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Research article

Post-quantum Simpson's type inequalities for coordinated convex functions

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Abstract: In this paper, we prove some new Simpson's type inequalities for partial (p, q)-differentiable convex functions of two variables in the context of (p, q)-calculus. We also show that the findings in this paper are generalizations of comparable findings in the literature.

Keywords: Simpson's inequalities; (p, q)-integrals; post quantum calculus; co-ordinated convexity
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1. Introduction

Thomas Simpson has evolved essential techniques for the numerical integration and estimation of definite integrals taken into consideration as Simpson's rule during (1710-1761). Nevertheless, a comparable approximation became utilized by J. Kepler nearly earlier than 10 decades, so it's also called Kepler's rule. Simpson's rule consists of the 3-point Newton-Cotes quadrature rule, so estimation primarily based totally on 3 steps quadratic kernel is every so often known as Newton-type results.

1) Simpson's quadrature formula (Simpson's 1/3 rule)

$$\int_{\pi_1}^{\pi_2} F(x)dx \approx \frac{\pi_2 - \pi_1}{6}\left[F(\pi_1) + 4F\left(\frac{\pi_1 + \pi_2}{2}\right) + F(\pi_2)\right].$$

2) Simpson's second formula or Newton-Cotes quadrature formula (Simpson's 3/8 rule).

$$\int_{\pi_1}^{\pi_2} F(x)dx \approx \frac{\pi_2 - \pi_1}{8}\left[F(\pi_1) + 3F\left(\frac{2\pi_1 + \pi_2}{3}\right) + 3F\left(\frac{\pi_1 + 2\pi_2}{3}\right) + F(\pi_2)\right].$$

There are a huge variety of estimations associated with those quadrature rules inside the literature, certainly considered one among them is the subsequent estimation called Simpson's inequality:

Theorem 1.1. *Suppose that $F : [\pi_1, \pi_2] \to \mathbb{R}$ is a four times continuously differentiable mapping on (π_1, π_2), and let $\left\|F^{(4)}\right\|_\infty = \sup_{x\in(\pi_1,\pi_2)} \left|F^{(4)}(x)\right| < \infty$. Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{F(\pi_1) + F(\pi_2)}{2} + 2F\left(\frac{\pi_1 + \pi_2}{2}\right)\right] - \frac{1}{\pi_2 - \pi_1}\int_{\pi_1}^{\pi_2} F(x)dx\right| \le \frac{1}{2880}\left\|F^{(4)}\right\|_\infty (\pi_2 - \pi_1)^4 .$$

In recent years, many writers have focused on Simpson's type inequality in various categories of mappings. Specifically, some mathematicians have worked on the results of Simpson's and Newton's type in obtaining a convex map, because convexity theory is an effective and powerful way to solve a large number of problems from different branches of pure and applied mathematics. For example, Dragomir et al. [1] presented the new Simpson's inequalities and their applications in quadrature formulas for numerical integration. In addition, some inequalities of Simpson's type of s-convex functions were determined by Alomari et al. in [2]. Subsequently, Sarikaya et al. note the variance of Simpson's type inequality based on convexity in [3]. For the further studies of this area, one can consult [4–6].

On the other side, in the domain of q-analysis, many works are being carried out initiating from Euler in order to attain adeptness in mathematics that constructs quantum computing q-calculus considered as a relationship between physics and mathematics. In different areas of mathematics, it has numerous applications such as combinatorics, number theory, basic hypergeometric functions, orthogonal polynomials, and other sciences, mechanics, the theory of relativity, and quantum theory [7, 8]. Quantum calculus also has many applications in quantum information theory which is an interdisciplinary area that encompasses computer science, information theory, philosophy, and cryptography, among other areas [9, 10]. Apparently, Euler invented this important mathematics branch. He used the q parameter in Newton's work on infinite series. Later, in a methodical manner, the q-calculus that knew without limits calculus was firstly given by F. H. Jackson [11,12]. In 1966, W. Al-Salam [13] introduced a q-analogue of the q-fractional integral and q-Riemann-Liouville fractional. Since then, the related research has gradually increased. In particular, in 2013, J. Tariboon and S. K. Ntouyas introduced ${}_{\pi_1}D_q$-difference operator and q_{π_1}-integral in [14]. In 2020, S. Bermudo et al. introduced the notion of ${}^{\pi_2}D_q$ derivative and q^{π_2}-integral in [15]. P. N. Sadjang generalized to quantum calculus and introduced the notions of post-quantum calculus or shortly (p, q)-calculus in [16]. Later, Soontharanon and Sitthiwirattham [17] introduced the fractional (p, q)-calculus. In [18], M. Tunç and E. Göv gave the post-quantum variant of ${}_{\pi_1}D_q$-difference operator and q_{π_1}-integral. Recently, in 2021, Y.-M. Chu et al. introduced the notions of ${}^{\pi_2}D_{p,q}$ derivative and $(p, q)^{\pi_2}$-integral in [19].

Many integral inequalities have been studied using quantum and post-quantum integrals for various types of functions. For example, in [20–28], the authors used ${}_{\pi_1}D_q, {}^{\pi_2}D_q$-derivatives and q_{π_1}, q^{π_2}-integrals to prove Hermite-Hadamard integral inequalities and their left-right estimates for convex and coordinated convex functions. In [29], M. A. Noor et al. presented a generalized version of quantum integral inequalities. For generalized quasi-convex functions, E. R. Nwaeze et al. proved certain parameterized quantum integral inequalities in [30]. M. A. Khan et al. proved quantum Hermite-Hadamard inequality using the green function in [31]. H. Budak et al. [32], M. A. Ali et al. [33,34] and M. Vivas-Cortez et al. [35] developed new quantum Simpson's and quantum Newton's type inequalities for convex and coordinated convex functions. For quantum Ostrowski's inequalities for convex and co-ordinated convex functions, one can consult [36–39]. M. Kunt et al. [40] generalized the results of [22] and proved Hermite-Hadamard type inequalities and their left estimates using ${}_{\pi_1}D_{p,q}$-difference operator and $(p,q)_{\pi_1}$-integral. Recently, M. A. Latif et al. [41] found the right estimates of Hermite-Hadamard type inequalities proved by M. Kunt et al. [40]. To prove Ostrowski's inequalities, Y.-M. Chu et al. [19] used the concepts of ${}^{\pi_2}D_{p,q}$-difference operator and $(p,q)^{\pi_2}$-integral.

In the context of post-quantum calculus, we establish several Simpson's type inequalities for post-quantum differentiable co-ordinated convex functions. The findings in this paper are generalizations of the findings in [33], which is the primary motivation for this paper.

The following is the structure of this paper: Section 2 provides a brief explanation of q-calculus concepts as well as some related works in this field. Section 3 goes over the fundamental concepts of post-quantum calculus and related inequalities. Section 4 establishes an important identity, and Section 5 establishes some new Simpson's type inequalities for coordinated convex functions in the context of post-quantum calculus. In Section 6, we prove three different integral identities in the context of post-quantum calculus, as well as some more Simpson's type inequalities for coordinated convex functions. Section 7 discusses some of the findings as well as potential future research directions.

2. Quantum calculus and some inequalities

In this section, we present some required definitions and inequalities.

In [12], F. H. Jackson gave the q-Jackson integral from 0 to π_2 for $0 < q < 1$ as follows:

$$\int_0^{\pi_2} F(x)\ d_qx = (1-q)\pi_2 \sum_{n=0}^{\infty} q^n F(\pi_2 q^n) \tag{2.1}$$

provided the sum converge absolutely. Moreover, he gave the q-Jackson integral in an arbitrary interval $[\pi_1, \pi_2]$ as:

$$\int_{\pi_1}^{\pi_2} F(x)\ d_qx = \int_0^{\pi_2} F(x)\ d_qx - \int_0^{\pi_1} F(x)\ d_qx .$$

Definition 2.1. *[14] For a continuous function $F : [\pi_1, \pi_2] \to \mathbb{R}$, then q_{π_1}-derivative of F at $x \in [\pi_1, \pi_2]$ is characterized by the expression:*

$$ {}_{\pi_1}D_q F(x) = \frac{F(x) - F(qx + (1-q)\pi_1)}{(1-q)(x-\pi_1)},\ x \neq \pi_1. \tag{2.2}$$

For $x = \pi_1$, we state ${}_{\pi_1}D_q F(\pi_1) = \lim_{x\to\pi_1}\ {}_{\pi_1}D_q F(x)$ if it exists and it is finite.

Definition 2.2. *[15] For a continuous function $F : [\pi_1, \pi_2] \to \mathbb{R}$, then q^{π_2}-derivative of F at $x \in [\pi_1, \pi_2]$ is characterized by the expression:*

$$ {}^{\pi_2}D_q F(x) = \frac{F(qx + (1-q)\pi_2) - F(x)}{(1-q)(\pi_2 - x)}, \ x \neq \pi_2. \tag{2.3} $$

For $x = \pi_2$, we state ${}^{\pi_2}D_q F(\pi_2) = \lim_{x \to \pi_2} {}^{\pi_2}D_q F(x)$ if it exists and it is finite.

Definition 2.3. *[14] Let $F : [\pi_1, \pi_2] \to \mathbb{R}$ be a continuous function. Then, the q_{π_1}-definite integral on $[\pi_1, \pi_2]$ is defined as:*

$$ \begin{aligned} \int_{\pi_1}^{\pi_2} F(x) \ {}_{\pi_1}d_q x &= (1-q)(\pi_2 - \pi_1) \sum_{n=0}^{\infty} q^n F(q^n \pi_2 + (1-q^n)\pi_1) \qquad (2.4) \\ &= (\pi_2 - \pi_1) \int_0^1 F((1-\tau)\pi_1 + \tau\pi_2) \ d_q\tau \ . \end{aligned} $$

On the other hand, S. Bermudo et al. gave the following new definition:

Definition 2.4. *[15] Let $F : [\pi_1, \pi_2] \to \mathbb{R}$ be a continuous function. Then, the q^{π_2}-definite integral on $[\pi_1, \pi_2]$ is defined as:*

$$ \begin{aligned} \int_{\pi_1}^{\pi_2} F(x) \ {}^{\pi_2}d_q x &= (1-q)(\pi_2 - \pi_1) \sum_{n=0}^{\infty} q^n F(q^n \pi_1 + (1-q^n)\pi_2) \qquad (2.5) \\ &= (\pi_2 - \pi_1) \int_0^1 F(\tau\pi_1 + (1-\tau)\pi_2) \ d_q\tau \ . \end{aligned} $$

For more details about q^{π_2}-integrals and corresponding inequalities one can see [15].

Now, let's give the following notation which will be used many times in the next sections (see, [8]):

$$ [n]_q = \frac{q^n - 1}{q - 1}. $$

Moreover, we give the following Lemma for our main results:

Lemma 2.1. *[14] With the notation above, we have the equality*

$$ \int_{\pi_1}^{\pi_2} (x - \pi_1)^{\alpha} \ {}_{\pi_1}d_q x = \frac{(\pi_2 - \pi_1)^{\alpha+1}}{[\alpha + 1]_q} $$

for $\alpha \in \mathbb{R} \backslash \{-1\}$.

In [39], H. Budak et al. proved the following variant of quantum Ostrowski inequality by using the q_{π_1} and q^{π_2}-integrals:

Theorem 2.1. *[39] Let* $F:[\pi_1,\pi_2]\subset\mathbb{R}\to\mathbb{R}$ *be a function and* ${}^{\pi_2}D_qF$, ${}_{\pi_1}D_qF$ *be two continuous and integrable functions on* $[\pi_1,\pi_2]$. *If* $\left|{}^{\pi_2}D_q\ F(\tau)\right|,\left|{}_{\pi_1}D_q\ F(\tau)\right|\leq M$ *for all* $\tau\in[\pi_1,\pi_2]$, *then we have the following quantum Ostrowski type inequality:*

$$\left|F(x)-\frac{1}{\pi_2-\pi_1}\left[\int_{\pi_1}^{x}F(\tau)\ {}_{\pi_1}d_q\tau+\int_{x}^{\pi_2}F(\tau)\ {}^{\pi_2}d_q\tau\right]\right| \tag{2.6}$$

$$\leq\frac{qM}{(\pi_2-\pi_1)}\left[\frac{(x-\pi_1)^2+(\pi_2-x)^2}{[2]_q}\right]$$

for all $x\in[\pi_1,\pi_2]$ *where* $0<q<1$.

