Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(4): 6396–6403.
DOI: 10.3934/math.2022356
Received: 17 November 2021
Revised: 31 December 2021
Accepted: 14 January 2022
Published: 19 January 2022

Research article

The general two-dimensional divisor problems involving Hecke eigenvalues

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Abstract: We consider the general two-dimensional divisor problems involving Hecke eigenvalues, and are able to improve the previous results in this direction.

Keywords: Hecke eigenvalue; divisor problem; Fourier coefficients
Mathematics Subject Classification: 11N37, 11A25

1. Introduction

As usual, let $\tau(m)$ be the divisor function. The famous Dirichlet divisor problem on $\tau(m)$ has attracted many authors. For example, the best result to date was established by Bourgain and Watt [2], who obtained that

$$\sum_{m\leq y}\tau(m) = y\log y + (2\gamma-1)y + O\left(y^{\frac{517}{1648}+\varepsilon}\right)$$

with Euler's constant γ.

Let $1\leq k<l$ be fixed integers. Denote by $\tau(m;k,l)$ the number of representations of m as $m=m_1^k m_2^l$, where m_1, m_2 are natural numbers, that is,

$$\tau(m;k,l) = \sum_{m=m_1^k m_2^l} 1.$$

In 1969, Krätzel [13] proved that

$$\sum_{m\leq y}\tau(m;k,l) = \zeta\left(\frac{l}{k}\right)y^{\frac{1}{k}} + \zeta\left(\frac{k}{l}\right)y^{\frac{1}{l}} + \Delta,$$

where

$$\Delta = -\sum_{m^{k+l}\le y}\left\{\psi\Big(\big(\frac{y}{m^l}\big)^{\frac{1}{k}}\Big)+\psi\Big(\big(\frac{y}{m^k}\big)^{\frac{1}{l}}\Big)\right\}+O(1),$$

ζ is the Riemann zeta function, $\psi(z) = z - [z] - \frac{1}{2}$ and $[z]$ denotes the integer part of z. After that, a lot of results have been established in this direction. We refer to Ivić [8, Chapter 14] for details.

Now we draw attention to the Hecke eigenvalues. Denote by $SL_2(\mathbb{Z})$ the full modular group and by $\mathcal{H}_\kappa$ the set of primitive holomorphic cusp forms $g(z)$ of weight κ for $SL_2(\mathbb{Z})$, respectively, where $\kappa \ge 2$ is an even integer. It is known that $\mathcal{H}_\kappa$ is composed of the eigenfunctions of all Hecke operators. And at the cusp ∞, $g(z)$ has the Fourier expansion:

$$g(z) = \sum_{m=1}^{\infty}\lambda_g(m)m^{(\kappa-1)/2}e^{2\pi imz} \quad (\mathrm{Im}\, z > 0),$$

where $\lambda_g(m)$ is the m-th normalized Hecke eigenvalue. For prime number p, one has

$$\lambda_g(p) = \alpha_p + \beta_p \quad \text{and} \quad \alpha_p\beta_p = |\alpha_p| = |\beta_p| = 1.$$

Then define the Hecke L-function $L(g, s)$ attached to g as

$$L(g, s) = \sum_{m=1}^{\infty}\lambda_g(m)m^{-s} = \prod_p\left(1-\alpha_p p^{-s}\right)^{-1}\left(1-\beta_p p^{-s}\right)^{-1} \quad (\mathrm{Re}\, s > 1).$$

Further, define the Rankin-Selberg L-function as

$$L(g\times g, s) = \prod_p (1-p^{-s})^{-2}\left(1-\alpha_p^2 p^{-s}\right)^{-1}\left(1-\beta_p^2 p^{-s}\right)^{-1} \quad (\mathrm{Re}\, s > 1).$$

Therefore, we have

$$L(g\times g, s) = \zeta(2s)\sum_{m=1}^{\infty}\lambda_g(m)^2 m^{-s} := \sum_{m=1}^{\infty}\lambda_{g\times g}(m)m^{-s} \quad (\mathrm{Re}\, s > 1).$$

