Mathematics

AIMS Mathematics, 7(3): 4338–4358.
DOI: 10.3934/math.2022241
Received: 01 October 2021
Revised: 05 December 2021
Accepted: 13 December 2021
Published: 20 December 2021

http://www.aimspress.com/journal/Math

Research article

Some new Jensen, Schur and Hermite-Hadamard inequalities for log convex fuzzy interval-valued functions

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Abstract: The inclusion relation and the order relation are two distinct ideas in interval analysis. Convexity and nonconvexity create a significant link with different sorts of inequalities under the inclusion relation. For many classes of convex and nonconvex functions, many works have been devoted to constructing and refining classical inequalities. However, it is generally known that log-convex functions play a significant role in convex theory since they allow us to deduce more precise inequalities than convex functions. Because the idea of log convexity is so important, we used fuzzy order relation($\preccurlyeq$) to establish various discrete Jensen and Schur, and Hermite-Hadamard (H-H) integral inequality for log convex fuzzy interval-valued functions (L-convex F-I-V-Fs). Some nontrivial instances are also offered to bolster our findings. Furthermore, we show that our conclusions include as special instances some of the well-known inequalities for L-convex F-I-V-Fs and their variant forms. Furthermore, we show that our conclusions include as special instances some

of the well-known inequalities for L-convex F-I-V-Fs and their variant forms. These results and different approaches may open new directions for fuzzy optimization problems, modeling, and interval-valued functions.

Keywords: log convex fuzzy interval-valued function; Riemann integral operator; Jensen type inequality; Schur type inequality; Hermite-Hadamard type inequality; Hermite-Hadamard-Fejér type inequality
Mathematics Subject Classification: 26A33, 26A51, 26D10

1. Introduction

Convex functions are well-known for their importance and superior applications in a variety of domains, particularly in integral inequalities, variational inequalities, and optimization. As a result, substantial effort has gone into analyzing and describing many aspects of the traditional concept of convexity. Several writers have recently examined several extensions and generalizations of convex functions, such as generalized convexity [1], s-convexity in the second sense [2], $\hbar$-convexity [3], P-convexity [4], and so on.

In classical approach, a real mapping $\mathfrak{A}: K \to \mathbb{R}$ is named as convex if

$$\mathfrak{A}(\varsigma\partial + (1-\varsigma)y) \le \varsigma\mathfrak{A}(\partial) + (1-\varsigma)\mathfrak{A}(y), \tag{1}$$

for all $\partial, y \in K, \varsigma \in [0, 1]$. If $\mathfrak{A}$ is a concave, then inequality (1) is flipped.

The integral problem and the idea of convexity is a fascinating subject for investigation. As a result, several inequalities have been offered as convex function applications. Among these, the H-H inequality is a fascinating convex analytic result. The H-H inequality [5,6] for convex function $\mathfrak{A}: K \to \mathbb{R}$ on an interval $K = [\mu, \upsilon]$

$$\mathfrak{A}\left(\frac{\mu+\upsilon}{2}\right) \le \frac{1}{\upsilon-\mu}\int_{\mu}^{\upsilon}\mathfrak{A}(\partial)d\partial \le \frac{\mathfrak{A}(\mu)+\mathfrak{A}(\upsilon)}{2}, \tag{2}$$

for all $\partial \in K$. If $\mathfrak{A}$ is a concave, then inequality (2) is flipped.

We should notice that H-H inequality is a refinement of the idea of convexity, and it readily follows from Jensen's inequality. In the recent few decades, H-H inequality has drawn a large number of authors to this topic. It's worth noting that (2) may be used to generate some of the standard inequalities for function $\mathfrak{A}$ selection.

If $\mathfrak{A}$ is concave, both inequalities hold in the opposite direction. Ostrowski inequality [7,8], Jensen type inequality [9], and H-H type inequalities are examples of inequalities that generalize, enhance, and extend the inequality (2). Fejér created the H-H Fejér inequality as the most important weighted extension of H-H inequality in [10].

Let $\mathfrak{A}: [\mu, \upsilon] \to \mathbb{R}^+$ be a convex function on a convex set K and $\mu, \upsilon \in K$ with $\mu \le \upsilon$. Then,

$$\mathfrak{A}\left(\frac{\mu+\upsilon}{2}\right) \le \frac{1}{\int_{\mu}^{\upsilon}\mathfrak{Q}(\partial)d\partial}\int_{\mu}^{\upsilon}\mathfrak{A}(\partial)\mathfrak{Q}(\partial)d\partial \le \frac{\mathfrak{A}(\mu)+\mathfrak{A}(\upsilon)}{2}\int_{\mu}^{\upsilon}\mathfrak{Q}(\partial)d\partial. \tag{3}$$

If $\mathfrak{A}$ is concave, then inequality (3) is flipped. If $\mathfrak{Q}(\partial) = 1$, then we obtain (2) from (3) with the assistance of inequality (3), many inequalities can be obtained through special symmetric function $\mathfrak{Q}(\partial)$ for convex functions.

It is well knowledge that log-convex functions are important in convex theory because they allow us to construct more precise inequalities than convex functions. Some writers have recently studied other classes of log-convex and log-nonconvex functions, such as $\hbar$-convexity [11], s-log convexity [12], and log-preinvexity [13,14], among others. The log-convex functions introduced by Pecari'c et al. [15] are a significant subclass of convex functions. Dragomir further looked into several log convex function features and developed H-H and Jensen [16–22] type inequalities for various log convex function classes.

On the other hand, to improve the accuracy of measurement findings and perform error analysis automatically, Moore [23] and, Kulish and Miranker [24] have proposed and examined the notion of interval analysis, replacing interval operations with simple operations. It is a field in which an uncertain variable is represented by a range of real numbers. Robotics, computer graphics, error analysis, experimental and computational physics, and many more fields have applications. Following their research, other writers turned to the literature to present several key generalized convex classes and inequalities for set-valued and interval-valued functions. With the use of fuzzy variational inequality, Nanda and Kar [25] and Chang [26] studied the concept of convex fuzzy mapping and discovered its optimality condition. Fuzzy convexity generalization and extension play an important role in a variety of applications. Let us mention that preinvex fuzzy mapping is one of the most well studied nonconvex fuzzy mapping classes. Noor [27] presented this concept and demonstrated certain findings using a fuzzy variational-like inequality to identify the fuzzy optimality condition of differentiable fuzzy preinvex mappings. We suggest readers to [28–32] and the references therein for more examination of literature on the applications and properties of variational-like inequalities and generalized convex fuzzy mappings. Román-Flores et al. found Beckenbaches inequality for interval-valued functions in [33,34]. Chalco-Cano et al. derived Ostrowski type inequalities for interval-valued functions using the Hukuhara derivative in [35,36]. Zhang et al. [37] used a pseudo order relation to establish a novel version of Jensen's inequalities for set-valued and fuzzy set-valued functions, proving that these Jensen's inequalities are an expanded form of Costa Jensen's inequalities [38]. In addition, for interval-valued functions (I-V-Fs) and F-I-V-Fs, fuzzy-interval and interval inequalities [39–41], fuzzy differential inequalities [42], are some related inequalities. See [43–58] for further details.

The goal of this study is to use fuzzy Riemann integrals to establish novel Jensen, Schur, H-H, and H-H Fejér type inequalities for L-convex F-I-V-Fs. For some exceptional circumstances, such as log convex functions, we establish our results. We further demonstrate the validity of our major findings using nontrivial examples. A brief conclusion is presented at the end.

2. Preliminaries

In this section, we introduces some preliminary notions, elementary concepts, and results as a pre-work, including operations, orders, and distance between interval and fuzzy numbers, Riemannian integrals, and fuzzy Riemann integrals. Some new definitions and results are also provided, which will be helpful to prove our main results.

Let $\mathbb{R}$ be the set of real numbers and $\mathcal{K}_C$ be the space of all closed and bounded intervals of $\mathbb{R}$, and $\varpi \in \mathcal{K}_C$ be established by

$$\varpi = [\varpi_*, \varpi^*] = \{\partial \in \mathbb{R} |\ \varpi_* \leq \partial \leq \varpi^*\}, (\varpi_*, \varpi^* \in \mathbb{R}).$$

If $\varpi_* = \varpi^*$, then ϖ is named as degenerate. If $\varpi_* \geq 0$, then $[\varpi_*, \varpi^*]$ is named as positive interval. The set of all positive interval is denoted by $\mathcal{K}_C^+$ and established as $\mathcal{K}_C^+ = \{[\varpi_*, \varpi^*]: [\varpi_*, \varpi^*] \in \mathcal{K}_C \text{ and } \varpi_* \geq 0\}$.

Let $\mathrm{s} \in \mathbb{R}$ and $\mathrm{s}\varpi$ be established by

$$\mathrm{s}.\varpi = \begin{cases} [\mathrm{s}\varpi_*, \mathrm{s}\varpi^*] \text{ if } \mathrm{s} > 0, \\ \{0\} \quad \text{ if } \mathrm{s} = 0, \\ [\mathrm{s}\varpi^*, \mathrm{s}\varpi_*] \text{ if } \mathrm{s} < 0. \end{cases}$$

Then the Minkowski difference $\xi - \varpi$, addition $\varpi + \xi$ and $\varpi \times \xi$ for $\varpi, \xi \in \mathcal{K}_C$ are established by

$$[\xi_*, \xi^*] - [\varpi_*, \varpi^*] = [\xi_* - \varpi_*,\ \xi^* - \varpi^*],$$
$$[\xi_*, \xi^*] + [\varpi_*, \varpi^*] = [\xi_* + \varpi_*,\ \xi^* + \varpi^*],$$

and

$$[\xi_*, \xi^*] \times [\varpi_*, \varpi^*] = [min\{\xi_*\varpi_*, \xi^*\varpi_*, \xi_*\varpi^*, \xi^*\varpi^*\}, max\{\xi_*\varpi_*, \xi^*\varpi_*, \xi_*\varpi^*, \xi^*\varpi^*\}].$$

The inclusion " $\subseteq$ " means that $\xi \subseteq \varpi$ if and only if, $[\xi_*, \xi^*] \subseteq [\varpi_*, \varpi^*]$, if and only if

$$\varpi_* \leq \xi_*,\ \xi^* \leq \varpi^*. \tag{4}$$

Remark 2.1. [24] The relation " $\leq_I$ " established on $\mathcal{K}_C$ by $[\xi_*, \xi^*] \leq_I [\varpi_*, \varpi^*]$ if and only if

$$\xi_* \leq \varpi_*,\ \xi^* \leq \varpi^*, \tag{5}$$

for all $[\xi_*, \xi^*]$, $[\varpi_*, \varpi^*] \in \mathcal{K}_C$, it is an order relation.

