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Research article

Automorphism groups of representation rings of the weak Sweedler Hopf algebras

Dong Su[1] **and Shilin Yang**[2,*]

[1] School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023, China

[2] Faculty of Science, Beijing University of Technology, Beijing 100124, China

* **Correspondence:** Email: slyang@bjut.edu.cn.

Abstract: Let $\mathfrak{w}_{2,2}^{s}(s=0,1)$ be two classes of weak Hopf algebras corresponding to the Sweedler Hopf algebra, and $r(\mathfrak{w}_{2,2}^{s})$ be the representation rings of $\mathfrak{w}_{2,2}^{s}$. In this paper, we investigate the automorphism groups $\mathrm{Aut}(r(\mathfrak{w}_{2,2}^{s}))$ of $r(\mathfrak{w}_{2,2}^{s})$, and discuss some properties of $\mathrm{Aut}(r(\mathfrak{w}_{2,2}^{s}))$. We obtain that $\mathrm{Aut}(r(\mathfrak{w}_{2,2}^{0}))$ is isomorphic to K_4, where K_4 is the Klein four-group. It is shown that $\mathrm{Aut}(r(\mathfrak{w}_{2,2}^{1}))$ is a non-commutative infinite solvable group, but it is not nilpotent. In addition, $\mathrm{Aut}(r(\mathfrak{w}_{2,2}^{1}))$ is isomorphic to $(\mathbb{Z}\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2$, and its centre is isomorphic to $\mathbb{Z}_2$.

Keywords: automorphism group; representation ring; weak Hopf algebra; Sweedler Hopf algebra
Mathematics Subject Classification: 16W20, 19A22

1. Introduction

The study of representation rings has attracted extensive attention of mathematicians. Chen et al. [5] described the structure of the representation rings of the Taft algebra $H_n(q)$. Li and Zhang [12] determined the representation rings of the generalized Taft Hopf algebras $H_{n,d}(q)$, and determined all nilpotent elements in the representation ring of $H_{n,d}(q)$. In [14], we constructed two classes of weak Hopf algebras $\mathfrak{w}_{n,d}^{s}(s=0,1)$ corresponding to generalized Taft algebra $H_{n,d}$, and investigate the representation rings $r(\mathfrak{w}_{n,d}^{s})$ of $\mathfrak{w}_{n,d}^{s}$. More conclusions related to representation rings can be seen in [4, 15].

Many significant researches focused on studying automorphisms of algebras or rings. van der Kulk [11], Zhao [18], Yu [17], Vesselin and Yu [16] did some significant contributions on the automorphisms of polynomial algebras. Alperin [2] investigated the homology of the group of automorphisms of $k[x,y]$ over a field k. Furthermore, Dicks [7] researched automorphisms of polynomial ring in two variables. Chen et al. [3, 6] considered the coalgebra automorphism groups of the Hopf algebras . Han and Su [8] studied automorphism group of Witt algebras. Jia et al. [10] proved that the automorphism group of representation ring of Sweedler Hopf algebra is isomorphic to the Klein four-group. Motivated by the above works, in this paper, we investigate the automorphism groups $\mathrm{Aut}(r(\mathfrak{w}^s_{2,2}))$ of representation rings $r(\mathfrak{w}^s_{2,2})$ of two classes of weak Sweedler Hopf algebra $\mathfrak{w}^s_{2,2}(s = 0, 1)$. It is shown that $\mathrm{Aut}(r(\mathfrak{w}^0_{2,2}))$ is isomorphic to K_4, where K_4 is the Klein four-group. Through calculation, the structure of automorphism group $\mathrm{Aut}(r(\mathfrak{w}^1_{2,2}))$ has been constructed. We prove that $\mathrm{Aut}(r(\mathfrak{w}^1_{2,2}))$ is isomorphic to $(\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$. Its centre is isomorphic to $\mathbb{Z}_2$. In addition, $\mathrm{Aut}(r(\mathfrak{w}^1_{2,2}))$ is a non-commutative infinite solvable group, but it is not nilpotent.

The paper is organized as follows. In Section 1, we recall some relative background and knowledge in detail. In Section 2, the structures of automorphism groups $\mathrm{Aut}(r(\mathfrak{w}^s_{2,2}))(s = 0, 1)$ are described. We obtain that $\mathrm{Aut}(r(\mathfrak{w}^1_{2,2}))$ is a non-commutative infinite group, and the automorphism group $\mathrm{Aut}(r(\mathfrak{w}^0_{2,2}))$ is isomorphic to K_4. In Section 3, the properties of the automorphism groups $\mathrm{Aut}(r(\mathfrak{w}^s_{2,2}))$ are discussed. It is shown that $\mathrm{Aut}(r(\mathfrak{w}^1_{2,2}))$ is isomorphic to $(\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, and its centre is isomorphic to $\mathbb{Z}_2$. Finally, we prove that $\mathrm{Aut}(r(\mathfrak{w}^1_{2,2}))$ is a solvable and non-nilpotent group. It is interesting that although both $\mathfrak{w}^0_{2,2}$ and $\mathfrak{w}^1_{2,2}$ are weak Hopf algebras of Sweedler Hopf algebra, the automorphism groups of their representation rings are strongly different.