On the other hand, the authors gave the following definitions of $q_{\pi_1\pi_3}$, $q_{\pi_1}^{\pi_4}$, $q_{\pi_2}^{\pi_3}$ and $q^{\pi_2\pi_4}$ integrals and related inequalities of Hermite-Hadamard type:

Definition 2.5. *[25,42] Suppose that* $F:[\pi_1,\pi_2]\times[\pi_3,\pi_4]\subset\mathbb{R}^2\to\mathbb{R}$ *is a continuous function. Then, the following* $q_{\pi_1\pi_3}$, $q_{\pi_1}^{\pi_4}$, $q_{\pi_3}^{\pi_2}$ *and* $q^{\pi_2\pi_4}$ *integrals on* $[\pi_1,\pi_2]\times[\pi_3,\pi_4]$ *are defined by*

$$\int_{\pi_1}^{x}\int_{\pi_3}^{y}F(\tau,s)\ {}_{\pi_3}d_{q_2}s\ {}_{\pi_1}d_{q_1}\tau=(1-q_1)(1-q_2)(x-\pi_1)(y-\pi_3)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^mF\left(q_1^nx+(1-q_1^n)\pi_1,q_2^my+(1-q_2^m)\pi_3\right)$$

$$\int_{\pi_1}^{x}\int_{y}^{\pi_4}F(\tau,s)\ {}^{\pi_4}d_{q_2}s\ {}_{\pi_1}d_{q_1}\tau=(1-q_1)(1-q_2)(x-\pi_1)(\pi_4-y)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^mF\left(q_1^nx+(1-q_1^n)\pi_1,q_2^my+(1-q_2^m)\pi_4\right) \tag{2.7}$$

$$\int_{x}^{\pi_2}\int_{\pi_3}^{y}F(\tau,s)\ {}_{\pi_3}d_{q_2}s\ {}^{\pi_2}d_{q_1}\tau=(1-q_1)(1-q_2)(\pi_2-x)(y-\pi_3)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^mF\left(q_1^nx+(1-q_1^n)\pi_2,q_2^my+(1-q_2^m)\pi_3\right) \tag{2.8}$$

and

$$\int_{x}^{\pi_2}\int_{y}^{\pi_4}F(\tau,s)\ {}^{\pi_4}d_{q_2}s\ {}^{\pi_2}d_{q_1}\tau=(1-q_1)(1-q_2)(\pi_2-x)(\pi_4-y)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^mF\left(q_1^nx+(1-q_1^n)\pi_2,q_2^my+(1-q_2^m)\pi_4\right) \tag{2.9}$$

respectively, for $(x,y)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$.

Definition 2.6. *[42,43] Let $F : [\pi_1, \pi_2] \times [\pi_3, \pi_4] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a continuous function of two variables. Then, the partial q_1-derivatives, q_2-derivatives and q_1q_2-derivatives at $(x, y) \in [\pi_1, \pi_2] \times [\pi_3, \pi_4]$ can be given as follows:*

$$
\begin{aligned}
\frac{{}_{\pi_1}\partial_{q_1} F(x,y)}{{}_{\pi_1}\partial_{q_1} x} &= \frac{F(q_1 x + (1-q_1)\pi_1, y) - F(x,y)}{(1-q_1)(x-\pi_1)}, \; x \neq \pi_1 \\
\frac{{}_{\pi_3}\partial_{q_2} F(x,y)}{{}_{\pi_3}\partial_{q_2} y} &= \frac{F(x, q_2 y + (1-q_2)\pi_3) - F(x,y)}{(1-q_2)(y-\pi_3)}, \; y \neq \pi_3 \\
\frac{{}_{\pi_1,\,\pi_3}\partial^2_{q_1,q_2} F(x,y)}{{}_{\pi_1}\partial_{q_1} x \; {}_{\pi_3}\partial_{q_2} y} &= \frac{1}{(x-\pi_1)(y-\pi_3)(1-q_1)(1-q_2)} \\
&\quad \times \left[F(q_1 x + (1-q_1)\pi_1, q_2 y + (1-q_2)\pi_3) - F(q_1 x + (1-q_1)\pi_1, y)\right. \\
&\quad \left. - F(x, q_2 y + (1-q_2)\pi_3) + F(x,y)\right], \; x \neq \pi_1, \; y \neq \pi_3 \\
\frac{{}^{\pi_2}\partial_{q_1} F(x,y)}{{}^{\pi_2}\partial_{q_1} x} &= \frac{F(q_1 x + (1-q_1)\pi_2, y) - F(x,y)}{(1-q_1)(\pi_2 - x)}, \; x \neq \pi_2 \\
\frac{{}^{\pi_4}\partial_{q_2} F(x,y)}{{}^{\pi_4}\partial_{q_2} y} &= \frac{F(x, q_2 y + (1-q_2)\pi_4) - F(x,y)}{(1-q_2)(\pi_4 - y)}, \; y \neq \pi_4
\end{aligned}
$$

$$
\begin{aligned}
\frac{{}^{\pi_4}_{\pi_1}\partial^2_{q_1,q_2} F(x,y)}{{}_{\pi_1}\partial_{q_1} x \; {}^{\pi_4}\partial_{q_2} y} &= \frac{1}{(x-\pi_1)(\pi_4 - y)(1-q_1)(1-q_2)} \\
&\quad \times \left[F(q_1 x + (1-q_1)\pi_1, q_2 y + (1-q_2)\pi_4) - F(q_1 x + (1-q_1)\pi_1, y)\right. \\
&\quad \left. - F(x, q_2 y + (1-q_2)\pi_4) + F(x,y)\right], \; x \neq \pi_1, \; y \neq \pi_4, \\
\frac{{}^{\pi_2}_{\pi_3}\partial^2_{q_1,q_2} F(x,y)}{{}^{\pi_2}\partial_{q_1} x \; {}_{\pi_3}\partial_{q_2} y} &= \frac{1}{(\pi_2 - x)(y-\pi_3)(1-q_1)(1-q_2)} \\
&\quad \times \left[F(q_1 x + (1-q_1)\pi_2, q_2 y + (1-q_2)\pi_3) - F(q_1 x + (1-q_1)\pi_2, y)\right. \\
&\quad \left. - F(x, q_2 y + (1-q_2)\pi_3) + F(x,y)\right], \; x \neq \pi_2, \; y \neq \pi_3, \\
\frac{{}^{\pi_2,\,\pi_4}\partial^2_{q_1,q_2} F(x,y)}{{}^{\pi_2}\partial_{q_1} x \; {}^{\pi_4}\partial_{q_2} y} &= \frac{1}{(\pi_2 - x)(\pi_4 - y)(1-q_1)(1-q_2)} \\
&\quad \times \left[F(q_1 x + (1-q_1)\pi_2, q_2 y + (1-q_2)\pi_4) - F(q_1 x + (1-q_1)\pi_2, y)\right. \\
&\quad \left. - F(x, q_2 y + (1-q_2)\pi_4) + F(x,y)\right], \; x \neq \pi_2, \; y \neq \pi_4.
\end{aligned}
$$

3. Post-quantum calculus and some inequalities

In this section, we review some fundamental notions and notations of (p,q)-calculus.

The $[n]_{p,q}$ is said to be (p,q)-integers and expressed as:

$$[n]_{p,q}=\frac{p^n-q^n}{p-q}$$

with $0<q<p\leq 1$. The $[n]_{p,q}!$ and $\left[\begin{array}{c} n \\ k \end{array}\right]!$ are called (p,q)-factorial and (p,q)-binomial, respectively, and expressed as:

$$\begin{aligned}
[n]_{p,q}! &= \prod_{k=1}^{n}[k]_{p,q}\,,\ n\geq 1,\ [0]_{p,q}!=1,\\
\left[\begin{array}{c} n \\ k \end{array}\right]! &= \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}.
\end{aligned}$$

Definition 3.1. *[16] The (p,q)-derivative of mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ is given as:*

$$D_{p,q}F(x)=\frac{F(px)-F(qx)}{(p-q)x},\ x\neq 0$$

with $0<q<p\leq 1$.

Definition 3.2. *[18] The $(p,q)_{\pi_1}$-derivative of mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ is given as:*

$$_{\pi_1}D_{p,q}F(x)=\frac{F(px+(1-p)\pi_1)-F(qx+(1-q)\pi_1)}{(p-q)(x-\pi_1)},\ x\neq\pi_1 \tag{3.1}$$

with $0<q<p\leq 1$. For $x=\pi_1$, we state $_{\pi_1}D_{p,q}F(\pi_1)=\lim_{x\to\pi_1}\ {}_{\pi_1}D_{p,q}F(x)$ if it exists and it is finite.

Definition 3.3. *[19] The $(p,q)^{\pi_2}$-derivative of mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ is given as:*

$$^{\pi_2}D_{p,q}F(x)=\frac{F(qx+(1-q)\pi_2)-F(px+(1-p)\pi_2)}{(p-q)(\pi_2-x)},\ x\neq\pi_2. \tag{3.2}$$

with $0<q<p\leq 1$. For $x=\pi_2$, we state $^{\pi_2}D_{p,q}F(\pi_2)=\lim_{x\to\pi_2}\ {}^{\pi_2}D_{p,q}F(x)$ if it exists and it is finite.

Remark 3.1. *It is clear that if we use $p=1$ in (3.1) and (3.2), then the equalities (3.1) and (3.2) reduce to (2.2) and (2.3), respectively.*

Definition 3.4. *[18] The definite $(p,q)_{\pi_1}$-integral of mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ on $[\pi_1,\pi_2]$ is stated as:*

$$\int_{\pi_1}^{x}F(\tau)\ {}_{\pi_1}d_{p,q}\tau=(p-q)(x-\pi_1)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}F\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)\pi_1\right) \tag{3.3}$$

with $0<q<p\leq 1$.

Definition 3.5. *[19] The definite $(p,q)^{\pi_2}$-integral of mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ on $[\pi_1,\pi_2]$ is stated as:*

$$\int_x^{\pi_2} F(\tau)\ {}^{\pi_2}d_{p,q}\tau = (p-q)(\pi_2 - x)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}F\left(\frac{q^n}{p^{n+1}}x + \left(1-\frac{q^n}{p^{n+1}}\right)\pi_2\right) \tag{3.4}$$

with $0<q<p\le 1$.

Remark 3.2. *It is evident that if we pick $p=1$ in (3.3) and (3.4), then the equalities (3.3) and (3.4) change into (2.4) and (2.5), respectively.*

Remark 3.3. *If we take $\pi_1=0$ and $x=\pi_2=1$ in (3.3), then we have*

$$\int_0^1 F(\tau)\ {}_0d_{p,q}\tau = (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}F\left(\frac{q^n}{p^{n+1}}\right).$$

Similarly, by taking $x=\pi_1=0$ and $\pi_2=1$ in (3.4), then we obtain that

$$\int_0^1 F(\tau)\ {}^1d_{p,q}\tau = (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}F\left(1-\frac{q^n}{p^{n+1}}\right).$$

Lemma 3.1. *[44] We have the following equalities*

$$\int_{\pi_1}^{\pi_2}(\pi_2-x)^{\alpha}\ {}^{\pi_2}d_{p,q}x = \frac{(\pi_2-\pi_1)^{\alpha+1}}{[\alpha+1]_{p,q}}$$

$$\int_{\pi_1}^{\pi_2}(x-\pi_1)^{\alpha}\ {}_{\pi_1}d_{p,q}x = \frac{(\pi_2-\pi_1)^{\alpha+1}}{[\alpha+1]_{p,q}},$$

where $\alpha\in\mathbb{R}-\{-1\}$.

In [40], M. Kunt et al. proved the following Hermite-Hadamard type inequalities for convex functions via $(p,q)_{\pi_1}$-integral:

Theorem 3.1. *[40] For a convex mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ which is differentiable on $[\pi_1,\pi_2]$, the following inequalities hold for $(p,q)_{\pi_1}$-integral:*

$$F\left(\frac{q\pi_1+p\pi_2}{[2]_{p,q}}\right)\le\frac{1}{p(\pi_2-\pi_1)}\int_{\pi_1}^{p\pi_2+(1-p)\pi_1}F(x)\ {}_{\pi_1}d_{p,q}x\le\frac{qF(\pi_1)+pF(\pi_2)}{[2]_{p,q}}, \tag{3.5}$$

where $0<q<p\le 1$.

Recently, M. Vivas-Cortez et al. [44] proved the following Hermite-Hadamard type inequalities for convex functions using the $(p,q)^{\pi_2}$-integral:

Theorem 3.2. *[44] For a convex mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ which is differentiable on $[\pi_1,\pi_2]$, the following inequalities hold for $(p,q)^{\pi_2}$-integral:*

$$F\left(\frac{p\pi_1+q\pi_2}{[2]_{p,q}}\right)\le\frac{1}{p(\pi_2-\pi_1)}\int_{p\pi_1+(1-p)\pi_2}^{\pi_2}F(x)\ {}^{\pi_2}d_{p,q}x\le\frac{pF(\pi_1)+qF(\pi_2)}{[2]_{p,q}}, \tag{3.6}$$

where $0<q<p\le 1$.

In [45] and [46], the authors gave the following notions of post-quantum integrals for the functions of two variables.

Definition 3.6. *[45, 46] For a function $F : [\pi_1, \pi_2] \times [\pi_3, \pi_4] \to \mathbb{R}$,*

1. *the $(p, q)^{\pi_4}_{\pi_1}$ integral of F is given as:*

$$\int_{\pi_1}^{x} \int_{y}^{\pi_4} F(\tau, s)\ {}^{\pi_4}d_{p_2,q_2} s\ {}_{\pi_1}d_{p_1,q_1}\tau = (p_1 - q_1)(p_2 - q_2)(x - \pi_1)(\pi_4 - y)$$
$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} F\left(\frac{q_1^n}{p_1^{n+1}} x + \left(1 - \frac{q_1^n}{p_1^{n+1}}\right)\pi_1, \frac{q_2^m}{p_2^{m+1}} y + \left(1 - \frac{q_2^m}{p_2^{m+1}}\right)\pi_4\right),$$

where $x, y \in [\pi_1, p_1\pi_2 + (1 - p_1)\pi_1] \times [p_2\pi_3 + (1 - p_2)\pi_4, \pi_4]$.

2. *the $(p, q)^{\pi_2}_{\pi_3}$ integral of F is given as:*

$$\int_{x}^{\pi_2} \int_{\pi_3}^{y} F(\tau, s)\ {}_{\pi_3}d_{p_2,q_2} s\ {}^{\pi_2}d_{p_1,q_1}\tau = (p_1 - q_1)(p_2 - q_2)(\pi_2 - x)(y - \pi_3)$$
$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} F\left(\frac{q_1^n}{p_1^{n+1}} x + \left(1 - \frac{q_1^n}{p_1^{n+1}}\right)\pi_2, \frac{q_2^m}{p_2^{m+1}} y + \left(1 - \frac{q_2^m}{p_2^{m+1}}\right)\pi_3\right)$$

where $x, y \in [p_1\pi_1 + (1 - p_1)\pi_1, \pi_2] \times [\pi_3, p_2\pi_4 + (1 - p_2)\pi_3]$.