For $[\xi_*, \xi^*]$, $[\varpi_*, \varpi^*] \in \mathcal{K}_C$, the Hausdorff-Pompeiu distance between intervals $[\xi_*, \xi^*]$ and $[\varpi_*, \varpi^*]$ is established by

$$d([\xi_*, \xi^*], [\varpi_*, \varpi^*]) = max\{|\xi_* - \varpi_*|, |\xi^* - \varpi^*|\}. \tag{6}$$

It is familiar fact that $(\mathcal{K}_C, d)$ is a complete metric space.

A fuzzy subset T of $\mathbb{R}$ is characterize by a mapping $\xi: \mathbb{R} \to [0,1]$ named as the membership function, for each fuzzy set and $\theta \in (0, 1]$, then θ-cut sets of ξ is denoted and established as follows $\xi_\theta = \{\mu \in \mathbb{R} |\ \xi(\mu) \geq \theta\}$. If $\theta = 0$, then $supp(\xi) = \{\partial \in \mathbb{R} |\ \xi(\partial) > 0\}$ is named as support of ξ.

Let $\mathbb{F}(\mathbb{R})$ be the collection of all fuzzy sets and $\xi \in \mathbb{F}(\mathbb{R})$ be a fuzzy set. Then, we establish the following:

(1) ξ is named as normal if there exists $\partial \in \mathbb{R}$ and $\xi(\partial) = 1$;

(2) ξ is named as upper semi continuous on $\mathbb{R}$ if for provided $\partial \in \mathbb{R}$, there exist $\varepsilon > 0$ there exist $\delta > 0$ such that $\xi(\partial) - \xi(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|\partial - y| < \delta$;

(3) ξ is named as fuzzy convex if ξ_θ is convex for every $\theta \in [0, 1]$;

(4) ξ is compactly supported if $supp(\xi)$ is compact.

A fuzzy set is named as a fuzzy number or fuzzy interval if it has properties (1)–(4). We denote by $\mathbb{F}_C(\mathbb{R})$ the group of all fuzzy intervals.

Let $\xi \in \mathbb{F}_C(\mathbb{R})$ be a fuzzy-interval, if and only if, θ-cuts $[\xi]^\theta$ is a noempty compact convex set of $\mathbb{R}$. From these definitions, we have

$$[\xi]^\theta = [\xi_*(\theta), \xi^*(\theta)], \tag{7}$$

where

$$\xi_*(\theta) = inf\{\partial \in \mathbb{R}|\ \xi(\partial) \geq \theta\}, \xi^*(\theta) = sup\{\partial \in \mathbb{R}|\ \xi(\partial) \geq \theta\}.$$

Proposition 2.2. [39] If $\xi, \varpi \in \mathbb{F}_C(\mathbb{R})$, then relation " $\preccurlyeq$ " established on $\mathbb{F}_C(\mathbb{R})$ by $\xi \preccurlyeq \varpi$ if and only if,

$$[\xi]^\theta \leq_I [\varpi]^\theta, \text{ for all } \theta \in [0, 1], \tag{8}$$

this relation is known as partial order relation.

For $\xi, \varpi \in \mathbb{F}_C(\mathbb{R})$ and $\mathrm{s} \in \mathbb{R}$, the sum $\xi \tilde{+} \varpi$, product $\xi \tilde{\times} \varpi$, scalar product $\mathrm{s}.\xi$ and sum with scalar are established by:

Then, for all $\theta \in [0, 1]$, we have

$$[\xi \tilde{+} \varpi]^\theta = [\xi]^\theta + [\varpi]^\theta, \tag{9}$$

$$[\xi \tilde{\times} \varpi]^\theta = [\xi]^\theta \times [\varpi]^\theta, \tag{10}$$

$$[\mathrm{s}.\xi]^\theta = \mathrm{s}.[\xi]^\theta. \tag{11}$$

For $\psi \in \mathbb{F}_C(\mathbb{R})$ such that $\xi = \varpi \tilde{+} \psi$, then by this result we have existence of Hukuhara difference of ξ and ϖ, and we say that ψ is the H-difference of ξ and ϖ, and denoted by $\xi \tilde{-} \varpi$.

Definition 2.3. [39] A fuzzy-interval-valued map $\mathfrak{A}: K \subset \mathbb{R} \to \mathbb{F}_C(\mathbb{R})$ is named as F-I-V-F. For each $\theta \in (0, 1]$, θ-cuts establish the series of I-V-Fs $\mathfrak{A}_\theta: K \subset \mathbb{R} \to \mathcal{K}_C$ are provided by $\mathfrak{A}_\theta(\partial) = [\mathfrak{A}_*(\partial, \theta), \mathfrak{A}^*(\partial, \theta)]$ for all $\partial \in K$. Here, for each $\theta \in (0, 1]$, the end point real functions $\mathfrak{A}_*(., \theta), \mathfrak{A}^*(., \theta): K \to \mathbb{R}$ are named as lower and upper functions of $\mathfrak{A}$.

The following conclusions can be drawn from the preceding literature review [7,39,40,42]:

Definition 2.4. Let $\mathfrak{A}: [\mu, \upsilon] \subset \mathbb{R} \to \mathbb{F}_C(\mathbb{R})$ be a F-I-V-F. Then, fuzzy Riemann integral of $\mathfrak{A}$ over $[\mu, \upsilon]$, denoted by $(FR)\int_\mu^\upsilon \mathfrak{A}(\partial)d\partial$, it is provided by level-wise

$$\left[(FR)\int_\mu^\upsilon \mathfrak{A}(\partial)d\partial\right]^\theta = (IR)\int_\mu^\upsilon \mathfrak{A}_\theta(\partial)d\partial = \left\{\int_\mu^\upsilon \mathfrak{A}(\partial, \theta)d\partial : \mathfrak{A}(\partial, \theta) \in \mathcal{R}_{([\mu,\upsilon],\theta)}\right\}, \tag{12}$$

for all $\theta \in (0, 1]$, where $\mathcal{R}_{([\mu,\upsilon],\theta)}$ denotes the collection of Riemannian integrable functions of I-V-Fs. $\mathfrak{A}$ is FR-integrable over $[\mu, \upsilon]$ if $(FR)\int_\mu^\upsilon \mathfrak{A}(\partial)d\partial \in \mathbb{F}_C(\mathbb{R})$. Note that, if both end point functions are Lebesgue-integrable, then $\mathfrak{A}$ is fuzzy Aumann-integrable function over $[\mu, \upsilon]$, see [39].

Theorem 2.5. Let $\mathfrak{A}: [\mu, \upsilon] \subset \mathbb{R} \to \mathbb{F}_C(\mathbb{R})$ be a F-I-V-F and for all $\theta \in (0, 1]$, θ-cuts establish the series of I-V-Fs $\mathfrak{A}_\theta: [\mu, \upsilon] \subset \mathbb{R} \to \mathcal{K}_C$ are provided by $\mathfrak{A}_\theta(\partial) = [\mathfrak{A}_*(\partial, \theta), \mathfrak{A}^*(\partial, \theta)]$ for all $\partial \in [\mu, \upsilon]$. Then, $\mathfrak{A}$ is fuzzy Riemann integrable (FR-integrable) over $[\mu, \upsilon]$ if and only if, $\mathfrak{A}_*(\partial, \theta)$ and $\mathfrak{A}^*(\partial, \theta)$ both are Riemann integrable (R-integrable) over $[\mu, \upsilon]$. Moreover, if $\mathfrak{A}$ is FR-integrable over $[\mu, \upsilon]$, then

$$\left[(FR)\int_{\mu}^{v}\mathfrak{A}(\partial)d\partial\right]^{\theta} = \left[(R)\int_{\mu}^{v}\mathfrak{A}_*(\partial,\theta)d\partial,\ (R)\int_{\mu}^{v}\mathfrak{A}^*(\partial,\theta)d\partial\right]$$

$$= (IR)\int_{\mu}^{v}\mathfrak{A}_{\theta}(\partial)d\partial, \tag{13}$$

for all $\theta \in (0,1]$, where IR represent interval Riemann integration of $\mathfrak{A}_{\theta}(\partial)$. For all $\theta \in (0,1]$, $\mathcal{FR}_{([\mu,v],\theta)}$ denotes the collection of all FR-integrable F-I-V-Fs over $[\mu, v]$.

Definition 2.12. [15] A function $\mathfrak{A}: K \to \mathbb{R}$ is named as log-convex function if

$$\mathfrak{A}(\varsigma\partial + (1-\varsigma)y) \le \mathfrak{A}(\partial)^{\varsigma}\mathfrak{A}(y)^{1-\varsigma}, \forall\ \partial, y \in K, \varsigma \in [0,1], \tag{14}$$

where $\mathfrak{A}(\partial) \ge 0$, where K is a convex set. If (14) is flipped, then $\mathfrak{A}$ is named as log-concave.

Definition 2.13. [25] Let K be a convex set. Then F-I-V-F $\mathfrak{A}: K \to \mathbb{F}_C(\mathbb{R})$ is named as convex F-I-V-F on K if

$$\mathfrak{A}(\varsigma\partial + (1-\varsigma)y\ ) \preccurlyeq \varsigma\mathfrak{A}(\partial)\tilde{+}(1-\varsigma)\mathfrak{A}(y), \tag{15}$$

for all $\partial, y \in K, \varsigma \in [0,1]$, where $\mathfrak{A}(\partial) \succcurlyeq \tilde{0}$. If inequality (15) is flipped, then $\mathfrak{A}$ is named as concave F-I-V-F on $[\mu, v]$. $\mathfrak{A}$ is affine if and only if it is both convex I-V-F and concave I-V-F.

Definition 2.14. [25] Let K be a convex set. Then F-I-V-F $\mathfrak{A}: K \to \mathbb{F}_C(\mathbb{R})$ is named as log convex F-I-V-F (L-log convex F-I-V-F) on K if

$$\mathfrak{A}(\varsigma\partial + (1-\varsigma)y\ ) \preccurlyeq \mathfrak{A}(\partial)^{\varsigma}\ \tilde{\times}\ \mathfrak{A}(y)^{(1-\varsigma)}, \tag{16}$$

for all $\partial, y \in K, \varsigma \in [0,1]$, where $\mathfrak{A}(\partial) \succcurlyeq \tilde{0}$. If inequality (16) is flipped, then $\mathfrak{A}$ is named as L-concave F-I-V-F on $[\mu, v]$. $\mathfrak{A}$ is L-affine if and only if it is both L-convex F-I-V-F and L-concave F-I-V-F.