2. Preliminaries

Throughout, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0 unless otherwise stated. All algebras, Hopf algebras and weak Hopf algebras are defined over $\mathbb{K}$.

In the sequel, we fix two integers $n, d \geq 2$ such that $d \mid n$, and assume that $q \in \mathbb{K}$ is a primitive d-th root of unity. In [14], we constructed the weak Hopf algebras $\mathfrak{w}^s_{n,d}(s = 0, 1)$ corresponding to generalized Taft algebra [13], and investigated the representation rings $r(\mathfrak{w}^s_{n,d})(s = 0, 1)$ of $\mathfrak{w}^s_{n,d}(s = 0, 1)$. As an algebra $\mathfrak{w}^s_{n,d}(s = 0, 1)$ is generated by g, x subject to the relations

$$g^{n+1} = g, \quad xg = qgx, \quad x^d = 0.$$

The comultiplication, counit and weak antipode T are given by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = g^r \otimes x + x \otimes g, \quad \epsilon(g) = 1, \quad \epsilon(x) = 0,$$

$$T(1) = 1, \quad T(g) = g^{n-1}, \quad T(x) = -q^{-1} g^{n-1} x.$$

If $r = 0$, we get the weak Hopf algebra $\mathfrak{w}^1_{n,d}$, where $g^0 = 1$. If $r = n$ and $x = g^n x$, we get the weak Hopf algebra $\mathfrak{w}^0_{n,d}$. Let $E = g^n$, it is easy to see that the dimension of $\mathfrak{w}^s_{n,d}$ is $nd + (d-1)s + 1$, and the set

$$\{g^\iota x^\kappa E \mid 0 \leq \iota \leq n-1,\ 0 \leq \kappa \leq d-1\} \cup \{x^\kappa(1-E) | \kappa = 0, 1, \cdots, (d-1)s\}$$

forms a PBW basis for $\mathfrak{w}^s_{n,d}$.

In particular, when $n = d = 2$, $\mathfrak{w}^s_{2,2}(s = 0, 1)$ are exactly two classes of weak Hopf algebras corresponding to the Sweedler Hopf algebra (see also [1]).

Let H be a weak Hopf algebra, the representation ring $r(H)$ of H is defined as follows. Assume that $F(H)$ is a free abelian group generated by isomorphism classes $[V]$ of finite dimensional H-modules V. Let $r(H)$ be the quotient group $F(H)$ modulo the relations $[M \oplus V] = [M] + [V]$, We equip $r(H)$ with the multiplication $[M][V] = [M \otimes V]$. It is well known that $r(H)$ is an associative ring with $\mathbb{Z}$-basis $\{[V] \mid V \in \text{ind-}H\}$.

Theorem 2.1. *[14] The representation ring $r(\mathfrak{w}_{n,d}^s) \cong \mathbb{Z}\langle x_1, x_2, x_3 \rangle / I$ as ring isomorphisms, and the ideal I is generated by the relations*

$$x_1^n - 1, \quad (x_2 - x_1^m - 1)F_d(x_1^m, x_2), \quad x_1x_2 - x_2x_1, \quad x_1x_3 - x_3, \quad x_3x_1 - x_3,$$
$$(1-s)x_3x_2 - (1-s)x_2x_3, \quad x_2x_3 - 2x_3, \quad x_3^2 - x_3,$$

where $m = \frac{n}{d}$ and $F_t(y,z)$ are the generalized Fibonacci polynomials defined by $F_{t+2}(y,z) = zF_{t+1}(y,z) - yF_t(y,z), t > 1, F_0(y,z) = 0, F_1(y,z) = 1, F_2(y,z) = z$.

Corollary 2.2. *The representation ring $r(\mathfrak{w}_{2,2}^s) \cong \mathbb{Z}\langle x, y, z \rangle / I$ as ring isomorphisms, and the ideal I is generated by the relations*

$$x_1^2 - 1, \quad (x_2 - x_1 - 1)x_2, \quad x_1x_2 - x_2x_1, \quad x_1x_3 - x_3, \quad x_3x_1 - x_3,$$
$$(1-s)x_3x_2 - (1-s)x_2x_3, \quad x_2x_3 - 2x_3, \quad x_3^2 - x_3.$$

Notice that

$$\{1, x_1, x_2, x_3, x_1x_2\} \text{ and } \{1, x_1, x_2, x_3, x_1x_2, x_3x_2\}$$

are $\mathbb{Z}-$basis of $r(\mathfrak{w}_{2,2}^0)$ and $r(\mathfrak{w}_{2,2}^1)$, respectively.

Let $\mathbf{A}_f$ denote the corresponding coefficient matrix of $\mathbb{Z}-$linear map

$$f : r(\mathfrak{w}_{n,d}^s) \to r(\mathfrak{w}_{n,d}^s),$$

where $s = 0, 1$. And let $|\mathbf{A}_f|$ denote the determinant of $\mathbf{A}_f$.

3. Automorphism groups of representation rings $r(\mathfrak{w}_{2,2}^s)$

In this section, we will discuss the automorphism groups of representation rings of weak Sweedler Hopf algebras. By Corollary 2.2, one see that $r(\mathfrak{w}_{2,2}^0)$ is a commutative ring, but $r(\mathfrak{w}_{2,2}^1)$ is a non-commutative ring. We mainly consider the automorphism group of $r(\mathfrak{w}_{2,2}^1)$, the automorphism group of $r(\mathfrak{w}_{2,2}^0)$ will be stated directly.

