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Research article

On robust weakly ε-efficient solutions for multi-objective fractional programming problems under data uncertainty

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Abstract: In this study, we use the robust optimization techniques to consider a class of multi-objective fractional programming problems in the presence of uncertain data in both of the objective function and the constraint functions. The components of the objective function vector are reported as ratios involving a convex non-negative function and a concave positive function. In addition, on applying a parametric approach, we establish ε-optimality conditions for robust weakly ε-efficient solution. Furthermore, we present some theorems to obtain a robust ε-saddle point for uncertain multi-objective fractional problem.

Keywords: multi-objective fractional programming problem; robust optimization; robust weakly ε-efficient solution; robust ε-saddle point theorem; ε-optimality theorem
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1. Introduction

Nowadays, multi-objective fractional programming problem (MFP) is as a powerful tool to formulate optimization problems in management science and economic theory. MFP problem is a special type of optimization problems in which at least two fractional objective functions should be optimized subject to some certain constraints. The traditional MFP problems consider the situation that all data are reported as certain parameters; see [5, 9, 22, 24, 26] for more studies about MFP problems. However, this assumption can be violated due to the modelling errors, the estimation and the prediction ones which lead to the uncertainty in data of an optimization problem; see [2] for more details. Robust optimization (RO) technique is a method used to model optimization problems in the case of data uncertainty aiming at determining an optimal solution which is the best for all or the most possible realization of the uncertain parameters. Some characterizations of robust optimal solutions for uncertain fractional optimization and applications [29] is investigated by Sun et al. in 2017 by using the properties of subdifferential sum formulae and introducing some robust basic subdifferential constraint qualifications, also, they considered the multi-objective fractional programming problem in the case of data uncertainty in the objective function and the parameters of the constraints and used the closedness constraint qualification to present some conditions for determining the robust weakly efficient solutions. Debnath and Qin have studied the problem of robust optimality and duality for minimax fractional programming problems with support functions [6] in which they have considered a class of robust nondifferentiable minimax fractional programming problems containing support functions in both the objective functions and in the constraints by using the robust subdifferentiable constraint qualification. For more details; see [1, 3, 4, 11, 12, 27]. In many cases, it is practically impossible to find the exact optimal solution of an optimization problem. In this situation, the theory of approximate solutions is used to determine an approximation of the optimal solution of the optimization problem. Many scholars have presented the duality theorems and the optimality conditions for approximate solutions in the situation that all data has certain values; see [10, 15, 16, 18, 25] for more details.

In recent years, many studies have been presented on the optimality conditions and the duality results of the robust approximation solution in the uncertain optimization problems. For example, Lee and Lee [19, 20] proposed ε-duality and ε-optimality theorems for the convex optimization and uncertain convex-concave fractional optimization problems with the geometric constraint set. Sun et al. [28] used a robust type of the closed convex constraint qualification and investigated the necessary and sufficient conditions for the optimality of the robust approximate solutions of an uncertain convex programming problem. Also, they presented the strong and weak duality theorems for the robust approximate solutions by introducing the Wolfe-dual and Mond-Weir dual and generalized it to the multi-objective programming problems. For more studies about the approximate solutions of the uncertain optimization problems; see [30, 31, 33].

On the other hand, the saddle point theorems have attracted the attentions of many scholars due to their relationship with the optimal solution of the primal and dual problems. For example, [23, 25, 32] considered the ε-saddle point in the situation that all parameters have certain values and [8, 17] presented the weak vector saddle point theorems for the uncertain multi-objective optimization problems. Given the importance of the uncertain multi-objective fractional programming problems, the approximate solutions and the saddle points, this paper aims to present the robust weakly ε-efficient optimality conditions and the robust ε-saddle point theorems for the uncertain multi-objective fractional programming (UMFP) problems. For this purpose, we use a parametric approach to convert an uncertain multi-objective fractional programming problem into a non-fractional multi-objective programming problem and then closed convex constraint qualification and the scalarization of the results are used to generalize the robust ε-optimality and the robust ε-saddle point theorem of the uncertain convex programming problem to the uncertain multi-objective fractional programming problem.

The rest of this paper is organized as follows: In section 2, we review some preliminaries and basic definitions. In section 3, we consider the necessary and sufficient optimality conditions for the uncertain multi-objective fractional programming problems by using the convex closed constraint qualification. In section 4, we propose robust ε-saddle point theorem for the UMFP problems. Finally, in section 5, we submite the conclusion of the paper.

2. Preliminaries

In this section we review some preliminaries and basic concepts which are used throughout this paper.

Suppose that $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$. The function f is convex, if $f(\mu x+(1-\mu)x') \leq \mu f(x)+(1-\mu)f(x')$, for all $x, x' \in \mathbb{R}^n$ and any $\mu \in [0, 1]$. The domain (effective domain) and the epigraph of f are the nonempty sets which are defined by $\mathrm{dom}f = \{x \in \mathbb{R}^n : f(x) < +\infty\}$ and $\mathrm{epi}f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : r \geq f(x)\}$, respectively. If f is a proper lower semi-continuous convex function, then its conjugate function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $f^*(x^*) = sup\{\langle x^*, x\rangle - f(x)|x \in \mathbb{R}^n\}$, where $\langle ., .\rangle$ denotes the inner product on $\mathbb{R}^n$.

The indicator function of the nonempty set $C \subseteq X$, $\delta_C : X \to \mathbb{R} \cup \{+\infty\}$ is defined as follows:

$$\delta_C = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

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Let $\varepsilon \geq 0$, the ε-subdifferential of f at $a \in \mathrm{dom}f$ is defined as follows:

$$\partial_\varepsilon f(a) = \Big\{a^* \in \mathbb{R}^n : f(x) - f(a) \geq \langle a^*, x - a\rangle - \varepsilon, \forall x \in \mathbb{R}^n\Big\}.$$

If $\varepsilon = 0$, then $\partial_0 f(a)$ is the classical subdifferential of f at $a \in \mathrm{dom}f$.

Throughout this paper, the convex hull and the closure of $A \subseteq \mathbb{R}^n$ are denoted by $\mathrm{co}A$ and $\mathrm{cl}A$, respectively. For any closed convex set $C \subseteq \mathbb{R}^n$ and $\varepsilon \geq 0$, the ε-normal cone of C at $x \in \mathbb{R}^n$, denoted by $N^\varepsilon_C(x)$, is defined as follows:

$$N^\varepsilon_C(x) = \Big\{\bar{x} \in \mathbb{R}^n : \langle \bar{x}, y - x\rangle \leq \varepsilon, \ \forall y \in C\Big\}.$$

If $\varepsilon = 0$, then $N_C(x)$ is the classical normal cone of C at $x \in C$, also if C is a closed convex cone, then $N_C(0)$ is denoted by C^*. In the following, we present some lemmas which help us to prove our main results.

Lemma 2.1 ([13]). *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semi-continuous convex function; then*

$$epi(f + g)^* = epi f^* + epi\, g^*.$$

The following lemma shows that $\mathrm{epi}f^*$ can be expressed by ε-subdifferentials.

