbb{R})}, ..., ({}^cD_a^{\alpha,\psi}u, \mathcal{G}_{m-1}^{*\psi})_{H^2_{\psi}(J;\mathbb{R})}],$$

$$W_m^{\beta,\psi} = [({}^cD_a^{\beta,\psi}u, \mathcal{G}_o^{*\psi})_{H^2_{\psi}(J;\mathbb{R})}, ({}^cD_a^{\beta,\psi}u, \mathcal{G}_1^{*\psi})_{H^2_{\psi}(J;\mathbb{R})}, ..., ({}^cD_a^{\beta,\psi}u, \mathcal{G}_{m-1}^{*\psi})_{H^2_{\psi}(J;\mathbb{R})}].$$

By using the linearity of $\mathfrak{T}_m$ and (3.4), we have

$$\mathfrak{T}_m({}^cD_a^{\alpha,\psi}u + A\,{}^cD_a^{\beta,\psi}u) = W_m^{*\psi}\ \Omega_m^{*\psi}(x), \tag{6.1}$$

where A is constant and

$$W_m^{*\psi} = W_m^{\alpha,\psi} + AW_m^{\beta,\psi}.$$

Further, let us define

$$\mathfrak{T}_m({}^cI_a^{\delta,\psi}\Omega_m^{*\psi}) = S^{*\psi}_{m\times m}\ \Omega_m^{*\psi}, \tag{6.2}$$

and

$$\mathfrak{T}_m\Big(\sum_{k=0}^{m-1}\frac{u_\psi^{[k]}}{k!}(\psi(x)-\psi(a))^k\Big)=P_m^{*T}\ \Omega_m^{*\psi}(x),\tag{6.3}$$

where $P_m^* \in \mathbb{R}^m$ is a vector and $S_{m\times m}^{*\psi}$ is the square matrix.
Define the sequence, $\{u_m\} \subset H_\psi^2(J;\mathbb{R})$ as [26]

$$u_m(x)=(W_m^{*\psi}S_{m\times m}^{*\psi}+P_m^{*T})\Omega_m^{*\psi}(x).\tag{6.4}$$

Lemma 6.1. *For $m \in \mathbb{N}$,* $\lim_{m\to\infty}\|W_m^{*\psi}\mathfrak{T}_{\mathrm{m}}({}^cI_a^{\delta,\psi}\Omega_m^{*\psi})-{}^cI_a^{\delta,\psi}u\,{}^cD_a^{\delta,\psi}u\|_{H^2_{\psi(J;\mathbb{R})}}=0.$

The proof of Lemma 6.1 can be made on the same lines as proof of Lemma 5.2 in [26].

Theorem 6.2. *We have,* $\lim_{m\to\infty}\|u_m-u\|_{H_\psi^2(J;\mathbb{R})}=0$

Proof. Using Theorem 2.3 and (6.2–6.4), we have

$$\begin{aligned}
&\lim_{m\to\infty}\|u_m-u\|_{H_\psi^2(J;\mathbb{R})}\\
&=\lim_{m\to\infty}\left\|(W_m^{*\psi}S_{m\times m}^{*\psi}+P_m^{*T})\Omega_m^{*\psi}-u\right\|_{H_\psi^2(J;\mathbb{R})}\\
&=\lim_{m\to\infty}\left\|(W_m^{*\psi}S_{m\times m}^{*\psi}+P_m^{*T})\Omega_m^{*\psi}-{}^cI_a^{\delta,\psi}\,{}^cD_a^{\delta,\psi}u-\sum_{k=0}^{m-1}\frac{u_\psi^{[k]}(a)}{k!}(\psi(x)-\psi(a))^k\right\|_{H_\psi^2(J;\mathbb{R})}\\
&\le\lim_{m\to\infty}\left\|W_m^{*\psi}S_{m\times m}^{*\psi}\Omega_m^{*\psi}-{}^cI_a^{\delta,\psi}\,{}^cD_a^{\delta,\psi}u\right\|_{H_\psi^2(J;\mathbb{R})}\\
&\quad+\lim_{m\to\infty}\left\|P_m^{*T}\Omega_m^{*\psi}-\sum_{k=0}^{m-1}\frac{u_\psi^{[k]}(a)}{k!}(\psi(x)-\psi(a))^k\right\|_{H_\psi^2(J;\mathbb{R})}=0.
\end{aligned}$$

