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Research article

New results for the non-oscillatory asymptotic behavior of high order differential equations of Poincaré type

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Abstract: This paper discusses the study of asymptotic behavior of non-oscillatory solutions for high order differential equations of Poincaré type. We present two new and weaker hypotheses on the coefficients, which implies a well posedness result and a characterization of asymptotic behavior for the solution of the Poincaré equation. In our discussion, we use the scalar method: we define a change of variable to reduce the order of the Poincaré equation and thus demonstrate that a new variable can satisfies a nonlinear differential equation; we apply the method of variation of parameters and the Banach fixed-point theorem to obtain the well posedness and asymptotic behavior of the non-linear equation; and we establish the existence of a fundamental system of solutions and formulas for the asymptotic behavior of the Poincaré type equation by rewriting the results in terms of the original variable. Moreover we present an example to show that the results introduced in this paper can be used in class of functions where classical theorems fail to be applied.

Keywords: Poincaré-Perron problem; asymptotic behavior; Riccati equations
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1. Introduction

In this paper we are interested in the non-oscillatory asymptotic behavior of the following differential equation:

$$y^{(n)} + \sum_{i=0}^{n-1}[a_i + r_i(t)]y^{(i)} = 0, \quad n \in \mathbb{N}, n \geq 2, \tag{1.1}$$

where $a_i \in \mathbb{R}$ are given constants and r_i are real-valued functions. The Eq (1.1) is called of a Poincaré type, since its analysis was motivated and introduced by Poincaré [17], where the author obtained conditions guaranteeing that all nonzero solutions of (1.1) are such that $\lim_{t\to\infty}(y'/y)(t)$ exists as a finite number. Perron [16] completed the result of Poincaré in the sense of existence. Later on, the asymptotic behavior of (1.1) have been investigated by several authors with a long and rich history of results [3, 8, 12, 13]. A recent short review of the most relevant landmarks of the evolution of the Poincaré problem, is included in [5, 18, 19].

The purpose of this paper is to solve the Poincaré problem by introducing hypotheses, on perturbation functions r_i, that are weaker than the classical ones. Our analysis is based on the modified scalar method which consists of three big steps, for details on cases $n = 3, 4$ see [10] and [5, 6], respectively. In the first step, let us consider that y is a nontrivial solution of (1.1), we introduce the change of variable

$$z(t) = \frac{y'(t)}{y(t)} - \mu \quad \text{or equivalently} \quad y(t) = \exp\left(\int_{t_0}^{t} (z(s) + \mu)ds\right), \tag{1.2}$$

to reduce the order of (1.1). If $\mu \in \mathbb{R}$ is a simple characteristic root of (1.1) when $r_i = 0$ then z satisfies a non-linear differential equation of the following type

$$z^{(n-1)}(t) + \sum_{i=0}^{n-2} b_i(\mu) z^{(i)}(t) = \mathbb{P}(\mu, t, r_0(t), \ldots, r_{n-1}(t), z(t), z'(t), \ldots, z^{(n-2)}(t)), \tag{1.3}$$

where b_i are real-valued functions, $\mu \in \mathbb{R}$ is a given (fix) parameter, and $\mathbb{P} : \mathbb{R}^{2n+1} \to \mathbb{R}$ is a polynomial of n degree in the $n-1$ last variables and the coefficients depends on the first $n+2$ variables, i.e., $\mathbb{P}$ admits the representation

$$\mathbb{P}(\mathbf{e}, \mathbf{x}) = \sum_{|\alpha|=0}^{n} \Omega_\alpha(\mathbf{e})\mathbf{x}^\alpha \quad \text{with } \mathbf{e} = (e_1, \ldots, e_{n+2}) \text{ and } \mathbf{x} = (x_1, \ldots, x_{n-1}), \tag{1.4a}$$

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{N}_0^{n-1}, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{1.4b}$$

Here, we have used the standard multindex notation for $|\alpha|$ and $\mathbf{x}^\alpha$, i.e. $|\alpha| = \alpha_1 + \ldots + \alpha_{n-1}$ and $\mathbf{x}^\alpha = x_1^{\alpha_1} \ldots x_{n-1}^{\alpha_{n-1}}$. We observe the inclusion of $r_0(t), \ldots, r_{n-1}(t)$ as arguments of $\mathbb{P}$ is only by notational convenience and the Eq (1.3) is the abstract form of a more particular equation obtained in the specific case of change of variable (1.2). For instance, in the case $n = 3, 4$ there is several coefficients such that $\Omega_\alpha = 0$ [5, 10]. In the second step, we analyze the existence and asymptotic behavior of (1.3). Finally, we particularize the results of (1.3) to the specific equation obtained for μ characteristic root of (1.1) and using the definition of y in (1.3) we deduce the well posedness and asymptotic behavior of (1.1).

The main aims of the paper are the following

- (O_1) Provide the framework to analyze (1.3).
- (O_2) Precise the asymptotic behavior of (1.3).
- (O_3) Introduce the appropriate assumptions to solve Poincaré problem by particularize the results of (O_1) and (O_2).

Hence, our main results, related with the aims (O_1), (O_2) and (O_3) are given by Theorem 3.1–3.4, respectively.

The rest of the paper is organized as follows: we introduce some notation in section 2, we present the main results on section 3, we develop the proofs on section 4, and we present an example on section 5.

2. Assumptions on coefficients of (1.3) and (1.1)

2.1. Assumptions for the coefficients of (1.3)

Let us introduce some notation. Throughout the paper unless otherwise stated we assume the notation t_0 is a given (fix) real number. Given the Green function g (see (4.1)), we consider that the functions $\mathcal{R}$, $\mathcal{L}_k$ and Υ_ℓ are defined as follows

$$\mathcal{R}(t) = \sum_{j=0}^{n-2} \left| \int_{t_0}^{\infty} \frac{\partial^j g}{\partial t^j}(t,s)\Omega_{\mathbf{0}}\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s)\Big) ds \right|, \tag{2.1}$$

$$\mathcal{L}_k(t) = \int_{t_0}^{\infty} \left[\sum_{j=0}^{n-2} \left| \frac{\partial^j g}{\partial t^j}(t,s) \right| \right] \sum_{|\alpha|=k} \Big| \Omega_\alpha\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s)\Big) \Big| ds, \quad k = \overline{1,n}, \tag{2.2}$$

$$\Upsilon_0(y_1,\ldots,y_d) = \prod_{1\le i<j\le d} (y_j - y_i), \quad \Upsilon_\ell(y_1,\ldots,y_d) = \prod_{\substack{1\le i<j\le d,\\ i\neq \ell,\, j\neq \ell.}} (y_j - y_i) \quad \text{for } \ell = \overline{1,d}. \tag{2.3}$$

Hereinafter we consider that the integrals $\int_{t_0}^{\infty}$ defining $\mathcal{R}$ and $\mathcal{L}_k$ exists as improper Riemann integrals. Then, we assume the following hypotheses on $b_i(\mu)$ and $\mathbb{P}$:

(R1) The functions $b_i(\mu)$ for $i = \overline{0, n-2}$ are such that the set of characteristic roots for (1.3) when $\mathbb{P} = 0$ is given by $\Gamma_\mu = \Big\{ \gamma_i(\mu), i = \overline{1, n-1} \ : \ \gamma_1 > \gamma_2 > \ldots > \gamma_{n-1} \Big\} \subset \mathbb{R}$.

(R2) The functions Ω_α given on (1.4a) satisfy the following requirements

$$\lim_{t\to\infty} \mathcal{R}(t) = \lim_{t\to\infty} \mathcal{L}_1(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \sum_{k=2}^{n} \mathcal{L}_k(t) < 1, \tag{2.4}$$

where $\mathcal{R}$ and $\mathcal{L}_k$ are defined on (2.2).

(R3) The functions Ω_α on (1.4a) satisfy the following property: given Φ_1 and γ_i with

$$\Phi_1 = \Big[\Big| \Upsilon_0(\gamma_1,\ldots,\gamma_{n-1}) \Big| \Big]^{-1} \sum_{\ell=1}^{n-1} \Big| \Upsilon_\ell(\gamma_1,\ldots,\gamma_{n-1}) \Big| \sum_{j=0}^{n-2} (|\gamma_\ell|)^j, \tag{2.5}$$

and $\gamma_i \in \Gamma_\mu$, then there exists $\sigma_{\gamma_i} \in [0, 1/\Phi_1[$ such that the inequality

$$\int_{t_0}^{\infty} \exp(-\gamma_i(t-s)) \sum_{|\alpha|=1}^{n} \Big| \Omega_\alpha\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s)\Big) \Big| ds \le \sigma_{\gamma_i}$$

is satisfied for all $t \in [t_0, \infty[$.

The assumptions (R1) and (R2) are considered to study the existence of a solution for (1.3). Since we are interested in the non-oscillatory behavior, we considered the hypothesis (R1). Meanwhile, (R2) is the natural condition in order to get the application of Banach fixed point theorem. Now, the hypothesis (R3) is used in order to get the asymptotic behavior of the solution for (1.3).

2.2. *Assumptions for the coefficients of* (1.1)

Let us consider $\gamma_j(\lambda_i)$ and the function $\mathbb{H}$ defined by

$$\gamma_j(\lambda_i) = \begin{cases} \lambda_j - \lambda_i, & j = 1, \ldots, i-1, \\ \lambda_{j+1} - \lambda_i, & j = i, \ldots, n-1, \end{cases} \tag{2.6}$$

$$\begin{aligned} \mathbb{H}(t,\mu) &= (n-1) + a_2 + r_2(t)\Big(2 + 2\mu\Big) + r_1(t) + \sum_{m=1}^{n-3} \hat{S}_{m,n} \\ &\quad + \sum_{i=3}^{n-1} \left[a_i\Big((i-1) + \sum_{m=1}^{i-3} \hat{S}_{m,i} + 1 + \sum_{j=1}^{i-2} \binom{i}{j}\mu^j\Big) + r_i(t)\Big(\sum_{m=0}^{i-3} \hat{S}_{m,i}(t) + (1+\mu)^i\Big)\right], \end{aligned} \tag{2.7}$$

where $\hat{S}_{0,j}(t) = j + \mu(j-1)$ and for $m > 1$ we have that

$$\begin{aligned} \hat{S}_{m,j} = \sum_{\ell_1=0}^{j-(m+2)} \sum_{\ell_2=0}^{j-\ell_1-(m+2)} \cdots \sum_{\ell_m=0}^{j-\sum_{q=0}^{m-1}\ell_q-(m+2)} \binom{j-1}{\ell_1} \prod_{i=1}^{m-1} \binom{j-\sum_{q=1}^{i}\ell_q - i - 1}{\ell_{i+1}} \\ \times \left[1 + \Big(j - \sum_{q=1}^{m}\ell_q - m - 1\Big)\Big(1+\mu\Big)\right]. \end{aligned}$$

Then, related with the coefficients of (1.1), we consider that:

(H1) The constants a_i are selected such that the set of characteristic roots for (1.1) when $r_i = 0$ is given by $\Lambda = \Big\{\lambda_i, i = \overline{1,n} \ : \ \lambda_1 > \lambda_2 > \ldots > \lambda_n\Big\} \subset \mathbb{R}$.

(H2) The perturbation functions satisfy the asymptotic behavior:

$$\lim_{t\to\infty} \sum_{i=0}^{n-2} \left| \int_{t_0}^{\infty} \frac{\partial^i g}{\partial t^i}(t,s) r_j(s) ds \right| = 0 \quad \text{for } j = \overline{0, n-1}, \tag{2.8}$$

$$\lim_{t\to\infty} \int_{t_0}^{\infty} \sum_{i=0}^{n-2} \left| \frac{\partial^i g}{\partial t^i}(t,s) \right| \left| \sum_{\ell=0}^{n-1} \mu^\ell r_\ell(s) \right| ds = 0 \quad \text{for each } \mu \in \Lambda, \tag{2.9}$$

for a Green function g defined on (4.1).