Lemma 2.2 ([14]). *Assume that $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is a proper lower semi-continuous convex function and $a \in dom f$; then*

$$epi f^* = \bigcup_{\varepsilon\geq 0} \Big\{(b, \langle b, a\rangle + \varepsilon - f(a))\Big| b \in \partial_\varepsilon f(a)\Big\}.$$

Lemma 2.3 ([21]). *Let $f_j : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $j \in J$, be proper lower semi-continuous convex functions with $\sup\limits_{j\in J} f_j(x_0) < +\infty$, for some $x_0 \in X$; then*

$$epi\Big(\sup_{j\in J} f_j\Big)^* = cl\Big(co \bigcup_{j\in J} epi f^*_j\Big),$$

where J is an arbitrary index set.

Lemma 2.4 ([11]). *Suppose that $h_j : \mathbb{R}^n \times \mathbb{R}^{q_0} \to \mathbb{R}$, $j = 1, \dots, m$, are continuous functions such that, for any $w_j \in \mathcal{W}_j$, $h_j(., w_j)$ is a convex function; then*

$$\bigcup_{w_j \in \mathcal{W}_j, \lambda_j \geq 0} epi\Big(\sum_{j=1}^{m} \lambda_j h_j(., w_j)\Big)^*,$$

is a cone.

Lemma 2.5 ([11]). *Let $h_j : \mathbb{R}^n \times \mathbb{R}^{q_0} \to \mathbb{R}$, $j = 1, \dots, m$, be continuous functions and C be a closed convex cone on $\mathbb{R}^n$. Also, suppose that $\mathcal{W}_j \subseteq \mathbb{R}^{q_0}$, $j = 1, \dots, m$, are convex sets and for any $w_j \in \mathcal{W}_j$, $h_j(., w_j)$ is a convex function and for any $x \in \mathbb{R}^n$, $h_j(x, .)$ is a concave function; then,*

$$\bigcup_{w_j \in \mathcal{W}_j, \lambda_j \geq 0} epi\Big(\sum_{j=1}^{m} \lambda_j g_j(., w_j)\Big)^* + C^* \times \mathbb{R}_+,$$

is convex.

Lemma 2.6 ([11]). *Assume that $h_j : \mathbb{R}^n \times \mathbb{R}^{q_0} \to \mathbb{R}$, $j = 1, \dots, m$, are continuous functions such that for any $w_j \in \mathbb{R}^{q_0}$, $h_j(., w_j)$ is a convex function and C is a closed convex cone in $\mathbb{R}^n$. Furthermore, suppose that $\mathcal{W}_j$, $j = 1, \dots, m$, are compact and convex sets and there is $x_0 \in C$ such that*

$$h_j(x_0, w_j) < 0, \quad \forall w_j \in \mathcal{W}_j, \quad j = 1, \dots, m.$$

Then

$$\bigcup_{w_j \in \mathcal{W}_j, \lambda_j \geq 0} epi\Big(\sum_{j=1}^{m} \lambda_j g_j(., w_j)\Big)^* + C^* \times \mathbb{R}_+,$$

is a closed set.

3. ε-optimality theorems

In this section, we consider the uncertain multi-objective fractional programming (UMFP) problem with a geometric constraint set as follows:

$$\begin{aligned} \text{(UMFP)} \qquad & \min \ \Big(\frac{f_1(x, u_1)}{g_1(x, v_1)}, \dots, \frac{f_l(x, u_l)}{g_l(x, v_l)}\Big) \\ & s.t. \ \ h_j(x, w_j) \leq 0, \qquad j = 1, \dots, m, \\ & \qquad x \in C, \end{aligned}$$

where $C \subseteq \mathbb{R}^n$ is a closed convex cone. Assume that $f_i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \dots, l$ and $h_j : \mathbb{R}^n \times \mathbb{R}^{q_0} \to \mathbb{R}$, $j = 1, \dots, m$. Also, suppose that u_i, v_i, w_j are uncertain parameters which belong to the convex and compact uncertainty sets $\mathcal{U}_i \subseteq \mathbb{R}^p$, $\mathcal{V}_i \subseteq \mathbb{R}^q$ and $\mathcal{W}_j \subseteq \mathbb{R}^{q_0}$, respectively.

Throughout this paper, we assume that, for any $u_i \in \mathcal{U}_i$, $f_i(x, u_i)$ is a convex non-negative function and for any $v_i \in \mathcal{V}_i$, $g_i(x, v_i)$ is a concave positive function, for all $i = 1, \dots, l$. The robust counterpart of UMFP problem, namely RMFP, is formulated as follows:

$$\begin{aligned} \text{(RMFP)} \qquad & \min \ \Big(\frac{\max_{u_1 \in \mathcal{U}_1} f_1(x, u_1)}{\min_{v_1 \in \mathcal{V}_1} g_1(x, v_1)}, \dots, \frac{\max_{u_l \in \mathcal{U}_l} f_l(x, u_l)}{\min_{v_l \in \mathcal{V}_l} g_l(x, v_l)}\Big) \\ & s.t. \ \ h_j(x, w_j) \leq 0, \quad \forall w_j \in \mathcal{W}_j, \quad j = 1, \dots, m, \\ & \qquad x \in C. \end{aligned}$$

Clearly, $F = \{x \in C : h_j(x, w_j) \leq 0, \forall w_j \in \mathcal{W}_j, j = 1, \ldots, m\}$ is a feasible solution set for RMFP.

Definition 3.1. *Let $\varepsilon \in \mathbb{R}^l_+$. A point $\bar{x} \in F$ is a robust weakly ε-efficient solution of UMFP problem if and only if $\bar{x}$ is a weakly ε-efficient solution of RMFP.*

Definition 3.2. *Let $\varepsilon \in \mathbb{R}^l_+$. A point $\bar{x} \in F$ is a weakly ε-efficient solution of RMFP problem if and only if there does not exist any $x \in F$ such that*

$$\frac{\max_{u_i \in \mathcal{U}_i} f_i(x, u_i)}{\min_{v_i \in \mathcal{V}_i} g_i(x, v_i)} < \frac{\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)}{\min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i)} - \varepsilon_i \qquad \text{for all } \ i = 1, \ldots, l.$$

In the following, we use the parametric approach, introduced by Dinkelbach [7], to associate the corresponding RMFP model to the robust multi-objective convex optimization problem (RMCP) with a parameter vector $r \in \mathbb{R}^l_+$:

$$\begin{aligned}
&\text{(RMCP)} \\
&\min \Big(\max_{u_1 \in \mathcal{U}_1} f_1(x, u_1) - r_1 \min_{v_1 \in \mathcal{V}_1} g_1(x, v_1), \ldots, \max_{u_l \in \mathcal{U}_l} f_l(x, u_l) - r_l \min_{v_l \in \mathcal{V}_l} g_l(x, v_l) \Big) \\
&\text{s.t.} \qquad h_j(x, w_j) \leq 0, \quad \forall w_j \in W_j, \quad j = 1, \ldots, m, \\
&\qquad\qquad x \in C.
\end{aligned}$$

Definition 3.3. *Let $\varepsilon \in \mathbb{R}^l_+$. A point $\bar{x} \in F$ is a weakly ε-efficient solution of RMCP problem if and only if there does not exist any $x \in F$ such that*

$$\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - r_i \min_{v_i \in \mathcal{V}_i} g_i(x, v_i) < \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) - r_i \min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i) - \varepsilon_i,$$

for all $i = 1, \ldots, l$.

