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Research article

Stability analysis of a simple mathematical model for school bullying

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Abstract: School bullying is a highly concerned problem due to its effect on students' academic achievement. The effect might go beyond that to develop health problems, school drop out and, in some extreme cases, commit suicide for victims. On the other hand, adolescents who continuously bully over time are at risk of becoming involved in gang membership and other types of crime. Therefore, we propose a simple mathematical model for school bullying by considering two variables: the number of victims students and the number of bullies students. The main assumption employed to develop the mathematical model is that school policy bans bullying and expels students who practice this behavior to maintain a constructive educational environment within the school. We show that the model has two equilibrium points, and that both equilibrium points are locally and globally asymptotically stable under certain conditions. Also, we define a threshold parameter with a new criterion called the bullying index. Furthermore, we show that the model exhibits the phenomena of transcritical bifurcation subject to the bullying index. All the findings are supported with numerical simulations.

Keywords: nonlinear stability theory; local stability; global stability; transcritical bifurcation; numerical simulation; Lyapunov function
Mathematics Subject Classification: 34D20

1. Introduction

There are now clear indications of an increasing societal as well as research interest into bully/victim problems in several parts of the world [8]. All researches have shown that many bullies are socially intelligent, hyperactive, impulsive. However, they are experiencing more peer rejection, more academic difficulties, and more stressful and harsh home environments. Typically, bullies have high but unstable self-esteem. They pick on the weaker persons and put upon them some words or actions for various reasons; to invoke fear, for entertainment to regain that sense of high self-esteem during the periods of instability or to get others to comply or obey under duress. The victim of a bully is someone who is of smaller size or strength compared to the bully, someone who is quiet, will simply submit rather than fight back, and more than likely will not tell anyone. Bullies do not want any resistance.

Bullying takes different forms: physical bullying, verbal bullying, social bullying, or electronic harassment (cyber-bullying) [3, 4]. The more common bullying forms experienced by students are social and verbal bullying. However, physical and cyber-bullying are often of greatest concern. Schools policy prohibits physical harm between students, but verbal and social bullying are more difficult to identify.

Bullying is a repetitive and an aggressive behavior and usually involves an imbalance of power. It is intended to hurt another individual, and it is a wound that might never heals. Many of the wounds that are left on people begin to make them depressed or suicidal.

School bullying can have long-term effects on students' academic achievement. Some students (victims) develop health problems, school drop out, and in some extreme cases commit suicide. In addition, studies have found that adolescents who continuously bully over time are at risk of becoming involved in gang membership, and other types of crime.

In order to understand the factors that lead to the students' academic achievement we need to know about the social environment of children in school. However, due to the lack of access to information, bullying was ignored and limited acknowledged. Nowadays, bullying became as a phenomenon and it is widely agreed upon that it is common and detrimental, but, few mathematical models are available describing this phenomenon [9]. Mathematical modeling plays an important part for understanding and predicting the dynamics of various phenomena in biology, economy [1, 2, 5, 6], physics [7] and other fields.

This study uses a simulation model consisting of a system of two coupled nonlinear ordinary differential equations that is described in §2. The students in the class were divided into two roles: the bullies and the victims. The model possesses two equilibrium points that are defined in §2.1. A new criterion called the bullying index has been designed to analyze the local and global stability in §2.2 and §3.1 respectively. Afterward, in §3.2, our analysis discusses the existence of transcritical bifurcation phenomena according to the bullying index. Our research finding is summed up in §4. Finally, the improvement of this study is briefly pointed out in §5.

2. Mathematical model

In this section, we propose the school bullying model that is associated with the following assumptions:

- All students enrolling to the school have the ability to be educated.
- The rate of students graduating or moving to other schools for various reasons is assumed to be proportional to their number.
- The rate of students exposed to bullying is jointly proportional to their number and to the number of the bullies denoted by $x(t)$ and $y(t)$ respectively.
- School policy is to ban bullying and expel students who practice this behavior to maintain a constructive educational environment within the school.