Remark 2.15. If $\mathfrak{A}$ is L-convex F-I-V-F, then $\Upsilon\mathfrak{A}$ is also L-convex F-I-V-F for $\Upsilon \ge 0$.
If $\mathfrak{A}$ and $\mathcal{J}$ both are L-convex F-I-V-F s, then $\max(\mathfrak{A}(\partial), \mathcal{J}(\partial))$ is also L-convex F-I-V-F.

Theorem 2.16. Let K be a convex set and let $\mathfrak{A}: K \to \mathbb{F}_C(\mathbb{R})$ be a F-I-V-F with $\mathfrak{A}(\partial) \succcurlyeq \tilde{0}$, whose θ-cuts establish the series of I-V-Fs $\mathfrak{A}_{\theta}: K \subset \mathbb{R} \to \mathcal{K}_C^{\ +} \subset \mathcal{K}_C$ are provided by

$$\mathfrak{A}_{\theta}(\partial) = [\mathfrak{A}_*(\partial,\theta), \mathfrak{A}^*(\partial,\theta)], \tag{17}$$

for all $\partial \in K$ and for all $\theta \in (0,1]$. Then $\mathfrak{A}$ is L-convex (resp. concave) on K, if and only if, for all $\theta \in (0,1]$, $\mathfrak{A}_*(\partial,\theta)$ and $\mathfrak{A}^*(\partial,\theta)$ both are L-convex (resp. L-concave)**.**

Proof. Let $\mathfrak{A}$ be a L-convex F-I-V-F on K. Then, for all $\partial, y \in K$ and $\varsigma \in [0,1]$, we have

$$\mathfrak{A}(\varsigma\partial + (1-\varsigma)y) \preccurlyeq \mathfrak{A}(\partial)^{\varsigma}\ \tilde{\times}\ \mathfrak{A}(y)^{(1-\varsigma)}.$$

Therefore, from inequality (17) and Proposition 2.4, we have

$$[\mathfrak{A}_*(\varsigma\partial + (1-\varsigma)y,\theta), \mathfrak{A}^*(\varsigma\partial + (1-\varsigma)y,\theta)]$$

$$\le_I [\mathfrak{A}_*(\partial,\theta)^{\varsigma}, \mathfrak{A}^*(\partial,\theta)^{\varsigma}] \times \left[\mathfrak{A}_*(y,\theta)^{(1-\varsigma)},\ \mathfrak{A}^*(y,\theta)^{(1-\varsigma)}\right]. \tag{18}$$

It follows that $\mathfrak{A}_*(\varsigma\partial + (1-\varsigma)y,\theta) \le \mathfrak{A}_*(\partial,\theta)^{\varsigma}\mathfrak{A}_*(y,\theta)^{(1-\varsigma)}$, and $\mathfrak{A}^*(\varsigma\partial + (1-\varsigma)y,\theta) \le \mathfrak{A}^*(\partial,\theta)^{\varsigma}\ \mathfrak{A}^*(y,\theta)^{(1-\varsigma)}$, for each $\theta \in (0,1]$. This shows that $\mathfrak{A}_*(\partial,\theta)$ and $\mathfrak{A}^*(\partial,\theta)$ both are L-convex functions.

Conversely, suppose that $\mathfrak{A}_*(\partial,\theta)$ and $\mathfrak{A}^*(\partial,\theta)$ both are L-convex functions. Then from definition and above inequality (19), it follows that $\mathfrak{A}(\partial)$ is L-convex F-I-V-F.

Example 2.17. We consider the F-I-V-F $\mathfrak{A}: [1, 4] \to \mathbb{F}_C(\mathbb{R})$ established by,

$$\mathfrak{A}(\partial)(\mathrm{s}) = \begin{cases} \frac{\mathrm{s}}{\frac{1}{\partial}} & \mathrm{s} \in \left[0, \frac{1}{\partial}\right]; \\ \frac{\frac{2}{\partial} - \mathrm{s}}{\frac{1}{\partial}} & \mathrm{s} \in \left(\frac{1}{\partial}, \frac{2}{\partial}\right]; \\ 0 & oterwise. \end{cases} \tag{19}$$

Then, for each $\theta \in (0, 1]$, we have $\mathfrak{A}_\theta(\partial) = \left[\theta \frac{1}{\partial},\ (2-\theta)\frac{1}{\partial}\right]$. Since end point functions $\mathfrak{A}_*(\partial, \theta)$, $\mathfrak{A}^*(\partial, \theta)$ are L-convex functions for each $\theta \in (0, 1]$, then by Theorem 2.16, $\mathfrak{A}(\partial)$ is L-convex F-I-V-F.

Remark 2.18. If $\mathfrak{A}_*(\partial, \theta) = \mathfrak{A}^*(\partial, \theta)$ with $\theta = 1$, then L-convex F-I-V-F becomes classical L-convex function [15].

3. Jensen and Schur inequalities for log convex fuzzy interval-valued functions

Now, we prove the Jensen inequality for L-convex F-I-V-F.

Theorem 3.1. Let $\omega_j \in \mathbb{R}^+$, $\partial_j \in [\mu, v]$, $(j = 1, 2, 3, \dots, k, k \geq 2)$ and $\mathfrak{A}: [\mu, v] \to \mathbb{F}_C(\mathbb{R})$ be a L-convex F-I-V-F, whose θ-cuts establish the series of I-V-Fs $\mathfrak{A}_\theta: [\mu, v] \subset \mathbb{R} \to \mathcal{K}_C^{\ +}$ are provided by $\mathfrak{A}_\theta(\partial) = [\mathfrak{A}_*(\partial, \theta), \mathfrak{A}^*(\partial, \theta)]$ for all $\partial \in [\mu, v]$ and for all $\theta \in (0, 1]$. Then

$$\mathfrak{A}\left(\frac{1}{W_k}\sum_{j=1}^{k} \omega_j\, \partial_j\right) \preccurlyeq \prod_{j=1}^{k}\left[\mathfrak{A}(\partial_j)\right]^{\frac{\omega_j}{W_k}}, \tag{20}$$

where $W_k = \sum_{j=1}^{k} \omega_j$. If $\mathfrak{A}$ is L-concave then, inequality (20) is flipped.

Proof. When $k = 2$ inequality (20) is true. Consider inequality (20) is true for $k = n-1$, then

$$\mathfrak{A}\left(\frac{1}{W_{n-1}}\sum_{j=1}^{n-1} \omega_j\, \partial_j\right) \preccurlyeq \prod_{j=1}^{n-1}\left[\mathfrak{A}(\partial_j)\right]^{\frac{\omega_j}{W_{n-1}}}.$$

Now, let us prove that inequality (20) holds for $k = n$, we have

$$\mathfrak{A}\left(\frac{1}{W_n}\sum_{j=1}^{n} \omega_j\, \partial_j\right) = \mathfrak{A}\left(\frac{W_{n-2}}{W_n}\frac{1}{W_{n-2}}\sum_{j=1}^{n-2} \omega_j\, \partial_j + \frac{\omega_{n-1}+\omega_n}{W_n}\left(\frac{\omega_{n-1}}{\omega_{n-1}+\omega_n}\partial_{n-1} + \frac{\omega_n}{\omega_{n-1}+\omega_n}\partial_n\right)\right).$$

Therefore, for every $\theta \in (0, 1]$, we have

$$\mathfrak{A}_*\left(\frac{1}{W_n}\sum_{j=1}^{n} \omega_j\, \partial_j, \theta\right)$$

$$\leq \mathfrak{A}_*\left(\frac{W_{n-2}}{W_n}\frac{1}{W_{n-2}}\sum_{j=1}^{n-2} \omega_j\, \partial_j + \frac{\omega_{n-1}+\omega_n}{W_n}\left(\frac{\omega_{n-1}}{\omega_{n-1}+\omega_n}\partial_{n-1} + \frac{\omega_n}{\omega_{n-1}+\omega_n}\partial_n,\ \theta\right)\right),$$

$$\leq \prod_{j=1}^{n-2}\left[\mathfrak{A}_*(\partial_j,\ \theta)\right]^{\frac{\omega_j}{W_n}}\left[\mathfrak{A}_*\left(\frac{\omega_{n-1}}{\omega_{n-1}+\omega_n}\partial_{n-1} + \frac{\omega_n}{\omega_{n-1}+\omega_n}\partial_n,\ \theta\right)\right]^{\frac{\omega_{n-1}+\omega_n}{W_n}},$$

$$\leq \prod_{j=1}^{n-2}\left[\mathfrak{A}_*\left(\partial_j,\ \theta\right)\right]^{\frac{\omega_j}{W_n}}\left[\ [\mathfrak{A}_*(\partial_{n-1},\ \theta)]^{\frac{\omega_{n-1}}{\omega_{n-1}+\omega_n}}\ [\mathfrak{A}_*(\partial_n,\ \theta)]^{\frac{\omega_n}{\omega_{n-1}+\omega_n}}\right]^{\frac{\omega_{n-1}+\omega_n}{W_n}},$$

$$\leq \prod_{j=1}^{n-2}\left[\mathfrak{A}_*\left(\partial_j,\ \theta\right)\right]^{\frac{\omega_j}{W_n}}[\mathfrak{A}_*(\partial_{n-1},\ \theta)]^{\frac{\omega_{n-1}}{W_n}}\ [\mathfrak{A}_*(\partial_n,\ \theta)]^{\frac{\omega_n}{W_n}},$$

$$= \prod_{j=1}^{n}\left[\mathfrak{A}_*\left(\partial_j,\ \theta\right)\right]^{\frac{\omega_j}{W_n}}.$$

Similarly, for $\mathfrak{A}^*(\partial,\ \theta)$, we have

$$\mathfrak{A}^*\left(\frac{1}{W_n}\sum_{j=1}^{n}\omega_j\,\partial_j,\theta\right) \leq \prod_{j=1}^{n}\left[\mathfrak{A}^*\left(\partial_j,\ \theta\right)\right]^{\frac{\omega_j}{W_n}}.$$

From which, we have

$$\left[\mathfrak{A}_*\left(\frac{1}{W_n}\sum_{j=1}^{n}\omega_j\,\partial_j,\theta\right),\mathfrak{A}^*\left(\frac{1}{W_n}\sum_{j=1}^{n}\omega_j\,\partial_j,\theta\right)\right] \leq_I \left[\prod_{j=1}^{n}\left[\mathfrak{A}_*\left(\partial_j,\ \theta\right)\right]^{\frac{\omega_j}{W_n}},\prod_{j=1}^{n}\left[\mathfrak{A}^*\left(\partial_j,\ \theta\right)\right]^{\frac{\omega_j}{W_n}}\right],$$

that is,

$$\mathfrak{A}\left(\frac{1}{W_n}\sum_{j=1}^{n}\omega_j\,\partial_j\right) \preccurlyeq \prod_{j=1}^{n}\left[\mathfrak{A}\left(\partial_j\right)\right]^{\frac{\omega_j}{W_n}},$$

and the result follows.