Lemma 3.4. *Let $f_i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, i = 1, \ldots, l$, be functions such that $f_i(., u_i), u_i \in \mathcal{U}_i$, is a convex function and $g_i(., v_i), v_i \in \mathcal{V}_i$, is a concave function. Moreover, suppose that $\bar{x} \in F$ and $\varepsilon \in \mathbb{R}^l_+$. If $\bar{r}_i = \max_{(u_i, v_i) \in \mathcal{U}_i \times \mathcal{V}_i} \frac{f_i(\bar{x}, u_i)}{g_i(\bar{x}, v_i)} - \varepsilon_i \geq 0$, $i = 1, \ldots, l$, then the following statements are equivalent:*

(i) $\bar{x}$ is a weakly ε-efficient solution of RMFP;
(ii) $\bar{x}$ is a weakly $\bar{\varepsilon}$-efficient solution of RMCP;
(iii) there is $\bar{\mu} \in \Delta^l$, such that

$$\sum_{i=1}^{l} \bar{\mu}_i \Big[\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \bar{r}_i \min_{v_i \in \mathcal{V}_i} g_i(x, v_i) \Big] \geq \sum_{i=1}^{l} \bar{\mu}_i \Big[\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) - \bar{r}_i \min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i) \Big] - \sum_{i=1}^{l} \bar{\mu}_i \bar{\varepsilon}_i,$$

for all $x \in F$. Where

$$\bar{\varepsilon} = (\varepsilon_1 \min_{v_1 \in \mathcal{V}_1} g_1(\bar{x}, v_1), \ldots, \varepsilon_l \min_{v_l \in \mathcal{V}_l} g_l(\bar{x}, v_l)),$$

and

$$\Delta^l = \Big\{ \delta \in \mathbb{R}^l_+ : \sum_{i=1}^{l} \delta_i = 1 \Big\}.$$

Proof. In the following, the equivalence of (i) and (ii) is proved.
Suppose that $\bar{x} \in F$ is a weakly ε-efficient solution of RMFP, so there does not exist any $x \in F$ such that

$$\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \bar{r}_i \min_{v_i \in \mathcal{V}_i} g_i(x, v_i) < 0, \qquad i = 1, \ldots, l.$$

On the other hand,

$$\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) - \bar{r}_i \min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i) - \varepsilon_i \min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i) = 0, \qquad i = 1, \ldots, l,$$

hence, we have

$$\begin{aligned} &\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \bar{r}_i \min_{v_i \in \mathcal{V}_i} g_i(x, v_i) \\ &\qquad < \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) - \bar{r}_i \min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i) - \varepsilon_i \min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i), \qquad i = 1, \ldots, l. \end{aligned}$$

This means that $\bar{x} \in F$ is a weakly $\bar{\varepsilon}$-efficient solution of RMCP.
(ii) $\Rightarrow$ (iii)
Assume that,

$$\phi(x) = (\phi_1(x), \ldots, \phi_l(x)), \qquad \text{for all } x \in F,$$

where

$$\phi_i(x) = \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \bar{r}_i \min_{v_i \in \mathcal{V}_i} g_i(x, v_i), \qquad i = 1, \ldots, l.$$

Therefore, $\phi_i(x)$, $i = 1, \ldots, l$, are convex functions. On the other hand, since $\bar{x}$ is a weakly ε-efficient solution of RMCP, so there does not exist $x \in F$ such that $\phi_i(x) < 0$ for all $i = 1, \ldots, l$. By using the generalized Gordan theorem, there exist $\bar{\mu}_i \geq 0, i = 1, \ldots, l, \sum_{i=1}^{l} \bar{\mu}_i = 1$, such that

$$\sum_{i=1}^{l} \bar{\mu}_i \phi_i(x) \geq 0, \qquad \text{for all } x \in F.$$

This means that, the statement (iii) holds.
(iii) $\Rightarrow$ (ii)
Assume that the statement (ii) does not hold. Therefore, $\bar{x}$ is not a weakly robust $\bar{\varepsilon}$-efficient for RMCP. This means that the statement (iii) cannot be held. □

Lemma 3.5. *Assume that $f_i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}, i = 1, \ldots, l$, $h_j : \mathbb{R}^n \times \mathbb{R}^{q_0}$, $j = 1, \ldots, m$, are continuous functions such that $f_i(., u_i)$, $u_i \in \mathcal{U}_i$ and $h_j(., w_i)$, $w_j \in \mathcal{W}_j$ are convex functions and $f_i(x, .)$, $x \in \mathbb{R}^n$ is a concave function. Furthermore, let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, i = 1, \ldots, l$, be continuous concave-convex functions. Also, suppose that $\mathcal{U}_i \subseteq \mathbb{R}^p, \mathcal{V}_i \subseteq \mathbb{R}^q, i = 1, \ldots, l$, and $\mathcal{W}_j \subseteq \mathbb{R}^{q_0}$, $j = 1, \ldots, m$, are convex and compact sets. Let $(r, \mu) \in \mathbb{R}^l_+ \times \Delta^l$ and let $C \subseteq \mathbb{R}^n$ be a closed convex cone. If $F \neq 0$, then the following statements are equivalent:*

(i)

$$\Big\{x \in C \Big| h_j(x, w_j) \leq 0, \forall w_j \in \mathcal{W}_j, j = 1, \ldots, m\Big\}$$
$$\subseteq \Big\{x \in \mathbb{R}^n \Big| \sum_{i=1}^{l} \mu_i(\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - r_i \min_{v_i \in \mathcal{V}_i} g_i(x, v_i)) \geq 0\Big\};$$

(ii)

$$(0,0) \in \sum_{i=1}^{l} \Big[epi\Big(\max_{u_i \in \mathcal{U}_i} (\mu_i f_i(., u_i))\Big)^* + epi\Big(- r_i \min_{v_i \in \mathcal{V}_i} (\mu_i g_i(., v_i))\Big)^* \Big]$$
$$+ \, cl \, co\Big(\bigcup_{w_j \in \mathcal{W}_j, \lambda_j \geq 0} epi(\sum_{j=1}^{m} \lambda_j h_j(., w_j))^* + C^* \times \mathbb{R}_+ \Big);$$

(iii)

$$(0,0) \in \sum_{i=1}^{l} \Big[\bigcup_{u_i \in \mathcal{U}_i} epi(\mu_i f_i(., u_i))^* + \bigcup_{v_i \in \mathcal{V}_i} epi(- r_i \mu_i g_i(., v_i))^* \Big]$$
$$+ \, cl \, co\Big(\bigcup_{w_j \in \mathcal{W}_j, \lambda_j \geq 0} epi(\sum_{j=1}^{m} \lambda_j h_j(., w_j))^* + C^* \times \mathbb{R}_+ \Big).$$

Proof. It is very easy to verify and prove the equivalence of (i) and (ii) through the following few lines. Let $f(x) = \sum_{i=1}^{l} \mu_i(\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - r_i \min_{v_i \in \mathcal{V}_i} g_i(x, v_i))$; then applying [20, Lemma 2.1], the statement (i) is equivalent to

$$(0,0) \in \mathrm{epi} f^* + \mathrm{cl}\ \mathrm{co}\Big(\bigcup_{w_j \in \mathcal{W}_j, \lambda_j \geq 0} \mathrm{epi}(\sum_{j=1}^{m} \lambda_j h_j(., w_j))^* + C^* \times \mathbb{R}_+ \Big).$$