Thus, the dynamics of the model are governed by the following nonlinear system of ordinary differential equations (see Figure 1):

$$\begin{aligned} x'(t) &= A - \beta x(t) - bx(t)y(t), \\ y'(t) &= -cy(t) + \alpha x(t)y(t), \end{aligned} \tag{2.1}$$

where A, β, b, c and α are positive constants that are defined as:
A: Rate of increase in the number of students enrolling in the school.
β: Rate of the number of students who graduate or move to schools in their catchment area.
b: Rate of the number of students who leave school due to the harassment of the bullies.
c: Rate of the number of bullies who are expelled from school.
α: Rate of the number of bullying practiced by bullies on other students.

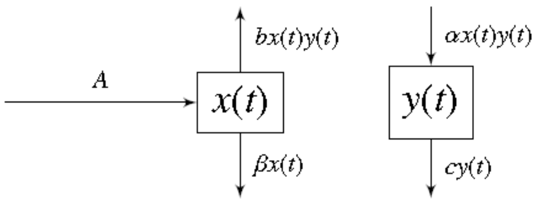


Figure 1. Flow diagram of the school bullying model.

Theorem 2.1. *If* $(x(t), y(t)) \in R_2^+$*, then the set defined by*

$$\Omega = \left\{(x(t), y(t)) : 0 \le x(t) \le \frac{A}{\gamma}, 0 \le y(t) \le \frac{A\alpha}{\gamma b}\right\}, \tag{2.2}$$

where $\gamma = \min(\beta, c)$*, is bounded and positively invariant.*

Proof. We note that

$$x'(t)|_{x(t)=0} = A > 0, \quad y'(t)|_{y(t)=0} = 0. \tag{2.3}$$

This implies that for $t \ge 0$, all solutions that are non-negative remain non-negative.

Now, if we combine the first two equations of model (2.1) we get

$$x'(t) + \frac{b}{\alpha}y'(t) = A - \beta x(t) - \frac{cb}{\alpha}y(t) \le A - \gamma\left(x(t) + \frac{b}{\alpha}y(t)\right), \tag{2.4}$$

where $\gamma = \min(\beta, c)$. By taking the limit supremum

$$\limsup_{t\to\infty}\left(x(t) + \frac{b}{\alpha}y(t)\right) \le \frac{A}{\gamma}. \tag{2.5}$$

This proves that all solutions of system (2.1) are bounded and do not exit the region Ω. Hence, Ω is positively invariant.

2.1. Equilibrium points analysis

In this section, we derive two equilibrium points for the bullying model (2.1).

Theorem 2.2. *let*

$$R_0 = \frac{\alpha A}{c\beta}. \tag{2.6}$$

The school bullying model (2.1) *has two equilibrium points: the bullying free school equilibrium* $Q_0 = \left(\frac{A}{\beta}, 0\right)$ *that exists for all values of* $R_0 \in \mathbb{R}^+$, *and the positive equilibrium* $Q^* = \left(\frac{A}{\beta R_0}, \frac{\beta}{b}(R_0 - 1)\right)$ *that exists only if* $R_0 > 1$.

Proof. The first equilibrium is obtained in the absence of the bullies ($y_0 = 0$) (see system (2.9)). It is known as the bullying free school equilibrium and denoted by

$$Q_0 = (x_0, y_0) = \left(\frac{A}{\beta}, 0\right). \tag{2.7}$$

The second one is a positive equilibrium denoted by

$$Q^* = (x^*, y^*) = \left(\frac{c}{\alpha}, \frac{1}{bc}(\alpha A - c\beta)\right). \tag{2.8}$$

Equilibria are obtained by solving the system of algebraic equations

$$\begin{aligned} A - \beta x - bxy &= 0, \\ -cy + \alpha xy &= 0, \end{aligned} \tag{2.9}$$

simultaneously.

The existence and stability of both equilibrium points Eqs (2.8) and (2.7) are subject to the threshold parameter R_0 called bullying index.

In terms of the bullying index R_0, the positive equilibrium point Eq (2.8) can be rewritten as:

$$Q^* = (x^*, y^*) = \left(\frac{A}{\beta R_0}, \frac{\beta}{b}(R_0 - 1)\right). \tag{2.10}$$

2.2. Local stability analysis

This section investigates the local stability of the equilibrium points Q_0 and Q^*. To conduct the analysis we state the following theorem:

Theorem 2.3. *If* $0 < R_0 < 1$, *then the bullying free school equilibrium* $Q_0 = \left(\frac{A}{\beta}, 0\right)$ *is locally asymptotically stable. If not, then* Q_0 *is unstable.*