□

Lemma 6.3. *For $m \in \mathbb{N}$, we have $u_m=\mathfrak{T}_m(u)$.*

Proof. Using Lemma 2.10 and Theorem 6.2, the result follows. □

Now, we will calculate the unknown vector $W_m^{*\psi}\in\mathbb{R}^m$. For this purpose, let $\mathcal{Z}_m\in\mathbb{R}^m$ be the known vector defined as

$$\mathfrak{T}_m(f)(x)=\mathcal{Z}_m^T\ \Omega_m^{*\psi}(x),\quad x\in J.$$

For $m \in \mathbb{N}$, A and B constants, Eq (1.1) gives

$$\mathfrak{T}_m({}^cD_a^{\alpha,\psi}u+A\,{}^cD_a^{\beta,\psi}u)+B\mathfrak{T}_m(u)=\mathfrak{T}_m(f).\tag{6.5}$$

By using Lemma 6.3 and Eq (6.1)

$$W_m^{*\psi}+B(W_m^{*\psi}S_{m\times m}^{*\psi}+P_m^{*T})=\mathcal{Z}_m^T.$$

Thus

$$W_m^{*\psi} = \mathfrak{X}_m\, \mathfrak{Y}_m^{-1},$$

where

$$\mathfrak{Y}_m = I_{m\times m} + BS^{*\psi}_{m\times m}, \;\; \mathfrak{X}_m = \mathcal{Z}_m^T - BP_m^{*T}.$$

The matrix $I_{m\times m}$ is the identity matrix of order m.

Remark 1. We observe that $\mathfrak{Y}_m$ is invertible, if it is not so, then the number of ψ-shifted Chebyshev coefficients are adjusted to make it invertible. Thus, the solution of (1.1) can be approximated by the sequence $\{u_m\}$.

7. Numerical results

In this section, we present some examples to justify our theoretical results. Comparison of both approximated and existing results for multiple values of α and β (both fractional and integer) reveals that the proposed method is very convenient. All the numerical computations are executed in Matlab.

Example 7.1. Consider the following differential equation with given boundary conditions

$$\begin{gathered} {}^cD_0^{\alpha,\psi}u(x) + A\,{}^cD_0^{\beta,\psi}u(x) + B(x)u(x) = f(x)\,, \quad x \in J, \\ u(0) = 0 \quad and \quad u(1) = 0, \end{gathered} \tag{7.1}$$

where $A = 3$, $B(x) = e^{-\lambda x}$. For simplicity, we take $\lambda = 1$. The exact solution is

$$u_{ex} = (\psi(x))^{4-\alpha} - (\psi(x))^{3-\alpha}.$$

Let $1 < \alpha \le 2$ and $0 \le \beta \le 1$, $n \in \mathbb{N}$, $J = [a, b]$, where $0 \le a, b \le 1$ and $\psi : J \to [0, 1]$. Let $\psi, f \in C^n(J)$, ψ is increasing with $\psi'(x) \neq 0$ for all $x \in J$. Then, $f(x)$ is defined as

$$\begin{aligned} f(x) =& \frac{\Gamma(5-\alpha)}{\Gamma(5-2\alpha)}(\psi(x))^{(4-2\alpha)} - \frac{\Gamma(4-\alpha)}{\Gamma(4-2\alpha)}(\psi(x))^{(3-2\alpha)} \\ &+ 3\left[\frac{\Gamma(5-\alpha)}{\Gamma(5-\alpha-\beta)}(\psi(x))^{(4-\alpha-\beta)} - \frac{\Gamma(4-\alpha)}{\Gamma(4-\alpha-\beta)}(\psi(x))^{(3-\alpha-\beta)}\right] \\ &+ B(x)((\psi(x))^{4-\alpha} - (\psi(x))^{3-\alpha}). \end{aligned}$$