3. *the $(p, q)^{\pi_2\pi_4}$ integral of F is given as:*

$$\int_{x}^{\pi_2} \int_{y}^{\pi_4} F(\tau, s)\ {}^{\pi_4}d_{p_2,q_2} s\ {}^{\pi_2}d_{p_1,q_1}\tau = (p_1 - q_1)(p_2 - q_2)(\pi_2 - x)(\pi_4 - y)$$
$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} F\left(\frac{q_1^n}{p_1^{n+1}} x + \left(1 - \frac{q_1^n}{p_1^{n+1}}\right)\pi_2, \frac{q_2^m}{p_2^{m+1}} y + \left(1 - \frac{q_2^m}{p_2^{m+1}}\right)\pi_4\right),$$

where $x, y \in [p_1\pi_1 + (1 - p_1)\pi_2, \pi_2] \times [p_2\pi_3 + (1 - p_2)\pi_4, \pi_4]$.

4. *the $(p, q)_{\pi_1\pi_3}$ integral of F is given as:*

$$\int_{\pi_1}^{x} \int_{\pi_3}^{y} F(\tau, s)\ {}_{\pi_3}d_{p_2,q_2} s\ {}_{\pi_1}d_{p_1,q_1}\tau = (p_1 - q_1)(p_2 - q_2)(x - \pi_1)(y - \pi_3)$$
$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} F\left(\frac{q_1^n}{p_1^{n+1}} x + \left(1 - \frac{q_1^n}{p_1^{n+1}}\right)\pi_1, \frac{q_2^m}{p_2^{m+1}} y + \left(1 - \frac{q_2^m}{p_2^{m+1}}\right)\pi_3\right)$$

where $x, y \in [\pi_1, p_1\pi_2 + (1 - p_1)\pi_3] \times [\pi_3, p_2\pi_4 + (1 - p_2)\pi_3]$.

Remark 3.4. *It is obvious that if we use $p_1 = p_2 = 1$, then Definition 3.6 transforms into Definition 2.5.*

In [45], H. Kalsoom et al. introduced the following notions of post-quantum partial derivatives.

Definition 3.7. *[45] Let $F : [\pi_1, \pi_2] \times [\pi_3, \pi_4] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a continuous function of two variables. Then the partial p_1q_1-derivatives, p_2q_2-derivatives and $p_1q_1p_2q_2$-derivatives at $(x, y) \in [\pi_1, \pi_2] \times [\pi_3, \pi_4]$ can be given as follows:*

$$\frac{{}_{\pi_1}\partial_{p_1,q_1} F(x, y)}{{}_{\pi_1}\partial_{p_1,q_1} x} = \frac{F(q_1 x + (1 - q_1)\pi_1, y) - F(p_1 x + (1 - p_1)\pi_1, y)}{(p_1 - q_1)(x - \pi_1)},\ x \neq \pi_1$$

$$\frac{{}_{\pi_3}\partial_{p_2,q_2}F(x,y)}{{}_{\pi_3}\partial_{p_2,q_2}y} = \frac{F(x,q_2y+(1-q_2)\pi_3)-F(x,p_2y+(1-p_2)\pi_3)}{(p_2-q_2)(y-\pi_3)},\ y\neq\pi_3$$

$$\begin{aligned}\frac{{}_{\pi_1,\,\pi_3}\partial^2_{p_1,q_1,p_2,q_2}F(x,y)}{{}_{\pi_1}\partial_{p_1,q_1}x\,{}_{\pi_3}\partial_{p_2,q_2}y} &= \frac{1}{(x-\pi_1)(y-\pi_3)(p_1-q_1)(p_2-q_2)}\\ &\times[F(q_1x+(1-q_1)\pi_1,q_2y+(1-q_2)\pi_3)\\ &-F(q_1x+(1-q_1)\pi_1,p_2y+(1-p_2)\pi_3)\\ &-F(p_1x+(1-p_1)\pi_1,q_2y+(1-q_2)\pi_3)\\ &+F(p_1x+(1-p_1)\pi_1,p_2y+(1-p_2)\pi_3)],\ x\neq\pi_1,\ y\neq\pi_3.\end{aligned}$$

Recently, Ali et al. gave the following notions:

Definition 3.8. *[47] Let $F:[\pi_1,\pi_2]\times[\pi_3,\pi_4]\subseteq\mathbb{R}^2\to\mathbb{R}$ be a continuous function of two variables. Then the partial p_1q_1-derivatives, p_2q_2-derivatives and $p_1q_1p_2q_2$-derivatives at $(x,y)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$ can be given as follows:*

$$\frac{{}^{\pi_2}\partial_{p_1,q_1}F(x,y)}{{}^{\pi_2}\partial_{p_1,q_1}x} = \frac{F(q_1x+(1-q_1)\pi_2,y)-F(p_1x+(1-p_1)\pi_2,y)}{(p_1-q_1)(\pi_2-x)},\ x\neq\pi_2$$

$$\frac{{}^{\pi_4}\partial_{p_2,q_2}F(x,y)}{{}^{\pi_4}\partial_{p_2,q_2}y} = \frac{F(x,q_2y+(1-q_2)\pi_4)-F(x,p_2y+(1-p_2)\pi_4)}{(p_2-q_2)(\pi_4-y)},\ y\neq\pi_4$$

$$\begin{aligned}\frac{{}^{\pi_4}_{\pi_1}\partial^2_{p_1,q_1,p_2,q_2}F(x,y)}{{}_{\pi_1}\partial_{p_1,q_1}x\,{}^{\pi_4}\partial_{p_2,q_2}y} &= \frac{1}{(x-\pi_1)(\pi_4-y)(p_1-q_1)(p_2-q_2)}[F(q_1x+(1-q_1)\pi_1,q_2y+(1-q_2)\pi_4)\\ &-F(q_1x+(1-q_1)\pi_1,p_2y+(1-p_2)\pi_4)-F(p_1x+(1-p_1)\pi_1,q_2y+(1-q_2)\pi_4)\\ &+F(p_1x+(1-p_1)\pi_1,p_2y+(1-p_2)\pi_4)],\ x\neq\pi_1,\ y\neq\pi_4,\end{aligned}$$

$$\begin{aligned}\frac{{}^{\pi_2}_{\pi_3}\partial^2_{p_1,q_1,p_2,q_2}F(x,y)}{{}^{\pi_2}\partial_{p_1,q_1}x\,{}_{\pi_3}\partial_{p_2,q_2}y} &= \frac{1}{(\pi_2-x)(y-\pi_3)(p_1-q_1)(p_2-q_2)}[F(q_1x+(1-q_1)\pi_2,q_2y+(1-q_2)\pi_3)\\ &-F(q_1x+(1-q_1)\pi_2,p_2y+(1-p_2)\pi_3)-F(p_1x+(1-p_1)\pi_2,q_2y+(1-q_2)\pi_3)\\ &+F(p_1x+(1-p_1)\pi_2,p_2y+(1-p_2)\pi_3)],\ x\neq\pi_2,\ y\neq\pi_3,\end{aligned}$$

$$\begin{aligned}\frac{{}^{\pi_2,\,\pi_4}\partial^2_{p_1,q_1,p_2,q_2}F(x,y)}{{}^{\pi_2}\partial_{p_1,q_1}x\,{}^{\pi_4}\partial_{p_2,q_2}y} &= \frac{1}{(\pi_2-x)(\pi_4-y)(p_1-q_1)(p_2-q_2)}[F(q_1x+(1-q_1)\pi_2,q_2y+(1-q_2)\pi_4)\\ &-F(q_1x+(1-q_1)\pi_2,p_2y+(1-p_2)\pi_4)-F(p_1x+(1-p_1)\pi_2,q_2y+(1-q_2)\pi_4)\end{aligned}$$

$$+F\left(p_1x+(1-p_1)\pi_2,p_2y+(1-p_2)\pi_4\right)\Big],\ x\neq\pi_2,\ y\neq\pi_4.$$

Remark 3.5. *It is obvious that if we set $p_1 = p_2 = 1$ in Definitions 3.7 and 3.8, then we obtain the Definition 2.6.*

4. An identity

In this section, we deal with identity, which is required to reach our main estimations.

We begin with the following lemma.

Lemma 4.1. *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially $(p_1,q_1)(p_2,q_2)$-differentiable function. If the partial $(p_1,q_1)(p_2,q_2)$-derivative $\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta$. Then following identity holds for $(p_1,q_1)(p_2,q_2)$-integrals.*

$$\begin{aligned}&{}^{b,\ d}\mathcal{I}_{(p_1,q_1),(p_2,q_2)}(F) \qquad (4.1)\\ =\ &(b-a)(d-c)\int_0^1\int_0^1\Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}d_{p_1,q_1}t\ d_{p_2,q_2}s,\end{aligned}$$

where $0<q_1<p_1\leq 1, 0<q_2<p_2\leq 1,$

$$\begin{aligned}{}^{b,\ d}\mathcal{I}_{(p_1,q_1),(p_2,q_2)}(F)\ =\ &\frac{F\left(\frac{a+b}{2},c\right)+F\left(\frac{a+b}{2},d\right)+4F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+F\left(a,\frac{c+d}{2}\right)+F\left(b,\frac{c+d}{2}\right)}{9}\\ &+\frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{36}\\ &-\frac{1}{6(b-a)}\int_{p_1a+(1-p_1)b}^{b}\left[F(x,c)+4F\left(x,\frac{c+d}{2}\right)+F(x,d)\right]\ {}^{b}d_{p_1,q_1}x\\ &-\frac{1}{6(d-c)}\int_{p_2c+(1-p_2)d}^{d}\left[F(a,y)+4F\left(\frac{a+b}{2},y\right)+F(b,y)\right]\ {}^{d}d_{p_2,q_2}y\\ &+\frac{1}{(b-a)(d-c)}\int_{p_1a+(1-p_1)b}^{b}\int_{p_2c+(1-p_2)d}^{d}F(x,y)\ {}^{b}d_{p_1,q_1}x\ {}^{d}d_{p_2,q_2}y\end{aligned}$$

and

$$\Lambda_{p_1,q_1}(t)=\begin{cases}\left(q_1t-\frac{1}{6}\right), & t\in\left[0,\frac{1}{2}\right),\\ \left(q_1t-\frac{5}{6}\right), & t\in\left[\frac{1}{2},1\right],\end{cases}$$

$$\Lambda_{p_2,q_2}(s)=\begin{cases}\left(q_2s-\frac{1}{6}\right), & s\in\left[0,\frac{1}{2}\right),\\ \left(q_2s-\frac{5}{6}\right), & s\in\left[\frac{1}{2},1\right].\end{cases}$$

Proof. Because of fundamental properties of (p,q)-integrals and the definition of $\Lambda_{p_1,q_1}(t)$ and $\Lambda_{p_2,q_2}(s)$, it is easy to see that

$$\int_0^1\int_0^1\Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}d_{p_1,q_1}t\ d_{p_2,q_2}s \qquad (4.2)$$

$$
\begin{aligned}
&= \frac{4}{9}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(ta+(1-t)b, sc+(1-s)d\right)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s} d_{p_1,q_1}t\, d_{p_2,q_2}s \\
&\quad +\frac{2}{3}\int_0^{\frac{1}{2}}\int_0^{1}\left(q_2 s-\frac{5}{6}\right) \frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(ta+(1-t)b, sc+(1-s)d\right)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s} d_{p_1,q_1}t\, d_{p_2,q_2}s \\
&\quad +\frac{2}{3}\int_0^{1}\int_0^{\frac{1}{2}}\left(q_1 t-\frac{5}{6}\right) \frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(ta+(1-t)b, sc+(1-s)d\right)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s} d_{p_1,q_1}t\, d_{p_2,q_2}s \\
&\quad +\int_0^{1}\int_0^{1}\left(q_1 t-\frac{5}{6}\right)\left(q_2 s-\frac{5}{6}\right) \frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(ta+(1-t)b, sc+(1-s)d\right)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s} d_{p_1,q_1}t\, d_{p_2,q_2}s. \\
&= I_1+I_2+I_3+I_4.
\end{aligned}
$$

From Definition 3.8, we have

$$
\begin{aligned}
&\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(ta+(1-t)b, sc+(1-s)d\right)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s} \\
&= \frac{1}{(p_1-q_1)(p_2-q_2)(b-a)(d-c)ts}\left[F\left(tq_1a+(1-tq_1)b, sq_2c+(1-sq_2)d\right)\right. \\
&\quad -F\left(tq_1a+(1-tq_1)b, sp_2c+(1-sp_2)d\right)-F\left(tp_1a+(1-tp_1)b, sq_2c+(1-sq_2)d\right) \\
&\quad \left.+F\left(tp_1a+(1-tp_1)b, sp_2c+(1-sp_2)d\right)\right].
\end{aligned}
$$

It is need to calculate the integrals in the right side of (4.2) in order to finish this proof. By using the definition of $(p_1, q_1)(p_2, q_2)$-integrals, we obtain that