Proof. If we construct the Jacobian matrix for model (2.1), and then evaluate it at the bullying free school equilibrium Q_0, we obtain

$$J(Q_0) = \begin{pmatrix} -\beta & -bx_0 \\ 0 & -c + \alpha x_0 \end{pmatrix}, \tag{2.11}$$

hence, the roots are

$$\begin{aligned}\lambda_1 &= -\beta,\\ \lambda_2 &= c(R_0 - 1).\end{aligned} \tag{2.12}$$

Clearly, the first root λ_1 is negative. The second root λ_2 is also negative provided that $R_0 < 1$. We conclude that the bullying free school equilibrium Q_0 is locally asymptotically stable if $R_0 < 1$, whereas, it is unstable if $R_0 > 1$.

Next, to determine the stability of the equilibrium Q^*, we state the following theorem:

Theorem 2.4. *The positive equilibrium $Q^* = (x^*, y^*)$ is locally asymptotically stable only when $R_0 > 1$.*

Proof. If we evaluate the Jacobian matrix for model (2.1) at the positive equilibrium Q^* Eq (2.10), we obtain

$$J(Q^*) = \begin{pmatrix} \frac{-\alpha A}{c} & \frac{-bc}{\alpha} \\ \frac{\alpha\beta}{b}(R_0 - 1) & 0 \end{pmatrix}, \tag{2.13}$$

which its characteristic equation is given by

$$\lambda^2 + \frac{\alpha A}{c}\lambda + c\beta(R_0 - 1) = 0. \tag{2.14}$$

It is clear that $(\lambda_1 + \lambda_2) = \frac{\alpha A}{c} > 0$. Furthermore, since Q^* exists only if $R_0 > 1$, we can deduce that $\lambda_1\lambda_2 = c\beta(R_0 - 1) > 0$. Therefore, all eigenvalues of the characteristic equation (2.14) have a negative real part. This shows that Q^* is locally asymptotically stable.

3. Numerical simulation

Choosing $A = 500$, $\beta = 0.34$, $c = 0.6$, $b = 0.02$, $\alpha = 2 \times 10^{-4}$, we have $R_0 \approx 0.4902 < 1$. Consider also, the initial conditions $x(0) = 900$ and $y(0) = 14$. One can notice, in Figure 2, that the bullies $y(t) \to 0$ as $t \to \infty$. This means that the school environment will be free of bullying and become appropriate for good learning.

Choosing $A = 500$, $\beta = 0.34$, $c = 0.6$, $b = 0.02$, $\alpha = 12 \times 10^{-4}$, we have $R_0 \approx 2.9412 > 1$. Consider also, the initial conditions $x(0) = 900$ and $y(0) = 14$, one can notice, in Figure 3, that the bullies $y(t) \to 33$ as $t \to \infty$. This means that the bullying behavior will persist and students will no longer remain in this school and will leave to look for a better educational environment.

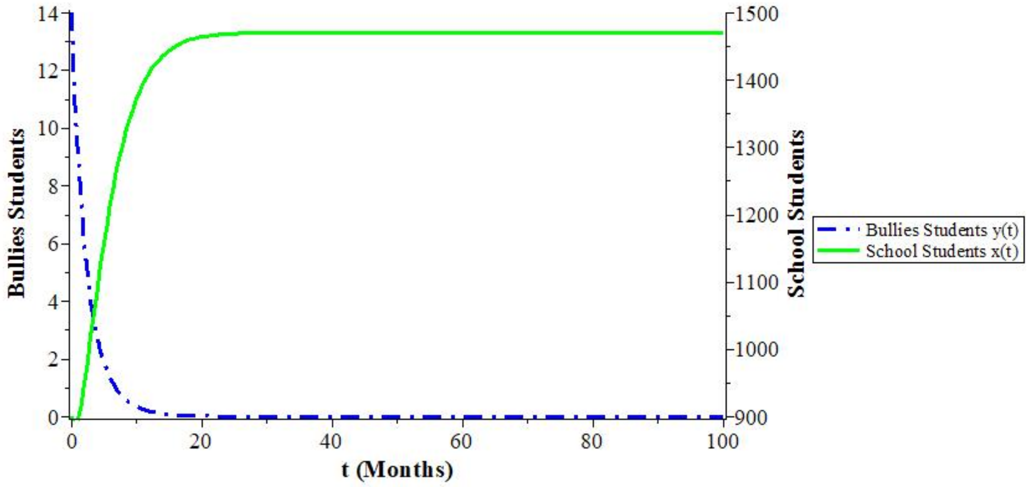


Figure 2. Simultaneous plot of both numerical solutions $x(t)$ and $y(t)$ of the school bullying model (2.1) versus time (months) for $R_0 < 1$ on a dual axis.