We will consider different cases for the chosen function ψ for different integral and fractional values of α and β, discussing the accuracy of proposed technique numerically and graphically by comparing the numerical solution with exact solution. For the purpose, consider three different choices of ψ.

- $\psi_1(x) = x$,
- $\psi_2(x) = \frac{1}{2}x(x + 1)$,
- $\psi_3(x) = \frac{x(e^x+2)}{e+2}$.

Table 1 shows the absolute error analysis of exact and numerical solution for derivatives of different orders for different choices of function $\psi(x)$ when $n = 10$. From the data in Table 1, it is evident that error varies in such a way that it becomes almost negligible at some points in the case $\alpha = 2, \beta = 0.5$ and in other case numerical solution is very close to exact solution. In the Figure 1, we see that the proposed solution is almost overlapping the exact solution for $\alpha = 2, \beta = 0.5$ thoroughly. When all derivatives are taken to be fractional i.e., in the case when, $\alpha = 1.5, \beta = 0.5$, there is a little deviation in the exact and numerical solution for a few points but overall the numerical solution remains close to exact solution, also both solutions are very close on the boundaries.

Table 1. Absolute error when n=10.

x	$\alpha = 2, \beta = 0.5$			$\alpha = 1.5, \beta = 0.5$		
	$\psi_1(x)$	$\psi_2(x)$	$\psi_3(x)$	$\psi_1(x)$	$\psi_2(x)$	$\psi_3(x)$
0.1	1.94e-16	9.71e-17	1.72e-11	1.07e-03	6.37e-04	7.26e-04
0.3	0.00	0.00	1.49e-10	1.14e-03	2.77e-03	7.58e-04
0.5	9.99e-16	9.99e-16	0.00	8.54e-04	2.61e-03	1.51e-03
0.7	0.00	0.00	1.61e-10	1.94e-04	2.98e-03	2.26e-03
0.9	0.00	0.00	2.01e-11	1.84e-05	1.20e-02	6.41e-03

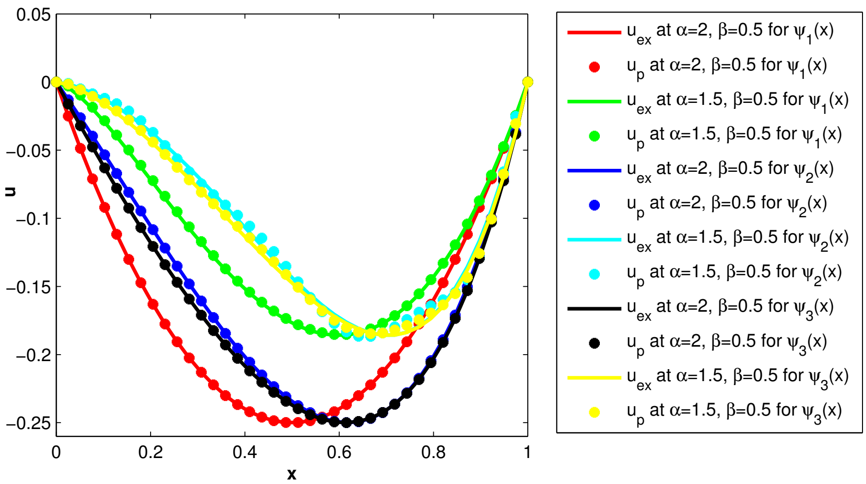

Figure 1. Comparison of exact and numerical solutions for $\alpha = 2, 1.5$ and $\beta = 0.5$.