(H3) The perturbation functions r_i and the characteristic set Λ satisfy the following property: given Φ_i^Λ and $\gamma_k(\lambda_i)$ with

$$\Phi_i^\Lambda = \Big[\Big|\Upsilon_0(\gamma_1(\lambda_i), \ldots, \gamma_{n-1}(\lambda_i))\Big|\Big]^{-1} \sum_{\ell=1}^{n-1} \Big|\Upsilon_\ell(\gamma_1(\lambda_i), \ldots, \gamma_{n-1}(\lambda_i))\Big| \sum_{j=0}^{n-2} \Big(|\gamma_\ell(\lambda_i)|\Big)^j,$$

and $\gamma_k(\lambda_i) \in \Gamma_{\lambda_i}$, then there exits $\sigma_{\gamma_k} \in [0, 1/\Phi_i^\Lambda[$ such that the inequality

$$\int_{t_0}^{\infty} \exp(-\gamma_k(\lambda_i)(t-s))\Big|\mathbb{H}(s,\lambda_i)\Big| ds \le \sigma_{\gamma_k}$$

is satisfied for all $t \in [t_0, \infty[$.

2.3. A brief discussion of the assumptions.

Related with the constant coefficients part of (1.1), i.e. when $r_i = 0$, there is a coincidence or common hypothesis used by the different researchers, in order to deduce the non-oscillatory asymptotic behavior. All of them consider the fact that

$$\left.\begin{array}{l}\text{There is a simple characteristic root } \mu \text{ such that} \\ \mathrm{Re}(\mu) \neq \mathrm{Re}(\mu_0) \text{ for any other characteristic root } \mu_0\end{array}\right\}. \tag{2.10}$$

Then, (H1) is an extension of the condition (2.10) for all characteristic roots of (1.1). Now, with respect to the regularity and the asymptotic behavior of perturbation functions r_i we have different assumptions. For instance, the seminal work of Poincaré [17] consider that r_i are rational functions, Perron [16] assumes that r_i are continuous functions, Levinson [15] considers that $r_i \in L^1([t_0, \infty[)$, and Hartman and Wintner in [14] select $r_i \in L^p([t_0, \infty[)$ for some $p \in]1, 2]$. There is a coincidence of the authors with respect to the asymptotic behavior of r_i, by considering that: $r_i(t) \to 0$ when $t \to \infty$. Then (H2) and (H3) are new and more weak than the previous assumptions. Indeed, we refer to the recent work [5], in the case $n = 4$, for an example of perturbation functions r_i which satisfy (H2) and (H3) and neither satisfy the assumptions of L^p regularity and nor satisfy the asymptotic behavior $r_i(t) \to 0$ when $t \to \infty$.

In principle, the asymptotic behavior of (1.1) can be obtained by application of the typical existing results for the asymptotic behavior for systems. However, it has some drawbacks as is specified below by synthesizing the system methodology in three steps:

(a) *Rewrite* (1.1) *as a diagonal system.* We consider the change of variable

$$x_i^{(1)}(t) = \begin{cases} y^{(i)}(t), & i = 1, \ldots, n-1, \\ -\sum_{j=0}^{n-1} [a_j + r_j(t)] y^{(j)}(t), & i = n, \end{cases} \tag{2.11}$$

and introduce the notation

$$X(t) = \Big[x_1(t) \ \ldots \ x_n(t)\Big]^T, \quad A = \begin{bmatrix} 0 & I_{n-1} \\ -a_0 & -\mathbf{a} \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ -\mathbf{r}(t) \end{bmatrix},$$

with $\mathbf{a} = [a_1 \ \ldots \ a_n]$, $\mathbf{r}(t) = [r_0(t) \ \ldots \ r_{n-1}(t)]$ and I_{n-1} the identity matrix of order $n-1$. Then, we observe that the Eq (1.1) is equivalently to the following system

$$X^{(1)}(t) = \Big[A + B(t)\Big] X(t). \tag{2.12}$$

We note that the eigenvalues of A are the characteristic roots of (1.1) when $\mathbf{r} = 0$. Then, if we assume that (H1) holds, we deduce that we can diagonalize the system (2.12). More precisely, under (H1) the system (2.12) can be rewritten in a diagonal form as follows

$$Y^{(1)}(t) = \Big[\mathrm{diag}(\lambda_1, \ldots, \lambda_n) + M_n^{-1} B(t) M_n\Big] Y(t), \quad \text{with} \quad Y = M_n^{-1} X M_n, \tag{2.13}$$

where M_n is the Vandermonde matrix of order n asocieted with $(\lambda_1, \ldots, \lambda_n)$.

(b) *Application of results for asymptotic behavior of systems.* The results for asymptotic behavior on systems require some assumptions on the perturbation matrix $M_n^{-1} B(t) M_n$ as specified below for

Levinson, Hartman-Wintner and Eastham Theorems. The *Levinson Theorem* (see [8, Theorem 1.3.1]), assume that if the perturbation matrix satisfies

$$\int_{t_0}^{\infty} |M_n^{-1}B(s)M_n|ds < \infty, \tag{2.14}$$

then we deduce that the system (2.13) has n solutions Y_i for $i = 1, \ldots, n$ with the following asymptotic behavior

$$Y_i(t) = (\mathbf{e}_i + o(1)) \exp\Big(\lambda_i(t - t_0)\Big) \text{ when } t \to \infty. \tag{2.15}$$

The notation $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is used to denote the vectors of the standard canonical base of $\mathbb{R}^n$. The Hartman-Wintner Theorem (see [8, Theorem 1.5.1]) consider that the perturbation matrix satisfies

$$\int_{t_0}^{\infty} |M_n^{-1}B(s)M_n|^p ds < \infty, \quad \text{for some } p \in]1, 2], \tag{2.16}$$

and deduce that the system (2.13) has n solutions Y_i for $i = 1, \ldots, n$ with the following asymptotic behavior

$$Y_i(t) = (\mathbf{e}_i + o(1)) \exp\Big(\int_{t_0}^{t} (\lambda_i + r_{ii}(s))ds\Big) \text{ when } t \to \infty, \tag{2.17}$$

where $r_{ii}(s)$ are the elements of the diagonal of $M_n^{-1}B(s)M_n$. The *Eastham Theorem* (see [8, Theorem 1.6.1]) suppose that $\mu_k(t)$ the eigenvalues of $\operatorname{diag}(\lambda_1, \ldots, \lambda_n) + M_n^{-1}B(t)M_n$ satisfies the dichotomy Levinson condition, then deduce the system (2.13) has n solutions Y_i for $i = 1, \ldots, n$ with the following asymptotic behavior

$$Y_i(t) = (\mathbf{e}_i + o(1)) \exp\Big(\int_{t_0}^{t} \mu_k(s)ds\Big) \text{ when } t \to \infty. \tag{2.18}$$

(c) *Translation of the results for the behavior of Y to the variable X*. By recalling that $X = M_n Y M_n^{-1}$ an using (2.11), we deduce the asymptotic behavior of (1.1).

Theoretically, the process (a)–(c) can be rigorously applied to study the asymptotic behavior of any specific equation of the form (1.1). However, it has some clear and practical computation disadvantages for higher order differential equations, among them we have the the conditions (2.14), (2.16) and dichotomy Levinson condition for $\mu_k(t)$ are hard to verify and the change of variable for the diagonalization $M_n^{-1}XM_n$ is expensive. Note also that the analytic computation of $\mu_k(t)$ is not always possible since, most of the time, $M_n^{-1}B(t)M_n$ is a full matrix. Consequently, in order to overcome this disadvantages, in this paper we apply the scalar method [1–3, 5, 6, 10, 11].

In section 5, we present an example to compare the application of the results in this paper in comparison with the classical results.

3. Main results

The main results are given by the following four theorems:

Theorem 3.1. *Consider the notation $C_0^{n-2}([t_0,\infty[)$ for set of functions*

$$C_0^{n-2}([t_0,\infty[) = \left\{ z \in C^{n-2}([t_0,\infty[,\mathbb{R}) \quad : \quad \lim_{t\to\infty} z^{(k)}(t) = 0 \, for \; k = \overline{0,n-2} \right\},$$

which is a Banach space with the norm $\|z\|_0 = \sup_{t\ge t_0} \sum_{i=0}^{n-2} |z^{(i)}(t)|$. Assume that the coefficients of the Eq (1.3) *satisfy (R1) and (R2). Then, exists $z \in C_0^{n-2}([t_0,\infty[)$ a solution of* (1.3)*.*

Theorem 3.2. *Let us introduce the notation*

$$\mathbb{G}_\mu = \left\{ (\gamma_1,\ldots,\gamma_{n-1}) \quad : \quad \gamma_i \in \Gamma_\mu, i = \overline{1,n-1} \right\}, \tag{3.1}$$

$$\mathbb{E}_i^{n-1} = \Big\{ (x_1,\ldots,x_{n-1}) \in \mathbb{R}^{n-1} \; : \; x_1 > x_2 > \ldots > x_{n-1} \quad and \\ \Big(x_1 < 0 \; if \; i = 1 \; or \; 0 \in]x_i, x_{i-1}[\; if \; i = \overline{2,n-1} \; or \; x_{n-1} > 0 \; if \; i = n \Big) \Big\}. \tag{3.2}$$

Consider that the hypotheses of Theorem 3.1 and the assumption (R3) are valid. Then, z_μ the solution of (1.3) *has the following asymptotic behavior*

$$z_\mu^{(j)}(t) = \begin{cases} O\left(\displaystyle\int_{t_0}^{t} e^{\beta(t-s)} \Big| \Omega_{\mathbf{0}}\big(\mu, s, r_0(s),\ldots,r_{n-1}(s)\big) \Big| ds \right), \\ \qquad\qquad when \; \mathbb{G}_\mu \subset \mathbb{E}_1^{n-1} \; and \; \beta \in [\gamma_1, 0[, \\ O\left(\displaystyle\int_{t_0}^{\infty} e^{\beta(t-s)} \Big| \Omega_{\mathbf{0}}\big(\mu, s, r_0(s),\ldots,r_{n-1}(s)\big) \Big| ds \right), \\ \qquad\qquad when \; \mathbb{G}_\mu \subset \mathbb{E}_i^{n-1} \; and \; \beta \in [\gamma_i, 0[, \quad i = \overline{2,n-1}, \\ O\left(\displaystyle\int_{t}^{\infty} e^{\beta(t-s)} \Big| \Omega_{\mathbf{0}}\big(\mu, s, r_0(s),\ldots,r_{n-1}(s)\big) \Big| ds \right), \\ \qquad\qquad when \; \mathbb{G}_\mu \subset \mathbb{E}_n^{n-1} \; and \; \beta \in]0, \gamma_{n-1}], \end{cases} \tag{3.3}$$

when $t \to \infty$, for all $j \in \{0,\ldots,n-2\}$.