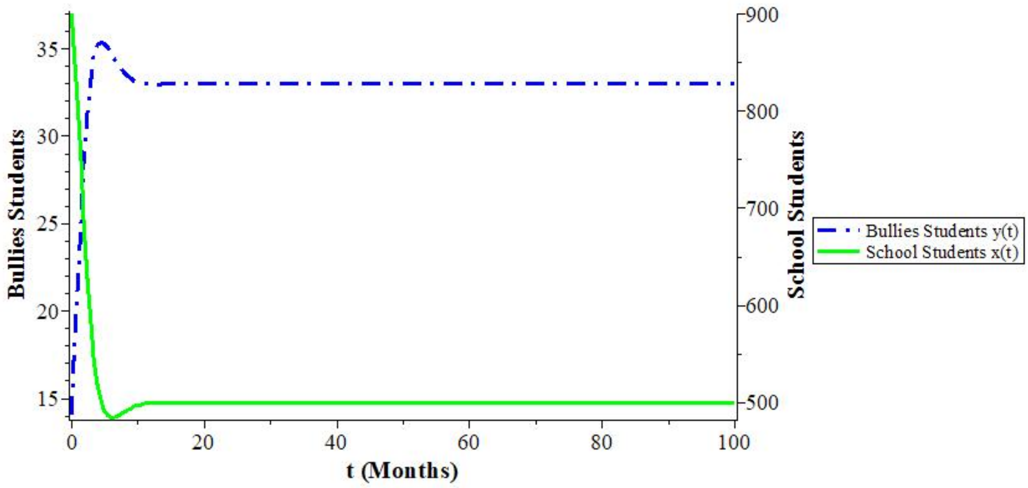


Figure 3. Simultaneous plot of both numerical solutions $x(t)$ and $y(t)$ of the school bullying model (2.1) versus time (months) for $R_0 > 1$ on a dual axis.

3.1. Global stability analysis

In this section, we focus on the global dynamics of system (2.1). We discuss, in Theorem 3.1 and Theorem 3.2, the global stability of the bullying free school equilibrium Q_0 and the positive equilibrium Q^* respectively.

Theorem 3.1. *If $R_0 < 1$, then the bullying free school equilibrium $Q_0 = (x_0, y_0) = \left(\frac{A}{\beta}, 0\right)$ is globally asymptotically stable.*

Proof. Consider the Lyapunov function $L_1(x(t), y(t))$ as:

$$L_1(x(t), y(t)) = x(t) - x_0 - x_0 \ln \frac{x(t)}{x_0} + \frac{2b}{\alpha} y(t) + \frac{\alpha}{2(c+\beta)} \left(x(t) - x_0 + \frac{b}{\alpha} y(t) \right)^2 . \tag{3.1}$$

If we compute the time derivative of L_1 and apply the derivative of the variables in system (2.1) we obtain

$$\begin{aligned} L_1'(t) &= \left(1 - \frac{x_0}{x}\right)(A - \beta x - bxy) - \frac{2cb}{\alpha} y + 2bxy \\ &+ \frac{\alpha}{(c+\beta)} \left(x - x_0 + \frac{b}{\alpha} y \right) \left(A - \beta x - \frac{cb}{\alpha} y \right), \\ &= -\beta \left(\frac{x - x_0}{x} + \frac{\alpha(x - x_0)}{(c+\beta)} + \frac{by}{(c+\beta)} \right) \left(x - \frac{A}{\beta} \right) + by\,(x + x_0) - \frac{2cb}{\alpha} y \\ &- \frac{cby(x - x_0)}{(c+\beta)} - \frac{cb^2y^2}{\alpha(c+\beta)}, \\ &= -\beta \left(\frac{1}{x} + \frac{\alpha}{(c+\beta)} \right) (x - x_0)^2 - \frac{by(x - x_0)(c+\beta)}{(c+\beta)} \\ &+ by\,(x + x_0) - \frac{2cb}{\alpha} y - \frac{cb^2y^2}{\alpha(c+\beta)}, \\ &= -\beta \left(\frac{1}{x} + \frac{\alpha}{(c+\beta)} \right) (x - x_0)^2 - \frac{2cby}{\alpha} (1 - R_0) - \frac{cb^2y^2}{\alpha(c+\beta)}. \end{aligned} \tag{3.2}$$