Example 7.2. Now, we will discuss problem in Example 7.1 taking $A = 0$ when $\psi(x) = x$. Thus, the problem of taking two different derivatives reduces to a single one. We will take fractional derivatives of different orders. In Figure 2, the behaviour of the solutions for different fractional order derivatives allow us to say that there is a convergence of fractional order solution to integer order solution which indicates the usefulness of method for multiple order derivatives. Table 2 shows the error analysis for different fractional and integer order derivatives. From the table, it is clear that the error increases with the increase in order of fractional derivative and is almost negligible for integer order derivatives.

Table 2. Error analysis of exact and proposed numerical solution for different choices of α when $\psi(x) = x$ and n=8.

x	$\alpha = 1.2$	$\alpha = 1.3$	$\alpha = 1.4$	$\alpha = 1.5$	$\alpha = 1.6$	$\alpha = 1.7$	$\alpha = 1.8$	$\alpha = 1.9$	$\alpha = 2.0$
0.2	1.64e-03	3.98e-03	7.59e-03	1.23e-02	1.75e-02	2.17e-02	2.24e-02	1.63e-02	0.0
0.4	2.87e-04	2.23e-03	5.91e-03	1.14e-02	1.81e-02	2.45e-02	2.71e-02	2.09e-02	0.0
0.6	8.02e-04	3.54e-04	3.06e-03	7.48e-03	1.32e-02	1.91e-02	2.22e-02	1.78e-02	0.0
0.8	2.31e-03	2.23e-03	9.77e-04	1.58e-03	5.25e-03	9.24e-03	1.18e-02	1.00e-02	8.33e-17

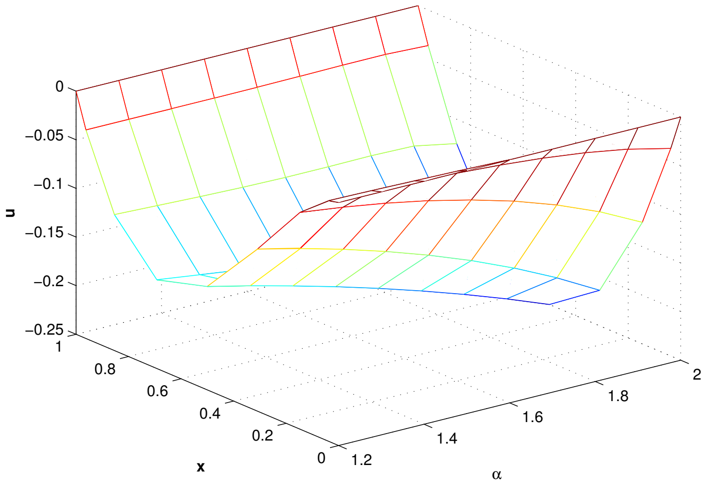

Figure 2. Exact and proposed numerical solution for different choices of α when $n = 8$.

Example 7.3. Consider the boundary value problem

$$
\begin{aligned}
{}^{c}D_{0}^{\alpha,\psi}u(x) &= {}^{c}D_{0}^{\beta,\psi}u(x) + f(x)\,, \qquad x \in [0,1], \\
u(0) &= 0 \quad and \quad u(1) = 0\,,
\end{aligned}
\tag{7.2}
$$

where, $f(x) = -1 - e^{\psi(x)-1}$. The exact solution at $\alpha = 2$ and $\beta = 1$ is $u_{ex} = \psi(x)(1 - e^{\psi(x)-1})$. In order to compare the results with existing techniques, we take $\psi(x) = x$. Y. G. Wang [40] solved this problem by Homotopy perturbation method. The same problem is solved by Gegenbauer wavelet method [41], Haar wavelet method [42]. Gegenbauer method gives good results as compared to Homotopy perturbation and Haar wavelet methods. So, we will compare the results obtained by proposed technique with Gegenbauer technique. Here, we take the dimension of fractional differentiation matrix 10×10 and in Gegenbauer technique the dimension of fractional differenciation matrix is $2^{k-1}M \times 2^{k-1}M$ where $k = 2$ and $M = 3$ which is very large as compare to proposed technique. The method gives reasonably accurate results for matrix with small dimension. Table 3 documents the absolute error of the results obtained by proposed and Gegenbauer technique. We observe that error by our method is less which shows that presented method works well for the problem under consideration. In Figure 3, the graphical analysis of exact, proposed and Gegenbauer technique is presented.

