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*Research article*

# The Lyapunov-Razumikhin theorem for the conformable fractional system with delay

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**Abstract:** This paper explicates the Razumikhin-type uniform stability and a uniform asymptotic stability theorem for the conformable fractional system with delay. Based on a Razumikhin-Lyapunov functional and some inequalities, a delay-dependent asymptotic stability criterion is in the term of a linear matrix inequality (LMI) for the conformable fractional linear system with delay. Moreover, an application of our theorem is illustrated via a numerical example.

**Keywords:** Lyapunov-Razumikhin technique; conformable fractional delayed system; linear matrix inequality; uniform asymptotic stability
**Mathematics Subject Classification:** 34K25, 34K40, 37B25, 58K25

## 1. Introduction

Fractional differential systems have been widely investigated due to their applications in science and engineering, including solving nonlinear equations, associative memory, data analysis, intelligent control, and optimization [3, 4, 6, 8, 11–14, 17, 18]. The advantages of fractional-order calculus are that it can increase the flexibility of a system with infinite memory and genetic characteristics. There have been studied and developments in the theoretic aspects such as controllability, periodicity, asymptotic behavior etc.

In [9], R. Khalil et al. defined the conformable fractional derivative. Recently, many researchers have studied definitions and properties of conformable fractional derivatives other than the Caputo, the GrunwaldLetnikov and the RiemannLiouville fractional derivatives for all of them do not satisfy the rules

$$T_{t_0}^{\alpha}\mu(t)\upsilon(t) = \mu(t)T_{t_0}^{\alpha}\upsilon(t) + \upsilon(t)T_{t_0}^{\alpha}\mu(t),$$
$$T_{t_0}^{\alpha}\frac{\mu(t)}{\upsilon(t)} = \frac{\upsilon(t)T_{t_0}^{\alpha}\mu(t) - \mu(t)T_{t_0}^{\alpha}\upsilon(t)}{\upsilon(t)^2},$$

where

$$T_{t_0}^{\alpha}\mu(t) = \lim_{\varsigma\to 0}\frac{\mu(t+\varsigma(t-t_0)^{1-\alpha}) - \mu(t)}{\varsigma},$$

for all $t > t_0$ and $\alpha \in (0, 1]$.

The aim of this paper is to construct Razumikhin-type uniform stability and a uniform asymptotic stability theorem for the conformable fractional system with delay. Moreover, a numerical example is given to show that our theorem can be applied in an uncomplicated way.

## 2. Problem formulation and preliminaries

In this section, we approach some preliminary definitions and necessary lemmas.

**Definition 2.1.** [10] For a function $\mu : [t_0, \infty) \to \mathbb{R}$, the conformable fractional derivative of $\mu$ of order $\alpha$ is defined by

$$T_{t_0}^{\alpha}\mu(t) = \lim_{\varsigma\to 0}\frac{\mu(t+\varsigma(t-t_0)^{1-\alpha}) - \mu(t)}{\varsigma}, \tag{2.1}$$

for all $t > t_0$ and $\alpha \in (0, 1]$.

If the conformable fractional derivative of $\mu(t)$ of order $\alpha$ exists on $(t_0, \infty)$, then the function $\mu(t)$ is said to be $\alpha$-differentiable on the interval $(t_0, \infty)$.

**Definition 2.2.** [10] Given a function $\mu : [t_0, \infty) \to \mathbb{R}$, the conformable fractional integral starting from $t_0$ of $\mu$ of order $\alpha$, where $0 < \alpha \leq 1$ is defined by

$$I_{t_0}^{\alpha}\mu(t) = \int_{t_0}^{t}(s-t_0)^{\alpha-1}\mu(s)ds. \tag{2.2}$$

**Lemma 2.3.** [10] Given $\alpha \in (0, 1)$ and a continuous function $\mu : [t_0, \infty) \to \mathbb{R}$, we have

$$T_{t_0}^{\alpha}(I_{t_0}^{\alpha}\mu(t)) = \mu(t), \tag{2.3}$$

for all $t > t_0$.