If $\omega_1 = \omega_2 = \omega_3 = \cdots = \omega_k = 1$, then Theorem 3.1 reduces to the following result:

Corollary 3.2. Let $\partial_j \in [\mu, \upsilon]$, $(j = 1, 2, 3, \ldots, k, k \geq 2)$ and $\mathfrak{A}: [\mu, \upsilon] \to \mathbb{F}_C(\mathbb{R})$ be a L-convex F-I-V-F, whose θ-cuts establish the series of I-V-Fs $\mathfrak{A}_\theta: [\mu, \upsilon] \subset \mathbb{R} \to {\mathcal{K}_C}^+$ are provided by $\mathfrak{A}_\theta(\partial) = [\mathfrak{A}_*(\partial, \theta), \mathfrak{A}^*(\partial, \theta)]$ for all $\partial \in [\mu, \upsilon]$ and for all $\theta \in (0, 1]$. Then,

$$\mathfrak{A}\left(\frac{1}{k}\sum_{j=1}^{k}\partial_j\right) \preccurlyeq \prod_{j=1}^{k}\left[\mathfrak{A}\left(\partial_j\right)\right]^{\frac{1}{k}}. \tag{21}$$

If $\mathfrak{A}$ is a L-concave then, inequality (21) is flipped.

Now we obtain Schur inequality for L-convex F-I-V-Fs.

Theorem 3.3. Let $\mathfrak{A}: [\mu, \upsilon] \to \mathbb{F}_C(\mathbb{R})$ be a F-I-V-F, whose θ-cuts establish the series of I-V-Fs $\mathfrak{A}_\theta: [\mu, \upsilon] \subset \mathbb{R} \to {\mathcal{K}_C}^+$ are provided by $\mathfrak{A}_\theta(\partial) = [\mathfrak{A}_*(\partial, \theta), \mathfrak{A}^*(\partial, \theta)]$ for all $\partial \in [\mu, \upsilon]$ and for all $\theta \in (0, 1]$. If $\mathfrak{A}$ be a L-convex F-I-V-F then, for $\partial_1, \partial_2, \partial_3 \in [\mu, \upsilon]$, $\partial_1 < \partial_2 < \partial_3$ such that $\partial_3 - \partial_1$, $\partial_3 - \partial_2, \partial_2 - \partial_1 \in [0, 1]$, we have

$$\mathfrak{A}(\partial_2)^{(\partial_3 - \partial_1)} \preccurlyeq \mathfrak{A}(\partial_1)^{\partial_3 - \partial_2}\mathfrak{A}(\partial_3)^{\partial_2 - \partial_1}. \tag{22}$$

If $\mathfrak{A}$ is a L-concave then, inequality (22) is flipped.

Proof. Let $\partial_1, \partial_2, \partial_3 \in [\mu, \upsilon]$ and $\partial_3 - \partial_1 > 0$. Taking $\lambda = \frac{\partial_3 - \partial_2}{\partial_3 - \partial_1}$, then $\partial_2 = \lambda\partial_1 + (1 - \lambda)\partial_3$. Since $\mathfrak{A}$ is a L-convex F-I-V-F then, by hypothesis, we have

$$\mathfrak{A}_*(\partial_2,\theta) \le [\mathfrak{A}_*(\partial_1,\theta)]^{\frac{\partial_3-\partial_2}{\partial_3-\partial_1}}[\mathfrak{A}_*(\partial_3,\theta)]^{\frac{\partial_2-\partial_1}{\partial_3-\partial_1}},$$
$$\mathfrak{A}^*(\partial_2,\theta) \le [\mathfrak{A}^*(\partial_1,\theta)]^{\frac{\partial_3-\partial_2}{\partial_3-\partial_1}}[\mathfrak{A}^*(\partial_3,\theta)]^{\frac{\partial_2-\partial_1}{\partial_3-\partial_1}}. \tag{23}$$

Taking "log" on the both sides of (23), we have

$$(\partial_3-\partial_1)ln\,\mathfrak{A}_*(\partial_2,\theta) \le (\partial_3-\partial_2)\,ln\,\mathfrak{A}_*(\partial_1,\theta)+(\partial_2-\partial_1)\,ln\,\mathfrak{A}_*(\partial_3,\theta),$$
$$(\partial_3-\partial_1)ln\,\mathfrak{A}^*(\partial_2,\theta) \le (\partial_3-\partial_2)ln\,\mathfrak{A}^*(\partial_1,\theta)+(\partial_2-\partial_1)\,ln\mathfrak{A}^*(\partial_3,\theta). \tag{24}$$

From (24), we have

$$\mathfrak{A}_*(\partial_2,\theta)^{(\partial_3-\partial_1)} \le [\mathfrak{A}_*(\partial_1,\theta)]^{(\partial_3-\partial_2)}[\mathfrak{A}_*(\partial_3,\theta)]^{(\partial_2-\partial_1)},$$
$$\mathfrak{A}^*(\partial_2,\theta)^{(\partial_3-\partial_1)} \le [\mathfrak{A}^*(\partial_1,\theta)]^{(\partial_3-\partial_2)}[\mathfrak{A}^*(\partial_3,\theta)]^{\partial_2-\partial_1}.$$

That is

$$\left[\mathfrak{A}_*(\partial_2,\theta)^{(\partial_3-\partial_1)},\ \mathfrak{A}^*(\partial_2,\theta)^{(\partial_3-\partial_1)}\right]$$
$$\le_I \left[[\mathfrak{A}_*(\partial_1,\theta)]^{(\partial_3-\partial_2)}[\mathfrak{A}_*(\partial_3,\theta)]^{(\partial_2-\partial_1)},\ [\mathfrak{A}^*(\partial_1,\theta)]^{(\partial_3-\partial_2)}[\mathfrak{A}^*(\partial_3,\theta)]^{(\partial_2-\partial_1)}\right],$$

Hence

$$\mathfrak{A}(\partial_2)^{(\partial_3-\partial_1)} \preccurlyeq \mathfrak{A}(\partial_1)^{(\partial_3-\partial_2)}\mathfrak{A}(\partial_3)^{(\partial_2-\partial_1)}.$$

Now, we obtain a refinement of Schur's inequality for L-convex F-I-V-F, which is provided in the following results.

Theorem 3.4. Let $\omega_j \in \mathbb{R}^+$, $\partial_j \in [\mu,\upsilon]$, $(j=1,2,3,\dots,k, k\ge 2)$ and $\mathfrak{A}:[\mu,\upsilon]\to\mathbb{F}_C(\mathbb{R})$ be a L-convex F-I-V-F, whose θ-cuts establish the series of I-V-Fs $\mathfrak{A}_\theta:[\mu,\upsilon]\subset\mathbb{R}\to{\mathcal{K}_C}^+$ are provided by $\mathfrak{A}_\theta(\partial)=[\mathfrak{A}_*(\partial,\theta),\mathfrak{A}^*(\partial,\theta)]$ for all $\partial\in[\mu,\upsilon]$ and for all $\theta\in(0,1]$. If $(L,U)\subseteq[\mu,\upsilon]$ then,

$$\prod_{j=1}^{k}[\mathfrak{A}(\partial_j)]^{\left(\frac{\omega_j}{W_k}\right)} \preccurlyeq \prod_{j=1}^{k}\left([\mathfrak{A}(L)]^{\left(\frac{U-\partial_j}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}[\mathfrak{A}(U)]^{\left(\frac{\partial_j-L}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}\right), \tag{25}$$

where $W_k=\sum_{j=1}^{k}\omega_j$. If $\mathfrak{A}$ is L-concave then, inequality (25) is flipped.

Proof. Consider $=\partial_1, \partial_j=\partial_2, (j=1,2,3,\dots,k)$, $U=\partial_3$ in inequality (23). Then, for each $\theta\in(0,1]$, then

$$\mathfrak{A}_*(\partial_j,\theta) \le [\mathfrak{A}_*(L,\theta)]^{\left(\frac{U-\partial_j}{U-L}\right)}[\mathfrak{A}_*(U,\theta)]^{\left(\frac{\partial_j-L}{U-L}\right)},$$
$$\mathfrak{A}^*(\partial_j,\theta) \le [\mathfrak{A}^*(L,\theta)]^{\left(\frac{U-\partial_j}{U-L}\right)}[\mathfrak{A}^*(U,\theta)]^{\left(\frac{\partial_j-L}{U-L}\right)}.$$

Above inequality can be written as,

$$\mathfrak{A}_*(\partial_j,\theta)^{\left(\frac{\omega_j}{W_k}\right)} \le [\mathfrak{A}_*(L,\theta)]^{\left(\frac{U-\partial_j}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}[\mathfrak{A}_*(U,\theta)]^{\left(\frac{\partial_j-L}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)},$$
$$\mathfrak{A}^*(\partial_j,\theta)^{\left(\frac{\omega_j}{W_k}\right)} \le [\mathfrak{A}^*(L,\theta)]^{\left(\frac{U-\partial_j}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}[\mathfrak{A}^*(U,\theta)]^{\left(\frac{\partial_j-L}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}. \tag{26}$$