Table 3. Comparison of Gegenbauer and proposed method when $\psi(x) = x$ and n=10.

x	$\alpha = 2, \beta = 1$				
	u_{ex}	u_{GWM}	u_p	E_{GWM}	E_p
0.1	0.059343034	0.05934302	0.059343034	1.58953244e-8	3.6921e-12
0.3	0.151024409	0.15102438	0.151024409	2.46505651e-8	2.3859e-12
0.5	0.196734670	0.19673463	0.196734670	3.86865598e-8	9.3800e-13
0.7	0.181427246	0.18142719	0.181427246	6.01022115e-8	8.3097e-13
0.9	0.085646324	0.08564623	0.085646324	9.17089691e-8	3.1104e-12

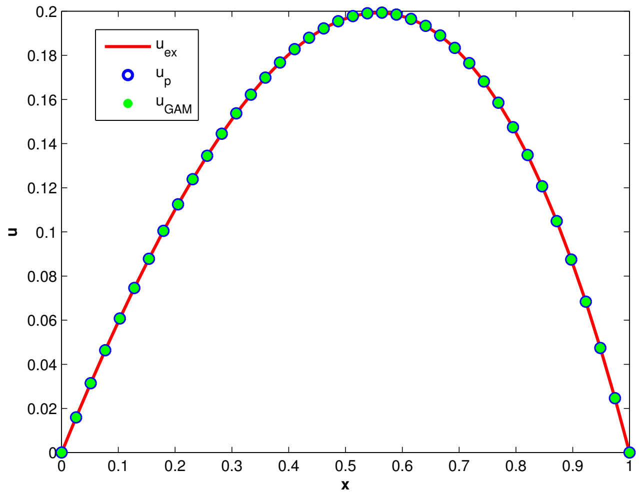

Figure 3. Comparison of exact, proposed and Gegenbauer solution for $\alpha = 2$ and $\beta = 1$.

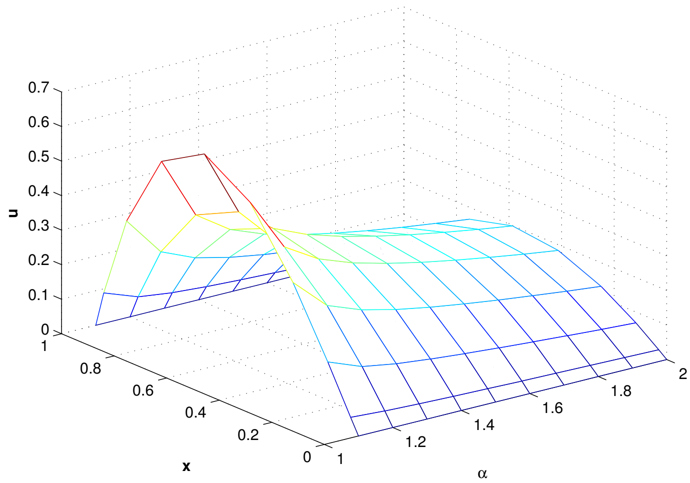

Figure 4. Numerical solution for $1 < \alpha \leq 2$ when $n = 9$.

Let us fix $\beta = 1$ and vary α i.e., $1 < \alpha \leq 2$ to see the behaviour of the solution. In Figure 4, we observe that solution for all values of α approaches to $\alpha = 2$ which shows the convergence of the fractional order solution to integer order solution.