Since $\max_{u_i \in \mathcal{U}_i} (\mu_i f_i(., u_i))$ and $-r_i \min_{v_i \in \mathcal{V}_i} (\mu_i g_i(., v_i))$ are continuous convex functions, so by using Lemma 2.1, we have

$$\mathrm{epi} f^* = \sum_{i=1}^{l} \Big[\mathrm{epi}\Big(\max_{u_i \in \mathcal{U}_i} (\mu_i f_i(., u_i))\Big)^* + \mathrm{epi}\Big(- r_i \min_{v_i \in \mathcal{V}_i} (\mu_i g_i(., v_i))\Big)^* \Big].$$

It means that, the statements (i) and (ii) are equivalent.
In the following, we prove the equivalence of (ii) and (iii). For this purpose, it is sufficient to show that

$$\mathrm{epi}(\max_{u_i \in \mathcal{U}_i} \mu_i f_i(., u_i))^* = \bigcup_{u_i \in \mathcal{U}_i} \mathrm{epi}(\mu_i f_i(., u_i))^*,$$

and

$$\mathrm{epi}(- r_i \min_{v_i \in \mathcal{V}_i} \mu_i g_i(., v_i))^* = \bigcup_{v_i \in \mathcal{V}_i} \mathrm{epi}(- r_i \mu_i g_i(., v_i))^*.$$

According to Lemma 2.3, we have

$$\operatorname{epi}(\max_{u_i\in\mathcal{U}_i}\mu_i f_i(.,u_i))^* = \operatorname{cl}\operatorname{co}\bigcup_{u_i\in\mathcal{U}_i}\operatorname{epi}(\mu_i f_i(.,u_i))^*.$$

Since f_i's are continuous convex-concave functions and g_i's are continuous concave-convex functions, therefore, it is easy to show that $\bigcup_{u_i\in\mathcal{U}_i}\operatorname{epi}(\mu_i f_i(.,u_i))^*$ and $\bigcup_{v_i\in\mathcal{V}_i}\operatorname{epi}(-r_i\mu_i g_i(.,v_i))^*$ are closed convex sets and this completes the proof. □

In the following theorem, we propose a necessary optimality condition for the robust weakly ε-efficient solution of UMFP problem.

Theorem 3.6. *Let $f_i:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}, i=1,\dots,l$, $h_j:\mathbb{R}^n\times\mathbb{R}^{q_0}$, $j=1,\dots,m$, are continuous functions such that $f_i(.,u_i)$, $u_i\in\mathcal{U}_i$ and $h_j(.,w_j)$, $w_j\in\mathcal{W}_j$ are convex functions and $f_i(x,.)$, $x\in\mathbb{R}^n$ is a concave function. Furthermore, let $g_i:\mathbb{R}^n\times\mathbb{R}^q\to\mathbb{R}, i=1,\dots,l$, are continuous concave-convex functions. Assume that $\varepsilon\in\mathbb{R}^l_+$ and $\bar r_i=\max_{(u_i,v_i)\in\mathcal{U}_i\times\mathcal{V}_i}\frac{f_i(\bar x,u_i)}{g_i(\bar x,v_i)}-\varepsilon_i\geq 0$. If $\bar x\in F$ is a weakly ε-efficient solution of RMFP and $\bigcup_{w_j\in\mathcal{W}_j,\lambda_j\geq 0}(\sum_{i=1}^m\lambda_j h_j(.,w_j))^*+C^*\times\mathbb{R}_+$ is a closed convex set, then there exist $(\bar u,\bar v,\bar w)\in\mathcal{U}\times\mathcal{V}\times\mathcal{W}$ and $(\bar\mu,\bar\lambda,\alpha,\beta,\gamma)\in\Delta^l\times\mathbb{R}^m_+\times\mathbb{R}^l_+\times\mathbb{R}^l_+\times\mathbb{R}^{m+1}_+$, such that*

$$0\in\sum_{i=1}^l\Big[\partial_{\alpha_i}(\bar\mu_i f_i(.,\bar u_i))(\bar x)+\partial_{\beta_i}(-\bar r_i\bar\mu_i g_i(.,\bar v_i))(\bar x)\Big]+\sum_{j=1}^m\partial_{\gamma_j}(\bar\lambda_j h_j(.,\bar w_j))(\bar x)+N_C^{\gamma_{m+1}}(\bar x),\tag{3.1}$$

$$\max_{u_i\in\mathcal{U}_i}f_i(\bar x,u_i)-\bar r_i\min_{v_i\in\mathcal{V}_i}g_i(\bar x_i,v_i)=\varepsilon_i\min_{v_i\in\mathcal{V}_i}g_i(\bar x,v_i),\quad i=1,\dots,l,\tag{3.2}$$

$$\sum_{i=1}^l(\alpha_i+\beta_i)-\sum_{i=1}^l\varepsilon_i\bar\mu_i\min_{v_i\in\mathcal{V}_i}g_i(\bar x,v_i)+\sum_{k=1}^{m+1}\gamma_{k+1}\leq\sum_{j=1}^m\bar\lambda_j h_j(\bar x,\bar w_j),\tag{3.3}$$

where $\mathcal{U}=\mathcal{U}_1\times\cdots\times\mathcal{U}_l$, $\mathcal{V}=\mathcal{V}_1\times\cdots\times\mathcal{V}_l$, and $\mathcal{W}=\mathcal{W}_1\times\cdots\times\mathcal{W}_m$.

Proof. Assume that $\bar x$ is a weakly ε-efficient solution of RMFP. Regarding the statement (iii) in Lemma 3.4, there exists $\bar\mu\in\Delta^l$, such that

$$\begin{aligned}&\sum_{i=1}^l\bar\mu_i[\max_{u_i\in\mathcal{U}_i}f_i(x,u_i)-\bar r_i\min_{v_i\in\mathcal{V}_i}g_i(x,v_i)]\\&\qquad\geq\sum_{i=1}^l\bar\mu_i[\max_{u_i\in\mathcal{U}_i}f_i(\bar x,u_i)-\bar r_i\min_{v_i\in\mathcal{V}_i}g_i(\bar x,v_i)]+\sum_{i=1}^l\bar\mu_i\varepsilon_i\min_{v_i\in\mathcal{V}_i}g_i(x,v_i),\ \forall x\in F,\end{aligned}\tag{3.4}$$

so relation (3.4) can be rewritten as follows:

$$\sum_{i=1}^l\bar\mu_i[\max_{u_i\in\mathcal{U}_i}f_i(x,u_i)-\bar r_i\min_{v_i\in\mathcal{V}_i}g_i(x,v_i)]\geq 0,$$

thus, using Lemma 3.5, we have

$$(0,0)\in\sum_{i=1}^l\Big[\bigcup_{u_i\in\mathcal{U}_i}\operatorname{epi}(\bar\mu_i f_i(.,u_i))^*+\bigcup_{v_i\in\mathcal{V}_i}\operatorname{epi}(-\bar r_i\bar\mu_i g_i(.,v_i))^*\Big]$$

$$+\,\mathrm{cl\,co}\Big(\bigcup_{w_j\in\mathcal{W}_j,\lambda_j\geq 0}\mathrm{epi}(\sum_{j=1}^m\lambda_j h_j(.,w_j))^*+C^*\times\mathbb{R}_+\Big),$$

according to the assumption,

$$(0,0)\in\sum_{i=1}^l\Big[\bigcup_{u_i\in\mathcal{U}_i}\mathrm{epi}(\bar\mu_i f_i(.,u_i))^*+\bigcup_{v_i\in\mathcal{V}_i}\mathrm{epi}(-\bar r_i\bar\mu_i g_i(.,v_i))^*\Big]$$
$$+\bigcup_{w_j\in\mathcal{W}_j,\lambda_j\geq 0}\mathrm{epi}(\sum_{j=1}^m\lambda_j h_j(.,w_j))^*+C^*\times\mathbb{R}_+,$$

thus, there are $\bar u_i\in\mathcal{U}_i,\bar v_i\in\mathcal{V}_i,\bar w_j\in\mathcal{W}_j$ and $\bar\lambda_j\geq 0$, for all $i=1,\ldots,l$, $j=1,\ldots,m$, such that

$$(0,0)\in\sum_{i=1}^l\Big[\mathrm{epi}(\bar\mu_i f_i(.,\bar u_i))^*+\mathrm{epi}(-\bar r_i\bar\mu_i g_i(.,\bar v_i))^*\Big]+\mathrm{epi}(\sum_{j=1}^m\bar\lambda_j h_j(.,\bar w_j))^*+\mathrm{epi}\delta_C^*.$$