Many scholars have studied $\lambda_g(m)$ and $\lambda_{g\times g}(m)$ in various ways and established a lot of results (for example, see [3, 4, 6, 9, 11, 12, 14–25], etc.). In addition, one may prefer to consider

$$\lambda_{g\times g}^{k,l}(m) = \sum_{m=m_1^k m_2^l}\lambda_{g\times g}(m_1)\lambda_{g\times g}(m_2).$$

Recently, Huang, Liu and Xu [7] studied the general two-dimensional divisor problems involving Hecke eigenvalues and proved that for any $\varepsilon > 0$,

$$\Gamma_{g\times g}(y;k,l) := \sum_{m\le y}\lambda_{g\times g}^{k,l}(m) = \begin{cases} C_1 y^{\frac{1}{k}} + O\Big(y^{1-\frac{84(2k-1)}{486k-131l}+\varepsilon}\Big), & \text{if } 318k^2+131l-402k-131kl<0, 2k\ge l;\\ O\Big(y^{1-\frac{84(2k-1)}{486k-131l}+\varepsilon}\Big), & \text{if } 318k^2+131l-402k-131kl>0, 2k\ge l;\\ C_2 y^{\frac{1}{2}} + O\Big(y^{\frac{7}{16}+\varepsilon}\Big), & \text{if } k=2, 2k<l;\\ O\Big(y^{1-\frac{3(2k-1)}{8k}+\varepsilon}\Big), & \text{if } k\ge 3, 2k<l,\end{cases}$$

where $C_1 = L(\mathrm{sym}^2 g, 1)L\big(g\times g, \frac{l}{k}\big)$ and $C_2 = L(\mathrm{sym}^2 g, 1)L\big(g\times g, \frac{l}{2}\big)$. In this paper, we are able to improve the above result by proving the following theorem.

Theorem 1.1. *Let $1 \le k < l$ be any fixed integers. Then for any $\varepsilon > 0$, we have*

$$\Gamma_{g\times g}(y;k,l) = \begin{cases} R_1 y^{\frac{1}{k}} + R_2 y^{\frac{1}{l}} + O\left(y^{\frac{1}{k}-\frac{4}{20k-5l}+\varepsilon}\right), & \text{if } l \le 2k; \\ R_1 y^{\frac{1}{k}} + O\left(y^{\frac{3}{5k}+\varepsilon}\right), & \text{if } l > 2k, \end{cases}$$

where

$$R_1 = L\left(g\times g, \frac{l}{k}\right)L(\mathrm{sym}^2 g, 1), \quad R_2 = L\left(g\times g, \frac{k}{l}\right)L(\mathrm{sym}^2 g, 1).$$

To prove Theorem 1.1, we mainly use the Perron's formula and the individual and averaged subconvexity bounds for the Riemann zeta-function and the symmetric square L-function.

2. Preliminary lemmas

Firstly, we introduce the symmetric square L-function $L(\mathrm{sym}^2 g, s)$ defined by

$$L(\mathrm{sym}^2 g, s) := \prod_p (1-p^{-s})^{-1}\left(1-\alpha_p^2 p^{-s}\right)^{-1}\left(1-\beta_p^2 p^{-s}\right)^{-1} \qquad (\mathrm{Re}\, s > 1).$$

Write $s = \sigma + it$. Let ε be a sufficiently small positive constant, whose value is not necessarily the same at each occurrence.

Lemma 2.1. *We have*

$$L(g\times g, s) = \zeta(s)L(\mathrm{sym}^2 g, s) \quad (\mathrm{Re}\, s > 1). \tag{2.1}$$

Proof. We can find this lemma in [7]. □

Lemma 2.2. *We have, uniformly for $T \ge 1$,*

$$\int_1^T |L(\mathrm{sym}^2 g, s)|^2 dt \ll T^{3(1-\sigma)+\varepsilon}, \tag{2.2}$$

and for $\frac{1}{2} < \sigma \le 1$, $|t| \ge 1$,

$$L(\mathrm{sym}^2 g, s) \ll (1+|t|)^{\max\{0,\frac{5}{4}(1-\sigma)\}+\varepsilon}. \tag{2.3}$$

Proof. We can obtain the former result (2.2) from the properties of $L(\mathrm{sym}^2 g, s)$ with standard arguments. The latter result (2.3) was proved by Nunes [18]. □