Example 7.4. Consider following boundary value problem

$$\begin{aligned} &{}^{c}D_0^{\alpha,\psi}u(x) + A(x)u(x) = f(x)\,, \quad x \in [0,1], \\ &u(0) = 0 \quad and \quad u(1) = 0\,, \end{aligned} \tag{7.3}$$

where $1 < \alpha \leq 2$, $A(x) = \sin(\psi(x)) * \cos(\psi(x))$, $f(x) = \frac{\Gamma(\alpha+2)\psi(x)}{\Gamma(2)} - \frac{\Gamma(3)(\psi(x))^{2-\alpha}}{\Gamma(3-\alpha)} + [(\psi(x))^{\alpha+1} - (\psi(x))^2]\sin(\psi(x))\cos(\psi(x))$. The exact solution is $u_{ex} = [(\psi(x))^{\alpha+1} - \psi(x)^2]$.

Saeed et al. in [43] derived the solution of the problem 7.4 by using Green-CAS wavelet method for $\psi(x) = x$. In Table 4, we have tabulated the maximum absolute error of proposed technique for different choices of α and n along with errors calculated by Green-CAS wavelet method. Figure 5 describes the graphical analysis of problem 7.4 using proposed numerical technique for different values of α. In Figure 5, again we see that proposed solution is very close to exact solution and convergence of fractional order solution to integer order solution can be accessed.

Table 4. Maximum absolute error for different choices of α and n.

α	*Green − CAS*		*Proposed*	
	$M = 7, k = 3$	$M = 7, k = 5$	$n = 10$	$n = 20$
1.1	2.81392e-4	6.18848e-5	1.80628e-5	2.12659e-6
1.3	3.67537e-4	6.11330e-5	1.73368e-5	7.36332e-7
1.5	2.44023e-4	3.06953e-5	1.43800e-5	6.53296e-7
1.7	1.25754e-4	1.19623e-5	9.10221e-6	5.21434e-7
1.9	5.50011e-5	4.18048e-6	1.93939e-6	1.10295e-7

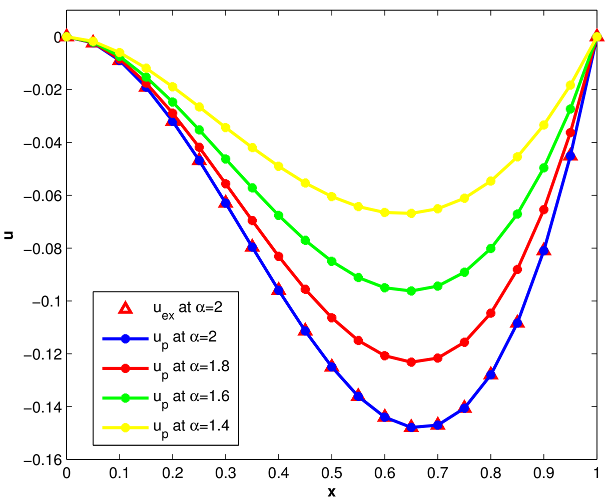

Figure 5. Comparison of solutions for different choices of α.

Example 7.5. Consider the following differential equation with given boundary conditions

$$A\,{}^{c}D_0^{\alpha,\psi}u(x) + B\,{}^{c}D_0^{\beta,\psi}u(x) + Cu(x) = f(x),$$
$$u(0) = 0 \quad and \quad u(1) = 1,$$

where $A = B = C = 1$, $\alpha = 2, \beta = 1.5$, the exact solution is $u_{ex} = (\psi(x))^2$ and $f(x) = \frac{2}{\Gamma(3-\alpha)}(\psi(x))^{(2-\alpha)} + \frac{2}{\Gamma(3-\beta)}(\psi(x))^{(2-\beta)} + (\psi(x))^2$.
For $\psi(x) = x$, the above problem reduces to,

$$A\,{}^{c}D_0^{2}u(x) + B\,{}^{c}D_0^{3/2}u(x) + Cu(x) = f(x)\,,$$
$$u(0) = 0 \quad and \quad u(1) = 1,$$

with $A = B = C = 1$, the exact solution $u_{ex} = x^2$ and $f(x) = 2 + 4\sqrt{\frac{x}{\pi}} + x^2$.