According to Lemma 2.2, we have

$$(0,0)\in\sum_{i=1}^l\Big[\Big\{\bigcup_{\alpha_i\geq 0}(a^{1i},\langle a^{1i},\bar x\rangle+\alpha_i-\bar\mu_i f_i(\bar x,\bar u_i))\Big|a^{1i}\in\partial_{\alpha_i}(\bar\mu_i f_i(.,\bar u_i))(\bar x)\Big\}$$
$$+\Big\{\bigcup_{\beta_i\geq 0}(a^{2i},\langle a^{2i},\bar x\rangle+\beta_i-\bar r_i\bar\mu_i g_i(\bar x,\bar v_i))\Big|a^{2i}\in\partial_{\beta_i}(-\bar r_i\bar\mu_i g_i(.,\bar v_i))(\bar x)\Big\}\Big]$$
$$+\Big\{\bigcup_{\gamma_i\geq 0}(a^{3j},\langle a^{3j},\bar x\rangle+\gamma_j-\sum_{j=1}^m\bar\lambda_j h_j(\bar x,\bar w_j))\Big|a^{3j}\in\partial_{\gamma_j}(\sum_{j=1}^m\bar\lambda_j h_j(.,\bar w_j))(\bar x)\Big\}$$
$$+\Big\{\bigcup_{\gamma_{m+1}}(a^{m+1},\langle a^{m+1},\bar x\rangle+\gamma_{m+1}-\delta_C(\bar x))\Big|a^{m+1}\in\partial_{\gamma_{m+1}}\delta_C(\bar x)\Big\}.$$

So, there are $\bar{a^{1j}}\in\partial_{\alpha_i}(\bar\mu_i f_i(.,\bar u_i))(\bar x),\bar{a^{2j}}\in\partial_{\beta_i}(-\bar r_i\bar\mu_i g_i(.,\bar v_i))(\bar x)$, $\bar{a^{3j}}\in\partial_{\gamma_j}(\sum_{j=1}^m\bar\lambda_j h_j(.,\bar w_j))(\bar x)$, $\bar a_{m+1}\in\partial_{\gamma_{m+1}}\delta_C(\bar x)$ and $\alpha_i,\beta_i,\gamma_j,\gamma_{m+1}\geq 0, i=1,\ldots,l$, $j=1,\ldots,m$, such that

$$0\in\sum_{i=1}^m\Big(\partial_{\alpha_i}(\bar\mu_i f_i(.,\bar u_i))(\bar x)+\partial_{\beta_i}(-\bar r_i\bar\mu_i g_i(.,\bar v_i))(\bar x)\Big)+\sum_{j=1}^m\partial_{\gamma_j}(\bar\lambda_j h_j(.,\bar w_j))(\bar x)+N_C^{\gamma_{m+1}}(\bar x),$$

and

$$\sum_{i=1}^l(\alpha_i+\beta_i)+\sum_{j=1}^{m+1}\gamma_j=\sum_{i=1}^l[\bar\mu_i f_i(\bar x,\bar u_i)-\bar r_i\bar\mu_i g_i(\bar x,\bar v_i)]+\sum_{j=1}^m\bar\lambda_j h_j(\bar x,\bar w_j)$$
$$\leq\sum_{i=1}^l\bar\mu_i(\max_{u_i\in\mathcal{U}_i}f_i(\bar x,u_i)-\bar r_i\min_{v_i\in\mathcal{V}_i}g_i(\bar x,v_i))+\sum_{j=1}^m\bar\lambda_j h_j(\bar x,\bar w_j)$$
$$=\sum_{i=1}^l\bar\mu_i\varepsilon_i\min_{v_i\in\mathcal{V}_i}g_i(\bar x,v_i)+\sum_{j=1}^m\bar\lambda_j h_j(\bar x,\bar w_j).$$

Thus, the proof is completed. □

Definition 3.7. *Let $\varepsilon \in \mathbb{R}^l_+$. The vector $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{u}, \bar{v}, \bar{w}) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \Delta^l \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$ satisfies the robust ε-KKT for UMFP problem, if there is $(\alpha, \beta, \gamma) \in \mathbb{R}^l_+ \times \mathbb{R}^l_+ \times \mathbb{R}^{m+1}_+$, such that the conditions* (3.1)-(3.3) *hold.*

In the following, we present a sufficient optimality condition for a robust weakly ε-efficient solution of UMFP problem.

Theorem 3.8. *Let $\varepsilon \in \mathbb{R}^l_+$ and let $\bar{r}_i = \max_{(u_i, v_i) \in \mathcal{U}_i \times \mathcal{V}_i} \frac{f_i(\bar{x}, u_i)}{g_i(\bar{x}, v_i)} - \varepsilon_i \geq 0, i = 1, \dots, l$. If $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{u}, \bar{v}, \bar{w}) \in F \times \mathbb{R}^m_+ \times \Delta^l \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$ is a robust ε-KKT for UMFP problem and $\frac{\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)}{\min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i)} = \frac{f_i(\bar{x}, \bar{u}_i)}{g_i(\bar{x}, \bar{v}_i)}$, then $\bar{x} \in F$ is a robust weakly ε-efficient solution for UMFP problem.*

Proof. Suppose that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{u}, \bar{v}, \bar{w}) \in F \times \mathbb{R}^m_+ \times \Delta^l \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$ is a robust ε-KKT; then there is $(\alpha, \beta, \gamma) \in \mathbb{R}^l_+ \times \mathbb{R}^l_+ \times \mathbb{R}^{m+1}_+$, such that the conditions (3.1)-(3.3) hold. Therefore, there are $u_i^* \in \partial_{\alpha_i}(\bar{\mu}_i f_i(., \bar{u}_i))(\bar{x})$, $v_i^* \in \partial_{\beta_i}(-\bar{r}_i \bar{\mu}_i g_i(., \bar{v}_i))(\bar{x})$, $w_j^* \in \partial_{\gamma_j}(\bar{\lambda}_j h_j(., \bar{w}_j))(\bar{x})$ and $n^* \in N_C^{\gamma_{m+1}}$ such that

$$\sum_{i=1}^{l} u_i^* + \sum_{i=1}^{l} v_i^* + \sum_{j=1}^{m} w_j^* + n^* = 0. \tag{3.5}$$