Lemma 2.3. *We have, uniformly for $T \ge 1$ and $\frac{1}{2} \le \sigma < 1$,*

$$\int_1^T \left|\zeta\left(\sigma + it\right)\right|^4 dt \sim T^{1+\varepsilon}, \tag{2.4}$$

and for $\frac{1}{2} < \sigma \le 1, |t| \ge 1$,

$$\zeta(s) \ll (1+|t|)^{\max\{0,\frac{13}{42}(1-\sigma)\}+\varepsilon}. \tag{2.5}$$

Proof. The result (2.4) with $\sigma = \frac{1}{2}$ is the classical result of Ingham. We can find the result (2.4) in [8]. The third result (2.5) was derived by Bourgain [1]. □

3. Proof of Theorem 1.1

Note that

$$L(g\times g, ks)L(g\times g, ls)=\sum_{m=1}^{\infty}\frac{\lambda_{g\times g}^{k,l}(m)}{m^s}. \tag{3.1}$$

Then from (3.1) and Perron's formula (see [10, Proposition 5.54]), with a similar argument to [8, page 411] we get

$$\Gamma_{g\times g}(y;k,l)=(2\pi i)^{-1}\int_{\xi-iT}^{\xi+iT}L(g\times g, ks)L(g\times g, ls)\frac{y^s}{s}ds+O\left(\frac{y^{\frac{1}{k}+\varepsilon}}{T}\right), \tag{3.2}$$

where $\xi=\frac{1}{k}+\varepsilon$ and T is a parameter to be determined later. Then we shift the integral line of (3.2) to the parallel line Re $s=\frac{1}{2k}$. From Gelbart-Jacquet [5], we note that $L(\mathrm{sym}^2 g, s)$ is holomorphic at $s=1$. Considering the sizes of l and $2k$, we see that $s=\frac{1}{k}$ and $s=\frac{1}{l}$ will be the only possible simple poles in $\mathbb{R}_T:=\{s=\sigma+it:\frac{1}{2k}\le\sigma\le\xi, \mid t\mid\le T\}$ according to (2.1), and the corresponding residues at $s=\frac{1}{k}$ and $s=\frac{1}{l}$ are

$$R_1:=L\left(g\times g,\frac{l}{k}\right)L(\mathrm{sym}^2 g,1),\quad R_2:=L\left(g\times g,\frac{k}{l}\right)L(\mathrm{sym}^2 g,1),$$

respectively.

In the following argument, we still carry out the discussion by two cases $2k\ge l$ and $2k<l$. In the case $2k\ge l$, both $s=\frac{1}{k}$ and $s=\frac{1}{l}$ are simple poles in $\mathbb{R}_T$. Then we derive from Cauchy's residue theorem,

$$\begin{aligned}\Gamma_{g\times g}(y;k,l) &= \left(\operatorname*{Res}_{s=\frac{1}{k}}+\operatorname*{Res}_{s=\frac{1}{l}}\right)L(g\times g, ks)L(g\times g, ls)\frac{y^s}{s}+O\left(\frac{y^{\frac{1}{k}+\varepsilon}}{T}\right)\\ &\quad+\frac{1}{2\pi i}\left(\int_{\frac{1}{2k}-iT}^{\frac{1}{2k}+iT}+\int_{\frac{1}{2k}+iT}^{\xi+iT}+\int_{\xi-iT}^{\frac{1}{2k}-iT}\right)L(g\times g, ks)L(g\times g, ls)\frac{y^s}{s}ds\\ &:= R_1 y^{\frac{1}{k}}+R_2 y^{\frac{1}{l}}+J_1+J_2+J_3+O\left(\frac{y^{\frac{1}{k}+\varepsilon}}{T}\right).\end{aligned} \tag{3.3}$$

To estimate J_2 and J_3, we also need to divide the integral interval into two arcs $\mathbb{A}'_1, \mathbb{A}'_2$ and draw support from Lemmas 2.2 and 2.3.