Theorem 3.3. *Let us consider the new variable z satisfying* (1.2)*. If y is a solution of* (1.1)*, the following assertions are valid:*

(a) If $\mu \in \Lambda$, then z satisfy the differential equation

$$z^{(n-1)} + \sum_{i=2}^{n-1} \left[\sum_{k=i}^{n} a_k \binom{k}{i} \mu^{k-i} \right] z^{(i-1)} + \left(n\mu^{n-1} + \sum_{i=1}^{n-1} i a_i \mu^{i-1} \right) z = -\mathbb{F}, \tag{3.4}$$

where

$$\begin{aligned} \mathbb{F} &= (n-1) z^{(n-2)}(t) z(t) + \sum_{m=1}^{n-3} S_{m,n}(t) \\ &+ \sum_{i=3}^{n-1} \Bigg[a_i \Big((i-1) z^{(i-2)}(t) z(t) + \sum_{m=1}^{i-3} S_{m,i}(t) + z^{(i-1)}(t) z^{(1)}(t) + \sum_{j=1}^{i-2} \binom{i}{j} z^{(i-j)}(t) \mu^j \Big) \\ &\qquad + r_i(t) \left(\sum_{m=0}^{i-3} S_{m,i}(t) + z^{(i-1)}(t) z^{(1)}(t) + (z(t)+\mu)^i \right) \Bigg] \end{aligned}$$

$$+ a_2 z^2(t) + r_2(t)\Big(z^{(1)}(t) + (z(t)+\mu)^2\Big) + r_1(t)\Big(z(t)+\mu\Big) + r_0(t), \tag{3.5}$$

$$S_{0,j}(t) = z^{(j-1)}(t) + (j-1)\, z^{(j-2)}(t)\,(z(t)+\mu), \tag{3.6}$$

$$\begin{aligned} S_{m,j}(t) = & \sum_{\ell_1=0}^{j-(m+2)} \sum_{\ell_2=0}^{j-\ell_1-(m+2)} \cdots \sum_{\ell_m=0}^{j-\sum_{q=0}^{m-1}\ell_q-(m+2)} \binom{j-1}{\ell_1} \prod_{i=1}^{m-1} \binom{j-\sum_{q=1}^{i}\ell_q - i - 1}{\ell_{i+1}} \\ & \times \prod_{i=1}^{m} (z+\mu)^{(\ell_i)}(t) \Big[(z+\mu)^{(j-\sum_{q=1}^{m}\ell_q-m-1)}(t) \\ & \qquad + \Big(j - \sum_{q=1}^{m}\ell_q - m - 1\Big)(z+\mu)^{(j-\sum_{q=1}^{m}\ell_q-m-2)}(t)\Big(z(t)+\mu\Big)\Big]. \end{aligned} \tag{3.7}$$

(b) If the hypothesis (H1) is satisfied, then

$$\Gamma_{\lambda_i} = \Big\{\gamma_j \;:\; \gamma_j(\lambda_i) \text{ given on } (2.6), \quad \gamma_1 > \gamma_2 > \ldots > \gamma_{n-1}\Big\} \subset \mathbb{R} \tag{3.8}$$

is the characteristic set of (3.4) *when* $\mu = \lambda_i$.

(c) If the hypothesis (H1) and (H2) are satisfied and $\mu = \lambda_i$. *Then, exists* $z_{\lambda_i} \in C_0^{n-2}([t_0,\infty[)$ *a solution of* (3.4).

(d) If the hypothesis (H1)–(H3) are satisfied. Then z_{λ_i}, *the solution of* (3.4) *with* $\mu = \lambda_i$, *satisfies the following asymptotic behavior*

$$z^{(j)}_{\lambda_i}(t) = \begin{cases} O\left(\displaystyle\int_t^{\infty} e^{-\beta(t-s)}\Big|\sum_{\ell=0}^{n-1}\lambda_1^{\ell} r_\ell(s)\Big|ds\right), & i=1, \;\; \beta \in [\lambda_2-\lambda_1, 0[, \\ O\left(\displaystyle\int_{t_0}^{\infty} e^{-\beta(t-s)}\Big|\sum_{\ell=0}^{n-1}\lambda_i^{\ell} r_\ell(s)\Big|ds\right), & i=\overline{2,n-1}, \;\; \beta \in [\lambda_{i+1}-\lambda_i, 0[, \\ O\left(\displaystyle\int_{t_0}^{t} e^{-\beta(t-s)}\Big|\sum_{\ell=0}^{n-1}\lambda_n^{\ell} r_\ell(s)\Big|ds\right), & i=n, \;\; \beta \in]0,\gamma_{n-1}], \end{cases} \tag{3.9}$$

when $t \to \infty$, *for all* $j \in \{0,\ldots,n-2\}$.

Theorem 3.4. *Let us consider that (H1) and (H2) are satisfied. Then, the Eq* (1.1) *has a fundamental system of solutions given by*

$$y_i(t) = \exp\left(\int_{t_0}^{t} [\lambda_i + z_{\lambda_i}(s)]ds\right), \quad \text{with } z_{\lambda_i} \text{ solution of } (3.4) \text{ with } \mu = \lambda_i \in \Lambda. \tag{3.10}$$

Moreover, the following properties about the asymptotic behavior

$$\lim_{t\to\infty} \frac{y_i^{(j)}(t)}{y_i(t)} = (\lambda_i)^j \quad j = \overline{1,n}, \tag{3.11}$$

$$W[y_1,\ldots,y_n](t) = \prod_{1\le k<\ell\le n} (\lambda_\ell - \lambda_k) \prod_{i=1}^{n} y_i(t)(1+o(1)), \tag{3.12}$$

is satisfied when $t \to \infty$. *Furthermore, if* $\pi_i = \prod_{1\le i<j\le n, j\ne i}(\lambda_j - \lambda_i)$ *and (H3) is satisfied, then*

$$y_i^{(j)}(t) = \Big(\lambda_i^j + o(1)\Big)e^{\lambda_i(t-t_0)}$$

$$\times \exp\left(\frac{1}{\pi_i}\int_{t_0}^{t}\mathbb{F}\Big(\lambda_i, s, r_0(s), \ldots, r_{n-1}(s), z_{\lambda_i}(s), \ldots, z_{\lambda_i}^{(n-2)}(s)\Big)ds\right), \tag{3.13}$$

holds, when $t \to \infty$ *with* $z_{\lambda_i}^{(j)}$, $j = \overline{0, n-2}$ *given asymptotically by* (3.9) *and* $\mathbb{F}$ *defined on* (3.5).

4. Proof of main results

4.1. Proof of Theorem 3.1

The proof is mainly based on variation of parameters technique and Banach fixed point Theorem. More specifically we proceed as follows: we introduce the precise notation of Green functions for (1.3) when $\mathbb{P} = 0$, we apply the method of variation of parameters to get the operator equation, and we deduce that the operator satisfies the hypotheses of Banach fixed point Theorem.

The Green function g for (1.3) when $\mathbb{P} = 0$ is defined by

$$g(t, s) = \frac{-g_\mu(t, s)}{\Upsilon_0(\gamma_1, \ldots, \gamma_{n-1})} \quad \text{when } \mathbb{G}_\mu \subset \mathbb{E}_i^{n-1} \text{ for } i = \overline{1, n}, \tag{4.1}$$

where Υ_0 is the notation on (2.3), $\mathbb{G}_\mu$ is the set defined on (3.1), the notation $\mathbb{E}_i^{n-1}$ is given on (3.2), and $g_\mu : \mathbb{R}^2 \to \mathbb{R}$ are the functions defined as follows

$$g_\mu(t, s) = \begin{cases} \displaystyle\sum_{\ell=1}^{n-1} G_\ell e^{\gamma_\ell(t-s)} H(t-s), & \mathbb{G}_\mu \subset \mathbb{E}_1^{n-1}, \\ \displaystyle\sum_{\ell=1}^{k-1} G_\ell e^{\gamma_\ell(t-s)} H(s-t) + \sum_{\ell=k}^{n-1} G_\ell e^{\gamma_\ell(t-s)} H(t-s), & \mathbb{G}_\mu \subset \mathbb{E}_k^{n-1}, \quad k = \overline{2, n-1}, \\ \displaystyle\sum_{\ell=1}^{n-1} G_\ell e^{\gamma_\ell(t-s)} H(s-t), & \mathbb{G}_\mu \subset \mathbb{E}_n^{n-1}, \end{cases}$$

where $G_\ell = (-1)^\ell \Upsilon_\ell(\gamma_1, \ldots, \gamma_{n-1})$ with Υ_ℓ defined on (2.3) and H is the Heaviside function, i.e., $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ for $x < 0$. For further details on Green functions the reader may be consult [3] (see also [9] for $n = 2$, [10] for $n = 3$ and [5] for $n = 4$).

We apply the method of variation of parameters to get that (1.3) is equivalent to the following integral equation

$$z(t) = \int_{t_0}^{\infty} g(t, s)\mathbb{P}\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s), z(s), \ldots, z^{(n-2)}(s)\Big)ds, \tag{4.2}$$

where g is the Green function defined on (4.1). Thus, if we define the operator T from $C_0^{n-2}([t_0, \infty[)$ to $C_0^{n-2}([t_0, \infty[)$ as follows

$$Tz(t) = \int_{t_0}^{\infty} g(t, s)\mathbb{P}\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s), z(s), \ldots, z^{(n-2)}(s)\Big)ds, \tag{4.3}$$

we note that (4.2) can be rewritten as the operator equation

$$Tz = z \qquad \text{over} \qquad D_\eta := \Big\{ z \in C_0^{n-2}([t_0, \infty[) \quad : \quad \|z\|_0 \leq \eta \Big\}, \tag{4.4}$$

where $\eta \in \mathbb{R}^+$ will be selected in order to apply the Banach fixed point theorem. Indeed, we have that
(a) *T is well defined from $C_0^{n-2}([t_0,\infty[)$ to $C_0^{n-2}([t_0,\infty[)$.* Let us consider an arbitrary $z \in C_0^{n-2}([t_0,\infty[)$, by the definition of the operator T we deduce that

$$T^{(j)}z(t) = \int_{t_0}^{\infty} \frac{\partial^j g}{\partial t^j}(t,s)\mathbb{P}\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s), z(s), \ldots, z^{(n-2)}(s)\Big)ds, \quad j = \overline{0, n-2}.$$

Then, considering the hypothesis (R2) and by using the notation (1.4), we can deduce the following estimates

$$\begin{aligned}
\sum_{j=0}^{n-2} |T^{(j)}z(t)| &= \sum_{j=0}^{n-2} \left| \int_{t_0}^{\infty} \frac{\partial^j g}{\partial t^j}(t,s) \sum_{|\alpha|=0}^{n} \Omega_\alpha\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s)\Big)\Big(z(s), \ldots, z^{(n-2)}(s)\Big)^\alpha ds \right| \\
&\le \sum_{j=0}^{n-2} \left| \int_{t_0}^{\infty} \frac{\partial^j g}{\partial t^j}(t,s)\Omega_{\mathbf{0}}\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s)\Big)ds \right| \\
&\quad + \int_{t_0}^{\infty} \sum_{j=0}^{n-2} \left|\frac{\partial^j g}{\partial t^j}(t,s)\right| \left| \sum_{|\alpha|=1}^{n} \Omega_\alpha\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s)\Big)(z(s))^{\alpha_1} \ldots (z^{(n-2)}(s))^{\alpha_{n-1}} \right| ds \\
&\le \sum_{j=0}^{n-2} \left| \int_{t_0}^{\infty} \frac{\partial^j g}{\partial t^j}(t,s)\Omega_{\mathbf{0}}\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s)\Big)ds \right| \\
&\quad + \sum_{k=1}^{n} \Big(\|z\|_0\Big)^k \int_{t_0}^{\infty} \sum_{j=0}^{n-2} \left|\frac{\partial^j g}{\partial t^j}(t,s)\right| \sum_{|\alpha|=k} \Big|\Omega_\alpha\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s)\Big)\Big| ds \qquad (4.5) \\
&\le \mathcal{R}(t) + \|z\|_0 \mathcal{L}_1(t) + \Big(\|z\|_0\Big)^2 \max\left\{1, \|z\|_0, \ldots, \Big(\|z\|_0\Big)^{n-2}\right\} \sum_{k=2}^{n} \mathcal{L}_k(t). \qquad (4.6)
\end{aligned}$$