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(ta+(1-t)b, sc+(1-s)d\right)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s} d_{p_1,q_1}t\, d_{p_2,q_2}s \qquad (4.3)\\
&= \frac{1}{(p_1-q_1)(p_2-q_2)(b-a)(d-c)}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\frac{1}{ts}\left\{F\left(tq_1a+(1-tq_1)b, sq_2c+(1-sq_2)d\right)\right. \\
&\quad -F\left(tq_1a+(1-tq_1)b, sp_2c+(1-sp_2)d\right)-F\left(tp_1a+(1-tp_1)b, sq_2c+(1-sq_2)d\right) \\
&\quad \left.+F\left(tp_1a+(1-tp_1)b, sp_2c+(1-sp_2)d\right)\right\} d_{p_1,q_1}t\, d_{p_2,q_2}s \\
&= \frac{1}{(b-a)(d-c)}\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F\left(\frac{q_1^{n+1}}{2p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{2p_1^{n+1}}\right)b, \frac{q_2^{m+1}}{2p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{2p_2^{m+1}}\right)d\right)\right. \\
&\quad -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F\left(\frac{q_1^{n+1}}{2p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{2p_1^{n+1}}\right)b, \frac{q_2^{m}}{2p_2^{m}}c+\left(1-\frac{q_2^{m}}{2p_2^{m}}\right)d\right) \\
&\quad -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F\left(\frac{q_1^{n}}{2p_1^{n}}a+\left(1-\frac{q_1^{n}}{2p_1^{n}}\right)b, \frac{q_2^{m+1}}{2p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{2p_2^{m+1}}\right)d\right) \\
&\quad \left.+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F\left(\frac{q_1^{n}}{2p_1^{n}}a+\left(1-\frac{q_1^{n}}{2p_1^{n}}\right)b, \frac{q_2^{m}}{2p_2^{m}}c+\left(1-\frac{q_2^{m}}{2p_2^{m}}\right)d\right)\right\}
\end{aligned}
$$

$$= \frac{1}{(b-a)(d-c)}\left[F(b,d)-F\left(\frac{a+b}{2},d\right)-F\left(b,\frac{c+d}{2}\right)+F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right].$$

So

$$I_1 = \frac{4}{9(b-a)(d-c)}\left[F(b,d)-F\left(\frac{a+b}{2},d\right)-F\left(b,\frac{c+d}{2}\right)+F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]$$

Now from Definition 2.5, we obtain the following

$$\int_0^{\frac{1}{2}}\int_0^1 s\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(ta+(1-t)b, sc+(1-s)d)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}d_{p_1,q_1}t\;d_{p_2,q_2}s \tag{4.4}$$

$$\begin{aligned}
&= \frac{1}{(p_1-q_1)(p_2-q_2)(b-a)(d-c)}\int_0^{\frac{1}{2}}\int_0^1 \frac{1}{t}\{F(tq_1a+(1-tq_1)b, sq_2c+(1-sq_2)d)\\
&\quad -F(tq_1a+(1-tq_1)b, sp_2c+(1-sp_2)d)-F(tp_1a+(1-tp_1)b, sq_2c+(1-sq_2)d)\\
&\quad +F(tp_1a+(1-tp_1)b, sp_2c+(1-sp_2)d)\}\,d_{p_1,q_1}t\;d_{p_2,q_2}s\\
&= \frac{1}{(b-a)(d-c)}\times\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}\left(F\left(\frac{q_1^{n+1}}{2p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{2p_1^{n+1}}\right)b, \frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)\right)\right.\\
&\quad -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}\left(F\left(\frac{q_1^{n+1}}{2p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{2p_1^{n+1}}\right)b, \frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\right)\\
&\quad -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}\left(F\left(\frac{q_1^{n}}{2p_1^{n}}a+\left(1-\frac{q_1^{n}}{2p_1^{n}}\right)b, \frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)\right)\\
&\quad \left.+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}\left(F\left(\frac{q_1^{n}}{2p_1^{n}}a+\left(1-\frac{q_1^{n}}{2p_1^{n}}\right)b, \frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\right)\right\}\\
&= \frac{1}{(b-a)(d-c)}\left\{\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}\left[\sum_{n=0}^{\infty}F\left(\frac{q_1^{n+1}}{2p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{2p_1^{n+1}}\right)b, \frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)\right.\right.\\
&\quad \left.-\sum_{n=0}^{\infty}\left(F\left(\frac{q_1^{n}}{2p_1^{n}}a+\left(1-\frac{q_1^{n}}{2p_1^{n}}\right)b, \frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)\right)\right]\\
&\quad +\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}\left[\sum_{n=0}^{\infty}F\left(\frac{q_1^{n}}{2p_1^{n}}a+\left(1-\frac{q_1^{n}}{2p_1^{n}}\right)b, \frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\right.\\
&\quad \left.\left.-\sum_{n=0}^{\infty}F\left(\frac{q_1^{n+1}}{2p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{2p_1^{n+1}}\right)b, \frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\right]\right\}\\
&= \frac{1}{(b-a)(d-c)}\left\{\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}F\left(b,\frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)-\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}F\left(b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\right.\\
&\quad \left.+\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{a+b}{2},\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)-\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{a+b}{2},\frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)\right\}\\
&= \frac{1}{(b-a)(d-c)}\left\{\frac{p_2-q_2}{q_2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}F\left(b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)-\frac{1}{q_2}F(b,c)\right.
\end{aligned}$$

$$-\frac{p_2-q_2}{q_2}\sum_{m-0}^{\infty}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{a+b}{2},\frac{q_2^m}{p_2^m}c+\left(1-\frac{q_2^m}{p_2^m}\right)d\right)+\frac{1}{q_2}F\left(\frac{a+b}{2},c\right)$$
$$= \frac{1}{q_2(b-a)(d-c)}\left[\frac{1}{d-c}\int_{p_2c+(1-p_2)d}^{d}F(b,y)\ {}^{d}d_{p_2,q_2}y\right.$$
$$\left.-\frac{1}{d-c}\int_{p_2c+(1-p_2)d}^{d}F\left(\frac{a+b}{2},y\right)\ {}^{d}d_{p_2,q_2}y-F(b,c)+F\left(\frac{a+b}{2},y\right)\right].$$

By using the similar operations used in (4.3), we have

$$\int_0^{\frac{1}{2}}\int_0^1\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}d_{p_1,q_1}t\ d_{p_2,q_2}s \tag{4.5}$$
$$= \frac{1}{(b-a)(d-c)}\left[F(b,d)-F\left(\frac{a+b}{2},d\right)-F(b,c)+F\left(\frac{a+b}{2},c\right)\right].$$

From (4.4) and (4.5), we obtain that

$$I_2 = \frac{2}{3(b-a)(d-c)}\left\{\frac{1}{d-c}\int_{p_2c+(1-p_2)d}^{d}F(b,y)\ {}^{d}d_{p_2,q_2}y\right.$$
$$\left.-\frac{1}{d-c}\int_{p_2c+(1-p_2)d}^{d}F\left(\frac{a+b}{2},y\right)\ {}^{d}d_{p_2,q_2}y-F(b,c)+F\left(\frac{a+b}{2},c\right)\right\}$$
$$-\frac{10p_1p_2}{18(b-a)(d-c)}\left\{F(b,d)-F\left(\frac{a+b}{2},d\right)-F(b,c)+F\left(\frac{a+b}{2},c\right)\right\}.$$

Similarly, we have

$$I_3 = \frac{2}{3(b-a)(d-c)}\left\{\frac{1}{b-a}\int_{p_1a+(1-p_1)b}^{b}F(x,d)\ {}^{b}d_{p_1,q_1}x-\frac{1}{b-a}\int_{p_1a+(1-p_1)b}^{b}F\left(x,\frac{c+d}{2}\right)\ {}^{b}d_{p_1,q_1}x\right.$$
$$\left.-F(a,d)+F\left(a,\frac{c+d}{2}\right)\right\}-\frac{10p_1p_2}{18(b-a)(d-c)}\left\{F(b,d)-F\left(b,\frac{c+d}{2}\right)-F(a,d)+F\left(a,\frac{c+d}{2}\right)\right\}.$$

Also, we have

$$\int_0^1\int_0^1\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}d_{p_1,q_1}t\ d_{p_2,q_2}s \tag{4.6}$$
$$= \frac{1}{(b-a)(d-c)}\left[F(b,d)-F(a,d)-F(b,c)+F(a,c)\right],$$

$$\int_0^1\int_0^1 s\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}d_{p_1,q_1}t\ d_{p_2,q_2}s \tag{4.7}$$
$$= \frac{1}{(b-a)(d-c)}\left\{\frac{1}{q_2(d-c)}\int_{p_2c+(1-p_2)d}^{d}F(b,y)\ {}^{d}d_{p_2,q_2}y\right.$$
$$\left.-\frac{1}{q_2(d-c)}\int_{p_2c+(1-p_2)d}^{d}F(a,y)\ {}^{d}d_{p_2,q_2}y-\frac{1}{q_2}F(b,c)+\frac{1}{q_2}F(a,c)\right\},$$

$$\int_0^1\int_0^1 t\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}d_{p_1,q_1}t\ d_{p_2,q_2}s \tag{4.8}$$
$$= \frac{1}{(b-a)(d-c)}\left\{\frac{1}{q_1(b-a)}\int_{p_1a+(1-p_1)b}^{b}F(x,d)\ {}^{b}d_{p_1,q_1}x\right.$$
$$\left.-\frac{1}{q_1(b-a)}\int_{p_1a+(1-p_1)b}^{b}F(x,c)\ {}^{b}d_{p_1,q_1}x-\frac{1}{q_1}F(a,d)+\frac{1}{q_1}F(a,c)\right\}$$

and

$$\int_0^1\int_0^1 ts\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}d_{p_1,q_1}t\ d_{p_2,q_2}s \tag{4.9}$$
$$= \frac{1}{(b-a)(d-c)}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^{n+1}}{p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{p_1^{n+1}}\right)b,\frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)\right.$$
$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^{n+1}}{p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{p_1^{n+1}}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)$$
$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^{n}}{p_1^{n}}a+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)b,\frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)$$
$$\left.+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^{n}}{p_1^{n}}a+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\right\}$$
$$= \frac{1}{(b-a)(d-c)}\left\{\frac{p_1p_2}{q_1q_2}\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^{n}}{p_1^{n}}a+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\right.\right.$$
$$\left.-\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}F\left(a,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)-\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}F\left(\frac{q_1^{n}}{p_1^{n}}a+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)b,c\right)+F(a,c)\right]$$
$$-\frac{p_1}{q_1}\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^{n}}{p_1^{n}}a+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\right.$$
$$\left.-\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}F\left(a,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\right]$$
$$-\frac{p_2}{q_2}\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^{n}}{p_1^{n}}a+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)-\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}F\left(\frac{q_1^{n}}{p_1^{n}}a+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)b,c\right)\right]$$
$$\left.+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^{n}}{p_1^{n}}a+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\right\}$$

$$= \frac{1}{(b-a)(d-c)}\left\{\frac{(p_1-q_1)(p_2-q_2)}{q_1q_2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^{n}}{p_1^{n}}a+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\right.$$
$$-\frac{p_2-q_2}{q_1q_2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}F\left(a,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)$$

$$-\frac{p_1-q_1}{q_1q_2}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}F\left(\frac{q_1^n}{p_1^n}a+\left(1-\frac{q_1^n}{p_1^n}\right)b,c\right)+\frac{1}{q_1q_2}F(a,c)\Bigg\}$$

$$= \frac{1}{(b-a)(d-c)}\left\{\frac{1}{q_1q_2(b-a)(d-c)}\int_{p_1a+(1-p_1)b}^{b}\int_{p_2c+(1-p_2)d}^{d}F(x,y)\ {}^bd_{p_1,q_1}x\ {}^dd_{p_2,q_2}y\right.$$
$$\left.-\frac{1}{q_1q_2(b-a)}\int_{pa+(1-p)b}^{b}F(x,c)\ {}^bd_{p_1,q_1}x-\frac{1}{q_1q_2(d-c)}\int_{p_2c+(1-p_2)d}^{d}F(a,x)\ {}^dd_{p_2,q_2}y+\frac{1}{q_1q_2}F(a,c)\right\}.$$

From (4.6)-(4.9), we obtain that

$$I_4$$
$$= \frac{1}{(b-a)(d-c)}\left\{\frac{1}{(b-a)(d-c)}\int_{p_1a+(1-p_1)b}^{b}\int_{p_2c+(1-p_2)d}^{d}F(x,y)\ {}^bd_{p_1,q_1}x\ {}^dd_{p_2,q_2}y\right.$$
$$\left.-\frac{1}{b-a}\int_{p_1a+(1-p_1)b}^{b}F(x,c)\ {}^bd_{p_1,q_1}x-\frac{1}{d-c}\int_{p_2c+(1-p_2)d}^{d}F(a,y)\ {}^dd_{p_2,q_2}y+F(a,c)\right\}$$
$$-\frac{5}{6(b-a)(d-c)}\left\{\frac{1}{b-a}\int_{p_1a+(1-p_1)b}^{b}F(x,d)\ {}^bd_{p_1,q_1}x\right.$$
$$-\frac{1}{b-a}\int_{p_1a+(1-p_1)b}^{b}F(x,c)\ {}^bd_{p_1,q_1}x+F(a,c)-F(a,d)+\frac{1}{d-c}\int_{p_2c+(1-p_2)d}^{d}F(b,y)\ {}^dd_{p_2,q_2}y$$
$$\left.-\frac{1}{d-c}\int_{p_2c+(1-p_2)d}^{d}F(a,y)\ {}^dd_{p_2,q_2}y+F(a,c)-F(b,c)\right\}$$
$$+\frac{25}{36(b-a)(d-c)}\{F(b,d)-F(a,d)-F(b,c)+F(a,c)\}.$$

Now using the calculated integrals $(I_1)-(I_4)$ in (4.2) and multiplying the resulting one with $(b-a)(d-c)$, then we have desired equality (4.1) which accomplishes the proof. □