Since we assume that $R_0 < 1$, this implies that

$$L_1'(t) < 0, \ \ for\ all\ (x(t), y(t)) \neq (x_0, y_0). \tag{3.3}$$

Therefore, the proof of Theorem 3.1 has been accomplished, and we conclude that the bullying free school equilibrium Q_0 is globally asymptotically stable.

Next, we give the global asymptotic stable theorem regarding the positive equilibrium Q^*.

Theorem 3.2. *If $R_0 > 1$, then the positive equilibrium Q^* is globally asymptotically stable.*

Proof. Here we chose the Lyapunov function $L_2(x(t), y(t))$ as:

$$\begin{aligned} L_2(x(t), y(t)) &= x(t) - x^* - x^* \ln \frac{x(t)}{x^*} + \frac{2b}{\alpha} \left(y(t) - y^* - y^* \ln \frac{y(t)}{y^*} \right) \\ &+ \frac{\alpha}{2(c+\beta)} \left(x(t) - x^* + \frac{b}{\alpha} (y(t) - y^*) \right)^2 . \end{aligned} \tag{3.4}$$

By differentiating L_2 with respect to time, and applying the derivative of the variables in system (2.1), we get

$$\begin{aligned} L_2'(t) &= \left(1 - \frac{x^*}{x}\right)(A - \beta x - bxy) + \frac{2b}{\alpha} \left(1 - \frac{y^*}{y}\right)(\alpha xy - cy) \\ &+ \frac{\alpha}{(c+\beta)} \left(x - x^* + \frac{b}{\alpha}(y - y^*) \right) \left(A - \beta x - \frac{cby}{\alpha} \right). \end{aligned} \tag{3.5}$$

Substituting the positive equilibrium $Q^* = (x^*, y^*)$ Eq (2.8) in system (2.9) gives the relation $A = \beta x^* + bx^*y^*$, which can be used in Eq (3.5) as

$$\begin{aligned}
L_2'(t) &= \left(\frac{x-x^*}{x}\right)(-\beta(x-x^*) + bx^*y^* - bxy) + 2b(y-y^*)(x-x^*) \\
&+ \frac{\alpha}{(c+\beta)}\left(x - x^* + \frac{b}{\alpha}(y-y^*)\right)\left(-\beta(x-x^*) - \frac{cb}{\alpha}(y-y^*)\right), \\
&= -\frac{(x-x^*)^2}{x}(\beta + by^*) + b(y-y^*)(x-x^*) \\
&+ \frac{\alpha}{(c+\beta)}\left(-\beta(x-x^*)^2 - \frac{cb^2(y-y^*)^2}{\alpha^2} - \frac{b(y-y^*)(x-x^*)(c+\beta)}{\alpha}\right), \\
&= -\frac{\beta R_0(x-x^*)^2}{x} - \frac{\alpha}{(c+\beta)}\left(\beta(x-x^*)^2 + \frac{cb^2(y-y^*)^2}{\alpha^2}\right).
\end{aligned} \tag{3.6}$$