Now, by (2.4) and the fact that $z \in C_0^{n-2}([t_0,\infty[)$, we have that the right hand side of (4.6) tend to 0 when $t \to \infty$. Thus, for all $j = 0, \ldots, n-2$ we have that $T^{(j)}z(t) \to 0$ when $t \to \infty$ or equivalently $Tz \in C_0^{n-2}([t_0,\infty[)$ for all $z \in C_0^{n-2}([t_0,\infty[)$.
(b) *For all $\eta \in]0,1[$, the set D_η is invariant under T.* Let us consider $z \in D_\eta$. From (4.5), we can deduce the following estimate

$$\begin{aligned}
\sum_{j=0}^{n-2} |T^{(j)}z(t)| &\le \mathcal{R}(t) + \sum_{k=1}^{n} \Big(\|z\|_0\Big)^k \mathcal{L}_k(t) \\
&\le \mathcal{R}(t) + \eta\mathcal{L}_1(t) + \eta^2 \left[\mathcal{L}_2(t) + \sum_{k=3}^{n} \Big(\eta\Big)^{k-2} \mathcal{L}_k(t)\right] \\
&\le \mathcal{R}(t) + \eta\mathcal{L}_1(t) + \eta^3, \qquad (4.7)
\end{aligned}$$

in a right neighborhood of $\eta = 0$. Here we notice that $\mathcal{L}_2(t) + \sum_{k=3}^{n} \Big(\eta\Big)^{k-2} \mathcal{L}_k(t) \le \eta$ in a right neighborhood of $\eta = 0$. Now, by (2.4) we deduce that the first two terms on the right side of (4.7) tend to 0 when $t \to \infty$. Hence, by (4.7) and (R2) we have that $\|Tz\|_0 \le \eta^3 \le \eta$, or equivalently $Tz \in D_\eta$ for all $z \in D_\eta$.

(c) *T is a contraction for* $\eta \in]0,1[$. Let $z_1, z_2 \in D_\eta$. We can prove that

$$\begin{aligned}
&\left|\sum_{|\alpha|=1}^{n} \Omega_\alpha\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s)\Big)\Big[\Big(z_1(s), \ldots, z_1^{(n-2)}(s)\Big)^\alpha - \Big(z_2(s), \ldots, z_2^{(n-2)}(s)\Big)^\alpha\right| \\
&\quad \le \sum_{|\alpha|=1}^{n} \left|\Omega_\alpha\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s)\Big)\right| \left(|\alpha|\eta^{|\alpha|-1} \sum_{i=0}^{n-2} \left|z_1^{(i)}(s) - z_2^{(i)}(s)\right|\right).
\end{aligned}$$

Then, by the hypothesis (R2) and some algebraic rearrangements, we obtain

$$\begin{aligned}
&\sum_{j=0}^{n-2} \left|T^{(j)} z_1(t) - T^{(j)} z_2(t)\right| \\
&\le \int_{t_0}^{\infty} \sum_{j=0}^{n-2} \left|\frac{\partial^j g}{\partial t^j}(t,s)\right| \sum_{|\alpha|=1}^{n} \left|\Omega_\alpha\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s)\Big)\right| \left(|\alpha|\eta^{|\alpha|-1} \sum_{i=0}^{n-2} |z_1^{(i)}(s) - z_2^{(i)}(s)|\right) ds \\
&\le \|z_1 - z_2\|_0 \int_{t_0}^{\infty} \sum_{j=0}^{n-2} \left|\frac{\partial^j g}{\partial t^j}(t,s)\right| \sum_{|\alpha|=1}^{n} |\alpha|\eta^{|\alpha|-1} \left|\Omega_\alpha\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s)\Big)\right| ds \\
&\le \|z_1 - z_2\|_0 \sum_{k=1}^{n} k\eta^{k-1} \mathcal{L}_k(t) \quad \le \quad \|z_1 - z_2\|_0 \max\left\{1, 2\eta, 3\eta^2, \ldots, n\eta^{n-1}\right\} \sum_{k=1}^{n} \mathcal{L}_k(t).
\end{aligned}$$

Then, by the application of (2.4), we deduce that T is a contraction, since, for an arbitrary $\eta \in]0, 1/2[$, we have that $\max\left\{1, 2\eta, 3\eta^2, \ldots, n\eta^{n-1}\right\} = 1$.

Hence, from (a)–(c) and by the application of Banach fixed point theorem, we deduce that there is a unique $z \in D_\eta \subset C_0^{n-2}([t_0, \infty[)$ solution of (4.4).

4.2. Proof of Theorem 3.2

The proof is based on the operator Eq (1.3) and the invariant and contraction properties of T. Indeed, let us first introduce some notation. We denote by z_μ the solution of the Eq (1.3) given by Theorem 3.1. Moreover, on D_η with $\eta \in]0, 1/n[$, we define recursively the sequence $\{\omega_m\}_{m\in\mathbb{N}}$ by assuming that $\omega_{m+1} = T\omega_m$ with $\omega_0 = 0$. We note that $\omega_m \to z_\mu$ when $m \to \infty$ as a consequence of the contraction property of T.

The Green function g defined on (4.1) is given in terms of g_μ and naturally, the operator T defined in (4.3) can be rewritten equivalently as follows

$$\begin{aligned}
Tz_\mu(t) &= \frac{-1}{\Upsilon_0(\gamma_1, \ldots, \gamma_{n-1})} \int_{t_0}^{\infty} g_\mu(t,s)\mathbb{P}\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s), z_\mu(s), \ldots, z_\mu^{(n-2)}(s)\Big) ds \\
&= \frac{-1}{\Upsilon_0(\gamma_1, \ldots, \gamma_{n-1})}
\end{aligned}$$

$$
\times \begin{cases} \displaystyle\sum_{\ell=1}^{n-1} G_\ell \int_{t_0}^{t} e^{\gamma_\ell(t-s)} \mathbb{P}\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s), z_\mu(s), \ldots, z_\mu^{(n-2)}(s)\Big) ds, \\ \hfill \mathbb{G}_\mu \subset \mathbb{E}_1^{n-1}, \\ \displaystyle\sum_{\ell=1}^{k-1} G_\ell \int_{t_0}^{t} e^{\gamma_\ell(t-s)} \mathbb{P}\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s), z_\mu(s), \ldots, z_\mu^{(n-2)}(s)\Big) ds \\ \quad + \displaystyle\sum_{\ell=k}^{n-1} G_\ell \int_{t}^{\infty} e^{\gamma_\ell(t-s)} \mathbb{P}\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s), z_\mu(s), \ldots, z_\mu^{(n-2)}(s)\Big) ds, \\ \hfill \mathbb{G}_\mu \subset \mathbb{E}_k^{n-1}, \quad k = \overline{2, n-1}, \\ \displaystyle\sum_{\ell=1}^{n-1} G_\ell \int_{t}^{\infty} e^{\gamma_\ell(t-s)} \mathbb{P}\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s), z_\mu(s), \ldots, z_\mu^{(n-2)}(s)\Big) ds, \\ \hfill \mathbb{G}_\mu \subset \mathbb{E}_n^{n-1}. \end{cases} \tag{4.8}
$$

Here we have used the definition of Heaviside function, for instance in the case $\mathbb{G}_\mu \subset \mathbb{E}_1^{n-1}$ the integration is on $[t_0, t[$ since $g_\mu(t, s) = 0$ for $s \in [t, \infty]$.

Thus, the proof of (3.3) is reduced to get the existence of the sequence $\{\Phi_m\} \subset \mathbb{R}_+$ such that the following two assertions are valid:

$$
\sum_{j=0}^{n-2} |\omega_m^{(j)}(t)| \le \Phi_m \times \begin{cases} \displaystyle\int_{t_0}^{t} e^{\beta(t-s)} \Big|\Omega_{\mathbf{0}}\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s)\Big)\Big| ds, \\ \hfill \mathbb{G}_\mu \subset \mathbb{E}_1^{n-1}, \quad \beta \in [\gamma_1, 0[, \\ \displaystyle\int_{t_0}^{\infty} e^{\beta(t-s)} \Big|\Omega_{\mathbf{0}}\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s)\Big)\Big| ds, \\ \hfill \mathbb{G}_\mu \subset \mathbb{E}_k^{n-1}, \quad \beta \in [\gamma_k, 0[, \quad k = \overline{2, n-1}, \\ \displaystyle\int_{t}^{\infty} e^{\beta(t-s)} \Big|\Omega_{\mathbf{0}}\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s)\Big)\Big| ds, \\ \hfill \mathbb{G}_\mu \subset \mathbb{E}_n^{n-1}, \quad \beta \in]0, \gamma_{n-1}], \end{cases} \tag{4.9}
$$

$$
\{\Phi_m\} \text{ is convergent.} \tag{4.10}
$$

Hence, to complete the proof of (3.3), we proceed to prove (4.9) and (4.10). Note that, the proof of (3.3) is concluded by passing to the limit the sequence $\{\Phi_m\}$ when $m \to \infty$ in the topology of $C_0^{n-2}([t_0, \infty])$.

Proof of (4.9). We apply mathematical induction on m and construct the sequences $\{\Phi_m\}$ for each $k = 1, \ldots, n$. Indeed, the induction step for $m = 1$ is proved as follows. From (4.8), fact that $\omega_0 = 0$, and (R1) we have that the following estimate

$$
\begin{aligned} \sum_{j=0}^{n-2} |\omega_1^{(j)}(t)| &= \sum_{j=0}^{n-2} |T\omega_0^{(j)}(t)| \\ &\le \frac{1}{\Big|\Upsilon_0(\gamma_1, \ldots, \gamma_{n-1})\Big|} \end{aligned}
$$

$$\times\begin{cases}\displaystyle\sum_{\ell=1}^{n-1}|G_\ell|(|\gamma_\ell|)^j\int_{t_0}^{t}e^{\beta(t-s)}\Big|\mathbb{P}\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s),0,\ldots,0\Big)\Big|ds,\\ \qquad\qquad\qquad\qquad\qquad\qquad \mathbb{G}_\mu\subset\mathbb{E}_1^{n-1},\ \ \beta\in[\gamma_1,0[,\\ \displaystyle\sum_{\ell=1}^{k-1}|G_\ell|(|\gamma_\ell|)^j\int_{t_0}^{t}e^{\beta(t-s)}\Big|\mathbb{P}\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s),0,\ldots,0\Big)\Big|ds\\ \quad\displaystyle+\sum_{\ell=k}^{n-1}|G_\ell|(|\gamma_\ell|)^j\int_{t}^{\infty}e^{\beta(t-s)}\Big|\mathbb{P}\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s),0,\ldots,0\Big)\Big|ds,\\ \qquad\qquad\qquad\qquad\qquad\qquad \mathbb{G}_\mu\subset\mathbb{E}_k^{n-1},\ \ \beta\in[\gamma_k,0[,\quad k=\overline{2,n-1},\\ \displaystyle\sum_{\ell=1}^{n-1}|G_\ell|(|\gamma_\ell|)^j\int_{t}^{\infty}e^{\gamma_\ell(t-s)}\Big|\mathbb{P}\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s),0,\ldots,0\Big)\Big|ds,\\ \qquad\qquad\qquad\qquad\qquad\qquad \mathbb{G}_\mu\subset\mathbb{E}_n^{n-1},\ \ \beta\in]0,\gamma_{n-1}[,\end{cases}$$