$\mathbb{A}'_1:=\left\{s=\sigma+iT:\frac{1}{2k}\le\sigma\le\frac{1}{l}\right\}$. Then in this arc we have

$$\begin{aligned}&\frac{1}{T}\int_{\mathbb{A}'_1}y^{\sigma}\mid\zeta(k\sigma+ikt)L(\mathrm{sym}^2 g,k\sigma+ikt)\zeta(l\sigma+ilt)L(\mathrm{sym}^2 g,l\sigma+ilt)\mid dt\\ \ll&\max_{\frac{1}{2k}\le\sigma\le\frac{1}{l}}y^{\sigma}T^{(\frac{13}{42}+\frac{5}{4})(1-k\sigma)+(\frac{13}{42}+\frac{5}{4})(1-l\sigma)}T^{-1+\varepsilon}\\ \ll&\max_{\frac{1}{2k}\le\sigma\le\frac{1}{l}}T^{\frac{89}{42}+\varepsilon}\left(\frac{y}{T^{\frac{131}{84}(k+l)}}\right)^{\sigma}\\ \ll& y^{\frac{1}{2k}}T^{\frac{225}{168}-\frac{131l}{168k}+\varepsilon}+y^{\frac{1}{l}}T^{\frac{47}{84}-\frac{131k}{84l}+\varepsilon}.\end{aligned} \tag{3.4}$$

$\mathbb{A}'_2 := \left\{ s = \sigma + iT : \frac{1}{l} < \sigma \le \frac{1}{k} \right\}$. Then in this arc we can get

$$\begin{aligned}
&\frac{1}{T}\int_{\mathbb{A}'_2} y^{\sigma} \mid \zeta(k\sigma + ikt)L(\mathrm{sym}^2 g, k\sigma + ikt)\zeta(l\sigma + ilt)L(\mathrm{sym}^2 g, l\sigma + ilt) \mid d\sigma \\
\ll\ & \max_{\frac{1}{l}<\sigma\le\frac{1}{k}} y^{\sigma} T^{(\frac{13}{42}+\frac{5}{4})(1-k\sigma)} T^{-1+\varepsilon} \\
\ll\ & \max_{\frac{1}{l}<\sigma\le\frac{1}{k}} T^{\frac{47}{84}} \left(\frac{y}{T^{\frac{131}{84}k}} \right)^{\sigma} \\
\ll\ & y^{\frac{1}{k}} T^{-1+\varepsilon} + y^{\frac{1}{l}} T^{\frac{47}{84}-\frac{131k}{84l}+\varepsilon}. \qquad (3.5)
\end{aligned}$$

From (3.4) and (3.5) we get

$$\begin{aligned}
& |J_2 + J_3| \\
\ll\ & T^{-1}\int_{\frac{1}{2k}}^{\xi} y^{\sigma} \mid \zeta(k\sigma + ikt)L(\mathrm{sym}^2 g, k\sigma + ikt)\zeta(l\sigma + ilt)L(\mathrm{sym}^2 g, l\sigma + ilt) \mid d\sigma \\
=\ & T^{-1}\int_{\mathbb{A}'_1\cup\mathbb{A}'_2} y^{\sigma} \mid \zeta(k\sigma + ikt)L(\mathrm{sym}^2 g, k\sigma + ikt)\zeta(l\sigma + ilt)L(\mathrm{sym}^2 g, l\sigma + ilt) \mid d\sigma \\
\ll\ & y^{\frac{1}{2k}} T^{\frac{225}{168}-\frac{131l}{168k}+\varepsilon} + y^{\frac{1}{l}} T^{\frac{47}{84}-\frac{131k}{84l}+\varepsilon} + y^{\frac{1}{k}+\varepsilon} T^{-1+\varepsilon}. \qquad (3.6)
\end{aligned}$$

While for J_1, we have

$$\begin{aligned}
|J_1| \ll\ & y^{\frac{1}{2k}} \int_1^T \left| \zeta\left(\frac{1}{2} + ikt\right) L\left(\mathrm{sym}^2 g, \frac{1}{2} + ikt\right) \zeta\left(\frac{l}{2k} + ilt\right) L\left(\mathrm{sym}^2 g, \frac{l}{2k} + ilt\right) \right| \\
& \times t^{-1} dt + y^{\frac{1}{2k}+\varepsilon} \\
\ll\ & y^{\frac{1}{2k}} \log T \max_{1\le T_1\le T} T_1^{-1} \int_{\frac{T_1}{2}}^{T_1} \left| \zeta\left(\frac{1}{2} + ikt\right) L\left(\mathrm{sym}^2 g, \frac{1}{2} + ikt\right) \zeta\left(\frac{l}{2k} + ilt\right) \right. \\
& \left. \times L\left(\mathrm{sym}^2 g, \frac{l}{2k} + ilt\right) \right| dt + y^{\frac{1}{2k}+\varepsilon}.
\end{aligned}$$