**Lemma 2.4.** [10] Given a $\alpha$-differentiable function $\mu : [t_0, \infty) \to \mathbb{R}$ with $\alpha \in (0, 1]$, we have

$$I_{t_0}^{\alpha}(T_{t_0}^{\alpha}\mu(t)) = \mu(t) - \mu(t_0), \tag{2.4}$$

for all $t > t_0$.

**Lemma 2.5.** [10] Given a symmetric positive definite matrix $P$ and a $\alpha$-differentiable function $\mu : [t_0, \infty) \to \mathbb{R}$ with $\alpha \in (0, 1]$. Then $T_{t_0}^{\alpha}\mu^T(t)P\mu(t)$ exists on $[t_0, \infty)$ and

$$T_{t_0}^{\alpha}\mu^T(t)P\mu(t) = 2\mu^T(t)PT_{t_0}^{\alpha}\mu(t), \tag{2.5}$$

for all $t > t_0$.

Consider the conformable fractional system with delay

$$T^{\alpha}_{t_0}\mu(t) = g(t, \mu(t-\eta)), \quad t \geq t_0, \tag{2.6}$$

where $0 < \alpha \leq 1$, $\mu(t) \in \mathbb{R}^n$ is the state vector, and $g : \mathbb{R} \times C([-\eta, 0], \mathbb{R}^n) \to \mathbb{R}^n$. For each solution $\mu(t)$ of (2.6), we assume the initial condition

$$\mu(t_0 + s) = \phi(s), \quad s \in [-\eta, 0],$$

where $\phi \in C([-\eta, 0], \mathbb{R}^n)$.

## 3. Main results

**Theorem 3.1.** Suppose that $\kappa_1, \kappa_2, \kappa_3 : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous non-decreasing functions, $\kappa_1(s)$ and $\kappa_2(s)$ are positive for $s > 0$, $\kappa_1(0) = \kappa_2(0) = 0$, and $\kappa_2$ is strictly increasing. If there exists a differentiable functional $\vee : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ such that

$$\kappa_1(\|\mu\|) \leq \vee(t, \mu) \leq \kappa_2(\|\mu\|), \tag{3.1}$$

for $t \in \mathbb{R}, \mu \in \mathbb{R}^n$, and for any given $t_0 \in \mathbb{R}$ the conformable fractional derivative of $\vee$ along the solution $\mu(t)$ of conformable system (2.6) satisfies

$$T^{\alpha}_{t_0} \vee (t, \mu(t)) \leq -\kappa_3(\|\mu(t)\|), \tag{3.2}$$

whenever $\vee(t + \theta, \mu(t + \theta)) \leq \vee(t, \mu(t))$ for all $\theta \in [-\eta, 0]$, then conformable system (2.6) is uniformly stable. If $\kappa_3(s) > 0$ for $s > 0$ and there exists a continuous non-decreasing function $\zeta(s) > s$ for $s > 0$ such that

$$T^{\alpha}_{t_0} \vee (t, \mu(t)) \leq -\kappa_3(\|\mu(t)\|), \tag{3.3}$$

whenever $\vee(t+\theta, \mu(t+\theta)) \leq \zeta(\vee(t, \mu(t)))$ for all $\theta \in [-\eta, 0]$, then conformable system (2.6) is uniformly asymptotically stable.

*Proof of Theorem 1.* Suppose that $\mu(t) = \mu(t, t_0, \phi)$, $\vee(t) = \vee(t, \mu(t))$, and

$$\vee^*(t) = \sup_{-\eta \leq \theta \leq 0} \vee(t + \theta, \mu(t + \theta)).$$

There exists $\hat{\theta} \in [-\eta, 0]$ such that $\vee^*(t) = \vee(t + \hat{\theta}, \mu(t + \hat{\theta}))$, and either $\hat{\theta} = 0$ or $\hat{\theta} < 0$ and

$$\vee\, (t + \theta, \mu(t + \theta)) \leq \vee(t + \hat{\theta}, \mu(t + \hat{\theta})), \tag{3.4}$$

for $\hat{\theta} \leq \theta \leq 0$.