Taking multiplication of all inequalities (26) for $j = 1, 2, 3, \ldots, k,$ we have

$$\prod_{j=1}^{k}\mathfrak{A}_*(\partial_j,\theta)^{\left(\frac{\omega_j}{W_k}\right)} \le \prod_{j=1}^{k}\left([\mathfrak{A}_*(L,\theta)]^{\left(\frac{U-\partial_j}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}[\mathfrak{A}_*(U,\theta)]^{\left(\frac{\partial_j-L}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}\right),$$
$$\prod_{j=1}^{k}\mathfrak{A}^*(\partial_j,\theta)^{\left(\frac{\omega_j}{W_k}\right)} \le \prod_{j=1}^{k}\left([\mathfrak{A}^*(L,\theta)]^{\left(\frac{U-\partial_j}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}[\mathfrak{A}^*(U,\theta)]^{\left(\frac{\partial_j-L}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}\right),$$

that is

$$\prod_{j=1}^{k}\mathfrak{A}_\theta(\partial_j)^{\left(\frac{\omega_j}{W_k}\right)} = \left[\prod_{j=1}^{k}\mathfrak{A}_*(\partial_j,\theta)^{\left(\frac{\omega_j}{W_k}\right)},\ \prod_{j=1}^{k}\mathfrak{A}^*(\partial_j,\theta)^{\left(\frac{\omega_j}{W_k}\right)}\right]$$

$$\le_I \left[\prod_{j=1}^{k}\left([\mathfrak{A}_*(L,\theta)]^{\left(\frac{U-\partial_j}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}[\mathfrak{A}_*(U,\theta)]^{\left(\frac{\partial_j-L}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}\right),\right.$$

$$\left.\prod_{j=1}^{k}\left([\mathfrak{A}^*(L,\theta)]^{\left(\frac{U-\partial_j}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}[\mathfrak{A}^*(U,\theta)]^{\left(\frac{\partial_j-L}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}\right)\right],$$

$$\le_I \prod_{j=1}^{k}\left(\left[[\mathfrak{A}_*(L,\theta)]^{\left(\frac{U-\partial_j}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)},\ [\mathfrak{A}^*(L,\theta)]^{\left(\frac{U-\partial_j}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}\right]\right).\prod_{j=1}^{k}\left(\left[[\mathfrak{A}_*(U,\theta)]^{\left(\frac{\partial_j-L}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)},\right.\right.$$

$$\left.\left.[\mathfrak{A}^*(U,\theta)]^{\left(\frac{\partial_j-L}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}\right]\right),$$

$$= \prod_{j=1}^{k}[\mathfrak{A}_\theta(L)]^{\left(\frac{U-\partial_j}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}.\prod_{j=1}^{k}[\mathfrak{A}_\theta(U)]^{\left(\frac{\partial_j-L}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}.$$

Thus,

$$\prod_{j=1}^{k}[\mathfrak{A}(\partial_j)]^{\left(\frac{\omega_j}{W_k}\right)} \preccurlyeq \prod_{j=1}^{k}\left([\mathfrak{A}(L)]^{\left(\frac{U-\partial_j}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}[\mathfrak{A}(U)]^{\left(\frac{\partial_j-L}{U-L}\right)\left(\frac{\omega_j}{W_k}\right)}\right),$$

this completes the proof.

Remark 3.5. If $\mathfrak{A}_*(\partial,\theta) = \mathfrak{A}^*(\partial,\theta)$ with $\theta = 1$, then Theorem 3.1, Theorem 3.3 and Theorem 3.4 reduce to the result for convex function, see [21].

4. Hermite-Hadamard type inequalities

Now, we will establish some integral inequalities of H-H type for L-convex F-I-V-F using fuzzy order relation.

Theorem 4.1. Let $\mathfrak{A}: [\mu, \upsilon] \to \mathbb{F}_C(\mathbb{R})$ be a L-convex F-I-V-F, whose θ-cuts establish the series of I-V-Fs $\mathfrak{A}_\theta: [\mu, \upsilon] \subset \mathbb{R} \to \mathcal{K}_C^{\ +}$ are provided by $\mathfrak{A}_\theta(\partial) = [\mathfrak{A}_*(\partial, \theta), \mathfrak{A}^*(\partial, \theta)]$ for all $\partial \in [\mu, \upsilon]$ and for all $\theta \in (0, 1]$. If $\mathfrak{A} \in \mathcal{FR}_{([\mu,\upsilon],\theta)}$, then

$$\mathfrak{A}\left(\frac{\mu+\upsilon}{2}\right) \preccurlyeq exp\left[\frac{1}{\upsilon-\mu}\ (FR) \int_\mu^\upsilon ln\ \mathfrak{A}(\partial) d\partial\right] \preccurlyeq \sqrt{\mathfrak{A}(\mu)\ \tilde{\times}\ \mathfrak{A}(\upsilon)}. \tag{27}$$

If $\mathfrak{A}$ is L-concave then, inequality (27) is flipped.

Proof. Let $\mathfrak{A}: [\mu, \upsilon] \to \mathbb{F}_C(\mathbb{R})$, L-convex F-I-V-F. Then, by hypothesis, we have

$$\mathfrak{A}\left(\frac{\mu+\upsilon}{2}\right) \preccurlyeq [\mathfrak{A}(\varsigma\mu + (1-\varsigma)\upsilon)]^{\frac{1}{2}}\ \tilde{\times}\ \left[\mathfrak{A}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\right]^{\frac{1}{2}}.$$

Therefore, for every $\theta \in (0, 1]$, we have

$$\begin{aligned}\mathfrak{A}_*\left(\frac{\mu+\upsilon}{2}, \theta\right) &\le [\mathfrak{A}_*(\varsigma\mu + (1-\varsigma)\upsilon, \theta)]^{\frac{1}{2}} \times \left[\mathfrak{A}_*\big((1-\varsigma)\mu + \varsigma\upsilon, \theta\big)\right]^{\frac{1}{2}},\\ \mathfrak{A}^*\left(\frac{\mu+\upsilon}{2}, \theta\right) &\le [\mathfrak{A}^*(\varsigma\mu + (1-\varsigma)\upsilon, \theta)]^{\frac{1}{2}} \times \left[\mathfrak{A}^*\big((1-\varsigma)\mu + \varsigma\upsilon, \theta\big)\right]^{\frac{1}{2}}.\end{aligned} \tag{28}$$

Taking logarithms on both sides of (28), then we obtain

$$\begin{aligned}2\ ln\ \mathfrak{A}_*\left(\frac{\mu+\upsilon}{2}, \theta\right) &\le ln\ \mathfrak{A}_*(\varsigma\mu + (1-\varsigma)\upsilon, \theta) + ln\ \mathfrak{A}_*\big((1-\varsigma)\mu + \varsigma\upsilon, \theta\big),\\ 2\ ln\ \mathfrak{A}^*\left(\frac{\mu+\upsilon}{2}, \theta\right) &\le ln\ \mathfrak{A}^*(\varsigma\mu + (1-\varsigma)\upsilon, \theta) + ln\ \mathfrak{A}^*\big((1-\varsigma)\mu + \varsigma\upsilon, \theta\big).\end{aligned}$$

Then,

$$\begin{aligned}2\int_0^1 ln\ \mathfrak{A}_*\left(\frac{\mu+\upsilon}{2}, \theta\right) d\varsigma &\le \int_0^1 ln\ \mathfrak{A}_*(\varsigma\mu + (1-\varsigma)\upsilon, \theta) d\varsigma + \int_0^1 ln\ \mathfrak{A}_*\big((1-\varsigma)\mu + \varsigma\upsilon, \theta\big) d\varsigma,\\ 2\int_0^1 ln\ \mathfrak{A}^*\left(\frac{\mu+\upsilon}{2}, \theta\right) d\varsigma &\le \int_0^1 ln\ \mathfrak{A}^*(\varsigma\mu + (1-\varsigma)\upsilon, \theta) d\varsigma + \int_0^1 ln\ \mathfrak{A}^*\big((1-\varsigma)\mu + \varsigma\upsilon, \theta\big) d\varsigma.\end{aligned}$$

It follows that

$$\begin{aligned}ln\ \mathfrak{A}_*\left(\frac{\mu+\upsilon}{2}, \theta\right) &\le \frac{1}{\upsilon-\mu} \int_\mu^\upsilon ln\ \mathfrak{A}_*(\partial, \theta) d\partial,\\ ln\ \mathfrak{A}^*\left(\frac{\mu+\upsilon}{2}, \theta\right) &\le \frac{1}{\upsilon-\mu} \int_\mu^\upsilon ln\ \mathfrak{A}^*(\partial, \theta) d\partial,\end{aligned}$$

which implies that

$$\begin{aligned}\mathfrak{A}_*\left(\frac{\mu+\upsilon}{2}, \theta\right) &\le exp\left(\frac{1}{\upsilon-\mu} \int_\mu^\upsilon ln\ \mathfrak{A}_*(\partial, \theta) d\partial\right),\\ \mathfrak{A}^*\left(\frac{\mu+\upsilon}{2}, \theta\right) &\le exp\left(\frac{1}{\upsilon-\mu} \int_\mu^\upsilon ln\ \mathfrak{A}^*(\partial, \theta) d\partial\right).\end{aligned}$$

That is

$$\left[\mathfrak{A}_*\left(\frac{\mu+\upsilon}{2}, \theta\right), \mathfrak{A}^*\left(\frac{\mu+\upsilon}{2}, \theta\right)\right] \le_I \left[exp\left(\frac{1}{\upsilon-\mu} \int_\mu^\upsilon ln\ \mathfrak{A}_*(\partial, \theta) d\partial\right), exp\left(\frac{1}{\upsilon-\mu} \int_\mu^\upsilon ln\ \mathfrak{A}^*(\partial, \theta) d\partial\right)\right].$$

Thus,

$$\mathfrak{A}\left(\frac{\mu+\upsilon}{2}\right) \preccurlyeq exp\left[\frac{1}{\upsilon-\mu}\ (FR)\int_{\mu}^{\upsilon} ln\ \mathfrak{A}(\partial)d\partial\right]. \tag{29}$$

In a similar way as above, we have

$$exp\left[\frac{1}{\upsilon-\mu}\ (FR)\int_{\mu}^{\upsilon} ln\ \mathfrak{A}(\partial)d\partial\right] \preccurlyeq \sqrt{\mathfrak{A}(\mu)\ \tilde{\times}\ \mathfrak{A}(\upsilon)}. \tag{30}$$

Combining (29) and (30), we have

$$\mathfrak{A}\left(\frac{\mu+\upsilon}{2}\right) \preccurlyeq exp\left[\frac{1}{\upsilon-\mu}\ (FR)\int_{\mu}^{\upsilon} ln\ \mathfrak{A}(\partial)d\partial\right] \preccurlyeq \sqrt{\mathfrak{A}(\mu)\ \tilde{\times}\ \mathfrak{A}(\upsilon)},$$

the required result.