is satisfied. By the application of (R2) and the property

$$\mathbb{P}(\mu,s,r_0(s),\ldots,r_{n-1}(s),0,\ldots,0)=\mathbf{\Omega_0}(\mu,s,r_0(s),\ldots,r_{n-1}(s)),$$

we deduce that (4.9) is valid with Φ_1 given on (2.5). Then, the induction step for $m=1$ is valid. Now, we prove the general induction step: by assuming that (4.9) is valid for $m=h$, we will deduce that (4.9) is also valid for $m=h+1$. Indeed, by the rewritten form of the operator T given on (4.8), we have that

$$\begin{aligned}&\sum_{j=0}^{n-2}|\omega_{h+1}^{(j)}(t)|=\sum_{j=0}^{n-2}|T\omega_h^{(j)}(t)|\\ &\le\Phi_1\times\begin{cases}\displaystyle\int_{t_0}^{t}e^{\gamma_1(t-s)}\Big|\mathbb{P}\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s),\omega_h(s),\ldots,\omega_h^{(n-2)}(s)\Big)\Big|ds,\quad\mathbb{G}_\mu\subset\mathbb{E}_1^{n-1},\\ \displaystyle\int_{t_0}^{t}e^{\gamma_\ell(t-s)}\mathbb{P}\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s),\omega_h(s),\ldots,\omega_h^{(n-2)}(s)\Big)ds\\ \qquad\displaystyle+\int_{t}^{\infty}e^{\gamma_\ell(t-s)}\mathbb{P}\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s),\omega_h(s),\ldots,\omega_h^{(n-2)}(s)\Big)ds,\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\mathbb{G}_\mu\subset\mathbb{E}_k^{n-1},k=\overline{2,n-1},\\ \displaystyle\int_{t}^{\infty}e^{\gamma_n(t-s)}\Big|\mathbb{P}\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s),\omega_h(s),\ldots,\omega_h^{(n-2)}(s)\Big)\Big|ds.\quad\mathbb{G}_\mu\subset\mathbb{E}_n^{n-1},\end{cases}\end{aligned}$$

with Φ_1 defined on (2.5). We study the following three cases separately: $\mathbb{G}_\mu\subset\mathbb{E}_1^{n-1}$, $\mathbb{G}_\mu\subset\mathbb{E}_k^{n-1}$ for $k=\overline{2,n-1}$, and $\mathbb{G}_\mu\subset\mathbb{E}_n^{n-1}$, respectively. Firstly, for $\mathbb{G}_\mu\subset\mathbb{E}_1^{n-1}$, by application of the hypothesis (R3) and $\beta\in[\gamma_1,0[$, we have that

$$\begin{aligned}\sum_{j=0}^{n-2}|\omega_{h+1}^{(j)}(t)|&\le\Phi_1\int_{t_0}^{t}\exp(\gamma_1(t-s))\Big|\mathbb{P}\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s),\omega_h(s),\ldots,w_h^{(n-2)}(s)\Big)\Big|ds\\ &\le\Phi_1\int_{t_0}^{t}\exp(\gamma_1(t-s))\sum_{|\alpha|=0}^{n}\Big|\Omega_\alpha\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s)\Big)\Big|\Big|\Big(\omega_h(s),\ldots,\omega_h^{(n-2)}(s)\Big)^\alpha\Big|ds\\ &\le\Phi_1\Big\{\int_{t_0}^{t}\exp(\gamma_1(t-s))\Big|\mathbf{\Omega_0}\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s)\Big)\Big|ds\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_0}^{t}\exp(\gamma_1(t-s))\sum_{|\alpha|=1}^{n}\Big|\Omega_\alpha\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s)\Big)\Big|\Big(\eta^{|\alpha|-1}\sum_{j=0}^{n-2}|\omega_h^{(j)}(s)|\Big)ds\Bigg\}\\
&\leq \Phi_1\Bigg\{\int_{t_0}^{t}\exp(\gamma_1(t-s))\Big|\Omega_{\mathbf{0}}\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s)\Big)\Big|ds+\Phi_h\int_{t_0}^{t}\exp(\gamma_1(t-s))\\
&\times\sum_{|\alpha|=1}^{n}\Big|\Omega_\alpha\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s)\Big)\Big|\int_{s}^{t}\exp(\beta(t-\tau))\Big|\Omega_{\mathbf{0}}\Big(\mu,\tau,r_0(\tau),\ldots,r_{n-1}(\tau)\Big)\Big|d\tau ds\Bigg\}\\
&\leq \Phi_1\Bigg\{1+\Phi_h\int_{t_0}^{t}\exp(\gamma_1(t-s))\sum_{|\alpha|=1}^{n}\Big|\Omega_\alpha\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s)\Big)\Big|ds\Bigg\}\\
&\qquad\times\int_{t_0}^{t}\exp(\beta(t-\tau))\Big|\Omega_{\mathbf{0}}\Big(\mu,\tau,r_0(\tau),\ldots,r_{n-1}(\tau)\Big)\Big|d\tau\\
&\leq \Phi_1\Big(1+\sigma_{\gamma_1}\Phi_h\Big)\int_{t_0}^{t}\exp(\beta(t-s))\Big|\Omega_{\mathbf{0}}\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s)\Big)\Big|ds. \qquad (4.11)
\end{aligned}
$$

For $\mathbb{G}_\mu\subset\mathbb{E}_k^{n-1}$ with $k=\overline{2,n-1}$, from (H3) and $\beta\in[\gamma_k,0[$, by a similar arguments we deduce the estimate

$$
\begin{aligned}
\sum_{j=0}^{n-2}|\omega_{h+1}^{(j)}(t)|&\leq\Phi_1\Bigg[\int_{t_0}^{t}e^{\gamma_\ell(t-s)}\mathbb{P}\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s),\omega_h(s),\ldots,\omega_h^{(n-2)}(s)\Big)ds\\
&\qquad+\int_{t}^{\infty}e^{\gamma_\ell(t-s)}\mathbb{P}\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s),\omega_h(s),\ldots,\omega_h^{(n-2)}(s)\Big)ds\Bigg]\\
&\leq\Phi_1\Bigg\{1+\Phi_h\int_{t_0}^{\infty}\exp(\gamma_k(t-s))\sum_{|\alpha|=1}^{n}\Big|\Omega_\alpha\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s)\Big)\Big|ds\Bigg\}\\
&\qquad\times\int_{t_0}^{\infty}\exp(\beta(t-\tau))\Big|\Omega_{\mathbf{0}}\Big(\mu,\tau,r_0(\tau),\ldots,r_{n-1}(\tau)\Big)\Big|d\tau\\
&\leq\Phi_1\Big(1+\sigma_{\gamma_k}\Phi_h\Big)\int_{t_0}^{\infty}\exp(\beta(t-s))\Big|\Omega_{\mathbf{0}}\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s)\Big)\Big|ds. \qquad (4.12)
\end{aligned}
$$

The case $\mathbb{G}_\mu\subset\mathbb{E}_n^{n-1}$ is similar to the case $\mathbb{G}_\mu\subset\mathbb{E}_1^{n-1}$ and we get

$$
\sum_{j=0}^{n-2}|\omega_{h+1}^{(j)}(t)|\leq\Phi_1\Big(1+\sigma_{\gamma_n}\Phi_h\Big)\int_{t}^{\infty}\exp(\beta(t-s))\Big|\Omega_{\mathbf{0}}\Big(\mu,s,r_0(s),\ldots,r_{n-1}(s)\Big)\Big|ds, \qquad (4.13)
$$

for $\beta\in]0,\gamma_{n-1}[$. Then, from (4.11)-(4.13), we deduce that the induction process is valid and (4.9) is satisfied with

$$
\Phi_h=\Phi_1\Big(1+\Phi_{h-1}\,\sigma_{\gamma_k}\Big),\quad k=1,\ldots,n, \qquad (4.14)
$$

where Φ_1 is defined on (2.5).

Proof of (4.10). From (4.14), using recursively the definition of $\Phi_{h-1},\ldots,\Phi_2$, we can rewrite Φ_h as the sum of the terms of a geometric progression where the common ratio is given by $\sigma_{\gamma_i}\Phi_1$. Then, the hypothesis (R3) implies that $\sigma_{\gamma_i}\Phi_1 \in]0,1[$, and we can deduce that

$$\lim_{m\to\infty} \Phi_m = \Phi_1 \lim_{m\to\infty} \sum_{i=0}^{m-1} \left(\Phi_1\sigma_{\gamma_i}\right)^i = \Phi_1 \lim_{m\to\infty} \frac{\left[(\Phi_1\sigma_{\gamma_i})^m - 1\right]}{\Phi_1\sigma_{\gamma_i} - 1} = \frac{\Phi_1}{1-\Phi_1\sigma_{\gamma_i}} = \Phi_{\gamma_i} > 0,$$

or equivalently Φ_m converges to Φ_{γ_i}.

4.3. *Proof of Theorem 3.3*

[*(a)*] In order to prove (3.4), we first construct a formula for $y^{(j)}$ and then we use that formula in (1.1). Indeed, we have that the j-th derivative for y is given by

$$y^{(j)}(t) = \begin{cases} (z(t)+\mu)y(t), & j=1,\\ \left[(z(t)+\mu)^{(1)} + (z(t)+\mu)^2\right]y(t), & j=2,\\ \left[\sum_{m=0}^{j-3} S_{m,j}(t) + z^{j-2}(t)z^{(1)}(t) + (z(t)+\mu)^j\right]y(t), & j\geq 3. \end{cases} \tag{4.15}$$

The proof of (4.15) for $j = 1, 2$ is given by direct differentiation of y in (1.2). Now, for $j \geq 3$, we proceed inductively by using the Leibniz formula for differentiation. Indeed, by using the relation for $y^{(1)}$ we have that

$$\begin{aligned} y^{(j)}(t) &= (y^{(1)}(t))^{(j-1)} = \Big((z(t)+\mu)y(t)\Big)^{(j-1)} = \sum_{\ell_1=0}^{j-1}\binom{j-1}{\ell_1}(z(t)+\mu)^{(\ell_1)}y^{(j-1-\ell_1)}(t).\\ &= \left[(z(t)+\mu)^{(j-1)} + (j-1)(z(t)+\mu)^{(j-2)}(z(t)+\mu)\right]y(t)\\ &\quad + \sum_{\ell_1=0}^{j-3}\binom{j-1}{\ell_1}(z(t)+\mu)^{(\ell_1)}y^{(j-\ell_1-1)}(t). \end{aligned} \tag{4.16}$$

The order of derivatives for y is strictly decreasing with respect to ℓ_1, since $j-0-1 > j-1-1 > \ldots > j-(j-3)-1 = 2$. Then, if $j-1 > 2$ we can again apply the Leibniz formula to compute $y^{(j-\ell_1-1)} = (y')^{(j-\ell_1-2)}$ and from (4.16) we get

$$\begin{aligned} y^{(j)}(t) &= \left[(z(t)+\mu)^{(j-1)} + (j-1)(z(t)+\mu)^{(j-2)}(z(t)+\mu)\right]y(t)\\ &+ \sum_{\ell_1=0}^{j-3}\binom{j-1}{\ell_1}(z(t)+\mu)^{(\ell_1)}\left[(z(t)+\mu)^{(j-\ell_1-2)} + (j-\ell_1-2)(z(t)+\mu)^{(j-\ell_1-3)}(z(t)+\mu)\right]y(t)\\ &+ \sum_{\ell_1=0}^{j-4}\sum_{\ell_2=0}^{j-\ell_1-4}\binom{j-1}{\ell_1}\binom{j-\ell_1-2}{\ell_2}(z(t)+\mu)^{(\ell_1)}(z(t)+\mu)^{(\ell_2)}y^{(j-\ell_1-\ell_2-2)}(t). \end{aligned} \tag{4.17}$$

Similarly, by observing the order of derivatives for y, we deduce that if $j-2 > 2$ we can use the Leibniz formula to compute $y^{(j-\ell_1-\ell_2-2)}$. We note that, we can apply this strategy $j-2$ times to obtain the desired result given in (4.15) for $j \geq 3$ and conclude the proof of item (a).