Next, we show that

$$T^{\alpha}_{t_0} \vee^* (t, \mu(t)) \leq 0. \tag{3.5}$$

For $\hat{\theta} < 0$, we have $\vee^*(t+\Delta t, \mu(t+\Delta t)) = \vee^*(t, \mu(t))$ for sufficiently small $\Delta t > 0$, and thus $T^{\alpha}_{t_0}\vee^*(t, \mu(t)) = 0$.

For $\hat{\theta} = 0$, we have $\vee^*(t) = \vee(t, \mu(t))$ and $T^{\alpha}_{t_0} \vee^* (t, \mu(t)) = T^{\alpha}_{t_0} \vee (t, \mu(t)) \leq 0$ by (3.2), So (3.5) holds. Moreover, we have

$$\kappa_1(\|\mu(t)\|) \leq \vee(t, \mu(t)) \leq \vee^*(t, \mu(t)) \leq \vee(t_0, \mu(t_0)) \leq \kappa_2(\|\mu(t_0)\|). \tag{3.6}$$

Given $\epsilon > 0$, we can choose a sufficiently small $\delta > 0$ with $\kappa_2(\delta) < \kappa_1(\epsilon)$.

Assume that $\|\mu_{t_0}\| < \delta$. Then from (3.6), it follows that

$$\kappa_1(\|\mu(t)\|) \leq \kappa_2(\|\mu(t_0)\|) \leq \kappa_2(\|\mu_{t_0}\|) \leq \kappa_2(\delta) < \kappa_1(\epsilon), \tag{3.7}$$

which implies that $\|\mu(t)\| < \epsilon$. This shows that conformable system (2.6) is uniformly stable.

Suppose $\delta > 0$ and $H > 0$ are such that $\varrho(\delta) = u(H)$. Since $\|\phi\| \leq \delta$, we have $\|\mu_{t_0}\| \leq H$ and $\vee(t, \mu(t)) < \varrho(\delta)$ for $t \geq t_0 - \eta$.

Suppose that $\beta$ with $0 < \beta \leq H$ is arbitrary. From the properties of the function $\zeta(s)$, there exists $\iota > 0$ such that $\zeta(s) - s > \iota$ for $u(\beta) \leq s \leq \varrho(\delta)$. Let $M$ be the smallest integer such that $u(\beta) + M\iota \geq \varrho(\delta)$ and let $T = \dfrac{M\varrho(\delta)}{\gamma}$ when $\gamma = \inf\limits_{\beta \leq s < H} \kappa_3(s)$.

Next, we will show that $\vee(t, \mu(t)) \leq u(\beta) + (M-1)\iota$ for $t \geq t_0 + \varrho(\delta)/\gamma$. If $u(\beta) + (M-1)\iota < \vee(t, \mu(t))$ for $t_0 - \eta \leq t < t_0 + \varrho(\delta)/\gamma$, then the fact that $\vee(t, \mu(t)) \leq \varrho(\delta)$ for all $t \geq t_0 - \eta$ yields

$$\zeta(\vee(t, \mu(t))) > \vee(t, \mu(t)) + \iota \geq u(\beta) + M\iota \geq \varrho(\delta) \geq \vee(t + \zeta, \mu(t + \zeta)),$$

for $t_0 - \eta \geq t \geq t_0 + \varrho(\delta)/\gamma$ and $\zeta \in [-\eta, 0]$. Thus $T^{\alpha}_{t_0}\varrho(t, \mu(t)) \leq -\kappa_3(|\mu(t)|) \leq -\gamma$ for $t_0 \leq t < t_0 + \varrho(\delta)/\gamma$. Consequently, we have

$$\vee(t, \mu(t)) \leq \vee(t_0, \mu(t_0)) - \gamma(t - t_0) \leq \varrho(\delta) - \gamma(t - t_0).$$

Then $\vee(t, \mu(t)) \leq u(\beta) + (M-1)\iota$ at $t_1 = t_0 + \varrho(\delta)/\gamma$. This implies that $\vee(t, \mu(t)) \leq u(\beta) + (M-1)\iota$ for all $t \geq t_0 + \varrho(\delta)/\gamma$, since $T^{\alpha}_{t_0} \vee (t, \mu(t))$ is negative when $\vee(t, \mu(t)) = u(\beta) + (M-1)\iota$.