Remark 4.1. *In Lemma 4.1, if we set $p_1=p_2=1$, then the Lemma 4.1 reduces to the [[33] Lemma 3].*

Remark 4.2. *In Lemma 4.1, if we use $p_1=p_2=1$ and $q_1,q_2\to 1^-$, then the Lemma 4.1 becomes [[6] Lemma 1].*

5. Some new $(p_1,q_1)(p_2,q_2)$-Simpson's type inequalities

For the sake of brevity, we present some calculated integrals before providing new estimates.

$$A_1(p,q)=\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|t\,d_{p,q}t=\begin{cases}\frac{p^2-2pq-2q^2}{24[2]_{p,q}[3]_{p,q}} & 0<q<\frac{1}{3}\\ \frac{-7p^2+18pq+18q^2}{216[2]_{p,q}[3]_{p,q}} & \frac{1}{3}\le q<1,\end{cases}\tag{5.1}$$

$$A_2(p,q)=\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|(1-t)\,d_qt=\begin{cases}\frac{[2]_{p,q}\left(2[3]_{p,q}+3q\right)-[3]_{p,q}(1+6q)}{24[2]_{p,q}[3]_{p,q}} & 0<q<\frac{1}{3}\\ \frac{-[2]_{p,q}\left(6[3]_{p,q}+25q\right)+7[3]_{p,q}(1+6q)}{216[2]_{p,q}[3]_{p,q}} & \frac{1}{3}\le q<1,\end{cases}\tag{5.2}$$

$$A_3(p,q)=\int_{\frac{1}{2}}^{1}\left|qt-\frac{5}{6}\right|td_qt=\begin{cases}\frac{15p^2-6pq-6q^2}{24[2]_{p,q}[3]_{p,q}} & 0<q<\frac{5}{6}\\ \frac{25p^2+18pq+18q^2}{216[2]_{p,q}[3]_{p,q}} & \frac{5}{6}\leq q<1\end{cases}\tag{5.3}$$

$$A_4(p,q)=\int_{\frac{1}{2}}^{1}\left|qt-\frac{5}{6}\right|(1-t)\,d_qt=\begin{cases}\frac{[2]_{p,q}\left(10[3]_{p,q}+21q\right)-[3]_{p,q}(5+6q)}{24[2]_{p,q}[3]_{p,q}} & 0<q<\frac{5}{6}\\ \frac{[2]_{p,q}\left(30[3]_{p,q}+7q\right)-5[3]_{p,q}(5+6q)}{216[2]_{p,q}[3]_{p,q}} & \frac{5}{6}\leq q<1,\end{cases}\tag{5.4}$$

$$A_5(p,q)=\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|d_qt=\begin{cases}\frac{3q-[2]_{p,q}}{12[2]_q} & 0<q<\frac{1}{3}\\ \frac{7q-[2]_{p,q}}{36[2]_{p,q}} & \frac{1}{3}\leq q<1,\end{cases},\tag{5.5}$$

$$A_6(p,q)=\int_{\frac{1}{2}}^{1}\left|qt-\frac{5}{6}\right|d_qt=\begin{cases}\frac{-9q+5[2]_{p,q}}{12[2]_{p,q}} & 0<q<\frac{5}{6}\\ \frac{-5q+5[2]_{p,q}}{36[2]_{p,q}} & \frac{5}{6}\leq q<1.\end{cases}.\tag{5.6}$$

Using the identity from the previous section, we now provide some new quantum estimates.

Theorem 5.1. *Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially $(p_1,q_1)(p_2,q_2)$-differentiable function on Δ° such that partial $(p_1,q_1)(p_2,q_2)$-derivative $\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. Then we have following inequality provided that $\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|$ is convex on $[a,b]\times[c,d]$.*

$$\begin{aligned}&\left|{}^{b,d}\mathcal{I}_{(p_1,q_1),(p_2,q_2)}(F)\right| \qquad (5.7)\\ \leq\ & (b-a)(d-c)\left[\left(A_1(p_1,q_1)+A_3(p_1,q_1)\right)\left(A_1(p_2,q_2)+A_3(p_2,q_2)\right)\right.\\ &\times\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|\\ &+\left(A_1(p_1,q_1)+A_3(p_1,q_1)\right)\left(A_2(p_2,q_2)+A_4(p_2,q_2)\right)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|\\ &+\left(A_2(p_1,q_1)+A_4(p_1,q_1)\right)\left(A_1(p_2,q_2)+A_3(p_2,q_2)\right)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|\\ &\left.+\left(A_2(p_1,q_1)+A_4(p_1,q_1)\right)\left(A_2(p_2,q_2)+A_4(p_2,q_2)\right)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|\right],\end{aligned}$$

where $0<q_1<p_1\leq 1, 0<q_2<p_2\leq 1$.

Proof. On taking modulus of the identity of Lemma 4.1, because of the properties of modulus, we find that

$$\left|{}^{b,d}\mathcal{I}_{(p_1,q_1),(p_2,q_2)}(F)\right|\leq(b-a)(d-c)\int_0^1\int_0^1\left|\Lambda_{q_1}(t)\,\Lambda_{q_2}(s)\right|\tag{5.8}$$

$$\times\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(ta+(1-t)\,b,sc+(1-s)\,d\right)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\;d_{p_1,q_1}t\;d_{p_2,q_2}s$$

Now using the convexity of $\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|$, then (5.8) becomes

$$\begin{aligned}&\left|{}^{b,d}\mathcal{I}_{(p_1,q_1),(p_2,q_2)}(F)\right|\\ \leq\;&(b-a)(d-c)\int_0^1\Lambda_{q_2}(s)\left[\int_0^1\Lambda_{q_1}(t)\left\{t\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(a,sc+(1-s)\,d\right)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\right.\right.\\&\left.\left.+(1-t)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(b,sc+(1-s)\,d\right)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\right\}d_{p_1,q_1}t\right]d_{p_2,q_2}s.\end{aligned}\tag{5.9}$$

Now we compute the integrals appear in right side of inequality (5.9)

$$\begin{aligned}&\int_0^1\Lambda_{q_1}(t)\left\{t\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(a,sc+(1-s)\,d\right)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|+(1-t)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(b,sc+(1-s)\,d\right)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\right\}d_{p_1,q_1}t\\ =\;&\int_0^{\frac{1}{2}}t\left|q_1t-\frac{1}{6}\right|\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(a,sc+(1-s)\,d\right)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|d_{p_1,q_1}t\\&+\int_0^{\frac{1}{2}}(1-t)\left|q_1t-\frac{1}{6}\right|\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(b,sc+(1-s)\,d\right)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|d_{p_1,q_1}t\\&+\int_{\frac{1}{2}}^1t\left|q_1t-\frac{5}{6}\right|\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(a,sc+(1-s)\,d\right)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|d_{p_1,q_1}t\\&+\int_0^{\frac{1}{2}}(1-t)\left|q_1t-\frac{5}{6}\right|\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(b,sc+(1-s)\,d\right)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|d_{p_1,q_1}t.\end{aligned}$$

From (5.1)-(5.4), we obtain that

$$\begin{aligned}&\int_0^1\Lambda_{q_1}(t)\left\{t\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(a,sc+(1-s)\,d\right)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|+(1-t)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(b,sc+(1-s)\,d\right)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\right\}d_{p_1,q_1}t\\ =\;&\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(a,sc+(1-s)\,d\right)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\left(A_1(p_1,q_1)+A_3(p_1,q_1)\right)\\&+\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(b,sc+(1-s)\,d\right)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\left(A_2(p_1,q_1)+A_4(p_1,q_1)\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\left|{}^{b,\,d}\mathcal{I}_{q_1,q_2}(F)\right|\\ \leq\;&(b-a)(d-c)\int_0^1\Lambda_{q_2}(s)\left[\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(a,sc+(1-s)\,d\right)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\left(A_1(p_1,q_1)+A_3(p_1,q_1)\right)\right.\end{aligned}$$

$$+\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F\left(b,sc+(1-s)\,d\right)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\left(A_2\left(p_1,q_1\right)+A_4\left(p_1,q_1\right)\right)\Bigg]\,d_{p_2,q_2}s$$

$$\begin{aligned}
\leq\ & (b-a)(d-c)\int_0^1 \Lambda_{q_2}(s)\Bigg[\Bigg\{s\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right| \\
& +(1-s)\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\times\left(A_1(p_1,q_1)+A_3(p_1,q_1)\right)\Bigg\}+\Bigg\{s\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right| \\
& +(1-s)\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\times\left(A_2(p_1,q_1)+A_4(p_1,q_1)\right)\Bigg\}\Bigg]d_{p_2,q_2}s \\
=\ & (b-a)(d-c)\left(A_1(p_1,q_1)+A_3(p_1,q_1)\right)\Bigg[\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\int_0^{\frac{1}{2}} s\left|q_2s-\frac{1}{6}\right|d_{p_2,q_2}s \\
& +\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\int_0^{\frac{1}{2}}(1-s)\left|q_2s-\frac{1}{6}\right|d_{p_2,q_2}s+\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\int_{\frac{1}{2}}^{1} s\left|q_2s-\frac{5}{6}\right|d_{p_2,q_2}s \\
& +\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\int_{\frac{1}{2}}^{1}(1-s)\left|q_2s-\frac{5}{6}\right|d_{p_2,q_2}s\Bigg]+(b-a)(d-c)\left(A_2(p_1,q_1)+A_4(p_1,q_1)\right) \\
& \times\Bigg[\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\int_0^{\frac{1}{2}} s\left|q_2s-\frac{1}{6}\right|d_{p_2,q_2}s+\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\int_0^{\frac{1}{2}}(1-s)\left|q_2s-\frac{1}{6}\right|d_{p_2,q_2}s \\
& +\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\int_{\frac{1}{2}}^{1} s\left|q_2s-\frac{5}{6}\right|d_{p_2,q_2}s+\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\int_{\frac{1}{2}}^{1}(1-s)\left|q_2s-\frac{5}{6}\right|d_{p_2,q_2}s\Bigg].
\end{aligned}$$

From (5.1)-(5.4), we have

$$\begin{aligned}
& \left|{}^{b,\,d}\mathcal{I}_{q_1,q_2}(F)\right| \\
\leq\ & (b-a)(d-c)\Bigg[\left(A_1(p_1,q_1)+A_3(p_1,q_1)\right)\left(A_1(p_2,q_2)+A_3(p_2,q_2)\right)\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right| \\
& +\left(A_1(p_1,q_1)+A_3(p_1,q_1)\right)\left(A_2(p_2,q_2)+A_4(p_2,q_2)\right)\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right| \\
& +\left(A_2(p_1,q_1)+A_4(p_1,q_1)\right)\left(A_1(p_2,q_2)+A_3(p_2,q_2)\right)\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right| \\
& +\left(A_2(p_1,q_1)+A_4(p_1,q_1)\right)\left(A_2(p_2,q_2)+A_4(p_2,q_2)\right)\left|\frac{{}^{b,\,d}\partial^{2}_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^{b}\partial_{p_1,q_1}t\;{}^{d}\partial_{p_2,q_2}s}\right|\Bigg].
\end{aligned}$$

Hence the proof is completed. □

Remark 5.1. *If we take $p_1 = p_2 = 1$ in Theorem 5.1, then Theorem 5.1 reduces to [[33] Theorem 7].*

Remark 5.2. *In Theorem 5.1, if we take $p_1 = p_2 = 1$ and $q_1, q_2 \to 1^-$, then the Theorem 5.1 becomes [6, Theorem 3].*

Theorem 5.2. *Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially $(p_1,q_1)(p_2,q_2)$-differentiable function on Δ° such that partial $(p_1,q_1)(p_2,q_2)$-derivative $\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. If $\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p$ is convex on $[a,b]\times[c,d]$ for some $p>1$ and $\frac{1}{r}+\frac{1}{p}=1$. Then we have following inequality.*

$$\begin{aligned}
&\left|{}^{b,\,d}\mathcal{I}_{(p_1,q_1),(p_2,\,q_2)}(F)\right| \qquad (5.10)\\
\leq\ & (b-a)(d-c)\left(\int_0^1\int_0^1\left|\Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)\right|^r d_{p_1,q_1}t\,d_{p_2,q_2}s\right)^{\frac{1}{r}}\\
&\left[\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p+\frac{[2]_{p_2,q_2}-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\right.\\
&\left.+\frac{[2]_{p_1,q_1}-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p+\frac{\left([2]_{p_1,q_1}-1\right)\left([2]_{p_2,q_2}-1\right)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\right]^{\frac{1}{p}},
\end{aligned}$$

where $0<q_1<p_1\leq 1, 0<q_2<p_2\leq 1$.