Remark 4.2. If $\mathfrak{A}_*(\partial,\theta) = \mathfrak{A}^*(\partial,\theta)$ with $\theta = 1$, then Theorem 4.1 reduces to the result for L-convex function see [16]:

$$\mathfrak{A}\left(\frac{\mu+\upsilon}{2}\right) \leq exp\left[\frac{1}{\upsilon-\mu}\ (R)\int_{\mu}^{\upsilon} ln\ \mathfrak{A}(\partial)d\partial\right] \leq \sqrt{\mathfrak{A}(\mu)\ \times\ \mathfrak{A}(\upsilon)}.$$

Example 4.3. We consider the F-I-V-F $\mathfrak{A}: [1,4] \to \mathbb{F}_C(\mathbb{R})$ established by,

$$\mathfrak{A}(\partial)(\mathrm{s}) = \begin{cases} \dfrac{\mathrm{s}}{e^{\partial^2}}, & \mathrm{s} \in \left[0, e^{\partial^2}\right], \\ \dfrac{2e^{\partial^2}-\mathrm{s}}{e^{\partial^2}}, & \mathrm{s} \in \left(e^{\partial^2}, 2e^{\partial^2}\right], \\ 0, & oterwise, \end{cases}$$

Then, for each $\theta \in (0,1]$, we have $\mathfrak{A}_\theta(\partial) = \left[\theta e^{\partial^2},\ (2-\theta)e^{\partial^2}\right]$. Since end point functions $\mathfrak{A}_*(\partial,\theta), \mathfrak{A}^*(\partial,\theta)$ are L-convex functions for each $\theta \in (0,1]$ then, by Theorem 2.16, $\mathfrak{A}(\partial)$ is L-convex F-I-V-F. Since, $\mathfrak{A}_*(\partial,\theta) = \theta e^{\partial^2}$ and $\mathfrak{A}^*(\partial,\theta) = (2-\theta)e^{\partial^2}$ then, we have

$$\mathfrak{A}_*\left(\frac{\mu+\upsilon}{2},\theta\right) = \theta e^{\left(\frac{5}{2}\right)^2} = \theta e^{\frac{25}{4}},$$

$$exp\left(\frac{1}{\upsilon-\mu}\int_{\mu}^{\upsilon} ln\ \mathfrak{A}_*(\partial,\theta)d\partial\right) = exp\left(\frac{1}{3}\int_{1}^{4} ln\left(\theta e^{\partial^2}\right)d\partial\right) = e^{ln(\theta)+7},$$

$$\sqrt{\mathfrak{A}_*(\mu)\ \times\ \mathfrak{A}_*(\upsilon)} = \left[(\theta e)(4\theta e^{16})\right]^{\frac{1}{2}} = 2\ \theta e^{\frac{17}{2}},$$

for all $\theta \in (0,1]$. That means $\theta e^{\frac{25}{4}} \leq e^{ln(\theta)+7} \leq 2\theta e^{\frac{17}{2}}$.

Similarly, it can be easily show that

$$\mathfrak{A}^*\left(\frac{\mu+\upsilon}{2},\theta\right) \leq exp\left[\frac{1}{\upsilon-\mu}\int_{\mu}^{\upsilon} ln\ \mathfrak{A}^*(\partial,\theta)d\partial\right] \leq \sqrt{\mathfrak{A}^*(\mu,\theta) \times\ \mathfrak{A}^*(\upsilon,\theta)},$$

for all $\theta \in (0,1]$, such that

$$\mathfrak{A}^*\left(\frac{\mu+\upsilon}{2},\theta\right) = (2-\theta)e^{\left(\frac{5}{2}\right)^2} = (2-\theta)e^{\frac{25}{4}},$$

$$exp\left(\frac{1}{\upsilon-\mu}\int_\mu^\upsilon ln\, \mathfrak{A}^*(\partial,\theta)d\partial\right) = exp\left(\frac{1}{3}\int_1^4 ln\left((2-\theta)e^{\partial^2}\right)d\partial\right) = e^{ln(2-\theta)+7},$$

$$\sqrt{\mathfrak{A}^*(\mu,\theta)\times\mathfrak{A}^*(\upsilon,\theta)} = [(2-\theta)e.4(2-\theta)e^{16}]^{\frac{1}{2}} = 2(2-\theta)e^{\frac{17}{2}}.$$

From which, it follows that

$$(2-\theta)e^{\frac{25}{4}} \le e^{ln(2-\theta)+7} \le 2\,(2-\theta)e^{\frac{17}{2}},$$

that is

$$\left[\theta e^{\frac{25}{4}},(2-\theta)e^{\frac{25}{4}}\right] \le_I \left[e^{ln(\theta)+7},\ e^{ln(2-\theta)+7}\right] \le_I \left[2\theta e^{\frac{17}{2}},2(2-\theta)e^{\frac{1}{2}}\right], \text{ for all } \theta\in(0,1].$$

Hence, Theorem 4.1 is verified.

To obtain H-H Fejér inequality for L-convex F-I-V-F, firstly, we give the following results connected with the right part of (3).

Theorem 4.4. Let $\mathfrak{A}:[\mu,\upsilon]\to\mathbb{F}_C(\mathbb{R})$ be a L-convex F-I-V-F with $\mu<\upsilon$, whose θ-cuts establish the series of I-V-Fs $\mathfrak{A}_\theta:[\mu,\upsilon]\subset\mathbb{R}\to{\mathcal{K}_C}^+$ are provided by $\mathfrak{A}_\theta(\partial)=[\mathfrak{A}_*(\partial,\theta),\mathfrak{A}^*(\partial,\theta)]$ for all $\partial\in[\mu,\upsilon]$ and for all $\theta\in(0,1]$. If $\mathfrak{A}\in\mathcal{FR}_{([\mu,\upsilon],\theta)}$ and $\mathfrak{Q}:[\mu,\upsilon]\to\mathbb{R}, \mathfrak{Q}(\partial)\ge 0$, symmetric with respect to $\frac{\mu+\upsilon}{2}$, then

$$\frac{1}{\upsilon-\mu}\,(FR)\int_\mu^\upsilon[ln\,\mathfrak{A}(\partial)]\mathfrak{Q}(\partial)d\partial \preccurlyeq ln\,[\mathfrak{A}(\mu)\ \tilde{\times}\ \mathfrak{A}(\upsilon)]\int_0^1\varsigma\mathfrak{Q}\big((1-\varsigma)\mu+\varsigma\upsilon\big)d\varsigma. \qquad (31)$$

If $\mathfrak{A}$ is L-concave then, inequality (31) is flipped.

Proof. Let $\mathfrak{A}$ be a L-convex F-I-V-F. Then, for each $\theta\in(0,1]$, we have

$$\begin{aligned}&[ln\,\mathfrak{A}_*(\varsigma\mu+(1-\varsigma)\upsilon,\theta)]\mathfrak{Q}(\varsigma\mu+(1-\varsigma)\upsilon)\\&\quad\le\big(\varsigma ln\,\mathfrak{A}_*(\mu,\theta)+(1-\varsigma)ln\,\mathfrak{A}_*(\upsilon,\theta)\big)\mathfrak{Q}(\varsigma\mu+(1-\varsigma)\upsilon),\\&[ln\,\mathfrak{A}^*(\varsigma\mu+(1-\varsigma)\upsilon,\theta)]\mathfrak{Q}(\varsigma\mu+(1-\varsigma)\upsilon)\\&\quad\le\big(\varsigma ln\,\mathfrak{A}^*(\mu,\theta)+(1-\varsigma)ln\,\mathfrak{A}^*(\upsilon,\theta)\big)\mathfrak{Q}(\varsigma\mu+(1-\varsigma)\upsilon).\end{aligned} \qquad (32)$$

And

$$\begin{aligned}&\big[ln\,\mathfrak{A}_*\big((1-\varsigma)\mu+\varsigma\upsilon,\theta\big)\big]\mathfrak{Q}\big((1-\varsigma)\mu+\varsigma\upsilon\big)\\&\quad\le\big((1-\varsigma)ln\,\mathfrak{A}_*(\mu,\theta)+\varsigma ln\,\mathfrak{A}_*(\upsilon,\theta)\big)\mathfrak{Q}\big((1-\varsigma)\mu+\varsigma\upsilon\big),\\&\big[ln\,\mathfrak{A}^*\big((1-\varsigma)\mu+\varsigma\upsilon,\theta\big)\big]\mathfrak{Q}\big((1-\varsigma)\mu+\varsigma\upsilon\big)\\&\quad\le\big((1-\varsigma)ln\,\mathfrak{A}^*(\mu,\theta)+\varsigma ln\,\mathfrak{A}^*(\upsilon,\theta)\big)\mathfrak{Q}\big((1-\varsigma)\mu+\varsigma\upsilon\big).\end{aligned} \qquad (33)$$

After adding (32) and (33), and then integrating over $(0,1)$, we get

$$\int_0^1 [\ln \mathfrak{A}_*(\varsigma\mu + (1-\varsigma)\upsilon, \theta)]\mathfrak{Q}(\varsigma\mu + (1-\varsigma)\upsilon)\, d\varsigma$$
$$+ \int_0^1 \ln \mathfrak{A}_*\big((1-\varsigma)\mu + \varsigma\upsilon, \theta\big)\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\, d\varsigma$$
$$\leq \int_0^1 \begin{bmatrix} \ln \mathfrak{A}_*(\mu,\theta)\{\varsigma\mathfrak{Q}(\varsigma\mu + (1-\varsigma)\upsilon) + (1-\varsigma)\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\} \\ + \ln \mathfrak{A}_*(\upsilon,\theta)\{(1-\varsigma)\mathfrak{Q}(\varsigma\mu + (1-\varsigma)\upsilon) + \varsigma\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\} \end{bmatrix} d\varsigma,$$
$$\int_0^1 \big[\ln \mathfrak{A}^*\big((1-\varsigma)\mu + \varsigma\upsilon, \theta\big)\big]\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\, d\varsigma$$
$$+ \int_0^1 \ln \mathfrak{A}^*(\varsigma\mu + (1-\varsigma)\upsilon, \theta)\mathfrak{Q}(\varsigma\mu + (1-\varsigma)\upsilon)\, d\varsigma$$
$$\leq \int_0^1 \begin{bmatrix} \ln \mathfrak{A}^*(\mu,\theta)\{\varsigma\mathfrak{Q}(\varsigma\mu + (1-\varsigma)\upsilon) + (1-\varsigma)\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\} \\ + \ln \mathfrak{A}^*(\upsilon,\theta)\{(1-\varsigma)\mathfrak{Q}(\varsigma\mu + (1-\varsigma)\upsilon) + \varsigma\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\} \end{bmatrix} d\varsigma$$

$$= 2\ln \mathfrak{A}_*(\mu,\theta) \int_0^1 \varsigma\mathfrak{Q}(\varsigma\mu + (1-\varsigma)\upsilon)\, d\varsigma + 2\ln \mathfrak{A}_*(\upsilon,\theta) \int_0^1 \varsigma\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\, d\varsigma,$$
$$= 2\ln \mathfrak{A}^*(\mu,\theta) \int_0^1 \varsigma\mathfrak{Q}(\varsigma\mu + (1-\varsigma)\upsilon)\, d\varsigma + 2\ln \mathfrak{A}^*(\upsilon,\theta) \int_0^1 \varsigma\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\, d\varsigma.$$