Now, let $\bar{t}_j = j\varrho(\delta)/\gamma$ for $j = 1, 2, \ldots, M$, and let $\bar{t}_0 = 0$. Assume that, for some integer $k \geq 1$, in the interval $\bar{t}_{k-1} - r \leq t - t_0 \leq \bar{t}_k$, we have

$$u(\beta) + (M-k)\iota \leq \vee(t, \mu(t)) \leq u(\beta) + (M-k+1)\iota.$$

Then

$$T^{\alpha}_{t_0} \vee (t, \mu(t)) \leq -\gamma, \quad \bar{t}_{k-1} \leq t - t_0 \leq \bar{t}_k,$$

and

$$\begin{aligned} \vee(t, \mu(t)) &\leq \vee(t_0 + \bar{t}_{k-1}, \mu(t_0 + \bar{t}_{k-1})) - \gamma(t - t_0 - \bar{t}_{k-1}) \\ &\leq \varrho(\delta) - \gamma(t - t_0 - \bar{t}_{k-1}) \leq 0, \end{aligned}$$

when $t - t_0 - \bar{t}_{k-1} \geq \varrho(\delta)/\gamma$. Consequently, we have

$$\vee(t_0 + \bar{t}_{k-1}, \mu(t_0 + \bar{t}_{k-1})) \leq u(\beta) + (M-k)\iota,$$

which implies that $\vee(t, \mu(t)) \leq u(\beta) + (M-k)\iota$ for all $t \geq t_0 + \bar{t}_{k-1}$.

Finally, we have $\vee(t, \mu(t)) \leq u(\beta)$ for all $t \geq t_0 + M\varrho(\delta)/\gamma$. This shows that conformable system (2.6) is uniformly asymptotically stable. □

Consider the conformable fractional linear system with delay

$$T_{t_0}^{\alpha}\mu(t) = -A\mu(t) + Bf(\mu(t-\eta)), \quad t \geq t_0, \tag{3.8}$$

where $0 < \alpha \leq 1, \mu(t) \in \mathbb{R}^n$ is the state vector, $A, B$ are known real constant matrices and $\eta$ is a positive real constant. For each solution $\mu(t)$ of (3.8), we assume the initial condition

$$\mu(t) = \phi(t), \quad t \in [-\eta, 0],$$

where $\phi \in C([-\eta, 0]; \mathbb{R}^n)$. The uncertainty $f(\cdot)$ represents the nonlinear parameter perturbation with respect to the state $x(t)$ and is bounded in magnitude of the form

$$f^T(\mu(t-\eta))f(\mu(t-\eta)) \leq \delta^2\mu^T(t-\eta)\mu(t-\eta), \tag{3.9}$$

where $\delta$ is a given constant.

**Theorem 3.2.** Given a positive scalar $\delta$, system (3.8) is uniformly stable if there exists a symmetric positive definite matrix $K$ such that the following symmetric linear matrix inequality holds:

$$\begin{bmatrix} -2KA + \eta\alpha K & 0 & KB \\ * & \epsilon\delta^2 I - \eta\alpha K & 0 \\ * & * & -\epsilon I \end{bmatrix} < 0. \tag{3.10}$$

*Proof of Theorem 2.* Let $K$ be a symmetric positive definite matrices. Consider the Lyapunov-Razumikhin functional of the form

$$\vee(t) = \mu^T(t)K\mu(t).$$

Taking the conformable fractional derivative of $\vee(t)$ along the trajectory of system (3.8), we have

$$\begin{aligned} T_{t_0}^{\alpha} \vee (t) &= \mu^T(t)KT_{t_0}^{\alpha}\mu(t) \\ &= 2\mu^T(t)K[-A\mu(t) + Bf(\mu(t-\eta))]. \end{aligned} \tag{3.11}$$

Next, from (3.9), we obtain

$$0 \leq \epsilon\delta^2\mu^T(t-\eta)\mu(t-\eta) - \epsilon f^T(\mu(t-\eta))f(\mu(t-\eta)), \tag{3.12}$$

for $\epsilon > 0$. When $\vee(t+\theta, \mu(t+\theta)) \leq \vee(t, \mu(t))$ for all $\theta \in [-\eta, 0]$, we obtain

$$0 \leq \eta\alpha\mu^T(t)K\mu(t) - \eta\alpha\mu^T(t-\eta)K\mu(t-\eta). \tag{3.13}$$