Proof. Applying well-known Hölder's inequality for the integrals in right side of (5.8), it is found that

$$\begin{aligned}
&\left|{}^{b,\,d}\mathcal{I}_{(p_1,q_1),(p_2,\,q_2)}(F)\right| \qquad (5.11)\\
\leq\ & (b-a)(d-c)\left[\left(\int_0^1\int_0^1\left|\Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)\right|^r d_{p_1,q_1}t\,d_{p_2,q_2}s\right)^{\frac{1}{r}}\right.\\
&\left.\times\left(\int_0^1\int_0^1\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p d_{p_1,q_1}t\,d_{p_2,q_2}s\right)^{\frac{1}{p}}\right].
\end{aligned}$$

By applying convexity of $\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p$, then (5.11) becomes

$$\begin{aligned}
&\left|{}^{b,\,d}\mathcal{I}_{(p_1,q_1),(p_2,\,q_2)}(F)\right| \qquad (5.12)\\
\leq\ & (b-a)(d-c)\left[\left(\int_0^1\int_0^1\left|\Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)\right|^r d_{p_1,q_1}t\,d_{p_2,q_2}s\right)^{\frac{1}{r}}\right.\\
&\times\left(\int_0^1\int_0^1\left[ts\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p+t(1-s)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\right.\right.\\
&\left.\left.\left.+(1-t)s\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p+(1-t)(1-s)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\right]d_{q_1}t\,d_{q_2}s\right)^{\frac{1}{p}}\right].
\end{aligned}$$

Now, if we apply the concept of Lemma 2.1 for $a=0$ to the above quantum integrals, we attain

$$\int_0^1\int_0^1 ts\,d_{p_1,q_1}t\,d_{p_2,q_2}s=\left(\int_0^1 t\,d_{p_1,q_1}t\right)\left(\int_0^1 s\,d_{p_2,q_2}s\right) \qquad (5.13)$$

$$= \frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}},$$

$$\int_0^1\int_0^1 t(1-s)\,d_{p_1,q_1}t d_{p_2,q_2}s = \frac{[2]_{p_2,q_2}-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}, \tag{5.14}$$

$$\int_0^1\int_0^1 (1-t)\,s d_{p_1,q_1}t d_{p_2,q_2}s = \frac{[2]_{p_1,q_1}-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}, \tag{5.15}$$

$$\int_0^1\int_0^1 (1-t)(1-s)\,d_{p_1,q_1}t d_{p_2,q_2}s = \frac{\left([2]_{p_1,q_1}-1\right)\left([2]_{p_2,q_2}-1\right)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}. \tag{5.16}$$

By substituting the calculated integrals (5.13)-(5.16) in (5.12), then we obtain the desired inequality (5.10) which finishes the proof. □

Remark 5.3. *If we take $p_1 = p_2 = 1$ in Theorem 5.2, then Theorem 5.2 reduces to [[33] Theorem 8].*

Theorem 5.3. *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially $(p_1,q_1)(p_2,q_2)$-differentiable function on Δ° such that partial $(p_1,q_1)(p_2,q_2)$-derivative $\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}$ is continuous and integrable on $[a,b]\times[c,d] \subseteq \Delta^\circ$. If $\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p$ is convex on $[a,b]\times[c,d]$ for some $p \geq 1$. Then we have following inequality.*

$$\begin{aligned}
&\left|{}^{b,\,d}\mathcal{I}_{(p_1,q_1),(p_2,\,q_2)}(F)\right| \qquad (5.17)\\
\leq\ & (b-a)(d-c)\Bigg[A_5^{1-\frac{1}{p}}(p_1,q_1)\,A_5^{1-\frac{1}{p}}(p_2,q_2)\Bigg\{A_1(p_1,q_1)\left(\begin{array}{c} A_1(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p \\ +A_2(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p \end{array}\right)\\
&+A_2(p_1,q_1)\left(\begin{array}{c} A_1(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p \\ +A_2(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p \end{array}\right)\Bigg\}^{\frac{1}{p}}\\
&+A_5^{1-\frac{1}{p}}(p_1,q_1)\,A_6^{1-\frac{1}{p}}(p_2,q_2)\Bigg\{A_1(p_1,q_1)\left(\begin{array}{c} A_3(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p \\ +A_4(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p \end{array}\right)\\
&+A_2(p_1,q_1)\left(\begin{array}{c} A_3(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p \\ +A_4(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p \end{array}\right)\Bigg\}^{\frac{1}{p}}\\
&+A_6^{1-\frac{1}{p}}(p_1,q_1)\,A_5^{1-\frac{1}{p}}(p_2,q_2)\Bigg\{A_3(p_1,q_1)\left(\begin{array}{c} A_1(p_1,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p \\ +A_2(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p \end{array}\right)\\
&+A_4(p_1,q_1)\left(\begin{array}{c} A_1(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p \\ +A_2(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p \end{array}\right)\Bigg\}^{\frac{1}{p}}
\end{aligned}$$

$$+A_6^{1-\frac{1}{p}}(p_1,q_1)A_6^{1-\frac{1}{p}}(p_2,q_2)\left\{A_3(p_1,q_1)\left(\begin{array}{c} A_3(p_2,q_2)\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p \\ +A_4(p_2,q_2)\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p \end{array}\right)\right.$$

$$\left.\left.+A_4(p_1,q_1)\left(\begin{array}{c} A_3(p_2,q_2)\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p \\ +A_4(p_2,q_2)\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p \end{array}\right)\right\}^{\frac{1}{p}}\right]$$

where $0 < q_1 < p_1 \le 1, 0 < q_2 < p_2 \le 1.$

Proof. Applying well-known power mean inequality for integrals in right side of (5.8), it is found that

$$\begin{aligned}
&\left|{}^{b,\ d}\mathcal{I}_{(p_1,q_1),(p_2,\ q_2)}(F)\right| \qquad (5.18)\\
\le\ & (b-a)(d-c)\left[\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|q_1t-\frac{1}{6}\right|\left|q_2s-\frac{1}{6}\right|d_{p_1,q_1}t d_{p_2,q_2}s\right)^{1-\frac{1}{p}}\right.\\
&\times\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|q_1t-\frac{1}{6}\right|\left|q_2s-\frac{1}{6}\right|\times\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p d_{p_1,q_1}t\ d_{p_2,q_2}s\right)^{\frac{1}{p}}\\
&+\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left|q_1t-\frac{1}{6}\right|\left|q_2s-\frac{5}{6}\right|d_{p_1,q_1}t d_{p_2,q_2}s\right)^{1-\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left|q_1t-\frac{1}{6}\right|\left|q_2s-\frac{5}{6}\right|\right.\\
&\times\left.\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p d_{p_1,q_1}t\ d_{p_2,q_2}s\right)^{\frac{1}{p}}\\
&+\left(\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\left|q_1t-\frac{5}{6}\right|\left|q_2s-\frac{1}{6}\right|\ d_{p_1,q_1}t\ d_{p_2,q_2}s\right)^{1-\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\left|q_1t-\frac{5}{6}\right|\left|q_2s-\frac{1}{6}\right|\right.\\
&\times\left.\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p d_{p_1,q_1}t\ d_{p_2,q_2}s\right)^{\frac{1}{p}}\\
&+\left(\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1\left|q_1t-\frac{5}{6}\right|\left|q_2s-\frac{5}{6}\right|\ d_{p_1,q_1}t\ d_{p_2,q_2}s\right)^{1-\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1\left|q_1t-\frac{5}{6}\right|\left|q_2s-\frac{5}{6}\right|\ d_{p_1,q_1}t\ d_{p_2,q_2}s\right.\\
&\times\left.\left.\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p d_{p_1,q_1}t\ d_{p_2,q_2}s\right)^{\frac{1}{p}}\right].
\end{aligned}$$

By applying convexity of $\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p$, then we have

$$\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|q_1t-\frac{1}{6}\right|\left|q_2s-\frac{1}{6}\right|d_{p_1,q_1}t\ d_{p_2,q_2}s\right)^{1-\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|q_1t-\frac{1}{6}\right|\left|q_2s-\frac{1}{6}\right|\right. \qquad (5.19)$$

$$\times\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(ta+(1-t)b,sc+(1-s)d\right)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p d_{p_1,q_1}t\ d_{p_2,q_2}s\Bigg)^{\frac{1}{p}}$$

$$\begin{aligned}
\leq\ & \left(\left(\int_0^{\frac{1}{2}}\left|q_1t-\frac{1}{6}\right|d_{p_1,q_1}t\right)\left(\int_0^{\frac{1}{2}}\left|q_2s-\frac{1}{6}\right|d_{p_2,q_2}s\right)\right)^{1-\frac{1}{p}}\\
& \left[\int_0^{\frac{1}{2}}\left|q_2s-\frac{1}{6}\right|\left\{\int_0^{\frac{1}{2}}\left|q_1t-\frac{1}{6}\right|\left(t\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(a,sc+(1-s)d\right)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p\right.\right.\right.\\
& \left.\left.\left.+(1-t)\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(b,sc+(1-s)d\right)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p\right)d_{p_1,q_1}t\right\}d_{p_2,q_2}s\right]^{\frac{1}{p}}\\
=\ & A_5^{1-\frac{1}{p}}\left(p_1,q_1\right)A_5^{1-\frac{1}{p}}\left(p_2,q_2\right)\\
& \times\left[A_1\left(p_1,q_1\right)\int_0^{\frac{1}{2}}\left|q_2s-\frac{1}{6}\right|\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(a,sc+(1-s)d\right)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p d_{p_2,q_2}s\right.\\
& \left.+A_2\left(p_1,q_1\right)\int_0^{\frac{1}{2}}\left|q_2s-\frac{1}{6}\right|\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(b,sc+(1-s)d\right)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p d_{p_2,q_2}s\right]^{\frac{1}{p}}\\
\leq\ & A_5^{1-\frac{1}{p}}\left(p_1,q_1\right)A_5^{1-\frac{1}{p}}\left(p_2,q_2\right)\left[A_1\left(p_1,q_1\right)\int_0^{\frac{1}{2}}\left|q_2s-\frac{1}{6}\right|\left(s\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(a,c\right)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p\right.\right.\\
& \left.+(1-s)\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(a,d\right)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p\right)d_{p_2,q_2}s+A_2\left(p_1,q_1\right)\int_0^{\frac{1}{2}}\left|q_2s-\frac{1}{6}\right|\\
& \left.\times\left(s\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(a,c\right)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p+(1-s)\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(a,d\right)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p\right)d_{p_2,q_2}s\right]^{\frac{1}{p}}\\
=\ & A_5^{1-\frac{1}{p}}\left(p_1,q_1\right)A_5^{1-\frac{1}{p}}\left(p_2,q_2\right)\left[A_1\left(p_1,q_1\right)\left\{A_1\left(p_2,q_2\right)\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(a,c\right)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p\right.\right.\\
& \left.\left.+A_2\left(p_2,q_2\right)\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(a,d\right)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p\right\}+A_2\left(p_1,q_1\right)\left\{\begin{array}{c}A_1\left(p_2,q_2\right)\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p\\ +A_2\left(p_2,q_2\right)\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p\end{array}\right\}\right]^{\frac{1}{p}}.
\end{aligned}$$

By applying the similar operations, we obtain that

$$\begin{aligned}
& \left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left|q_1t-\frac{1}{6}\right|\left|q_2s-\frac{5}{6}\right|d_{p_1,q_1}t\ d_{p_2,q_2}s\right)^{1-\frac{1}{p}}\\
& \times\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left|q_1t-\frac{1}{6}\right|\left|q_2s-\frac{5}{6}\right|\right.\\
& \left.\times\left|\frac{{}^{b,\ d}\partial^2_{(p_1,q_1),(p_2,q_2)}F\left(ta+(1-t)b,sc+(1-s)d\right)}{{}^{b}\partial_{p_1,q_1}t\ {}^{d}\partial_{p_2,q_2}s}\right|^p d_{p_1,q_1}t\ d_{p_2,q_2}s\right)^{\frac{1}{p}}
\end{aligned}\tag{5.20}$$

$$\leq A_5^{1-\frac{1}{p}}(p_1,q_1)A_6^{1-\frac{1}{p}}(p_2,q_2)\left[A_1(p_1,q_1)\left\{A_3(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\right.\right.$$

$$\left.+A_4(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\right\}$$

$$\left.+A_2(p_1,q_1)\left\{A_3(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p+A_4(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\right\}^{\frac{1}{p}}\right],$$

$$\left(\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}}\left|q_1t-\frac{5}{6}\right|\left|q_2s-\frac{1}{6}\right|\,d_{p_1,q_1}t\,d_{p_2,q_2}s\right)^{1-\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}}\left|q_1t-\frac{5}{6}\right|\left|q_2s-\frac{1}{6}\right|\right. \tag{5.21}$$

$$\left.\times\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\,d_{p_1,q_1}t\,d_{p_2,q_2}s\right)^{\frac{1}{p}}$$

$$\leq A_6^{1-\frac{1}{p}}(p_1,q_1)A_5^{1-\frac{1}{p}}(p_2,q_2)\left[A_3(p_1,q_1)\left\{A_1(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\right.\right.$$

$$\left.+A_2(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\right\}+A_4(p_1,q_1)\left\{A_1(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\right.$$

$$\left.\left.+A_2(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\right\}\right]^{\frac{1}{p}},$$

$$\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left|q_1t-\frac{5}{6}\right|\left|q_2s-\frac{5}{6}\right|\,d_{p_1,q_1}t\,d_{p_2,q_2}s\right)^{1-\frac{1}{p}} \tag{5.22}$$

$$\times\left[\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left|q_1t-\frac{5}{6}\right|\left|q_2s-\frac{5}{6}\right|\,d_{p_1,q_1}t\,d_{p_2,q_2}s\right.$$

$$\left.\times\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\,d_{p_1,q_1}t\,d_{p_2,q_2}s\right]^{\frac{1}{p}}$$

$$\leq A_6^{1-\frac{1}{p}}(p_1,q_1)A_6^{1-\frac{1}{p}}(p_2,q_2)\left[A_3(p_1,q_1)\left\{A_3(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\right.\right.$$

$$\left.\left.+A_4(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\right\}\right]+A_4(p_1,q_1)\left\{A_3(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\right.$$

$$\left.\left.+A_4(p_2,q_2)\left|\frac{{}^{b,\,d}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}\right|^p\right\}\right]^{\frac{1}{p}}.$$

From (5.18)-(5.22), we obtain desired inequality and the proof is ended. □

Remark 5.4. *If we take $p_1 = p_2 = 1$ in Theorem 5.3, then Theorem 5.3 reduces to [[33] Theorem 9].*

6. Additional Simpson's inequalities

We prove some additional estimates for post-quantum Simpson's inequalities in this section.