Replacing (4.15) in (1.1) we deduce that

$$y^{(n)}(t) + \sum_{i=0}^{n-1}[a_i + r_i(t)]y^{(i)}(t) = y^{(n)}(t) + \sum_{i=3}^{n-1}[a_i + r_i(t)]y^{(i)}(t) + \sum_{i=0}^{2}[a_i + r_i(t)]y^{(i)}(t)$$

$$
\begin{aligned}
&= \Big\{\Big[z^{(n-1)}(t) + n\mu^{(n-1)}z(t) + (n-1)\mu\, z^{(n-2)}(t)\Big] + \mu^n + \Big[(n-1)z^{(n-2)}(t)z(t) + \sum_{m=1}^{n-3} S_{m,n}(t) \\
&\quad + z^{(n-1)}(t)z^{(1)}(t) + \sum_{j=0}^{n-2}\binom{n}{j} z^{n-j}(t)\mu^j\Big] \\
&\quad + \Big[\sum_{i=3}^{n-1} a_i\Big(z^{(i-1)}(t) + (i-1)z^{(i-2)}(t)\mu + i\mu^{i-1}z(t)\Big)\Big] + \sum_{i=3}^{n-1} a_i\mu^i + \Big[\sum_{i=3}^{n-1} a_i\Big((i-1)z^{(i-2)}(t)z(t) \\
&\quad + \sum_{m=1}^{i-3} S_{m,i}(t) + z^{(i-1)}(t)z^{(1)}(t) + \sum_{j=1}^{i-2}\binom{i}{j} z^{i-j}(t)\mu^j\Big) + r_i(t)\Big(\sum_{m=0}^{i-3} S_{m,i}(t) + z^{(i-1)}(t)z^{(1)}(t) \\
&\quad + (z(t)+\mu)^i\Big)\Big] + \Big[a_2 z^{(1)}(t) + 2a_2\mu z(t) + a_1 z(t)\Big] + \Big[a_2\mu^2 + a_1\mu + a_0\Big] \\
&\quad + \Big[a_2 z^2(t) + r_2(t)\Big(z^{(1)}(t) + (z(t)+\mu)^2\Big) + r_1(t)\Big(z(t)+\mu\Big) + r_0(t)\Big]\Big\} y(t) = 0.
\end{aligned}
$$

Thus, by using the fact that μ is a characteristic root we have that z is a solution of (3.4).

Remark 4.1. *We observe that the derivatives of $y(t) = \exp(\int_{t_0}^{t}(z(s)+\mu)ds)$ can be deduced by other methodologies instead of Leibniz formula, for instance by using the Faà di Bruno's formula and complete Bell polynomials [4, 7].*

[*(b)*] The proof of (3.8) is a consequence of the following claim: If $\lambda_i, \lambda_j \in \Lambda$ for $i \neq j$, then $\lambda_j - \lambda_i$ is a root of the characteristic polynomial associated with the differential Eq (3.4) when $\mu = \lambda_i$ and the right hand side is zero. Indeed, using the identity

$$
\sum_{k=0}^{i-1}\binom{i}{k}(u-v)^{i-1-k}v^k = \sum_{k=0}^{i-1} u^{i-1-k}v^k,
$$

we deduce that

$$
\begin{aligned}
&(u^n + \sum_{i=0}^{n-1} a_i u^i) - (v^n + \sum_{i=0}^{n-1} a_i v^i) = (u-v)\Big[\sum_{k=0}^{n-1} u^{n-1-k}v^k + \sum_{i=2}^{n-1} a_i \sum_{k=0}^{i-1} u^{i-1-k}v^k + a_1\Big] \\
&= (u-v)\Big[\sum_{k=0}^{n-1}\binom{n}{k}(u-v)^{n-1-k}v^k + \sum_{i=2}^{n-1}\sum_{k=0}^{i-1}\binom{i}{k} a_i (u-v)^{i-1-k}v^k + a_1\Big] \\
&= (u-v)\left[(u-v)^{(n-1)} + \sum_{i=2}^{n-1}\left[\sum_{k=i}^{n} a_k\binom{k}{i}v^{k-i}\right](u-v)^{(i-1)} + \left(n\mu^{n-1} + \sum_{i=1}^{n-1} i a_i \mu^{i-1}\right)(u-v)\right],
\end{aligned}
$$

and by selecting $u = \lambda_j$ and $v = \lambda_i$ we can prove the claim.

[*(c)*] We note that the Eq (3.4) represents the type (1.3) where (R1) and (R2) are satisfied, from the item (b) of Theorem 3.3 we have that (H1) implies (R1) and also clearly (H2) implies (R2). Thus, from Theorem 3.1, we deduce the existence of z_{λ_i} belong to $C_0^{n-2}([t_0, \infty[)$ satisfying (3.4).

[*(d)*] We note that $\sum_{|\alpha|=1}^{n}\left|\Omega_\alpha\Big(\mu, s, r_0(s), \ldots, r_{n-1}(s)\Big)\right|$ is the sum on the coefficients of $\mathbb{P}$ except of the independent term and $\mathbb{H}(s,\mu)$ is the coefficients of $\mathbb{F}$ except of the independent term. Also, we observe that we can obtain $\mathbb{H}(s,\mu)$ from (3.5) by considering that $z = z' = \ldots = z^{n-2} = 1$. Then, we have that (H3) implies (R3).

4.4. Proof of Theorem 3.4

From Theorem 3.3, we have that the fundamental system of solutions for (1.1) is given by (3.10). Moreover, from (4.15) we deduce that

$$\frac{y_i^{(j)}(t)}{y_i(t)} = \begin{cases} (z_{\lambda_i}(t) + \lambda_i), & j = 1, \\ \left[(z_{\lambda_i}(t) + \lambda_i)^{(1)} + (z_{\lambda_i}(t) + \lambda_i)^2\right], & j = 2, \\ \left[\sum_{m=0}^{j-3} S_{m,j}(t) + z_{\lambda_i}^{j-2}(t) z_{\lambda_i}^{(1)}(t) + (z_{\lambda_i}(t) + \lambda_i)^j\right], & j \geq 3. \end{cases} \tag{4.18}$$

Now, using the fact that $z_{\lambda_i} \in C_0^2([t_0, \infty[)$ is a solution of (3.10) with $\mu = \lambda_i$, we deduce the proof of (3.11). By the definition of the $W[y_1, \ldots, y_n]$, some algebraic rearrangements and (3.11), we deduce (3.12).

In order to prove (3.13) we use the relations (3.10), (4.3), (4.8) and the identity

$$\int_{t_0}^{t} e^{a\tau} \int_{\tau}^{\infty} e^{as} H(s) ds d\tau = \frac{1}{a}\left[\int_{t}^{\infty} e^{a(t-s)} H(s) ds - \int_{t_0}^{\infty} e^{a(t_0-s)} H(s) ds\right] + \frac{1}{a}\int_{t_0}^{t} H(\tau) d\tau. \tag{4.19}$$

Indeed, the relation (3.13) implies that

$$y_i(t) = \exp\left(\int_{t_0}^{t} (\lambda_i + z_i(\tau)) d\tau\right) = e^{\lambda_i(t-t_0)} \exp\left(\int_{t_0}^{t} z_{\lambda_i}(\tau) d\tau\right). \tag{4.20}$$

Now, from (H1) and Theorem 3.3(b) we can deduce that the sets defined on (3.1) and (3.2) satisfy the following inclusion $\mathbb{G}_{\lambda_i} \subset \mathbb{E}_i^{n-1}$. Then, we can rewrite (4.8) with $\mathbb{F}$ instead of $\mathbb{P}$ and $\mu = \lambda_i$, i.e.,

$$\begin{aligned} T z_{\lambda_i}(t) = {} & \frac{1}{\Upsilon_0\Big(\gamma_1(\lambda_i), \ldots, \gamma_{n-1}(\lambda_i)\Big)} \\ & \times \begin{cases} \displaystyle\sum_{\ell=2}^{n} G_\ell \int_{t_0}^{t} e^{(\lambda_\ell - \lambda_1)(t-s)} \mathbb{F}\Big(\lambda_1, s, r_0(s), \ldots, r_{n-1}(s), z_{\lambda_1}(s), \ldots, z_{\lambda_1}^{(n-2)}(s)\Big) ds, & i = 1 \\ \displaystyle\sum_{\ell=1}^{i-1} G_\ell \int_{t_0}^{t} e^{(\lambda_\ell - \lambda_i)(t-s)} \mathbb{F}\Big(\lambda_i, s, r_0(s), \ldots, r_{n-1}(s), z_{\lambda_i}(s), \ldots, z_{\lambda_i}^{(n-2)}(s)\Big) ds & \\ \quad \displaystyle + \sum_{\ell=i}^{n-1} G_\ell \int_{t}^{\infty} e^{(\lambda_{\ell+1} - \lambda_i)(t-s)} \mathbb{F}\Big(\lambda_i, s, r_0(s), \ldots, r_{n-1}(s), z_{\lambda_i}(s), \ldots, z_{\lambda_i}^{(n-2)}(s)\Big) ds, & \\ & i = \overline{2, n-1}, \\ \displaystyle\sum_{\ell=1}^{n-1} G_\ell \int_{t}^{\infty} e^{(\lambda_\ell - \lambda_n)(t-s)} \mathbb{F}\Big(\lambda_n, s, r_0(s), \ldots, r_{n-1}(s), z_{\lambda_n}(s), \ldots, z_{\lambda_n}^{(n-2)}(s)\Big) ds, & i = n, \end{cases} \end{aligned}$$

where $G_\ell = (-1)^\ell \Upsilon_\ell\Big(\gamma_1(\lambda_i), \ldots, \gamma_{n-1}(\lambda_i)\Big)$ and $\mathbb{F}$ is defined on (3.5). Using (4.3), we have that

$$\int_{t_0}^{t} z_{\lambda_i}(\tau) d\tau = \int_{t_0}^{t} T z_{\lambda_i}(\tau) d\tau = \frac{1}{\Upsilon_0(\lambda_1, \ldots, \lambda_n)}$$

$$\times\begin{cases} \sum_{\ell=2}^{n} G_\ell \int_{t_0}^{t}\int_{t_0}^{\tau} e^{(\lambda_\ell-\lambda_1)(\tau-s)}\mathbb{F}\Big(\lambda_1, s, r_0(s),\dots,r_{n-1}(s), z_{\lambda_1}(s),\dots,z_{\lambda_1}^{(n-2)}(s)\Big)dsd\tau, \quad i=1 \\ \sum_{\ell=1}^{i-1} G_\ell \int_{t_0}^{t}\int_{t_0}^{\tau} e^{(\lambda_\ell-\lambda_i)(\tau-s)}\mathbb{F}\Big(\lambda_i, s, r_0(s),\dots,r_{n-1}(s), z_{\lambda_i}(s),\dots,z_{\lambda_i}^{(n-2)}(s)\Big)dsd\tau \\ \quad +\sum_{\ell=i}^{n-1} G_\ell \int_{t_0}^{t}\int_{\tau}^{\infty} e^{(\lambda_{\ell+1}-\lambda_i)(\tau-s)}\mathbb{F}\Big(\lambda_i, s, r_0(s),\dots,r_{n-1}(s), z_{\lambda_i}(s),\dots,z_{\lambda_i}^{(n-2)}(s)\Big)dsd\tau, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=\overline{2,n-1}, \\ \sum_{\ell=1}^{n-1} G_\ell \int_{t_0}^{t}\int_{\tau}^{\infty} e^{(\lambda_\ell-\lambda_n)(\tau-s)}\mathbb{F}\Big(\lambda_n, s, r_0(s),\dots,r_{n-1}(s), z_{\lambda_n}(s),\dots,z_{\lambda_n}^{(n-2)}(s)\Big)dsd\tau, \quad i=n, \end{cases}$$

$$=\Big[\prod_{1\le j\le n, j\ne i}(\lambda_j-\lambda_i)\Big]^{-1}\int_{t_0}^{t}\mathbb{F}\Big(\lambda_i, s, r_0(s),\dots,r_{n-1}(s), z_{\lambda_i}(s),\dots,z_{\lambda_i}^{(n-2)}(s)\Big)ds+o(1).$$

Then, replacing in (4.20) we have that (3.13) is valid.