For any $i, j, k, z \in \mathbb{Z}$, let $\omega_i, \delta_j, \tau_k, \varphi_z$ be $\mathbb{Z}-$linear maps of $r(\mathfrak{w}_{2,2}^1)$. They are determined by the following maps:

$$\begin{aligned}
\omega_i :\ & 1 \to 1 & & x_1 \to x_1, \\
& x_2 \to x_2 - 4ix_3 + 2ix_3x_2, & & x_3 \to (1-2i)x_3 + ix_3x_2, \\
& x_1x_2 \to -4ix_3 + x_1x_2 + 2ix_3x_2, & & x_3x_2 \to -4ix_3 + (1+2i)x_3x_2; \\
\delta_j :\ & 1 \to 1 & & x_1 \to x_1, \\
& x_2 \to (4-4j)x_3 + x_1x_2 + (2j-2)x_3x_2, & & x_3 \to (1-2j)x_3 + jx_3x_2, \\
& x_1x_2 \to x_2 + (4-4j)x_3 + (2j-2)x_3x_2, & & x_3x_2 \to (4-4j)x_3 + (2j-1)x_3x_2; \\
\tau_k :\ & 1 \to 1 & & x_1 \to x_1, \\
& x_2 \to x_2 + (4-4k)x_3 + (2k-2)x_3x_2, & & x_3 \to (1-2k)x_3 + kx_3x_2, \\
& x_1x_2 \to (4-4k)x_3 + x_1x_2 + (2k-2)x_3x_2, & & x_3x_2 \to (4-4k)x_3 + (2k-1)x_3x_2; \\
\varphi_z :\ & 1 \to 1 & & x_1 \to x_1, \\
& x_2 \to -4zx_3 + x_1x_2 + 2zx_3x_2, & & x_3 \to (1-2z)x_3 + zx_3x_2, \\
& x_1x_2 \to x_2 - 4zx_3 + 2zx_3x_2, & & x_3x_2 \to -4zx_3 + (1+2z)x_3x_2.
\end{aligned}$$

It is easy to check that $\omega_i, \delta_j, \tau_k, \varphi_z$ are four classes of automorphisms of $r(\mathfrak{w}^1_{2,2})$. For any $i, j, k, z \in \mathbb{Z}$ $(i = 1, 2, 3, 4)$, we have

$$\omega_0 = id, \quad \omega_i^{-1} = \omega_{-i}, \quad \delta_j^{-1} = \delta_j, \quad \tau_k^{-1} = \tau_k, \quad \varphi_z^{-1} = \varphi_{-z}.$$

Let

$$G = \left\{\omega_i, \delta_j, \tau_k, \varphi_z \mid i, j, k, z \in \mathbb{Z}\right\},$$

then G is a group under the composition of functions. For any $i, j, k, z, i', j', k', z' \in \mathbb{Z}$ the multiplication is described as follows

$\circ$	ω_i	δ_j	τ_k	φ_z
$\omega_{i'}$	$\omega_{i'+i}$	$\delta_{i'+j}$	$\tau_{i'+k}$	$\varphi_{i'+z}$
$\delta_{j'}$	$\delta_{j'-i}$	$\omega_{j'-j}$	$\varphi_{j'-k}$	$\tau_{j'-z}$
$\tau_{k'}$	$\tau_{k'-i}$	$\varphi_{k'-j}$	$\omega_{k'-k}$	$\delta_{k'-z}$
$\varphi_{z'}$	$\varphi_{z'+i}$	$\tau_{z'+j}$	$\delta_{z'+k}$	$\omega_{z'+z}$

It follow that G is a non-commutative infinite group.

In the sequel, we shall show the automorphism group $\mathrm{Aut}(r(\mathfrak{w}^1_{2,2}))$ is just the group G.

Lemma 3.1. *Let g be an automorphism of $r(\mathfrak{w}^1_{2,2})$, then*

$$g(x_1) = x_1 \text{ or } g(x_1) = x_1 - (2 + 2a)x_3 + ax_3x_2 \text{ or}$$

$$g(x_1) = -x_1 \text{ or } g(x_1) = -x_1 + (2 - 2a)x_3 + ax_3x_2 \text{ or}$$

$$g(x_1) = 1 - (2 + 2a)x_3 + ax_3x_2 \text{ or } g(x_1) = -1 + (2 - 2a)x_3 + ax_3x_2,$$

where $a \in \mathbb{Z}$.

Proof. Indeed, we have $(g(x_1))^2 = 1$ since g is an automorphism of $r(\mathfrak{w}^1_{2,2})$ and $x_1^2 = 1$.