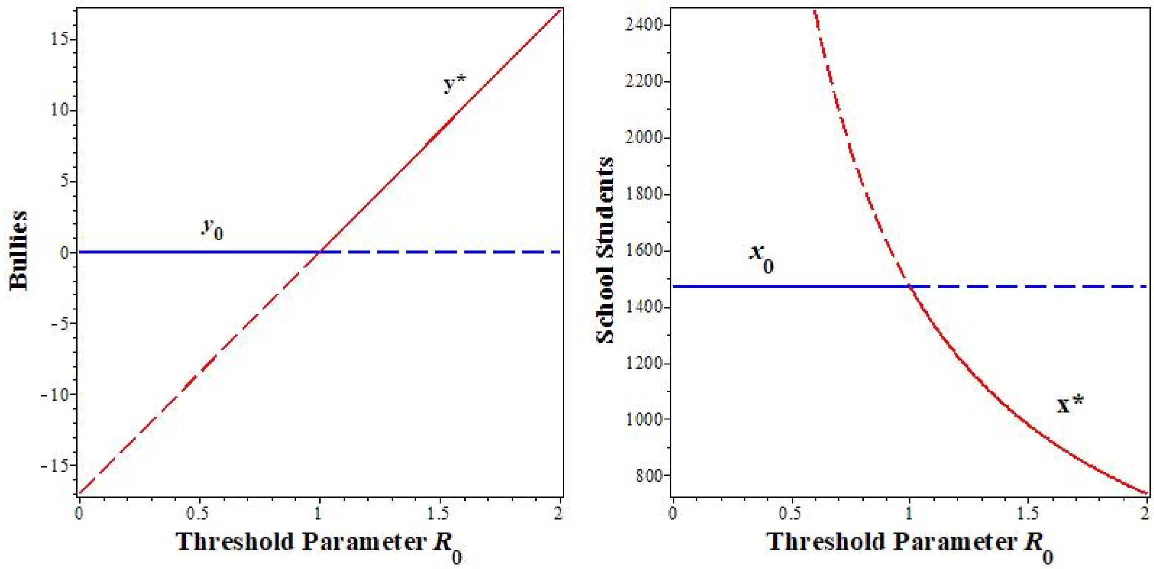
This proves that

$$L_2'(t) < 0, \ \textit{for all}\ (x(t), y(t)) \neq (x^*, y^*). \tag{3.7}$$

Therefore, the proof of theorem 3.2 has been accomplished, and we conclude that Q^* is globally asymptotically stable.

3.2. Bifurcation analysis

Our previous analysis indicates the existence of two equilibrium points: Q_0 that exists for all values of $R_0 > 0$, and the positive equilibrium Q^* that exists only if $R_0 > 1$ (for $R_0 < 1$, some of its coordinates are negative). Moreover, the analysis indicates that the equilibrium points change their stability from being asymptotically stable to unstable (lose their stability) as they cross through the bullying index $R_0 = 1$. This phenomenon is called transcritical bifurcation. For $R_0 < 1$, Q_0 is asymptotically stable while Q^* is not (bullies cannot invade the school. Note that the number of school students is very high, see Figure 4(a), (b) respectively). On the other hand, for $R_0 > 1$, Q^* is asymptotically stable while Q_0 is not (bullying is expected to become an epidemic. Note that the number of the school students is decreasing, see Figure 4(a) and (b) respectively).



(a) (b)

Figure 4. Transcritical Bifurcation diagram at $R_0 = 1$ (a) the bullying students $y(t)$ (b) the school students $x(t)$. The solid line indicates stability and the dashed line indicates instability.

4. Conclusions

We have proposed and analyzed a two dimensional mathematical model in order to addresses the issue of bullying in schools. Some assumptions such as the policy of schools should ban bullying and expel students who practice this behavior is required to formulate the model. The variables used are: the number of victims students and the number of bullies students. With the aid of the nonlinear stability theory, it is shown that the developed model has two equilibrium points. The first one is called the positive equilibrium. Its existence, which is subject to the bullying index R_0, considered to be a threat for the school environment. It is found that it is locally and globally asymptotically stable in the case $R_0 > 1$. The second equilibrium is interesting and called the bullying free school equilibrium. It is found that it is locally and globally asymptotically stable in the case $R_0 < 1$. Furthermore, the analysis has indicated that the model exhibits the phenomenon of Transcritical Bifurcation at $R_0 = 1$.

To check the feasibility of the stability analysis, we conducted some numerical experiments. The simulations have strongly supported the local and global asymptotic stability behavior of both equilibrium points.

You will never be able to prevent bullying, but you can deter it. Schools can best address the problem of bullying by providing anti-bullying programs and enhancing positive relationships between teachers and students, therefore, building a safe school environment with intensive supervision. Furthermore, bullying should be integrated into the curriculum, i.e., developing curriculum that include bullying definition, types of bullying, who get bullied, and strategies to stop such behavior.

5. Future work

The model of school bullying, generally, can still be further improved. For example, more variables, that have the effect of reducing the bullying rate, can be taken into consideration.

Conflict of interest

The author declare no conflict of interest.

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