5. An example

Let us consider the following fifth-order differential equation

$$y^{(5)}+\left(1+\frac{3^p}{\sqrt[p+1]{t}\,(\sin(t)+2)}\right)y^{(4)}-21y^{(3)}+11y^{(2)}+68y^{(1)}-60y=0, \quad p\ge 1. \tag{5.1}$$

Moreover, in this example, we consider that $t_0>0$. We observe that (5.1) is of the form (1.1) with

$$a_0=-60,\ \ a_1=68,\ \ a_2=11,\ \ a_3=-21,\ \ a_4=1,$$
$$r_0(t)=r_1(t)=r_2(t)=r_3(t)=0, \qquad \text{and} \quad r_4(t)=\frac{3^p}{t^{1/(p+1)}\,(\sin(t)+2)}.$$

Now, we show that the Levinson, Hartman-Wintner and Eastham theorems cannot be applied and to get the asymptotic behavior of (5.1) we appeal to Theorem 3.4.

5.1. *The hypothesis (H1) is satisfied.*

We observe that the hypothesis (H1) is satisfied. If all perturbations are null, the characteristic polynomial of (5.1) is given by $P(\lambda)=\lambda^5+\lambda^4-21\lambda^3+11\lambda^2+68\lambda-60$. It is not difficult to verify that the roots of P are given by $-5,-2,1,2$ and 3. Thus, clearly the hypothesis (H1) is satisfied with $\Lambda=\big\{3,2,1,-2,-5\}$. Notice that the notation $\lambda_1=3, \lambda_2=2, \lambda_3=1, \lambda_4=-2$, and $\lambda_5=-5$ will be needed.

5.2. *We cannot apply Levinson, Hartman-Wintner and Eastham theorems.*

In order to apply the Levinson, Hartman-Wintner and Eastham theorems, we rewrite (5.1) as a system of the form (2.13), i.e.,

$$Y^{(1)}(t)=\Big[\mathrm{diag}(3,2,1,-2,-5)+M_5^{-1}B(t)M_5\Big]Y(t), \quad \text{with} \quad Y=M_5^{-1}XM_5,$$

and where

$$M_5^{-1}B(t)M_5 = \begin{pmatrix} -\frac{51111}{50440} & -\frac{2524}{18915} & -\frac{631}{151320} & \frac{2524}{18915} & \frac{394375}{30264} \\ \frac{175203}{40352} & \frac{721}{1261} & \frac{721}{40352} & -\frac{721}{1261} & -\frac{2253125}{40352} \\ -\frac{68121}{10088} & -\frac{3364}{3783} & -\frac{841}{30264} & \frac{3364}{3783} & \frac{2628125}{30264} \\ -\frac{136161}{201760} & -\frac{1681}{18915} & -\frac{1681}{605280} & \frac{1681}{18915} & \frac{1050625}{121056} \\ \frac{243}{5044} & \frac{8}{1261} & \frac{}{5044} & -\frac{8}{1261} & -\frac{3125}{5044} \end{pmatrix} r_4(t).$$

We observe that $|M_5^{-1}B(t)M_5| = (2742967/15132)|r_4(t)|$ and also we have that

$$\frac{3^{p-1}}{t^{1/(p+1)}} \le r_4(t) \le \frac{3^p}{t^{1/(p+1)}}. \tag{5.2}$$

The integrals in (2.14) and (2.16) can be bounded below as follows

$$\begin{aligned} \int_{t_0}^{\infty} \left| M_n^{-1} B(s) M_n \right| ds &= \frac{2742967}{15132} \int_{t_0}^{\infty} |r_4(s)| ds = \frac{2742967}{15132} \int_{t_0}^{\infty} \left| \frac{3^p}{s^{1/(p+1)} (\sin(s) + 2)} \right| ds \\ &\ge \frac{(2742967) 3^{p-1}}{15132} \int_{t_0}^{\infty} \frac{1}{s^{1/(p+1)}} ds, \\ \int_{t_0}^{\infty} \left| M_n^{-1} B(s) M_n \right|^p ds &= \left(\frac{2742967}{15132} \right)^p \int_{t_0}^{\infty} |r_4(s)|^p ds \\ &= \left(\frac{2742967}{15132} \right)^p \int_{t_0}^{\infty} \left| \frac{3^p}{s^{1/(p+1)} (\sin(s) + 2)} \right|^p ds \\ &\ge \frac{(2742967)^p 3^{p(p-1)})}{(15132)^p} \int_{t_0}^{\infty} \frac{1}{s^{p/(p+1)}} ds. \end{aligned}$$

Then, the perturbation matrix does not satisfy neither the assumptions of Levinson Theorem (see [8, Theorem 1.3.1]) nor the ones of Hartman-Wintner Theorem (see [8, Theorem 1.5.1]). Meanwhile, related with Eastham Theorem, let us consider the notation $\bar{B}(t) = \mathrm{diag}(3, 2, 1, -2, -5) + M_5^{-1}B(t)M_5$, we have that the characteristic polynomial of $\bar{B}(t)$ is given by

$$\begin{aligned} p_{\bar{B}(t)}(\lambda) = &\frac{1}{763258080} \Big[-763258080\lambda^5 + 763258080 r_4(t) \lambda^4 - (827912072 r_4(t) - 15265161600)\lambda^3 \\ &+ (1725169056 r_4(t) - 22897742400)\lambda^2 - (2173489864 r_4(t) + 14501903520)\lambda \\ &+ 1022015280 r_4(t) + 22897742400 \Big]. \end{aligned}$$

Then the process of analytic calculus of the eigenvalues of the matrix $\bar{B}(t)$ is not possible, resulting that the verification of assumptions for Eastham Theorem (see [8, Theorem 1.6.1]) are not simple and consequently the result is not straightforward to apply in this particular case.

5.3. The hypothesis (H2) is satisfied.

To verify the hypothesis (H2) we proceed in three steps: construction of Green functions using the definition given (4.1), calculus of integrals and limits in (2.8), and calculus of integrals and limits in (2.9).

Step 1. *Construction of Green functions using the definition given* (4.1). We calculate $\mathbb{G}_\mu$ and Υ_i for $i = 0, \ldots, 4$ defined on (3.1) and (2.3). For instance, if we fix $\mu = \lambda_1 = 3$, by application of theorem 3.3*(b)*, we deduce that

$$\gamma_1 = \lambda_2 - \lambda_1 = -1, \quad \gamma_2 = \lambda_3 - \lambda_1 = -2,$$
$$\gamma_3 = \lambda_4 - \lambda_1 = -5, \quad \gamma_4 = \lambda_5 - \lambda_1 = -8,$$

or equivalently we have that $\mathbb{G}_{\lambda_1} = \{-1, -2, -5, -8\}$. The numerical values of Υ_i are given by

$$\begin{aligned}
\Upsilon_0 &= (\gamma_4 - \gamma_3)(\gamma_4 - \gamma_2)(\gamma_4 - \gamma_1)(\gamma_3 - \gamma_2)(\gamma_3 - \gamma_1)(\gamma_2 - \gamma_1) = 1512, \\
\Upsilon_1 &= (\gamma_4 - \gamma_3)(\gamma_4 - \gamma_2)(\gamma_3 - \gamma_2) = -54, \\
\Upsilon_2 &= (\gamma_4 - \gamma_3)(\gamma_4 - \gamma_1)(\gamma_3 - \gamma_1) = -84, \\
\Upsilon_3 &= (\gamma_4 - \gamma_2)(\gamma_4 - \gamma_1)(\gamma_2 - \gamma_1) = -42, \\
\Upsilon_4 &= (\gamma_3 - \gamma_2)(\gamma_3 - \gamma_1)(\gamma_2 - \gamma_1) = -12.
\end{aligned}$$

Then, by (4.1) we have that

$$\begin{aligned}
g(t,s) &= \frac{-g_{\lambda_1}(t,s)}{\Upsilon_0(\gamma_1,\gamma_2,\gamma_3,\gamma_4)} = \frac{-1}{1512}\sum_{\ell=1}^{4} G_\ell e^{\gamma_\ell(t-s)} H(t-s) \\
&= \frac{-1}{1512}\sum_{\ell=1}^{4}(-1)^\ell \Upsilon_\ell(\gamma_1,\gamma_2,\gamma_3,\gamma_4) e^{\gamma_\ell(t-s)} H(t-s) \\
&= \frac{-1}{1512}\Big[54e^{-(t-s)} - 84e^{-2(t-s)} + 42e^{-5(t-s)} - 12e^{-8(t-s)}\Big] H(t-s) \\
&= \begin{cases} -\frac{1}{28}e^{-(t-s)} + \frac{1}{18}e^{-2(t-s)} - \frac{1}{36}e^{-5(t-s)} + \frac{1}{126}e^{-8(t-s)}, & t \geq s, \\ 0, & t \leq s. \end{cases}
\end{aligned} \tag{5.3}$$

Proceeding similarly, we deduce that the Green functions for other cases of selection of μ are given by

$$g(t,s) = \begin{cases} \frac{1}{36}e^{-(t-s)} - \frac{1}{45}e^{-4(t-s)} + \frac{1}{144}e^{-7(t-s)}, & t \geq s, \\ -\frac{1}{80}e^{(t-s)}, & t \leq s, \end{cases} \tag{5.4}$$

$$g(t,s) = \begin{cases} -\frac{1}{60}e^{-3(t-s)} + \frac{1}{168}e^{-6(t-s)}, & t \geq s, \\ -\frac{1}{40}e^{2(t-s)} + \frac{1}{28}e^{(t-s)}, & t \leq s, \end{cases} \tag{5.5}$$

$$g(t,s), = \begin{cases} \frac{1}{336}e^{-3(t-s)}, & t \geq s, \\ -\frac{1}{16}e^{5(t-s)} + \frac{1}{7}e^{4(t-s)} - \frac{1}{12}e^{3(t-s)}, & t \leq s, \end{cases} \tag{5.6}$$

$$g(t,s), = \begin{cases} 0, & t \geq s, \\ -\frac{1}{10}e^{8(t-s)} + \frac{1}{4}e^{7(t-s)} - \frac{1}{6}e^{6(t-s)} + \frac{1}{60}e^{3(t-s)}, & t \leq s. \end{cases} \tag{5.7}$$

A complete calculus of the sets, the numerical values and Green functions is summarized on Table 1.