Then by Hölder's inequality, Lemmas 2.2 and 2.3, one can get

$$\begin{aligned}
|J_1| \ll\ & y^{\frac{1}{2k}} \log T \max_{T_1\le T} T_1^{-1} T_1^{\frac{5}{4}(1-\frac{l}{2k})+\varepsilon} \int_{\frac{T_1}{2}}^{T_1} \left| \zeta\left(\frac{1}{2} + ikt\right) L\left(\mathrm{sym}^2 g, \frac{1}{2} + ikt\right) \zeta\left(\frac{l}{2k} + ilt\right) \right| dt + y^{\frac{1}{2k}+\varepsilon} \\
\ll\ & y^{\frac{1}{2k}+\varepsilon} + y^{\frac{1}{2k}} \log T \max_{T_1\le T} T_1^{\frac{5}{4}(1-\frac{l}{2k})-1+\varepsilon} \left(\int_{\frac{T_1}{2}}^{T_1} \left| \zeta\left(\frac{1}{2} + ikt\right) \right|^4 dt \right)^{\frac{1}{4}} \\
& \times \left(\int_{\frac{T_1}{2}}^{T_1} \left| L\left(\mathrm{sym}^2 g, \frac{1}{2} + ikt\right) \right|^2 dt \right)^{\frac{1}{2}} \left(\int_{\frac{T_1}{2}}^{T_1} \left| \zeta\left(\frac{l}{2k} + ikt\right) \right|^4 dt \right)^{\frac{1}{4}} \\
\ll\ & y^{\frac{1}{2k}} \max_{T_1\le T} T_1^{\frac{3}{2}-\frac{5l}{8k}+\varepsilon} \\
\ll\ & y^{\frac{1}{2k}} T^{\frac{3}{2}-\frac{5l}{8k}+\varepsilon}. \qquad (3.7)
\end{aligned}$$

Therefore, from (3.3), (3.6) and (3.7), we can establish

$$\Gamma_{g\times g}(y;k,l) = R_1 y^{\frac{1}{k}} + R_2 y^{\frac{1}{l}} + O\left(y^{\frac{1}{2k}} T^{\frac{3}{2}-\frac{5l}{8k}+\varepsilon} + y^{\frac{1}{l}} T^{\frac{47}{84}-\frac{131k}{84k}+\varepsilon} + y^{\frac{1}{k}} T^{-1+\varepsilon}\right). \tag{3.8}$$

Taking $T = y^{\frac{4}{20k-5l}}$ in (3.8), we have

$$\Gamma_{g\times g}(y;k,l) = R_1 y^{\frac{1}{k}} + R_2 y^{\frac{1}{l}} + O\left(y^{\frac{1}{k}-\frac{4}{20k-5l}+\varepsilon}\right).$$

Thus, we prove the first part of Theorem 1.1.

For $2k < l$, $s = \frac{1}{k}$ is the only simple pole in the range $\mathbb{R}_T$ by nothing $\frac{1}{l} < \frac{1}{2k}$. Then from Cauchy's residue theorem, we can derive

$$\begin{aligned}
\Gamma_{g\times g}(y;k,l) &= \operatorname*{Res}_{s=\frac{1}{k}} L(g\times g, ks)L(g\times g, ls)\frac{y^s}{s} + O\left(\frac{y^{\frac{1}{k}+\varepsilon}}{T}\right) \\
&\quad + \frac{1}{2\pi i}\left(\int_{\frac{1}{2k}-iT}^{\frac{1}{2k}+iT} + \int_{\frac{1}{2k}+iT}^{\xi+iT} + \int_{\xi-iT}^{\frac{1}{2k}-iT}\right) L(g\times g, ks)L(g\times g, ls)\frac{y^s}{s}ds \\
&:= R_1 y^{\frac{1}{k}} + J_1' + J_2' + J_3' + O\left(\frac{y^{\frac{1}{k}+\varepsilon}}{T}\right).
\end{aligned} \tag{3.9}$$