Assume that

$$g(x_1) = a_0 + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_1x_2 + a_5x_3x_2, \quad a_i \in \mathbb{Z}\ (i = 0, 1, 2, 3, 4, 5).$$

Then we get

$$(a_0 + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_1x_2 + a_5x_3x_2)^2 = 1,$$

and

$$\begin{cases} a_0^2 + a_1^2 = 1, \\ 2a_0a_1 = 0, \\ 2a_0a_2 + 2a_1a_4 + a_2^2 + a_4^2 + 2a_2a_4 = 0, \\ 2a_0a_3 + 2a_1a_3 + 2a_2a_3 + 2a_3a_4 + 2a_3a_5 + a_3^2 = 0, \\ 2a_0a_4 + 2a_1a_2 + a_2^2 + 2a_2a_4 + a_4^2 = 0, \\ 2a_0a_5 + 2a_1a_5 + 4a_2a_5 + a_2a_3 + a_3a_4 + a_3a_5 + 4a_4a_5 + 2a_5^2 = 0. \end{cases}$$

Thanks to $a_0, a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}$, we obtain that the system of equations has eight distinct solutions $(a_0, a_1, a_2, a_3, a_4, a_5)$ as follows:

$$(0, 1, 0, 0, 0, 0), \quad (0, 1, 0, -2 - 2a, 0, a), \quad (0, -1, 0, 0, 0, 0), \quad (0, -1, 0, 2 - 2a, 0, a),$$

$$(1,0,0,0,0,0),\quad (1,0,0,-2-2a,0,a),\quad (-1,0,0,0,0,0),\quad (-1,0,0,2-2a,0,a),$$

where $a \in \mathbb{Z}$. Therefore, we get that only the fifth and seventh solutions are unreasonable, and

$$g(x_1) = x_1 \text{ or } g(x_1) = x_1 - (2+2a)x_3 + ax_3x_2 \text{ or}$$

$$g(x_1) = -x_1 \text{ or } g(x_1) = -x_1 + (2-2a)x_3 + ax_3x_2 \text{ or}$$

$$g(x_1) = 1 - (2+2a)x_3 + ax_3x_2 \text{ or } g(x_1) = -1 + (2-2a)x_3 + ax_3x_2.$$

The proof is finished. □

Lemma 3.2. *Let g be an automorphism of $r(\mathfrak{w}^1_{2,2})$, then*

$$g(x_3) = (1-2c)x_3 + cx_3x_2 \text{ or } g(x_3) = 1 - (1+2c)x_3 + cx_3x_2,$$

where $c \in \mathbb{Z}$.

Proof. Noting that $x_3^2 = x_3$, we have

$$(g(x_3))^2 = g(x_3).$$

Assume that

$$g(x_3) = b_0 + b_1x_1 + b_2x_2 + b_3x_3 + b_4x_1x_2 + b_5x_3x_2,\quad b_i \in \mathbb{Z}\ (i = 0,1,2,3,4,5).$$

Then we have

$$\begin{cases} b_0^2 + b_1^2 = b_0, \\ 2b_0b_1 = b_1, \\ 2b_0b_2 + 2b_1b_4 + b_2^2 + b_4^2 + 2b_2b_4 = b_2, \\ 2b_0b_3 + 2b_1b_3 + 2b_2b_3 + 2b_3b_4 + 2b_3b_5 + b_3^2 = b_3, \\ 2b_0b_4 + 2b_1b_2 + b_2^2 + 2b_2b_4 + b_4^2 = b_4, \\ 2b_0b_5 + 2b_1b_5 + 4b_2a_5 + b_2b_3 + b_3b_4 + b_3b_5 + 4b_4b_5 + 2b_5^2 = b_5. \end{cases}$$

It is easy to get that the system of equations has four distinct solutions $(b_0, b_1, b_2, b_3, b_4, b_5)$ as follows:

$$(0,0,0,0,0,0),\quad (0,0,0,1-2c,0,c),\quad (1,0,0,0,0,0),\quad (1,0,0,-1-2c,0,c),$$

where $c \in \mathbb{Z}$. Only the first and third solutions are unreasonable. Therefore

$$g(x_3) = (1-2c)x_3 + cx_3x_2 \text{ or } g(x_3) = 1 - (1+2c)x_3 + cx_3x_2.$$ □

Lemma 3.3. *Let g be an automorphism of $r(\mathfrak{w}^1_{2,2})$, we have*

1. *if $g(x_3) = (1-2c)x_3 + cx_3x_2$, then $g(x_1) = x_1$ or $g(x_1) = -x_1 + (2-4c)x_3 + 2cx_3x_2$ or $g(x_1) = -1 + (2-4c)x_3 + 2cx_3x_2$;*
2. *if $g(x_3) = 1 - (1+2c)x_3 + cx_3x_2$, then $g(x_1) = 1 - (2+4c)x_3 + 2cx_3x_2$;*

where $c \in \mathbb{Z}$.