μ	$\mathbb{G}_\mu$	Υ_0	Υ_1	Υ_2	Υ_3	Υ_4	$g(t,s)$
λ_1	$\{-1,-2,-5,-8\}$	1512	-54	-84	-42	-12	see (5.3)
λ_2	$\{1,-1,-4,-7\}$	4320	-54	-120	-96	-30	see (5.4)
λ_3	$\{2,1,-3,-6\}$	3360	-84	-120	-56	-20	see (5.5)
λ_4	$\{5,4,3,-3\}$	672	-42	-96	-56	-2	see (5.6)
λ_5	$\{8,7,6,3\}$	120	-12	-30	-20	-2	see (5.7)

Table 1. Summary of the Green functions.

Step 2. *Calculus of integrals and limits in* (2.8). Let us consider $\mu = \lambda_1$, then g is given by (5.3). We straightforward deduce that

$$\lim_{t\to\infty}\sum_{i=0}^{3}\left|\int_{t_0}^{\infty}\frac{\partial^i g}{\partial t^i}(t,s)r_j(s)ds\right| = 0 \quad \text{for } j=\overline{0,3}, \tag{5.8}$$

since $r_0 = r_1 = r_2 = r_3 = 0$. Using the fact that $|r_4(t)| \le 3^p/t^{1/(p+1)} \le 3^p/\sqrt{t}$ for all $t \ge t_0 > 0$, which is deduced from (5.2), we get the following bound

$$\begin{aligned}
&\sum_{i=0}^{3}\left|\int_{t_0}^{\infty}\frac{\partial^i g}{\partial t^i}(t,s)r_4(s)ds\right| \\
&= \left|\int_{t_0}^{t}\Big(-\frac{1}{28}e^{-1(t-s)}+\frac{1}{18}e^{-2(t-s)}-\frac{1}{36}e^{-5(t-s)}+\frac{1}{126}e^{-8(t-s)}\Big)r_4(s)ds\right| \\
&\quad+\left|\int_{t_0}^{t}\Big(\frac{1}{28}e^{-(t-s)}-\frac{2}{9}e^{-2(t-s)}+\frac{5}{36}e^{-5(t-s)}-\frac{4}{63}e^{-8(t-s)}\Big)r_4(s)ds\right| \\
&\quad+\left|\int_{t_0}^{t}\Big(-\frac{1}{28}e^{-(t-s)}+\frac{4}{9}e^{-2(t-s)}-\frac{25}{36}e^{-5(t-s)}+\frac{32}{63}e^{-8(t-s)}\Big)r_4(s)ds\right| \\
&\quad+\left|\int_{t_0}^{t}\Big(\frac{1}{28}e^{-(t-s)}-\frac{8}{9}e^{-2(t-s)}+\frac{125}{36}e^{-5(t-s)}-\frac{256}{63}e^{-8(t-s)}\Big)r_4(s)ds\right| \\
&\le 3^p\left[\frac{1}{7}\int_{t_0}^{t}\frac{e^{-(t-s)}}{\sqrt{s}}ds+\frac{5}{6}\int_{t_0}^{t}\frac{e^{-2(t-s)}}{\sqrt{s}}ds\right. \\
&\qquad\left.+\frac{13}{3}\int_{t_0}^{t}\frac{e^{-5(t-s)}}{\sqrt{s}}ds+\frac{65}{14}\int_{t_0}^{t}\frac{e^{-8(t-s)}}{\sqrt{s}}ds\right]\cdot
\end{aligned} \tag{5.9}$$

For any $m > 0$, we observe that

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{e^{-m(t-s)}}{\sqrt{s}}ds = \lim_{t\to\infty}e^{-mt}\int_{t_0}^{t}\frac{e^{ms}}{\sqrt{s}}ds = i\sqrt{\frac{\pi}{m}}\lim_{t\to\infty}\frac{\operatorname{erf}(\sqrt{mt}i)-\operatorname{erf}(\sqrt{mt_0}i)}{e^{mt}} = 0. \tag{5.10}$$

Using (5.10) in each integral of (5.9), we deduce that

$$\lim_{t\to\infty}\sum_{i=0}^{3}\left|\int_{t_0}^{\infty}\frac{\partial^i g}{\partial t^i}(t,s)r_j(s)ds\right| = 0 \quad \text{for } j=4. \tag{5.11}$$

Then, from (5.8) and (5.11), we deduce that (2.8) is satisfied for $u = \lambda_1$.

Le us consider that $\mu = \lambda_2$, then g is given by (5.4). A relation of the form (5.8) with g defined in (5.4) is satisfied, since $r_0 = r_1 = r_2 = r_3 = 0$. Proceeding similarly to the estimate (5.10), we deduce that

$$\begin{aligned}&\sum_{i=0}^{3}\left|\int_{t_0}^{\infty}\frac{\partial^i g}{\partial t^i}(t,s)r_4(s)ds\right|\\&\le 3^p\left[\frac{1}{20}\int_t^{\infty}\frac{e^{(t-s)}}{\sqrt{s}}ds+\frac{1}{9}\int_{t_0}^{t}\frac{e^{-(t-s)}}{\sqrt{s}}ds+\frac{17}{9}\int_{t_0}^{t}\frac{e^{-4(t-s)}}{\sqrt{s}}ds+\frac{25}{9}\int_{t_0}^{t}\frac{e^{-7(t-s)}}{\sqrt{s}}ds\right].\end{aligned}\tag{5.12}$$

For any $m > 0$ and for erf the error function, we observe that

$$\begin{aligned}\lim_{t\to\infty}\int_t^{\infty}\frac{e^{m(t-s)}}{\sqrt{s}}ds&=\lim_{t\to\infty}e^{mt}\int_t^{\infty}\frac{e^{-ms}}{\sqrt{s}}ds=\frac{2}{m}\lim_{t\to\infty}e^{mt}\int_{\sqrt{mt}}^{\infty}e^{-s^2}ds\\&=\frac{2}{m}\lim_{t\to\infty}e^{mt}\frac{\sqrt{\pi}}{2}(1-\operatorname{erf}(\sqrt{mt}))=\frac{\sqrt{\pi}}{m}\lim_{t\to\infty}\frac{1-\operatorname{erf}(\sqrt{mt})}{e^{-mt}}\\&=\frac{\sqrt{\pi}}{m}\lim_{t\to\infty}\frac{\operatorname{erf}'(\sqrt{mt})}{-m\,e^{-mt}}=\frac{\sqrt{\pi}}{m}\lim_{t\to\infty}\frac{2\,e^{-mt}}{\sqrt{\pi}(-m\,e^{-mt})}\frac{m}{2\sqrt{mt}}\\&=-\frac{1}{m\sqrt{m}}\lim_{t\to\infty}\frac{1}{\sqrt{t}}=0.\end{aligned}\tag{5.13}$$

From (5.13) and (5.10) we deduce that the integrals in the right hand side of (5.12) converges to 0 when $t\to\infty$. Then a limit of the form (5.11) with g defined in (5.4) is satisfied. Hence, we conclude that (2.8) is satisfied for $u = \lambda_2$.

The proof of the fact that (2.8) is satisfied for $u \in \{\lambda_3, \lambda_4, \lambda_5\}$ is analogous.

Step 3. *Calculus of integrals and limits in* (2.9). Let us consider that $\mu = \lambda_1$. Then, for g given by (5.3), by applying (5.10), we have that

$$\begin{aligned}&\lim_{t\to\infty}\int_{t_0}^{\infty}\sum_{i=0}^{3}\left|\frac{\partial^i g}{\partial t^i}(t,s)\right|\left|\sum_{\ell=0}^{n-1}\mu^{\ell}r_{\ell}(s)\right|ds\\&\le\lim_{t\to\infty}\int_{t_0}^{t}\left[\frac{1}{7}e^{-(t-s)}+\frac{5}{6}e^{-2(t-s)}+\frac{13}{3}e^{-5(t-s)}+\frac{65}{14}e^{-8(t-s)}\right]3^4|r_4(s)|ds\\&\le 3^{p+4}\lim_{t\to\infty}\int_{t_0}^{t}\left[\frac{1}{7}e^{-(t-s)}+\frac{5}{6}e^{-2(t-s)}+\frac{13}{3}e^{-5(t-s)}+\frac{65}{14}e^{-8(t-s)}\right]\frac{ds}{\sqrt{s}}=0.\end{aligned}$$

Then, we have that (2.9) is satisfied for $\mu = \lambda_1$. Analogously, using the definition of g and applying the limits in (5.10) and (5.13), we deduce that (2.9) is satisfied for $\mu \in \{\lambda_2, \lambda_3, \lambda_4, \lambda_5\}$.

From the results on Step 2 and Step 3 we deduce that the hypothesis (H2) is satisfied.

5.4. The hypothesis (H3) is satisfied.

Let us consider that $\mu = \lambda_1$. From the values of Table 1 we get

$$\Phi_1^{\Lambda}=\frac{1}{1512}\Big[54\sum_{j=0}^{3}(1)^j+84\sum_{j=0}^{3}(2)^j+42\sum_{j=0}^{3}(5)^j+12\sum_{j=0}^{3}(3)^j\Big]=\frac{209}{21}.$$

By follow a similar process to step 2 in section 5.3 we deduce that $\int_t^\infty \exp(t-s)\Big|\mathbb{H}(s,\lambda_1)\Big|ds \to 0$ when $t \to \infty$. Then for all $\epsilon > 0$, there is $N > 0$ such that $t > N$ implies that $\int_t^\infty \exp(t-s)\Big|\mathbb{H}(s,\lambda_1)\Big| < \epsilon$. Particularly, by selecting $\sigma_{\gamma_1} = \epsilon \in [0, 21/209]$ we deduce that the hypothesis (H3) is satisfied. We proceed analogously to prove the other cases.

5.5. Application of Theorem 3.4.

Since (H1) and (H2) are satisfied, by application of Theorem 3.4 we deduce that the Eq (5.1) has a fundamental system of solutions with following asymptotic behavior

$$\frac{y_1^{(j)}(t)}{y_1(t)} = (-5)^j,\ \frac{y_2^{(j)}(t)}{y_2(t)} = (-2)^j,\ \frac{y_3^{(j)}(t)}{y_3(t)} = 1,\ \frac{y_4^{(j)}(t)}{y_4(t)} = (2)^j,\ \frac{y_5^{(j)}(t)}{y_5(t)} = (3)^j, \quad j = \overline{1,5},$$
$$W[y_1,\ldots,y_n](t) = 12090\prod_{i=1}^{5} y_i(t)(1+o(1)),$$

when $t \to \infty$. Furthermore, an asymptotic formula of the type (3.13) is also valid since (H3) is satisfied.

6. Conclusions

In this paper, we have introduced and proved a new result on the asymptotic behavior of non-oscillatory solutions for high order differential equations of Poincaré type. The construction of the proof is based on the scalar method which was developed in [5, 9, 10] for differential equations with orders 2–4, respectively. The scalar method consist in three big steps: (i) a change of variable to reduce the order of the Poincaré equation and demonstrate that the new variable is a solution of a nonlinear differential equation; (ii) the application of the method of variation of parameters and the Banach fixed-point theorem to obtain the well posedness and asymptotic behavior of the non-linear equation; (iii) the proof of existence of a fundamental system of solutions and formulas for the asymptotic behavior of the Poincaré type equation by rewriting the results in terms of the original variable. Comparing with the classical results, the major contribution of the new result is the fact that the perturbation functions, appearing as coefficients of the equation, are more weak.

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Conflict of interest

The authors declare no conflict of interest.

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