On the other hand, according to the definition of ε-subdifferential, we have

$$\begin{aligned}
\bar{\mu}_i f_i(x, \bar{u}_i) &\geq \bar{\mu}_i f_i(\bar{x}, \bar{u}_i) + \langle u_i^*, x - \bar{x} \rangle - \alpha_i, && i = 1, \dots, l, \\
-\bar{r}_i \bar{\mu}_i g_i(x, \bar{v}_i) &\geq -\bar{r}_i \bar{\mu}_i g_i(\bar{x}, \bar{v}_i) + \langle v_i^*, x - \bar{x} \rangle - \beta_i, && i = 1, \dots, l, \\
\bar{\lambda}_j h_j(x, \bar{w}_i) &\geq \bar{\lambda}_j h_j(\bar{x}, \bar{w}_j) + \langle w_j^*, x - \bar{x} \rangle - \gamma_j, && j = 1, \dots, m, \\
\delta_C(x) &\geq \delta_C(\bar{x}) + \langle n^*, x - \bar{x} \rangle - \gamma_{m+1}.
\end{aligned}$$

So according to relations (3.2) and (3.5), it follows that

$$\begin{aligned}
&\sum_{i=1}^{l} \bar{\mu}_i [(f_i(x, \bar{u}_i) - \bar{r}_i g_i(x, \bar{v}_i)] + \sum_{j=1}^{m} \bar{\lambda}_j h_j(x, \bar{w}_j) \\
&\quad \geq \sum_{i=1}^{l} \bar{\mu}_i [(f_i(\bar{x}, \bar{u}_i) - \bar{r}_i g_i(\bar{x}, \bar{v}_i)] + \sum_{j=1}^{m} \bar{\lambda}_j h_j(\bar{x}, \bar{w}_j) - \sum_{i=1}^{l} (\alpha_i + \beta_i) - \sum_{k=1}^{m+1} \gamma_k \\
&\quad \geq \sum_{i=1}^{l} \bar{\mu}_i [(f_i(\bar{x}, \bar{u}_i) - \bar{r}_i g_i(\bar{x}, \bar{v}_i)] + \sum_{i=1}^{l} \bar{\mu}_i \varepsilon_i \min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i),
\end{aligned}$$

since $\sum_{j=1}^{m} \bar{\lambda}_j h_j(x, \bar{w}_j) \leq 0$, thus,

$$\sum_{i=1}^{l} \bar{\mu}_i [(f_i(x, \bar{u}_i) - \bar{r}_i g_i(x, \bar{v}_i)] \geq \sum_{i=1}^{l} \bar{\mu}_i [(f_i(\bar{x}, \bar{u}_i) - \bar{r}_i g_i(\bar{x}, \bar{v}_i)] + \sum_{i=1}^{l} \bar{\mu}_i \varepsilon_i \min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i).$$

On the other hand, since $\frac{\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)}{\min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i)} = \frac{f_i(\bar{x}, \bar{u}_i)}{g_i(\bar{x}, \bar{v}_i)}$, thus, we have

$$\sum_{i=1}^{l} \bar{\mu}_i [(\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \bar{r}_i \min_{v_i \in \mathcal{V}_i} g_i(x, v_i)]$$

$$\geq \sum_{i=1}^{l} \bar{\mu}_i [\max_{u_i \in \mathcal{U}_i} (f_i(\bar{x}, u_i) - \bar{r}_i \min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i)] + \sum_{i=1}^{l} \bar{\mu}_i \varepsilon_i \min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i).$$

Hence, according to Lemma 3.4, $\bar{x} \in F$ is a weakly ε-efficient solution of RMFP and it completes the proof. □

Corollary 3.9. *Let $f_i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}, i = 1, \ldots, l$, $h_j : \mathbb{R}^n \times \mathbb{R}^{q_0} \to \mathbb{R}, j = 1, \ldots, m$, are continuous convex-concave on $\mathbb{R}^n \times \mathcal{U}_i$ and $\mathbb{R}^n \times \mathcal{W}_j$, respectively. Moreover, assume that $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ is a continuous concave-convex on $\mathbb{R}^n \times \mathcal{V}_i$. Also, assume that $\varepsilon \in \mathbb{R}^l_+$ and $\bar{r}_i = \max_{(u_i.v_i) \in \mathcal{U}_i \times \mathcal{V}_i} \frac{f_i(\bar{x}, u_i)}{g_i(\bar{x}, v_i)} - \varepsilon_i \geq 0$. If $\bar{x} \in F$ is a weakly robust ε-efficient solution for UMFP problem, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{u}, \bar{v}, \bar{w}) \in F \times \mathbb{R}^m_+ \times \Delta^l \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$ is a robust ε-KKT of UMFP problem.*

Proof. We use Lemma 2.5 and Lemma 2.6 to show that $\bigcup_{w_j \in \mathcal{W}_j, \lambda_j \geq 0} \big(\sum_{i=1}^{m} \lambda_j h_j(., w_j) \big)^* + C^* \times \mathbb{R}_+$ is a closed and convex set. Finally, by the same argument similar to that of the Theorem 3.6 the proof is completed. □

4. Robust ε-saddle point theorems

In this section, we prove robust ε-saddle point theorem for UMFP problem.

The Lagrangian-type function associated to UMFP problem with respect to $(\mu, r) \in \Delta^l \times \mathbb{R}^l_+$, is defined as follow:

$$L_{\mu,r}(x, \lambda, u, v, w) = \sum_{i=1}^{l} \mu_i [f_i(x, u_i) - r_i g_i(x, v_i)] + \sum_{j=1}^{m} \lambda_j h_j(x, w_j),$$

where $(x, \lambda, u, v, w) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$.

Definition 4.1. *Let $\varepsilon \geq 0$. A point $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{w}) \in C \times \mathbb{R}^m_+ \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$ is a robust ε-saddle point of UMFP problem with respect to $(\bar{\mu}, \bar{r}) \in \Delta^l \times \mathbb{R}^l_+$, if the following two conditions hold:*

(i) $L_{\bar{\mu},\bar{r}}(\bar{x}, \lambda, u, v, w) \leq L_{\bar{\mu},\bar{r}}(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{w}) + \varepsilon, \quad \forall (\lambda, u, v, w) \in \mathbb{R}^m_+ \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}.$

(ii) $L_{\bar{\mu},\bar{r}}(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{w}) \leq L_{\bar{\mu},\bar{r}}(x, \bar{\lambda}, \bar{u}, \bar{v}, \bar{w}) + \varepsilon, \quad \forall x \in C.$

Theorem 4.2. *Let $\varepsilon \geq 0$. Suppose that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{u}, \bar{v}, \bar{w}) \in F \times \mathbb{R}^m_+ \times \Delta^l \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$ is a robust ε-KKT for UMFP problem. If $\frac{\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)}{\min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i)} = \frac{f_i(\bar{x}, \bar{u}_i)}{g_i(\bar{x}, \bar{v}_i)}$, then $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{w})$ is a robust ε^*-saddle point for UMFP problem with respect to $(\bar{\mu}, \bar{r}) \in \Delta^l \times \mathbb{R}^l_+$, where $\varepsilon^* = \sum_{i=1}^{l} \bar{\mu}_i \varepsilon_i \min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i)$ and $\bar{r}_i = \max_{(u_i, v_i) \times \mathcal{U}_i \times \mathcal{V}_i} \frac{f(\bar{x}, u_i)}{g(\bar{x}, v_i)} - \varepsilon_i, i = 1, \ldots, l$.*