To estimate J_2' and J_3', we also split the integral interval into two arcs $\mathbb{A}'{}_1{}^*, \mathbb{A}'{}_2{}^*$ but with different ranges from the case $2k \geq l$. By a similar argument, we can get

$$|J_2' + J_3'| \ll y^{\frac{1}{2k}} T^{-\frac{37}{168}+\varepsilon} + y^{\frac{1}{k}+\varepsilon} T^{-1+\varepsilon}.$$

Note that $\frac{l}{2k} > 1$. The estimate of J_1' becomes

$$\begin{aligned}
|J_1'| &\ll y^{\frac{1}{2k}} \int_1^T \left|\zeta\left(\frac{1}{2}+ikt\right) L\left(\mathrm{sym}^2 g, \frac{1}{2}+ikt\right) \zeta\left(\frac{l}{2k}+ilt\right) L\left(\mathrm{sym}^2 g, \frac{l}{2k}+ilt\right)\right| t^{-1} dt + y^{\frac{1}{2k}+\varepsilon} \\
&\ll y^{\frac{1}{2k}} \log T \max_{1\leq T_1\leq T} T_1^{-1} \int_{\frac{T_1}{2}}^{T_1} \left|\zeta\left(\frac{1}{2}+ikt\right) L\left(\mathrm{sym}^2 g, \frac{1}{2}+ikt\right)\right| dt + y^{\frac{1}{2k}+\varepsilon} \\
&\ll y^{\frac{1}{2k}} \log T \max_{T_1\leq T} T_1^{-1} \left(\int_{\frac{T_1}{2}}^{T_1} \left|\zeta\left(\frac{1}{2}+ikt\right)\right|^4 dt\right)^{\frac{1}{4}} \left(\int_{\frac{T_1}{2}}^{T_1} \left|L\left(\mathrm{sym}^2 g, \frac{1}{2}+ikt\right)\right|^2 dt\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\frac{T_1}{2}}^{T_1} 1 dt\right)^{\frac{1}{4}} + y^{\frac{1}{2k}+\varepsilon} \\
&\ll y^{\frac{1}{2k}} T^{\frac{1}{4}+\varepsilon}.
\end{aligned}$$

Therefore, recalling (3.9) we can get

$$\Gamma_{g\times g}(y;k,l) = R_1 y^{\frac{1}{k}} + O\left((y^{\frac{1}{2k}} T^{\frac{1}{4}} + y^{\frac{1}{k}} T^{-1}) T^{\varepsilon}\right). \tag{3.10}$$

Taking $T = y^{\frac{2}{5k}}$ in (3.10), we have

$$\Gamma_{g\times g}(y;k,l) = R_1 y^{\frac{1}{k}} + O\left(y^{\frac{3}{5k}+\varepsilon}\right).$$

Thus, the prove of Theorem 1.1 is finished.

4. Conclusions

In this paper, we investigate the average behaviors of the Fourier coefficients $\lambda_{g\times g}^{k,l}(m)$ and improve the previous estimates in this direction. Here, the condition $1 \leq k < l$ in Theorem 1.1 removes the complexity of discussing the sizes between k and l due to the symmetry. To give a sharper upper bounds for the sum $\sum_{m\leq y}\lambda_{g\times g}^{k,l}(m)$, we apply some analytic instruments such as Perron's formula, the decomposition of the Rankin-Selberg L-function, and the individual and averaged subconvexity bounds for the Riemann zeta-function and the symmetric square L-function. With the help of results in Theorem 1.1, we can understand the Fourier coefficients $\lambda_{g\times g}^{k,l}(m)$ on average more precisely.

Acknowledgments

This work was supported by National Natural Science Foundations of China (Grant Nos. 11801328, 11771256 and 11801318) and Young Scholars Program of Shandong University, Weihai (Grant No. 20820211012).

Conflict of interest

The authors declare no conflicts of interest.

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