Lemma 6.1. *Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially $(p_1,q_1)(p_2,q_2)$-differentiable function. If the partial $(p_1,q_1)(p_2,q_2)$-derivative $\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^b\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta$. Then following identity holds for $(p_1,q_1)(p_2,q_2)$-integrals.*

$$\begin{aligned}&{}_a^d\mathcal{I}_{(p_1,q_1),(p_2,q_2)}(F) \\ =\;&(b-a)(d-c)\\ &\times\int_0^1\int_0^1\Lambda_{p_1,q_1}(t)\,\Lambda_{p_2,q_2}(s)\,\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(tb+(1-t)a,sc+(1-s)d)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}d_{p_1,q_1}t\,d_{p_2,q_2}s,\end{aligned}\tag{6.1}$$

where $0<q_1<p_1\le1, 0<q_2<p_2\le1,$

$$\begin{aligned}{}_a^d\mathcal{I}_{(p_1,q_1),(p_2,q_2)}(F)\;=\;&\frac{F\left(\frac{a+b}{2},c\right)+F\left(\frac{a+b}{2},d\right)+4F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+F\left(a,\frac{c+d}{2}\right)+F\left(b,\frac{c+d}{2}\right)}{9}\\ &+\frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{36}\\ &-\frac{1}{6(b-a)}\int_a^{p_1b+(1-p_1)a}\left[F(x,c)+4F\left(x,\frac{c+d}{2}\right)+F(x,d)\right]{}_ad_{p_1,q_1}x\\ &-\frac{1}{6(d-c)}\int_{p_2c+(1-p_2)d}^{d}\left[F(a,y)+4F\left(\frac{a+b}{2},y\right)+F(b,y)\right]{}^dd_{p_2,q_2}y\\ &+\frac{1}{(b-a)(d-c)}\int_a^{p_1b+(1-p_1)a}\int_{p_2c+(1-p_2)d}^{d}F(x,y)\;{}_ad_{p_1,q_1}x\,{}^dd_{p_2,q_2}y\end{aligned}$$

and

$$\Lambda_{p_1,q_1}(t)=\begin{cases}\left(q_1t-\frac{1}{6}\right), & t\in\left[0,\frac{1}{2}\right),\\ \left(q_1t-\frac{5}{6}\right), & t\in\left[\frac{1}{2},1\right],\end{cases}$$

$$\Lambda_{p_2,q_2}(s)=\begin{cases}\left(q_2s-\frac{1}{6}\right), & s\in\left[0,\frac{1}{2}\right),\\ \left(q_2s-\frac{5}{6}\right), & s\in\left[\frac{1}{2},1\right].\end{cases}$$

Proof. The required inequality (6.1) may be obtained by applying the technique employed in the proof of Lemma 4.1 while taking into consideration the definition of $\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^b\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}$. □

Remark 6.1. *If we take $p_1=p_2=1$ in Lemma 6.1, then Lemma 6.1 reduces to [[33] Lemma 4].*

Theorem 6.1. *Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially $(p_1,q_1)(p_2,q_2)$-differentiable function on Δ° such that partial $(p_1,q_1)(p_2,q_2)$-derivative $\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. Then we have following inequality provided that $\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|$ is convex on $[a,b]\times[c,d]$.*

$$\left|{}_a^d\mathcal{I}_{(p_1,q_1),(p_2,q_2)}(F)\right|$$

$$\leq \quad (b-a)(d-c)\Bigg[(A_2(p_1,q_1)+A_4(p_1,q_1))(A_1(p_2,q_2)+A_3(p_2,q_2))\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|$$
$$+(A_2(p_1,q_1)+A_4(p_1,q_1))(A_2(p_2,q_2)+A_4(p_2,q_2))\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|$$
$$+(A_1(p_1,q_1)+A_3(p_1,q_1))(A_1(p_2,q_2)+A_3(p_2,q_2))\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|$$
$$+(A_1(p_1,q_1)+A_3(p_1,q_1))(A_2(p_2,q_2)+A_4(p_2,q_2))\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|\Bigg],$$

where $0<q_1<p_1\leq 1, 0<q_2<p_2\leq 1$.

Remark 6.2. *If we take* $p_1=p_2=1$ *in Theorem 6.1, then Theorem 6.1 reduces to [[33] Theorem 10].*

Theorem 6.2. *Let* $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a twice partially* $(p_1,q_1)(p_2,q_2)$*-differentiable function on* Δ° *such that partial* $(p_1,q_1)(p_2,q_2)$*-derivative* $\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}$ *is continuous and integrable on* $[a,b]\times[c,d]\subseteq\Delta^\circ$. *If* $\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p$ *is convex on* $[a,b]\times[c,d]$ *for some* $p>1$ *and* $\frac{1}{r}+\frac{1}{p}=1$. *Then we have following inequality.*

$$\left|{}_a^d\mathcal{I}_{(p_1,q_1),(p_2,\,q_2)}(F)\right|$$
$$\leq \quad (b-a)(d-c)\left(\int_0^1\int_0^1\left|\Lambda_{p_1,q_1}(t)\,\Lambda_{p_2,q_2}(s)\right|^r d_{p_1,q_1}t\,d_{p_2,q_2}s\right)^{\frac{1}{r}}$$
$$\Bigg[\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p+\frac{[2]_{p_2,q_2}-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p$$
$$+\frac{[2]_{p_1,q_1}-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p+\frac{\left([2]_{p_1,q_1}-1\right)\left([2]_{p_2,q_2}-1\right)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p\Bigg]^{\frac{1}{p}},$$

where $0<q_1<p_1\leq 1, 0<q_2<p_2\leq 1$.

Remark 6.3. *If we take* $p_1=p_2=1$ *in Theorem 6.2, then Theorem 6.2 reduces to [[33] Theorem 11].*

Theorem 6.3. *Let* $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a twice partially* $(p_1,q_1)(p_2,q_2)$*-differentiable function on* Δ° *such that partial* $(p_1,q_1)(p_2,q_2)$*-derivative* $\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}$ *is continuous and integrable on* $[a,b]\times[c,d]\subseteq\Delta^\circ$. *If* $\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p$ *is convex on* $[a,b]\times[c,d]$ *for some* $p\geq 1$. *Then we have following inequality.*

$$\left|{}_a^d\mathcal{I}_{(p_1,q_1),(p_2,\,q_2)}(F)\right|$$
$$\leq \quad (b-a)(d-c)\Big[A_5^{1-\frac{1}{p}}(p_1,q_1)\,A_5^{1-\frac{1}{p}}(p_2,q_2)$$

$$
\begin{aligned}
&\times\left\{A_1(p_1,q_1)\left(A_1(p_2,q_2)\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p+A_2(p_2,q_2)\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p\right)\right.\\
&\left.+A_2(p_1,q_1)\times\left(A_1(p_2,q_2)\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p+A_2(p_2,q_2)\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p\right)\right\}^{\frac{1}{p}}\\
&+A_5^{1-\frac{1}{p}}(p_1,q_1)\,A_6^{1-\frac{1}{p}}(p_2,q_2)\\
&\times\left\{A_1(p_1,q_1)\left(A_3(p_2,q_2)\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p+A_4(p_2,q_2)\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p\right)\right.\\
&\left.+A_2(p_1,q_1)\left(A_3(p_2,q_2)\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p+A_4(p_2,q_2)\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p\right)\right\}^{\frac{1}{p}}\\
&+A_6^{1-\frac{1}{p}}(p_1,q_1)\,A_5^{1-\frac{1}{p}}(p_2,q_2)\\
&\times\left\{A_3(p_1,q_1)\left(A_1(p_1,q_2)\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p+A_2(p_2,q_2)\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p\right)\right.\\
&\left.+A_4(p_1,q_1)\left(A_1(p_2,q_2)\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p+A_2(p_2,q_2)\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p\right)\right\}^{\frac{1}{p}}\\
&+A_6^{1-\frac{1}{p}}(p_1,q_1)\,A_6^{1-\frac{1}{p}}(p_2,q_2)\\
&\times\left\{A_3(p_1,q_1)\left(A_3(p_2,q_2)\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p+A_4(p_2,q_2)\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p\right)\right.\\
&\left.\left.+A_4(p_1,q_1)\left(A_3(p_2,q_2)\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p+A_4(p_2,q_2)\left|\frac{{}_a^d\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\right|^p\right)\right\}^{\frac{1}{p}}\right]
\end{aligned}
$$

where $0 < q_1 < p_1 \leq 1, 0 < q_2 < p_2 \leq 1$.

Remark 6.4. *If we take* $p_1 = p_2 = 1$ *in Theorem 6.3, then Theorem 6.3 reduces to [[33] Theorem 12].*

Lemma 6.2. *Let* $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a twice partially* $(p_1, q_1)(p_2, q_2)$*-differentiable function. If the partial* $(p_1, q_1)(p_2, q_2)$*-derivative* $\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^b\partial_{p_1,q_1}t\,{}_c\partial_{p_2,q_2}s}$ *is continuous and integrable on* $[a, b] \times [c, d] \subseteq \Delta$*. Then following identity holds for* $(p_1, q_1)(p_2, q_2)$*-integrals.*

$$
\begin{aligned}
&{}_c^b\mathcal{I}_{(p_1,q_1),(p_2,q_2)}(F) \qquad (6.2)\\
&= (b-a)(d-c)\int_0^1\int_0^1 \Lambda_{p_1,q_1}(t)\,\Lambda_{p_2,q_2}(s)\,\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(ta+(1-t)b, sd+(1-s)c)}{{}^b\partial_{p_1,q_1}t\,{}_c\partial_{p_2,q_2}s}d_{p_1,q_1}t\,d_{p_2,q_2}s,
\end{aligned}
$$

where $0 < q_1 < p_1 \leq 1, 0 < q_2 < p_2 \leq 1$,

$$
{}_c^b\mathcal{I}_{(p_1,q_1),(p_2,q_2)}(F) = \frac{F\left(\frac{a+b}{2}, c\right) + F\left(\frac{a+b}{2}, d\right) + 4F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + F\left(a, \frac{c+d}{2}\right) + F\left(b, \frac{c+d}{2}\right)}{9}
$$

$$+\frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{36}$$
$$-\frac{1}{6(b-a)}\int_{p_1a+(1-p_1)b}^{b}\left[F(x,c)+4F\left(x,\frac{c+d}{2}\right)+F(x,d)\right]\,{}^{b}d_{p_1,q_1}x$$
$$-\frac{1}{6(d-c)}\int_{c}^{p_2d+(1-p_2)c}\left[F(a,y)+4F\left(\frac{a+b}{2},y\right)+F(b,y)\right]\,{}_{c}d_{p_2,q_2}y$$
$$+\frac{1}{(b-a)(d-c)}\int_{p_1a+(1-p_1)b}^{b}\int_{c}^{p_2d+(1-p_2)c}F(x,y)\,{}^{b}d_{p_1,q_1}x\,{}_{c}d_{p_2,q_2}y$$

and

$$\Lambda_{p_1,q_1}(t)=\begin{cases}\left(q_1t-\frac{1}{6}\right), & t\in\left[0,\frac{1}{2}\right),\\ \left(q_1t-\frac{5}{6}\right), & t\in\left[\frac{1}{2},1\right],\end{cases}$$
$$\Lambda_{p_2,q_2}(s)=\begin{cases}\left(q_2s-\frac{1}{6}\right), & s\in\left[0,\frac{1}{2}\right),\\ \left(q_2s-\frac{5}{6}\right), & s\in\left[\frac{1}{2},1\right].\end{cases}$$

Proof. The required inequality (6.2) may be obtained by applying the technique employed in the proof of Lemma 4.1 while taking into consideration the definition of $\frac{{}_{c}^{b}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^{b}\partial_{p_1,q_1}t\,{}_{c}\partial_{p_2,q_2}s}$. □

Remark 6.5. *If we take $p_1 = p_2 = 1$ in Lemma 6.2, then Lemma 6.2 reduces to [[33] Lemma 5].*

Remark 6.6. *If we take $p_1 = p_2 = 1$ and $q_1, q_2 \to 1^-$ in Lemma 6.2, then Lemma 6.2 reduces to [[6] Lemma 1].*

Theorem 6.4. *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially $(p_1,q_1)(p_2,q_2)$-differentiable function on Δ° such that partial $(p_1,q_1)(p_2,q_2)$-derivative $\frac{{}_{c}^{b}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^{b}\partial_{p_1,q_1}t\,{}_{c}\partial_{p_2,q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. Then we have following inequality provided that $\left|\frac{{}_{c}^{b}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^{b}\partial_{p_1,q_1}t\,{}_{c}\partial_{p_2,q_2}s}\right|$ is convex on $[a,b]\times[c,d]$.*

$$\left|{}_{c}^{b}\mathcal{I}_{(p_1,q_1),(p_2,q_2)}(F)\right|$$
$$\leq (b-a)(d-c)\Bigg[(A_1(p_1,q_1)+A_3(p_1,q_1))(A_1(p_2,q_2)+A_3(p_2,q_2))\left|\frac{{}_{c}^{b}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^{b}\partial_{p_1,q_1}t\,{}_{c}\partial_{p_2,q_2}s}\right|$$
$$+(A_1(p_1,q_1)+A_3(p_1,q_1))(A_2(p_2,q_2)+A_4(p_2,q_2))\left|\frac{{}_{c}^{b}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^{b}\partial_{p_1,q_1}t\,{}_{c}\partial_{p_2,q_2}s}\right|$$
$$+(A_2(p_1,q_1)+A_4(p_1,q_1))(A_1(p_2,q_2)+A_3(p_2,q_2))\left|\frac{{}_{c}^{b}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^{b}\partial_{p_1,q_1}t\,{}_{c}\partial_{p_2,q_2}s}\right|$$
$$+(A_2(p_1,q_1)+A_4(p_1,q_1))(A_2(p_2,q_2)+A_4(p_2,q_2))\left|\frac{{}_{c}^{b}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^{b}\partial_{p_1,q_1}t\,{}_{c}\partial_{p_2,q_2}s}\right|\Bigg],$$

where $0 < q_1 < p_1 \leq 1, 0 < q_2 < p_2 \leq 1$.