Proof. Assume that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{u}, \bar{v}, \bar{w}) \in F \times \mathbb{R}^m_+ \times \Delta^l \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$ is a robust ε-KKT for UMFP problem; then there exists $(\alpha, \beta, \gamma) \in \mathbb{R}^l_+ \times \mathbb{R}^l_+ \times \mathbb{R}^{m+1}_+$, such that the conditions (3.1)-(3.3) hold. Hence, there are $u_i^* \in \partial_{\alpha_i}(\bar{\mu}_i f_i(., \bar{u}_i))(\bar{x}), v_i^* \in \partial_{\beta_i}(- \bar{r}_i \bar{\mu}_i g_i(., \bar{v}_i))(\bar{x})$, $w_j^* \in \partial_{\gamma_j}(\bar{\lambda}_j h_j(., \bar{w}_j))(\bar{x})$ and $n^* \in N_C^{\gamma_{m+1}}$ such that

$$\sum_{i=1}^{l} u_i^* + \sum_{i=1}^{l} v_i^* + \sum_{j=1}^{m} w_j^* + n^* = 0. \tag{4.1}$$

On the other hand, according to the definition of ε-subdifferential, we have

$$\begin{aligned}
\bar{\mu}_i f_i(x, \bar{u}_i) &\geq \bar{\mu}_i f_i(\bar{x}, \bar{u}_i) + \langle u_i^*, x - \bar{x}\rangle - \alpha_i, \qquad i = 1, \ldots, l,\\
-\bar{r}_i\bar{\mu}_i g_i(x, \bar{v}_i) &\geq -\bar{r}_i\bar{\mu}_i g_i(\bar{x}, \bar{v}_i) + \langle v_i^*, x - \bar{x}\rangle - \beta_i, \qquad i = 1, \ldots, l,\\
\bar{\lambda}_j h_j(x, \bar{w}_i) &\geq \bar{\lambda}_j h_j(\bar{x}, \bar{w}_j) + \langle w_j^*, x - \bar{x}\rangle - \gamma_j, \qquad j = 1, \ldots, m,\\
\delta_C(x) &\geq \delta_C(\bar{x}) + \langle n^*, x - \bar{x}\rangle - \gamma_{m+1}.
\end{aligned}$$

So by adding recent inequalities and using relation (4.1), it follows that

$$\begin{aligned}
&\sum_{i=1}^{l}\bar{\mu}_i[(f_i(x, \bar{u}_i) - \bar{r}_i g_i(x, \bar{v}_i)] + \sum_{j=1}^{m}\bar{\lambda}_j h_j(x, \bar{w}_j)\\
&\quad \geq \sum_{i=1}^{l}\bar{\mu}_i[(f_i(\bar{x}, \bar{u}_i) - \bar{r}_i g_i(\bar{x}, \bar{v}_i)] + \sum_{j=1}^{m}\bar{\lambda}_j h_j(\bar{x}, \bar{w}_j) - \sum_{i=1}^{l}(\alpha_i + \beta_i) - \sum_{k=1}^{m+1}\gamma_k\\
&\quad \geq \sum_{i=1}^{l}\bar{\mu}_i[(f_i(\bar{x}, \bar{u}_i) - \bar{r}_i g_i(\bar{x}, \bar{v}_i)] - \sum_{i=1}^{l}\bar{\mu}_i\varepsilon_i \min_{v_i\in\mathcal{V}_i} g_i(\bar{x}, v_i)\\
&\quad \geq \sum_{i=1}^{l}\bar{\mu}_i[(f_i(\bar{x}, \bar{u}_i) - \bar{r}_i g_i(\bar{x}, \bar{v}_i)] + \sum_{j=1}^{m}\lambda_j h_j(\bar{x}, \bar{w}_j) - \sum_{i=1}^{l}\bar{\mu}_i\varepsilon_i \min_{v_i\in\mathcal{V}_i} g_i(\bar{x}, v_i).
\end{aligned}$$

Hence, we have

$$L_{\bar{\mu},\bar{r}}(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{w}) \leq L_{\bar{\mu},\bar{r}}(x, \bar{\lambda}, \bar{u}, \bar{v}, \bar{w}) + \sum_{i=1}^{l}\varepsilon_i\bar{\mu}_i \min_{v_i\in\mathcal{V}_i} g_i(\bar{x}, v_i), \qquad \text{for all } x \in C.$$

On the other hand, according to the relation (3.3), we have

$$0 \leq \sum_{i=1}^{l}(\alpha_i + \beta_i) + \sum_{j=1}^{m+1}\gamma_{j+1} \leq \sum_{j=1}^{m}\bar{\lambda}_j h_j(\bar{x}, \bar{w}_j) + \sum_{i=1}^{l}\varepsilon_i\bar{\mu}_i \min_{v_i\in\mathcal{V}_i} g_i(\bar{x}, v_i),$$

since $\sum_{j=1}^{m}\lambda_j h_j(\bar{x}, w_j) \leq 0$, it follows that

$$\sum_{j=1}^{m}\bar{\lambda}_j h_j(\bar{x}, \bar{w}_j) + \sum_{i=1}^{l}\varepsilon_i\bar{\mu}_i \min_{v_i\in\mathcal{V}_i} g_i(\bar{x}, v_i) \geq \sum_{j=1}^{m}\lambda_j h_j(\bar{x}, w_j),$$

hence,

$$\begin{aligned}
&\sum_{i=1}^{l}\bar{\mu}_i[f_i(\bar{x}, \bar{u}_i) - \bar{r}_i g_i(\bar{x}, \bar{v}_i)] + \sum_{j=1}^{m}\bar{\lambda}_j h_j(\bar{x}, \bar{w}_j) + \sum_{i=1}^{l}\varepsilon_i\bar{\mu}_i \min_{v_i\in\mathcal{V}_i} g_i(\bar{x}, v_i)\\
&\quad \geq \sum_{i=1}^{l}\bar{\mu}_i[f_i(\bar{x}, \bar{u}_i) - \bar{r}_i g_i(\bar{x}, \bar{v}_i)] + \sum_{j=1}^{m}\lambda_j h_j(\bar{x}, w_j)\\
&\quad = \sum_{i=1}^{l}\bar{\mu}_i[\max_{u_i\in\mathcal{U}_i} f_i(\bar{x}, u_i) - \bar{r}_i \min_{v_i\in\mathcal{V}_i} g_i(\bar{x}, v_i)] + \sum_{j=1}^{m}\lambda_j h_j(\bar{x}, w_j)
\end{aligned}$$

$$\geq \sum_{i=1}^{l} \bar{\mu}_i [f_i(\bar{x}, u_i) - \bar{r}_i g_i(\bar{x}, v_i)] + \sum_{j=1}^{m} \lambda_j h_j(\bar{x}, w_j),$$

therefore,

$$L_{\bar{\mu},\bar{r}}(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{w}) + \sum_{i=1}^{l} \varepsilon_i \bar{\mu}_i \min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i) \geq L_{\bar{\mu},\bar{r}}(\bar{x}, \lambda, u, v, w).$$

This means that $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{w})$ is a robust ε^*-saddle point for UMFP problem with respect to $(\bar{\mu}, \bar{r}) \in \Delta^l \times \mathbb{R}^l_+$. □