Since $\mathfrak{Q}$ is symmetric, then

$$\begin{aligned} &= 2\ln[\mathfrak{A}_*(\mu,\theta) \times \mathfrak{A}_*(\upsilon,\theta)] \int_0^1 \varsigma\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\, d\varsigma, \\ &= 2\ln[\mathfrak{A}^*(\mu,\theta) \times \mathfrak{A}^*(\upsilon,\theta)] \int_0^1 \varsigma\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\, d\varsigma. \end{aligned} \tag{34}$$

Since

$$\begin{aligned} &\int_0^1 [\ln \mathfrak{A}_*(\varsigma\mu + (1-\varsigma)\upsilon, \theta)]\mathfrak{Q}(\varsigma\mu + (1-\varsigma)\upsilon)\, d\varsigma \\ &\quad = \int_0^1 \big[\ln \mathfrak{A}_*\big((1-\varsigma)\mu + \varsigma\upsilon, \theta\big)\big]\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\, d\varsigma \\ &\quad = \frac{1}{\upsilon-\mu} \int_\mu^\upsilon [\ln \mathfrak{A}_*(\partial,\theta)]\mathfrak{Q}(\partial) d\partial, \\ &\int_0^1 \big[\ln \mathfrak{A}^*\big((1-\varsigma)\mu + \varsigma\upsilon, \theta\big)\big]\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\, d\varsigma \\ &\quad = \int_0^1 [\ln \mathfrak{A}^*(\varsigma\mu + (1-\varsigma)\upsilon, \theta)]\mathfrak{Q}(\varsigma\mu + (1-\varsigma)\upsilon)\, d\varsigma \\ &\quad = \frac{1}{\upsilon-\mu} \int_\mu^\upsilon [\ln \mathfrak{A}^*(\partial,\theta)]\mathfrak{Q}(\partial) d\partial. \end{aligned} \tag{35}$$

From (34) and (35), we have

$$\frac{1}{\upsilon-\mu} \int_\mu^\upsilon [\ln \mathfrak{A}_*(\partial,\theta)]\mathfrak{Q}(\partial) d\partial \leq \ln[\mathfrak{A}_*(\mu,\theta) \times \mathfrak{A}_*(\upsilon,\theta)] \int_0^1 \varsigma\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\, d\varsigma,$$
$$\frac{1}{\upsilon-\mu} \int_\mu^\upsilon [\ln \mathfrak{A}^*(\partial,\theta)]\mathfrak{Q}(\partial) d\partial \leq \ln[\mathfrak{A}^*(\mu,\theta) \times \mathfrak{A}^*(\upsilon,\theta)] \int_0^1 \varsigma\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\, d\varsigma,$$

that is

$$\left[\frac{1}{\upsilon-\mu} \int_\mu^\upsilon [\ln \mathfrak{A}_*(\partial,\theta)]\mathfrak{Q}(\partial) d\partial, \frac{1}{\upsilon-\mu} \int_\mu^\upsilon [\ln \mathfrak{A}^*(\partial,\theta)]\mathfrak{Q}(\partial) d\partial\right]$$

$$\leq_I \big[\ln\, [\mathfrak{A}_*(\mu,\theta) \times \mathfrak{A}_*(\upsilon,\theta)],\ \ln\, [\mathfrak{A}^*(\mu,\theta) \times \mathfrak{A}^*(\upsilon,\theta)]\big] \int_0^1 \varsigma\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\, d\varsigma,$$

hence

$$\frac{1}{\upsilon-\mu}\ (FR)\int_{\mu}^{\upsilon}[\ln \mathfrak{A}(\partial)]\mathfrak{Q}(\partial)d\partial \preccurlyeq \ln[\mathfrak{A}(\mu)\ \tilde{\times}\ \mathfrak{A}(\upsilon)]\int_0^1 \varsigma\mathfrak{Q}\big((1-\varsigma)\mu+\varsigma\upsilon\big)d\varsigma.$$

This concludes the proof.

Now, we give the following result connected with the left part of (3) for L-convex F-I-V-F using fuzzy order relation.

Theorem 4.5. Let $\mathfrak{A}: [\mu, \upsilon] \to \mathbb{F}_C(\mathbb{R})$ be a L-convex F·-I·-V·-F with $\mu < \upsilon$, whose θ-cuts establish the series of I·-V·-Fs $\mathfrak{A}_\theta: [\mu, \upsilon] \subset \mathbb{R} \to {\mathcal{K}_C}^+$ are provided by $\mathfrak{A}_\theta(\partial) = [\mathfrak{A}_*(\partial, \theta), \mathfrak{A}^*(\partial, \theta)]$ for all $\partial \in [\mu, \upsilon]$ and for all $\theta \in (0, 1]$. If $\mathfrak{A} \in \mathcal{FR}_{([\mu,\upsilon],\theta)}$ and $\mathfrak{Q}: [\mu, \upsilon] \to \mathbb{R}, \mathfrak{Q}(\partial) \geq 0$, symmetric with respect to $\frac{\mu+\upsilon}{2}$, and $\int_\mu^\upsilon \mathfrak{Q}(\partial)d\partial > 0$, then

$$\ln \mathfrak{A}\left(\frac{\mu+\upsilon}{2}\right) \preccurlyeq \frac{1}{\int_\mu^\upsilon \mathfrak{Q}(\partial)d\partial}\ (FR)\int_\mu^\upsilon [\ln \mathfrak{A}(\partial)]\mathfrak{Q}(\partial)d\partial. \tag{36}$$

If $\mathfrak{A}$ is a L-concave then, inequality (36) is flipped.

Proof. Since $\mathfrak{A}$ is a L-convex then, for $\theta \in (0, 1]$ we have

$$\begin{aligned} 2\ln \mathfrak{A}_*\left(\frac{\mu+\upsilon}{2}, \theta\right) &\leq \ln \mathfrak{A}_*(\varsigma\mu + (1-\varsigma)\upsilon, \theta) + \ln \mathfrak{A}_*\big((1-\varsigma)\mu + \varsigma\upsilon, \theta\big), \\ 2\ln \mathfrak{A}^*\left(\frac{\mu+\upsilon}{2}, \theta\right) &\leq \ln \mathfrak{A}^*(\varsigma\mu + (1-\varsigma)\upsilon, \theta) + \ln \mathfrak{A}^*\big((1-\varsigma)\mu + \varsigma\upsilon, \theta\big). \end{aligned} \tag{37}$$

By multiplying (37) by $\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big) = \mathfrak{Q}(\varsigma\mu + (1-\varsigma)\upsilon)$ and integrate it by ς over $[0, 1]$, we obtain

$$\begin{aligned} &2\left[\ln \mathfrak{A}_*\left(\frac{\mu+\upsilon}{2}, \theta\right)\right]\int_0^1 \mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)d\varsigma \\ &\qquad \leq \int_0^1 [\ln \mathfrak{A}_*(\varsigma\mu + (1-\varsigma)\upsilon, \theta)]\mathfrak{Q}(\varsigma\mu + (1-\varsigma)\upsilon)d\varsigma \\ &\qquad\quad + \int_0^1 \big[\ln \mathfrak{A}_*\big((1-\varsigma)\mu + \varsigma\upsilon, \theta\big)\big]\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)d\varsigma, \\ &2\left[\ln \mathfrak{A}^*\left(\frac{\mu+\upsilon}{2}, \theta\right)\right]\int_0^1 \mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)d\varsigma \\ &\qquad \leq \int_0^1 [\ln \mathfrak{A}^*(\varsigma\mu + (1-\varsigma)\upsilon, \theta)]\mathfrak{Q}(\varsigma\mu + (1-\varsigma)\upsilon)d\varsigma \\ &\qquad\quad + \int_0^1 \big[\ln \mathfrak{A}^*\big((1-\varsigma)\mu + \varsigma\upsilon, \theta\big)\big]\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)d\varsigma. \end{aligned} \tag{38}$$

Since

$$\begin{aligned} &\int_0^1 [\ln \mathfrak{A}_*(\varsigma\mu + (1-\varsigma)\upsilon, \theta)]\mathfrak{Q}(\varsigma\mu + (1-\varsigma)\upsilon)\,d\varsigma \\ &\qquad = \int_0^1 \big[\ln \mathfrak{A}_*\big((1-\varsigma)\mu + \varsigma\upsilon, \theta\big)\big]\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\,d\varsigma, \\ &\qquad = \frac{1}{\upsilon-\mu}\int_\mu^\upsilon [\ln \mathfrak{A}_*(\partial, \theta)]\mathfrak{Q}(\partial)d\partial, \\ &\int_0^1 [\ln \mathfrak{A}^*(\varsigma\mu + (1-\varsigma)\upsilon, \theta)]\mathfrak{Q}(\varsigma\mu + (1-\varsigma)\upsilon)\,d\varsigma \\ &\qquad = \int_0^1 \big[\ln \mathfrak{A}^*\big((1-\varsigma)\mu + \varsigma\upsilon, \theta\big)\big]\mathfrak{Q}\big((1-\varsigma)\mu + \varsigma\upsilon\big)\,d\varsigma, \\ &\qquad = \frac{1}{\upsilon-\mu}\int_\mu^\upsilon [\ln \mathfrak{A}^*(\partial, \theta)]\mathfrak{Q}(\partial)d\partial. \end{aligned} \tag{39}$$

From (38) and (39), we have

$$ln\,\mathfrak{A}_*\left(\frac{\mu+\upsilon}{2},\theta\right) \le \frac{1}{\int_\mu^\upsilon \mathfrak{Q}(\partial)d\partial}\int_\mu^\upsilon [ln\,\mathfrak{A}_*(\partial,\theta)]\mathfrak{Q}(\partial)d\partial\,,$$
$$ln\,\mathfrak{A}^*\left(\frac{\mu+\upsilon}{2},\theta\right) \le \frac{1}{\int_\mu^\upsilon \mathfrak{Q}(\partial)d\partial}\int_\mu^\upsilon [ln\,\mathfrak{A}^*(\partial,\theta)]\mathfrak{Q}(\partial)d\partial\,.$$

From which, we have

$$\left[ln\,\mathfrak{A}_*\left(\frac{\mu+\upsilon}{2},\theta\right), ln\,\mathfrak{A}^*\left(\frac{\mu+\upsilon}{2},\theta\right)\right]$$
$$\le_I \frac{1}{\int_\mu^\upsilon \mathfrak{Q}(\partial)d\partial}\left[\int_\mu^\upsilon [ln\,\mathfrak{A}_*(\partial,\theta)]\mathfrak{Q}(\partial)d\partial,\ \int_\mu^\upsilon [ln\,\mathfrak{A}^*(\partial,\theta)]\mathfrak{Q}(\partial)d\partial\right],$$

that is

$$ln\,\mathfrak{A}\left(\frac{\mu+\upsilon}{2}\right) \preccurlyeq \frac{1}{\int_\mu^\upsilon \mathfrak{Q}(\partial)d\partial}\ (FR)\int_\mu^\upsilon [ln\,\mathfrak{A}(\partial)]\mathfrak{Q}(\partial)d\partial.$$

Then we complete the proof.