According to (3.11) and (3.13), it is straightforward to see that

$$T_{t_0}^{\alpha}\vee(t) \leq \xi^T(t)\begin{bmatrix} -2KA + \eta\alpha K & 0 & KB \\ * & \epsilon\delta^2 I - \eta\alpha K & 0 \\ * & * & -\epsilon I \end{bmatrix}\xi(t),$$

where $\xi(t) = col\{\mu(t), \mu(t-\eta), f(\mu(t-\eta))\}$. Note that if condition (3.10) holds, then system (3.8) is uniformly stable. □

## 4. A numerical example

In this section, a numerical example is given in order to present the effectiveness of our main results by showing the maximum upper bound of the parameter $\delta$.

**Example 4.1.** Consider the conformable linear system

$$T^{\alpha}_{t_0}\mu(t) = -A\mu(t) + Bf(\mu(t-\eta)). \tag{4.1}$$

Solving LMI (3.10) with $A = \begin{bmatrix} 2 & 0 \\ 0 & 0.9 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \eta = 0.3$ and $\alpha = 0.8$, we obtain the parameters $\epsilon = 0.9033$ and $K = \begin{bmatrix} 0.4522 & -0.0649 \\ -0.0649 & 0.4652 \end{bmatrix}$, which guarantee asymptotic stability of system (4.1) when $\delta = 0.3$.

Moreover, the maximum upper bound of the parameter $\delta$ which guarantees the asymptotical stability of system (4.1) is 0.4131 in Table 1. The permissible upper bounds $\delta$ for various $\eta$ and $\alpha$ are shown in Table 1.

**Table 1.** The least upper bound of $\delta$ for Example 4.1.

| | $\eta = 0.1$ | $\eta = 0.3$ | $\eta = 0.5$ |
|---|---|---|---|
| $\alpha = 0.6$ | 0.2172 | 0.3650 | 0.4536 |
| $\alpha = 0.8$ | 0.2495 | 0.4131 | 0.5073 |
| $\alpha = 1$ | 0.2774 | 0.4536 | 0.5490 |

We let $A = \begin{bmatrix} 2 & 0 \\ 0 & 0.9 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \eta = 0.5, \alpha = 1, f(t) = 0.01t$ and $\phi(t) = \begin{bmatrix} 2 & -4 \end{bmatrix}^T, \ \forall$ $t \in [-0.5, 0]$. Figure 1. shows the trajectories of solutions $\mu(t)$ of system (4.1).

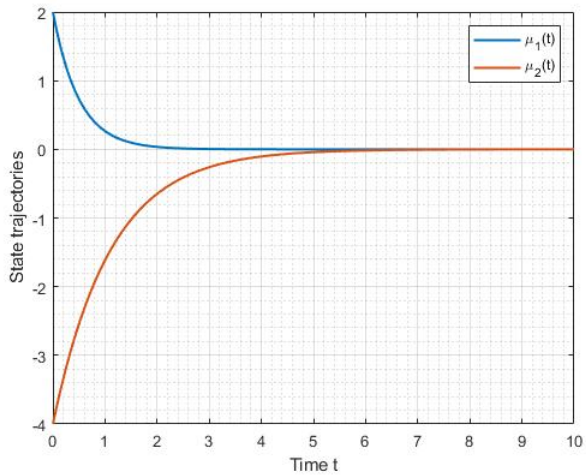


**Figure 1.** The trajectories of solutions $\mu(t)$ of system (4.1).

## 5. Conclusions

In this paper, an approach using the Lyapunov-Razumikhin theorem for the uniform stability and uniform asymptotic stability of the conformable fractional system with a delay has been presented. Some inequalities are adopted along with a Lyapunov-Razumikhin functional. Then we show a new delay-dependent asymptotic stability criterion of a conformable fractional linear system with delay. Finally, we give a numerical example to illustrate some advantages and applicability of our result. It will be important that future research investigate the asymptotic stability of the conformable fractional system with time-varying delay.

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## Conflict of interest

The authors declare no conflict of interest.

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