Proof. Noting that $x_1x_3 = x_3x_1 = x_3$, then we have

$$g(x_1)g(x_3) = g(x_3)g(x_1) = g(x_3).$$

1. Since $g(x_3) = (1-2c)x_3 + cx_3x_2$, and by the Lemma 3.1, then
   (a) if $g(x_1) = x_1$, we have
$$\begin{aligned} g(x_1)g(x_3) &= g(x_1)g(x_3) = x_1((1-2c)x_3 + cx_3x_2) \\ &= ((1-2c)x_3 + cx_3x_2)x_1 = (1-2c)x_3 + cx_3x_2 = g(x_3); \end{aligned}$$
   (b) if $g(x_1) = x_1 - (2+2a)x_3 + ax_2x_3$, we have
$$\begin{aligned} g(x_1)g(x_3) &= (x_1 - (2+2a)x_3 + ax_2x_3)((1-2c)x_3 + cx_3x_2) \\ &= -(1-2c)x_3 - cx_3x_2 \neq g(x_3); \end{aligned}$$
   (c) if $g(x_1) = -x_1$, we have
$$\begin{aligned} g(x_1)g(x_3) &= -x_1((1-2c)x_3 + cx_3x_2) \\ &= -(1-2c)x_3 - cx_3x_2 \neq g(x_3); \end{aligned}$$
   (d) if $g(x_1) = -x_1 + (2-2a)x_3 + ax_3x_2$, we have
$$\begin{aligned} g(x_1)g(x_3) &= (-x_1 + (2-2a)x_3 + ax_3x_2)((1-2c)x_3 + cx_3x_2) \\ &= (1-2c)x_3 + cx_3x_2 = g(x_3), \end{aligned}$$
$$\begin{aligned} g(x_3)g(x_1) &= ((1-2c)x_3 + cx_3x_2)(-x_1 + (2-2a)x_3 + ax_3x_2) \\ &= (2c-2a+1)x_3 + (a-c)x_3x_2. \end{aligned}$$
   Let $a = 2c$, then $g(x_1) = -x_1 + (2-4c)x_3 + 2cx_3x_2$ and $g(x_3)g(x_1) = g(x_3)$;
   (e) if $g(x_1) = 1 - (2+2a)x_3 + ax_2x_3$, we have
$$\begin{aligned} g(x_1)g(x_3) &= (1 - (2+2a)x_3 + ax_2x_3)((1-2c)x_3 + cx_3x_2) \\ &= -(1-2c)x_3 - cx_3x_2 \neq g(x_3); \end{aligned}$$
   (f) if $g(x_1) = -1 + (2-2a)x_3 + ax_3x_2$, we have
$$\begin{aligned} g(x_1)g(x_3) &= (-1 + (2-2a)x_3 + ax_3x_2)((1-2c)x_3 + cx_3x_2) \\ &= (1-2c)x_3 + cx_3x_2 = g(x_3), \end{aligned}$$
$$\begin{aligned} g(x_3)g(x_1) &= ((1-2c)x_3 + cx_3x_2)(-1 + (2-2a)x_3 + ax_3x_2) \\ &= (2c-2a+1)x_3 + (a-c)x_3x_2. \end{aligned}$$
   Let $a = 2c$, then $g(x_1) = -1 + (2-4c)x_3 + 2cx_3x_2$ and $g(x_3)g(x_1) = g(x_3)$.
2. Similar to the proof of 1. □

Proposition 3.4. *Let g be an automorphism of $r(\mathfrak{w}^1_{2,2})$, if $g(x_1) = x_1$ and $g(x_3) = (1-2c)x_3 + cx_3x_2$ ($c \in \mathbb{Z}$), then $g \in G$.*

Proof. Since g is an automorphism of $r(\mathfrak{w}^1_{2,2})$ and
$$\begin{cases} x_2^2 = x_2 + x_1x_2, \\ x_1x_2 = x_2x_1, \\ x_2x_3 = 2x_3. \end{cases}$$

Then we have

$$\begin{cases} (g(x_2))^2 = g(x_2) + g(x_1)g(x_2), \\ g(x_1)g(x_2) = g(x_2)g(x_1), \\ g(x_2)f(x_3) = 2g(x_3). \end{cases} \tag{3.1}$$

Assume

$$g(x_2) = c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_1x_2 + c_5x_3x_2, \quad c_i \in \mathbb{Z}\ (i = 0, 1, 2, 3, 4, 5).$$

Then we have

$$\begin{cases} c_0 + c_1 + 2c_2 + c_3 + 2c_4 + 2c_5 = 2, \\ c_0^2 + c_1^2 = c_0 + c_1, \\ 2c_0c_1 = c_0 + c_1, \\ 2c_0c_2 + 2c_1c_4 + c_2^2 + c_4^2 + 2c_2c_4 = c_2 + c_4, \\ 2c_0c_3 + 2c_1c_3 + 2c_2c_3 + 2c_3c_4 + 2c_3c_5 + c_3^2 = 2c_3, \\ 2c_0c_4 + 2c_1c_2 + c_2^2 + 2c_2c_4 + c_4^2 = c_2 + c_4, \\ 2c_0c_5 + 2c_1c_5 + 4c_2c_5 + c_2c_3 + c_3c_4 + c_3c_5 + 4c_4c_5 + 2c_5^2 = 2c_5, \end{cases}$$

Since $c_0^2 + c_1^2 = c_0 + c_1$ and $2c_0c_1 = c_0 + c_1$, then

$$c_0 = c_1 = 0 \text{ or } c_0 = c_1 = 1.$$

Therefore, we have following cases.
Case 1 $c_0 = c_1 = 0$.

$$\begin{cases} 2c_2 + c_3 + 2c_4 + 2c_5 = 2, \\ c_2^2 + c_4^2 + 2c_2c_4 = c_2 + c_4, \\ 2c_2c_3 + 2c_3c_4 + 2c_3c_5 + c_3^2 = 2c_3, \\ 4c_2c_5 + c_2c_3 + c_3c_4 + c_3c_5 + 4c_4c_5 + 2c_5^2 = 2c_5. \end{cases}$$