Corollary 4.3. *Suppose that $\bar{x} \in F$ is a weakly robust ε-efficient solution for UMFP problem and the assumptions of Theorem 3.6 hold; then there is $(\bar{\lambda}, \bar{\mu}, \bar{u}, \bar{v}, \bar{w}) \in \mathbb{R}^m_+ \times \Delta^l \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$, with $\frac{\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)}{\min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i)} = \frac{f_i(\bar{x}, \bar{u}_i)}{g_i(\bar{x}, \bar{v}_i)}$ such that $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{w})$ is a robust ε^*-saddle point for UMFP problem with respect to $(\bar{\mu}, \bar{r}) \in \Delta^l \times \mathbb{R}^l_+$; in which $\varepsilon^* = \sum_{i=1}^{l} \bar{\mu}_i \varepsilon_i \min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i)$ and $\bar{r}_i = \max_{(u_i, v_i) \times \mathcal{U}_i \times \mathcal{V}_i} \frac{f(\bar{x}, u_i)}{g(\bar{x}, v_i)} - \varepsilon_i, i = 1, \ldots, l$.*

Example 4.4. *Consider the following uncertain multi-objective fractional programming problem*

$$\begin{aligned} \min \quad & \Big(u_1 x_1, \frac{u_2 x_2}{x_1 + v_2}\Big) \\ s.t. \quad & -x_1 + w_1 \leq 0, \\ & -x_2 + w_2 \leq 0, \\ & x_1, x_2 \geq 0, \end{aligned} \tag{4.2}$$

where u_1, u_2, v_2, w_1, w_2 are the uncertain parameters belonging to their uncertainty sets $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{V}_2 = \mathcal{W}_1 = \mathcal{W}_2 = [0, 1]$.
Suppose that $f_1(x, u_1) = u_1 x_1, f_2(x, u_2) = u_2 x_2, g_1(x, v_1) = 1, g_2(x, v_2) = x_1 + v_2, h_1(x, w_1) = -x_1 + w_1, h_2(x, w_2) = -x_2 + w_2$ and $C = \mathbb{R}^2_+$. It is easy to show that $F = \{(x_1, x_2) \in \mathbb{R}^2_+ | x_1 \geq 1, x_2 \geq 1\}$. Let $\bar{x} = (\frac{3}{2}, \frac{15}{4})$ and $\varepsilon = (\varepsilon_1, \varepsilon_2) = (\frac{1}{2}, \frac{3}{2})$. It is clear that $\bar{x}$ is a weakly robust ε-efficient for model (4.2).
Suppose that, $(\bar{u}_1, \bar{u}_2, \bar{v}_2, \bar{w}_1, \bar{w}_2) = (1, 1, 0, 1, 1), (\bar{\mu}_1, \bar{\mu}_2, \bar{\lambda}_1, \bar{\lambda}_2, \bar{r}_1, \bar{r}_2) = (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, 1, 1)$ and $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = \gamma_3 = 0$; then we can obtain

$$\partial(\bar{\mu}_1 f_1(., \bar{u}_1))(\bar{x}) = \Big\{(\frac{1}{2}, 0)\Big\}, \qquad \partial(\bar{\mu}_2 f_2(., \bar{u}_2))(\bar{x}) = \Big\{(0, \frac{1}{2})\Big\},$$
$$\partial(-\bar{r}_2 \bar{\mu}_2 g_2(., \bar{v}_2))(\bar{x}) = \Big\{(-\frac{1}{2}, 0)\Big\}, \qquad \partial(\bar{\lambda}_2 h_2(., \bar{w}_2))(\bar{x}) = \Big\{(0, -\frac{1}{2})\Big\}.$$

Hence,

$$\sum_{i=1}^{2} \partial(\bar{\mu}_i f_i(., \bar{u}_i))(\bar{x}) + \partial(-\bar{r}_2 \bar{\mu}_2 g_2(., \bar{v}_2))(\bar{x}) + \sum_{j=1}^{2} \partial(\bar{\lambda}_j h_j(., \bar{w}_j))(\bar{x}) = \Big\{(0, 0)\Big\}.$$

On the other hand,

$$\sum_{j=1}^{2} (\bar{\lambda}_j h_j(\bar{x}, \bar{w}_j)) = -\frac{11}{8}, \qquad \sum_{i=1}^{2} \varepsilon_i \bar{\mu}_i \min_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i) = \frac{11}{8},$$

so we have

$$\sum_{i=1}^{2}(\alpha_i+\beta_i)+\sum_{k=1}^{3}\gamma_k-\sum_{i=1}^{2}\varepsilon_i\bar{\mu}_i\min_{v_i\in\mathcal{V}_i}g_i(\bar{x},v_i)=\sum_{j=1}^{2}\partial(\bar{\lambda}_jh_j(.,\bar{w}_j))(\bar{x}).$$

Thus, $(\bar{x},\bar{\lambda},\bar{u},\bar{v},\bar{w})$ *is a robust* ε*-KKT for model* (4.2) *with respect to* $(\bar{\mu},\bar{r})$.
Now, we verify the ε*-saddle point theorem. For any* $(x,\lambda,u,v,w)\in\mathbb{R}^2_+\times\mathbb{R}^2_+\times\mathcal{U}\times\mathcal{V}\times\mathcal{W}$*, we have*

$$L_{\bar{\mu},\bar{r}}(x,\lambda,u,v,w)=\frac{1}{2}(u_1x_1-1)+\frac{1}{2}(u_2x_2-x_1-v_2)+\lambda_1(-x_1+w_1)+\lambda_2(-x_2+w_2).$$

thus,

$$\begin{aligned}&L_{\bar{\mu},\bar{r}}(\bar{x},\bar{\lambda},\bar{u},\bar{v},\bar{w})=L_{\bar{\mu},\bar{r}}(x,\bar{\lambda},\bar{u},\bar{v},\bar{w})=0,\\&L_{\bar{\mu},\bar{r}}(\bar{x},\lambda,u,v,w)=\frac{3}{4}u_1+\frac{15}{8}u_2-\frac{1}{2}v_2-\frac{5}{4}+\lambda_1(-\frac{3}{4}+w_1)+\lambda_2(-\frac{15}{4}+w_2),\end{aligned}$$

Obviously,

$$L_{\bar{\mu},\bar{r}}(\bar{x},\lambda,u,v,w)\le L_{\bar{\mu},\bar{r}}(\bar{x},\bar{\lambda},\bar{u},\bar{v},\bar{w})+\varepsilon^*\qquad\forall(\lambda,u,v,w)\in\mathbb{R}^2_+\times\mathcal{U}\times\mathcal{V}\times\mathcal{W},$$

and

$$L_{\bar{\mu},\bar{r}}(\bar{x},\bar{\lambda},\bar{u},\bar{v},\bar{w})\le L_{\bar{\mu},\bar{r}}(x,\bar{\lambda},\bar{u},\bar{v},\bar{w})+\varepsilon^*\qquad\forall x\in\mathbb{R}^2_+.$$

Hence, Theorem 4.2 is applicable.

5. Conclusions

This study has considered the multi-objective fractional programming problem with a geometric constraint set in the presence of the uncertain parameters in the objective function and the constraint functions. The necessary and sufficient conditions for optimality of the approximate robust weakly ε-efficient were proposed by applying the robust optimization techniques. Also, the robust ε-saddle point theorems for UMFP problems were expressed. In addition, we applied a parametric approach to establish ε-optimality conditions for robust weakly ε-efficient solution. Furthermore, some theorems have been presented to obtain a robust ε-saddle point for UMFP problem. The numerical example in the end was illustrated the efficiency and correctness of our approach. In further research, we will consider the optimization conditions of the approximated solutions for the various optimization problems along with their applications for the real-world problems.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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