Remark 4.6. If $\mathfrak{A}_*(\mu,\theta) = \mathfrak{A}^*(\mu,\theta)$ with $\theta = 1$, then Theorem 4.4 and Theorem 4.5 reduces to classical first and second H-H Fejér inequality for L-convex function.

Example 4.7. We consider the F-I-V-F $\mathfrak{A}: [\mu,\upsilon] = [\frac{\pi}{4},\frac{\pi}{2}] \to \mathbb{F}_C(\mathbb{R})$ established by,

$$\mathfrak{A}(\partial)(\mathrm{s}) = \begin{cases} \frac{\mathrm{s}}{e^{\sin(\partial)}}, & \mathrm{s} \in \left[0, e^{\sin(\partial)}\right], \\ \frac{2e^{\sin(\partial)}-\mathrm{s}}{e^{\sin(\partial)}}, & \mathrm{s} \in \left(e^{\sin(\partial)}, 2e^{\sin(\partial)}\right], \\ 0, & oterwise. \end{cases}$$

Then, for each $\theta \in (0,1]$, we have

$$\mathfrak{A}_\theta(\partial) = \left[\theta e^{\sin(\partial)}, (2-\theta)e^{\sin(\partial)}\right].$$

Since end point functions $\mathfrak{A}_*(\partial,\theta) = \theta e^{\sin(\partial)}$, $\mathfrak{A}^*(\partial,\theta) = (2-\theta)e^{\sin(\partial)}$ are L-convex functions, for each $\theta \in (0,1]$ then, by Theorem 2.16, $\mathfrak{A}(\partial)$ is L-convex F-I-V-F. If

$$\mathfrak{Q}(\partial) = \begin{cases} \partial - \frac{\pi}{4}, & \mathrm{s} \in \left[\frac{\pi}{4},\frac{3\pi}{8}\right], \\ \frac{\pi}{2} - \partial, & \mathrm{s} \in \left(\frac{3\pi}{8},\frac{\pi}{2}\right]. \end{cases}$$

Then, we have

$$\frac{1}{\upsilon-\mu}\int_{\mu}^{\upsilon}[ln\,\mathfrak{A}_*(\partial,\theta)]\mathfrak{Q}(\partial)d\partial = \frac{4}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}[ln\,\mathfrak{A}_*(\partial,\theta)]\mathfrak{Q}(\partial)d\partial$$
$$= \frac{4}{\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{8}}[ln\,\mathfrak{A}_*(\partial,\theta)]\mathfrak{Q}(\partial)d\partial + \frac{4}{\pi}\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}}ln\,\mathfrak{A}_*(\partial,\theta)\mathfrak{Q}(\partial)d\partial\,,$$
$$\frac{1}{\upsilon-\mu}\int_{\mu}^{\upsilon}[ln\,\mathfrak{A}^*(\partial,\theta)]\mathfrak{Q}(\partial)d\partial = \frac{4}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}[ln\,\mathfrak{A}^*(\partial,\theta)]\mathfrak{Q}(\partial)d\partial$$
$$= \frac{4}{\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{8}}[ln\,\mathfrak{A}^*(\partial,\theta)]\mathfrak{Q}(\partial)d\partial + \frac{4}{\pi}\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}}ln\,\mathfrak{A}^*(\partial,\theta)\mathfrak{Q}(\partial)d\partial\,,$$

$$\begin{aligned} &= \frac{4}{\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{8}}\left[ln\left(\theta e^{\sin(\partial)}\right)\right]\left(\partial-\frac{\pi}{4}\right)d\partial + \frac{4}{\pi}\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}}\left[ln\left(\theta e^{\sin(\partial)}\right)\right]\left(\frac{\pi}{2}-\partial\right)d\partial \\ &\approx \frac{1}{25\pi}\left[\frac{31}{2}ln(\theta)+14\right], \\ &= \frac{4}{\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{8}}\left[ln(2-\theta)e^{\sin(\partial)}\right]\left(\partial-\frac{\pi}{2}\right)d\partial + \frac{4}{\pi}\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}}\left[ln\,\left((2-\theta)e^{\sin(\partial)}\right)\right]\left(\frac{\pi}{2}-\partial\right)d\partial \\ &\approx \frac{1}{25\pi}\left[\frac{31}{2}ln(2-\theta)+14\right], \end{aligned} \tag{40}$$

and

$$ln\,[\mathfrak{A}_*(\mu,\theta)\times\mathfrak{A}_*(\upsilon,\theta)]\int_0^1\varsigma\mathfrak{Q}\big((1-\varsigma)\mu+\varsigma\upsilon\big)\,d\varsigma$$
$$ln\,[\mathfrak{A}^*(\mu,\theta)\times\mathfrak{A}^*(\upsilon,\theta)]\int_0^1\varsigma\mathfrak{Q}\big((1-\varsigma)\mu+\varsigma\upsilon\big)\,d\varsigma$$

$$\begin{aligned} &= \left[2ln(\theta)+\frac{2+\sqrt{2}}{2}\right]\left[\int_0^{\frac{1}{2}}\varsigma^2 d\partial + \int_{\frac{1}{2}}^1\varsigma(1+\varsigma)d\varsigma\right] \approx \frac{17}{24\pi}\left[\frac{63}{10}ln(\theta)+\frac{2+\sqrt{2}}{2}\right], \\ &= \left[2ln(2-\theta)+\frac{2+\sqrt{2}}{2}\right]\left[\int_0^{\frac{1}{2}}\varsigma^2 d\partial + \int_{\frac{1}{2}}^1\varsigma(1+\varsigma)d\varsigma\right] \approx \frac{17}{24\pi}\left[\frac{63}{10}ln(2-\theta)+\frac{2+\sqrt{2}}{2}\right], \end{aligned} \tag{41}$$

From (40) and (41), we have

$$\left[\frac{1}{25\pi}\left[\frac{31}{2}ln(\theta)+14\right],\frac{1}{25\pi}\left[\frac{31}{2}ln(2-\theta)+14\right]\right]$$
$$\leq_I \left[\frac{17}{24\pi}\left[\frac{63}{10}ln(\theta)+\frac{2+\sqrt{2}}{2}\right],\frac{17}{24\pi}\left[\frac{63}{10}ln(2-\theta)+\frac{2+\sqrt{2}}{2}\right]\right],$$

for all $\theta \in (0,1]$. Hence, Theorem 4.4 is verified.
For Theorem 4.5, we have

$$\begin{aligned} ln\,\mathfrak{A}_*\left(\frac{\mu+\upsilon}{2},\theta\right) &= ln\,\mathfrak{A}_*\left(\frac{3\pi}{8},\theta\right) \approx ln\,\left(\frac{5}{2}\theta\right), \\ ln\,\mathfrak{A}^*\left(\frac{\mu+\upsilon}{2},\theta\right) &= ln\,\mathfrak{A}^*\left(\frac{3\pi}{8},\theta\right) \approx ln\,\left(\frac{5}{2}(2-\theta)\right), \end{aligned} \tag{42}$$

$$\int_{\mu}^{\upsilon}\mathfrak{Q}(\partial)d\partial = \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}}\left(\partial-\frac{\pi}{4}\right)d\partial + \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}}\left(\frac{\pi}{2}-\partial\right)d\partial \approx \frac{3}{20},$$

$$\frac{1}{\int_{\mu}^{\upsilon}\mathfrak{Q}(\partial)d\partial}\int_{\mu}^{\upsilon}[ln\,\mathfrak{A}_{*}(\partial,\theta)]\mathfrak{Q}(\partial)d\partial \approx \frac{2}{15}\left[\frac{31}{4}ln(\theta)+7\right],$$
$$\frac{1}{\int_{\mu}^{\upsilon}\mathfrak{Q}(\partial)d\partial}\int_{\mu}^{\upsilon}[ln\,\mathfrak{A}^{*}(\partial,\theta)]\mathfrak{Q}(\partial)d\partial \approx \frac{2}{15}\left[\frac{31}{4}ln(2-\theta)+7\right], \tag{43}$$

From (42) and (43), we have

$$\left[ln\left(\frac{5}{2}\theta\right), ln\left(\frac{5}{2}(2-\theta)\right)\right] \leq_{I} \left[\frac{2}{15}\left[\frac{31}{4}ln(\theta)+7\right], \frac{2}{15}\left[\frac{31}{4}ln(2-\theta)+7\right]\right].$$

Hence, Theorem 4.5 is verified.

5. Conclusions

In this research, some new Jensen, Schur, and H-H-Inequalities of L-convex F-I-V-Fs are offered using fuzzy order relation. We also prove that the results provided in this study generalize the results given for classical L-convex functions. It is an interesting and new problem that the upcoming researchers can obtain new results for different kinds of convexities and inequalities in their future investigations. In future, we will try to explore this concept for interval-valued functions. Morever, we will try to find fuzzy inequalities for L-convex F-I-V-Fs by using different fractional integral operators.

Acknowledgements

This work was supported by the Taif University Researchers Supporting Project (No. TURSP-2020/155), Taif University, Taif, Saudi Arabia, and the National Science, Research and Innovation Fund (NSRF), Thailand.

Conflict of interest

The authors declare that they have no competing interests.

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