Since $c_2^2 + c_4^2 + 2c_2c_4 = c_2 + c_4$, then we have

$$c_2 + c_4 = 0 \text{ or } c_2 + c_4 = 1.$$

1. $c_2 + c_4 = 0$.
   Since $c_0 = c_1 = 0$ and $c_2 + c_4 = 0$, then $c_3 = 2 - 2c_5, c_2 = -c_4$. We let $c_4 = d, c_5 = b$, then

$$\begin{cases} g(1) = 1, \\ g(x_1) = x_1, \\ g(x_2) = -dx_2 + (2 - 2b)x_3 + dx_1x_2 + bx_3x_2, \\ g(x_3) = (1 - 2c)x_3 + cx_3x_2, \\ g(x_1x_2) = -dx_1x_2 + (2 - 2b)x_3 + dx_2 + bx_3x_2, \\ g(x_3x_2) = (2 - 2b)x_3 + bx_3x_2. \end{cases}$$

And

$$\mathbf{A}_g = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -d & 0 & d & 0 \\ 0 & 0 & 2-2b & 1-2c & 2-2b & 2-2b \\ 0 & 0 & d & 0 & -d & 0 \\ 0 & 0 & b & c & b & b \end{pmatrix},$$

so $| A_g |= 0$, and hence, A_g is not invertible.

2. $c_2 + c_4 = 1$.

Since $c_0 = c_1 = 0$ and $c_2 + c_4 = 1$, then $c_3 = -2c_5, c_2 = 1 - c_4$. We let $c_4 = d, c_3 = b$ then

$$\begin{cases} g(1) = 1, \\ g(x_1) = x_1, \\ g(x_2) = (1-d)x_2 - 2bx_3 + dx_1x_2 + bx_3x_2, \\ g(x_3) = (1-2c)x_3 + cx_3x_2, \\ g(x_1x_2) = (1-d)x_1x_2 - 2bx_3 + dx_2 + bx_3x_2, \\ g(x_3x_2) = -2bx_3 + (1+b)x_3x_2. \end{cases}$$

And

$$\mathbf{A}_g = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-d & 0 & d & 0 \\ 0 & 0 & -2b & 1-2c & -2b & -2b \\ 0 & 0 & d & 0 & 1-d & 0 \\ 0 & 0 & b & c & b & 1+b \end{pmatrix},$$

so $| A_g |= (1-2d)(1+b-2c)$. Note that A_g is invertible if only if $| A_g |= \pm 1$, thus, we get that

$$\begin{cases} d = 0, \\ b = 2c, \end{cases} \text{or} \begin{cases} d = 1, \\ b = 2c-2, \end{cases} \text{or} \begin{cases} d = 0, \\ b = 2c-2, \end{cases} \text{or} \begin{cases} d = 1, \\ b = 2c. \end{cases}$$

For any $c \in \mathbb{Z}$, we have

(a) If $d = 0$ and $b = 2c$, then $g(x_2) = x_2 - 4cx_3 + 2cx_3x_2, g(x_1x_2) = x_1x_2 - 4cx_3 + 2cx_3x_2, g(x_3x_2) = -4cx_3 + (1+2c)x_3x_2$, thus $g = \omega_i$.

(b) If $d = 1$ and $b = 2c-2$, then $g(x_2) = (4-4c)x_3 + x_1x_2 + (2c-2)x_3x_2, g(x_1x_2) = (4-4c)x_3 + x_2 + (2c-2)x_3x_2, g(x_3x_2) = (4-4c)x_3 + (2c-1)x_3x_2$, thus $g = \delta_j$.

(c) If $d = 0$ and $b = 2c-2$, then $g(x_2) = x_2 + (4-4c)x_3 + (2c-2)x_3x_2, g(x_1x_2) = x_1x_2 + (4-4c)x_3 + (2c-2)x_3x_2, g(x_3x_2) = (4-4c)x_3 + (2c-1)x_3x_2$, thus $g = \tau_k$.

(d) If $d = 1$ and $b = 2c$, then $g(x_2) = -4cx_3 + x_1x_2 + 2cx_3x_2, g(x_1x_2) = -4cx_3 + x_2 + 2cx_3x_2, g(x_3x_2) = -4cx_3 + (2c+1)x_3x_2$, thus $g = \varphi_z$.