This is famous Bagley Torvik equation with real coefficients and some boundary conditions to get the unique solution. This equation is considered to be a prototype for a vast class of fractional differential equations that involves more than one differential operators. This problem is solved in [43] by Green-CAS wavelet method. The motion of a rigid plate immersed in a Newtonian fluid can be modelled by using Bagley-Torvik equation with real coefficients [44]. In the presented case, although the order of the matrix is smaller than the order of matrix in Green-CAS wavelet method, yet we obtain more accurate results. Table 5 shows the error analysis of Green-CAS wavelet method and proposed method for $\alpha = 2$ and different choices of β. We can observe that error is minimized in the presented scheme. Figure 6 shows the comparison of exact and proposed solution for $\alpha = 2$ and $\beta = 1.5, 0.5$. From the figure, it can be analyzed that proposed solution is very close to exact solution which shows the efficiency and applicability of purposed numerical scheme.

Table 5. Absolute error analysis of Green-CAS and proposed technique.

x	$\alpha = 2, \beta = 1.5$			$\alpha = 2, \beta = 0.5$		
	$k = 3, M = 5$ $Green-CAS$	$k = 3, M = 7$ $Green-CAS$	n=10 Proposed	$k = 3, M = 5$ $Green-CAS$	$k = 3, M = 7$ $Green-CAS$	n=10 Proposed
0.2	3.0041e-5	1.5569e-5	2.9976e-15	9.5247e-5	5.6492e-5	7.2442e-15
0.4	2.9087e-5	1.4928e-5	4.1356e-15	1.2568e-4	7.5422e-5	7.1332e-15
0.6	2.9455e-5	1.5227e-5	2.8866e-15	1.2312e-4	7.3839e-5	5.4401e-15
0.8	3.0625e-5	1.6108e-5	1.9846e-15	9.0324e-5	5.3451e-5	3.9968e-15

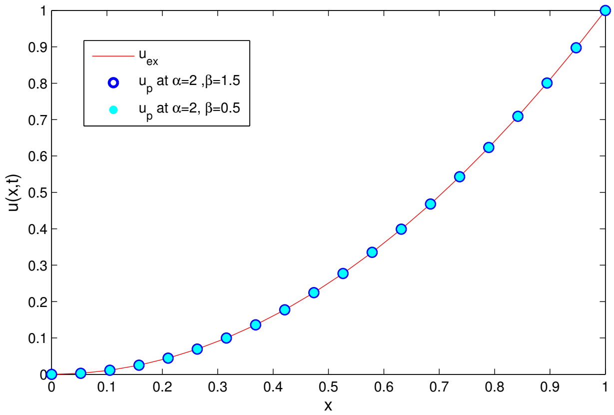

Figure 6. Comparison of exact and proposed numerical solution for $\alpha = 2$ and $\beta = 1.5, 0.5$.

8. Conclusions

In this paper, we presented a technique dealing with the numerical solution of fractional boundary value problems. First, we have presented the modified ψ-shifted Chebyshev polynomials from shifted Chebyshev polynomials. Next, we have formulated operational matrices of fractional differentiation using these modified polynomials. Then, we have applied these matrices for the solution of boundary value problem so that we have an algebraic system of equations which is then solved for numerical solution. Several examples are presented to elaborate the applicability of the method and comparison with different techniques show the effectiveness of the proposed method. Moreover, the convergence analysis for both integer and fractional order solutions is presented.

Conflict of interest

The authors declare no conflict of interest.

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