Remark 6.7. *If we take $p_1 = p_2 = 1$ in Theorem 6.4, then Theorem 6.4 reduces to [[33] Theorem 13].*

Theorem 6.5. *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially $(p_1,q_1)(p_2,q_2)$-differentiable function on Δ° such that partial $(p_1,q_1)(p_2,q_2)$-derivative $\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. If $\left|\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p$ is convex on $[a,b]\times[c,d]$ for some $p>1$ and $\frac{1}{r}+\frac{1}{p}=1$. Then we have following inequality.*

$$
\begin{aligned}
&\left|{}_c^b\mathcal{I}_{(p_1,q_1),(p_2,\ q_2)}(F)\right| \\
\leq\ & (b-a)(d-c)\left(\int_0^1\int_0^1\left|\Lambda_{p_1,q_1}(t)\,\Lambda_{p_2,q_2}(s)\right|^r d_{p_1,q_1}t d_{p_2,q_2}s\right)^{\frac{1}{r}} \\
&\left[\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left|\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+\frac{[2]_{p_2,q_2}-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left|\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\right. \\
&\left.+\frac{[2]_{p_1,q_1}-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left|\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+\frac{\left([2]_{p_1,q_1}-1\right)\left([2]_{p_2,q_2}-1\right)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left|\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\right]^{\frac{1}{p}},
\end{aligned}
$$

where $0<q_1<p_1\leq 1, 0<q_2<p_2\leq 1$.

Remark 6.8. *If we take $p_1 = p_2 = 1$ in Theorem 6.5, then Theorem 6.5 reduces to [[33] Theorem 14].*

Theorem 6.6. *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially $(p_1,q_1)(p_2,q_2)$-differentiable function on Δ° such that partial $(p_1,q_1)(p_2,q_2)$-derivative $\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. If $\left|\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p$ is convex on $[a,b]\times[c,d]$ for some $p\geq 1$. Then we have following inequality.*

$$
\begin{aligned}
&\left|{}_c^b\mathcal{I}_{(p_1,q_1),(p_2,\ q_2)}(F)\right| \\
\leq\ & (b-a)(d-c)\Bigg[A_5^{1-\frac{1}{p}}(p_1,q_1)\,A_5^{1-\frac{1}{p}}(p_2,q_2) \\
&\times\Bigg\{A_1(p_1,q_1)\left(A_1(p_2,q_2)\left|\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+A_2(p_2,q_2)\left|\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\right) \\
&+A_2(p_1,q_1)\left(A_1(p_2,q_2)\left|\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+A_2(p_2,q_2)\left|\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\right)\Bigg\}^{\frac{1}{p}} \\
&+A_5^{1-\frac{1}{p}}(p_1,q_1)\,A_6^{1-\frac{1}{p}}(p_2,q_2) \\
&\times\Bigg\{A_1(p_1,q_1)\left(A_3(p_2,q_2)\left|\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+A_4(p_2,q_2)\left|\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\right) \\
&+A_2(p_1,q_1)\left(A_3(p_2,q_2)\left|\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+A_4(p_2,q_2)\left|\frac{{}_c^b\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\right)\Bigg\}^{\frac{1}{p}} \\
&+A_6^{1-\frac{1}{p}}(p_1,q_1)\,A_5^{1-\frac{1}{p}}(p_2,q_2)
\end{aligned}
$$

$$\times\left\{A_3(p_1,q_1)\left(A_1(p_1,q_2)\left|\frac{{}^b_c\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+A_2(p_2,q_2)\left|\frac{{}^b_c\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\right)\right.$$

$$\left.+A_4(p_1,q_1)\left(A_1(p_2,q_2)\left|\frac{{}^b_c\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+A_2(p_2,q_2)\left|\frac{{}^b_c\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\right)\right\}^{\frac{1}{p}}$$

$$+A_6^{1-\frac{1}{p}}(p_1,q_1)A_6^{1-\frac{1}{p}}(p_2,q_2)$$

$$\times\left\{A_3(p_1,q_1)\left(A_3(p_2,q_2)\left|\frac{{}^b_c\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+A_4(p_2,q_2)\left|\frac{{}^b_c\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\right)\right.$$

$$\left.\left.+A_4(p_1,q_1)\left(A_3(p_2,q_2)\left|\frac{{}^b_c\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+A_4(p_2,q_2)\left|\frac{{}^b_c\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}^b\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\right)\right\}^{\frac{1}{p}}\right]$$

where $0<q_1<p_1\leq 1, 0<q_2<p_2\leq 1$.

Remark 6.9. *If we take* $p_1=p_2=1$ *in Theorem 6.6, then Theorem 6.6 reduces to [[33] Theorem 15].*

Lemma 6.3. *Let* $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a twice partially* $(p_1,q_1)(p_2,q_2)$*-differentiable function. If the partial* $(p_1,q_1)(p_2,q_2)$*-derivative* $\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}$ *is continuous and integrable on* $[a,b]\times[c,d]\subseteq\Delta$. *Then following identity holds for* $(p_1,q_1)(p_2,q_2)$*-integrals.*

$$\begin{aligned}&{}_{a,c}\mathcal{I}_{(p_1,q_1),(p_2,q_2)}(F)\\ =\ &(b-a)(d-c)\int_0^1\int_0^1\Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}d_{p_1,q_1}t\ d_{p_2,q_2}s,\end{aligned}$$

where $0<q_1<p_1\leq 1, 0<q_2<p_2\leq 1$,

$$\begin{aligned}{}_{a,c}\mathcal{I}_{(p_1,q_1),(p_2,q_2)}(F)\ =\ &\frac{F\left(\frac{a+b}{2},c\right)+F\left(\frac{a+b}{2},d\right)+4F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+F\left(a,\frac{c+d}{2}\right)+F\left(b,\frac{c+d}{2}\right)}{9}\\ &+\frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{36}\\ &-\frac{1}{6(b-a)}\int_a^{p_1b+(1-p_1)a}\left[F(x,c)+4F\left(x,\frac{c+d}{2}\right)+F(x,d)\right]{}_ad_{p_1,q_1}x\\ &-\frac{1}{6(d-c)}\int_c^{p_2d+(1-p_2)c}\left[F(a,y)+4F\left(\frac{a+b}{2},y\right)+F(b,y)\right]{}_cd_{p_2,q_2}y\\ &+\frac{1}{(b-a)(d-c)}\int_a^{p_1b+(1-p_1)a}\int_c^{p_2d+(1-p_2)c}F(x,y)\ {}_ad_{p_1,q_1}x\ {}_cd_{p_2,q_2}y\end{aligned}$$

and

$$\Lambda_{p_1,q_1}(t)=\begin{cases}\left(q_1t-\frac{1}{6}\right), & t\in\left[0,\frac{1}{2}\right),\\ \left(q_1t-\frac{5}{6}\right), & t\in\left[\frac{1}{2},1\right],\end{cases}$$

$$\Lambda_{p_2,q_2}(s)=\begin{cases}\left(q_2 s-\frac{1}{6}\right), & s\in\left[0,\frac{1}{2}\right),\\ \left(q_2 s-\frac{5}{6}\right), & s\in\left[\frac{1}{2},1\right].\end{cases}$$

Proof. The required inequality (6.2) may be obtained by applying the technique employed in the proof of Lemma 4.1 while taking into consideration the definition of $\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}$. □

Remark 6.10. *If we take $p_1=p_2=1$ and $q_1,q_2\to 1^-$ in Lemma 6.3, then Lemma 6.3 reduces to [[6] Lemma 1].*

Theorem 6.7. *Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially $(p_1,q_1)(p_2,q_2)$-differentiable function on Δ° such that partial $(p_1,q_1)(p_2,q_2)$-derivative $\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. Then we have following inequality provided that $\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|$ is convex on $[a,b]\times[c,d]$.*

$$\begin{aligned}&\left|{}_{a,c}\mathcal{I}_{(p_1,q_1),(p_2,q_2)}(F)\right|\\ \leq\ &(b-a)(d-c)\Bigg[\left(A_1(p_1,q_1)+A_3(p_1,q_1)\right)\left(A_1(p_2,q_2)+A_3(p_2,q_2)\right)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|\\ &+\left(A_1(p_1,q_1)+A_3(p_1,q_1)\right)\left(A_2(p_2,q_2)+A_4(p_2,q_2)\right)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|\\ &+\left(A_2(p_1,q_1)+A_4(p_1,q_1)\right)\left(A_1(p_2,q_2)+A_3(p_2,q_2)\right)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|\\ &+\left(A_2(p_1,q_1)+A_4(p_1,q_1)\right)\left(A_2(p_2,q_2)+A_4(p_2,q_2)\right)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|\Bigg],\end{aligned}$$

where $0<q_1<p_1\leq 1, 0<q_2<p_2\leq 1$.

Theorem 6.8. *Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially $(p_1,q_1)(p_2,q_2)$-differentiable function on Δ° such that partial $(p_1,q_1)(p_2,q_2)$-derivative $\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. If $\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p$ is convex on $[a,b]\times[c,d]$ for some $p>1$ and $\frac{1}{r}+\frac{1}{p}=1$. Then we have following inequality.*

$$\begin{aligned}&\left|{}_{a,c}\mathcal{I}_{(p_1,q_1),(p_2,\,q_2)}(F)\right|\\ \leq\ &(b-a)(d-c)\left(\int_0^1\int_0^1\left|\Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)\right|^r d_{p_1,q_1}t\,d_{p_2,q_2}s\right)^{\frac{1}{r}}\\ &\Bigg[\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+\frac{[2]_{p_2,q_2}-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\\ &+\frac{[2]_{p_1,q_1}-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+\frac{\left([2]_{p_1,q_1}-1\right)\left([2]_{p_2,q_2}-1\right)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\Bigg]^{\frac{1}{p}},\end{aligned}$$

where $0<q_1<p_1\leq 1, 0<q_2<p_2\leq 1$.

Theorem 6.9. *Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially $(p_1,q_1)(p_2,q_2)$-differentiable function on Δ° such that partial $(p_1,q_1)(p_2,q_2)$-derivative $\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. If $\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(t,s)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p$ is convex on $[a,b]\times[c,d]$ for some $p\geq1$. Then we have following inequality.*

$$
\begin{aligned}
&\left|{}_{a,c}\mathcal{I}_{(p_1,q_1),(p_2,\,q_2)}(F)\right|\\
\leq\ & (b-a)(d-c)\Bigg[A_5^{1-\frac{1}{p}}(p_1,q_1)A_5^{1-\frac{1}{p}}(p_2,q_2)\\
&\times\Bigg\{A_1(p_1,q_1)\Bigg(A_1(p_2,q_2)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+A_2(p_2,q_2)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\Bigg)\\
&+A_2(p_1,q_1)\Bigg(A_1(p_2,q_2)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+A_2(p_2,q_2)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\Bigg)\Bigg\}^{\frac{1}{p}}\\
&+A_5^{1-\frac{1}{p}}(p_1,q_1)A_6^{1-\frac{1}{p}}(p_2,q_2)\\
&\times\Bigg\{A_1(p_1,q_1)\Bigg(A_3(p_2,q_2)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+A_4(p_2,q_2)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\Bigg)\\
&+A_2(p_1,q_1)\Bigg(A_3(p_2,q_2)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+A_4(p_2,q_2)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\Bigg)\Bigg\}^{\frac{1}{p}}\\
&+A_6^{1-\frac{1}{p}}(p_1,q_1)A_5^{1-\frac{1}{p}}(p_2,q_2)\\
&\times\Bigg\{A_3(p_1,q_1)\Bigg(A_1(p_1,q_2)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+A_2(p_2,q_2)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\Bigg)\\
&+A_4(p_1,q_1)\Bigg(A_1(p_2,q_2)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+A_2(p_2,q_2)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\Bigg)\Bigg\}^{\frac{1}{p}}\\
&+A_6^{1-\frac{1}{p}}(p_1,q_1)A_6^{1-\frac{1}{p}}(p_2,q_2)\\
&\times\Bigg\{A_3(p_1,q_1)\Bigg(A_3(p_2,q_2)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,d)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+A_4(p_2,q_2)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(b,c)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\Bigg)\\
&+A_4(p_1,q_1)\Bigg(A_3(p_2,q_2)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,d)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p+A_4(p_2,q_2)\left|\frac{{}_{a,c}\partial^2_{(p_1,q_1),(p_2,q_2)}F(a,c)}{{}_a\partial_{p_1,q_1}t\ {}_c\partial_{p_2,q_2}s}\right|^p\Bigg)\Bigg\}^{\frac{1}{p}}\Bigg]
\end{aligned}
$$

where $0<q_1<p_1\leq1, 0<q_2<p_2\leq1$.

7. Conclusions

In this work, we proved several Simpson's type inequalities using mixed post-quantum partial derivatives and integrals in the context of (p,q)-calculus. We also demonstrated that the findings of this paper are refinements of comparable findings in the literature. Quantum information theory, an interdisciplinary topic that incorporates computer science, information theory, philosophy, cryptography, and entropy, can benefit from the findings of this study. It is a new and intriguing problem that upcoming researchers can use to establish similar inequalities for various types of convexity in their future work.

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Conflict of interest

The authors declare no conflict of interest.

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