Case 2 $c_0 = c_1 = 1$.

$$\begin{cases} 2 + 2c_2 + c_3 + 2c_4 + 2c_5 = 2, \\ 2c_2 + 2c_4 + c_2^2 + c_4^2 + 2c_2c_4 = c_2 + c_4, \\ 4c_3 + 2c_2c_3 + 2c_3c_4 + 2c_3c_5 + c_3^2 = 2c_3, \\ 4c_5 + 4c_2c_5 + c_2c_3 + c_3c_4 + c_3c_5 + 4c_4c_5 + 2c_5^2 = 2c_5. \end{cases}$$

Since $2c_2 + 2c_4 + c_2^2 + c_4^2 + 2c_2c_4 = c_2 + c_4$, then we have

$$c_2 + c_4 = 0 \text{ or } c_2 + c_4 = -1.$$

1. $c_2 + c_4 = 0$.
   Since $c_0 = c_1 = 1$ and $c_2 + c_4 = 0$, then $c_3 = c_5 = 0, c_2 = -c_4$. We let $c_4 = e$, then

$$\begin{cases} g(1) = 1, \\ g(x_1) = x_1, \\ g(x_2) = 1 + x_1 - ex_2 + ex_1x_2, \\ g(x_3) = (1-2c)x_3 + cx_3x_2, \\ g(x_1x_2) = 1 + x_1 + ex_2 - ex_1x_2, \\ g(x_3x_2) = 2(1-2c)x_3 + 2cx_3x_2. \end{cases}$$

   And

$$\mathbf{A}_g = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -e & 0 & e & 0 \\ 0 & 0 & 0 & 1-2c & 0 & 2-4c \\ 0 & 0 & e & 0 & -e & 0 \\ 0 & 0 & 0 & c & 0 & 2c \end{pmatrix},$$

   so $|A_g| = 0$, and hence, A_g is not invertible.
2. $c_2 + c_4 = -1$.
   Since $c_0 = c_1 = 1$ and $c_2 + c_4 = -1$, then $c_3 = 0, c_5 = 1, c_2 = -1 - c_4$. We let $c_4 = e$, then

$$\begin{cases} g(1) = 1, \\ g(x_1) = x_1, \\ g(x_2) = 1 + x_1 - (1+e)x_2 + ex_1x_2 + x_3x_2, \\ g(x_3) = (1-2c)x_3 + cx_3x_2, \\ g(x_1x_2) = 1 + x_1 + ex_2 - (1+e)x_1x_2 + x_3x_2, \\ g(x_3x_2) = 2(1-2c)x_3 + 2cx_3x_2. \end{cases}$$

   And

$$\mathbf{A}_g = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1-e & 0 & e & 0 \\ 0 & 0 & 0 & 1-2c & 0 & 2-4c \\ 0 & 0 & e & 0 & -1-e & 0 \\ 0 & 0 & 1 & c & 0 & 2c \end{pmatrix},$$

   so $|A_g| = 0$, and hence, A_g is not invertible.

In summary, if $g(x_1) = x_1$ and $g(x_3) = (1-2c)x_3 + cx_3x_2 (c \in \mathbb{Z})$, then $g \in G$. □

Theorem 3.5. *Let* $\mathrm{Aut}(r(\mathfrak{w}_{2,2}^1))$ *denote the automorphism group of* $r(\mathfrak{w}_{2,2}^1)$. *Then*

$$\mathrm{Aut}(r(\mathfrak{w}_{2,2}^1)) = G.$$

Proof. Let g be an automorphism of $r(\mathfrak{w}_{2,2}^1)$, by Lemma 3.3, we know that

- if $g(x_3) = (1-2c)x_3 + cx_3x_2$, then $g(x_1) = x_1$ or $g(x_1) = -x_1 + (2-4c)x_3 + 2cx_3x_2$ or $g(x_1) = -1 + (2-4c)x_3 + 2cx_3x_2$;
- if $g(x_3) = 1 - (1+2c)x_3 + cx_3x_2$, then $g(x_1) = 1 - (2+4c)x_3 + 2cx_3x_2$.

By Proposition 3.4, we have that if $g(x_1) = x_1$ and $g(x_3) = (1-2c)x_3 + cx_3x_2$, then $g \in G$. The similar arguments of Proposition 3.4 are applied to the remaining possibilities, show that $| A_g |= 0$, and hence, A_g is not invertible, in these cases.

Thus

$$\mathrm{Aut}(r(\mathfrak{w}_{2,2}^1)) = G.$$

The proof is finished. □

Let f_0, f_1, f_2 and f_3 are automorphisms of $r(\mathfrak{w}_{2,2}^0)$, determined by the following.

$$\begin{array}{lllllll}
f_0: & 1 \to 1 & x_1 \to x_1, & x_2 \to x_2, & x_3 \to x_3, & x_1x_2 \to x_1x_2, \\
f_1: & 1 \to 1 & x_1 \to x_1, & x_2 \to x_1x_2, & x_3 \to x_3, & x_1x_2 \to x_2, \\
f_2: & 1 \to 1 & x_1 \to x_1, & x_2 \to 1 + x_1 + 2x_3 - x_1x_2, & x_3 \to x_3, & x_1x_2 \to 1 + x_1 - x_2 + 2x_3, \\
f_3: & 1 \to 1 & x_1 \to x_1, & x_2 \to 1 + x_1 - x_2 + 2x_3, & x_3 \to x_3, & x_1x_2 \to 1 + x_1 + 2x_3 - x_1x_2,
\end{array}$$

where f_0 is the identity map. The set $\{f_0, f_1, f_2, f_3\}$ is a group under the composition of functions. The multiplication is described as follows.

$\circ$	f_0	f_1	f_2	f_3
f_0	f_0	f_1	f_2	f_3
f_1	f_1	f_0	f_3	f_2
f_2	f_2	f_3	f_0	f_1
f_3	f_3	f_2	f_1	f_0

Remark 3.6. *Similar to arguments of the proof of Theorem 3.5 show that*

$$\mathrm{Aut}(r(\mathfrak{w}_{2,2}^0)) = \{f_0, f_1, f_2, f_3\} \cong K_4,$$

where K_4 is the Klein four-group.

4. The properties of $\mathrm{Aut}(r(\mathfrak{w}_{2,2}^s))$

By Section 3, we have $\mathrm{Aut}(r(\mathfrak{w}_{2,2}^0)) \cong K_4$ and $\mathrm{Aut}(r(\mathfrak{w}_{2,2}^1)) = G$.

The infinite group $G = \left\{\omega_i, \delta_j, \tau_k, \varphi_z \mid i, j, k, z \in \mathbb{Z}\right\}$ is not abelian. The elements $\varphi_0, \delta_j, \tau_k$ $(j, k \in \mathbb{Z})$ of G have order 2, and other elements ω_i, φ_z $(i, z \in \mathbb{Z}, z \neq 0)$ have infinite order. In the sequel, we will discuss some properties of G. The definitions of solvable group, nilpotent group, and normal subgroups, etc. can be found in [9], they are used in the sequel.

Proposition 4.1. *Let $Z(G)$ be the centre of G, then $Z(G) \cong \mathbb{Z}_2$.*

Proof. For any $i, j, k, z \in \mathbb{Z}$, all subgroups of G, up to isomorphism, as follows

$$\begin{array}{lll}
\langle\, \omega_0 \,\rangle = \{id\}, & \langle\, \omega_1 \,\rangle = \{\omega_i \mid i \in \mathbb{Z}\}, & \langle\, \delta_j \,\rangle = \{id, \delta_j\}, \\
\langle\, \tau_k \,\rangle = \{id, \tau_k\}, & \langle\, \varphi_0 \,\rangle = \{id, \varphi_0\}, & \langle\, \varphi_1 \,\rangle = \{\omega_i, \varphi_z \mid i, z \in \mathbb{Z}\}, \\
\langle\, \delta_0, \delta_1 \,\rangle = \{\omega_i, \delta_j \mid i, j \in \mathbb{Z}\}, & \langle\, \tau_0, \tau_1 \,\rangle = \{\omega_i, \tau_k \mid i, k \in \mathbb{Z}\}, & \langle\, \delta_0, \varphi_1 \,\rangle = G,
\end{array}$$

and we have

$$\begin{array}{llll}
\omega_{i'}\omega_i\omega_{-i'} = \omega_i, & \delta_j\omega_i\delta_j = \omega_{-i}, & \tau_k\omega_i\tau_k = \omega_{-i}, & \varphi_z\omega_i\varphi_{-z} = \omega_i, \\
\omega_i\delta_j\omega_{-i} = \delta_{2i+j}, & \delta_{j'}\delta_j\delta_{j'} = \delta_{2j'-j}, & \tau_k\delta_j\tau_k = \delta_{2k-j}, & \varphi_z\delta_j\varphi_{-z} = \delta_{2z+j}, \\
\omega_i\tau_k\omega_{-i} = \tau_{2i+k}, & \delta_j\tau_k\delta_j = \tau_{2j-k}, & \tau_{k'}\tau_k\tau_{k'} = \tau_{2k'-k}, & \varphi_z\tau_k\varphi_{-z} = \tau_{2z+k}, \\
\omega_i\varphi_z\omega_{-i} = \varphi_z, & \delta_j\varphi_z\delta_j = \varphi_{-z}, & \tau_k\varphi_z\tau_k = \varphi_{-z}, & \varphi_z\varphi_z\varphi_{-z} = \varphi_z.
\end{array}$$

Hence $\langle\, \omega_0 \,\rangle, \langle\, \omega_1 \,\rangle, \langle\, \varphi_0 \,\rangle, \langle\, \varphi_1 \,\rangle, \langle\, \delta_0, \delta_1 \,\rangle, \langle\, \tau_0, \tau_1 \,\rangle, \langle\, \delta_0, \varphi_1 \,\rangle$ are normal subgroups of G. Furthermore

$$\begin{array}{llll}
\omega_i\delta_j \neq \delta_j\omega_i, & \omega_i\tau_k \neq \tau_k\omega_i, & \varphi_z\delta_j \neq \delta_j\varphi_z, & \varphi_z\tau_k \neq \tau_k\varphi_z, \\
\delta_{j'}\delta_j \neq \delta_j\delta_{j'}, & \tau_{k'}\tau_k \neq \tau_k\tau_{k'}, & \tau_k\delta_j \neq \delta_j\tau_k, &
\end{array}$$

for any $i, j, k, z, j', k' \in \mathbb{Z} \setminus \{0\}, j \neq j', k \neq k'$. Therefore

$$Z(G) = \langle\, \varphi_0 \,\rangle = \{id, \varphi_0\}.$$

It is easy to show that $Z(G) \cong \mathbb{Z}_2$, determined by the map $id \to 0, \varphi_0 \to 1$. □

Theorem 4.2. $G \cong (\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$.

Proof. We set

$$H = \langle\, \varphi_1 \,\rangle = \{\omega_i, \varphi_z \mid i, z \in \mathbb{Z}\}, \quad K = \langle\, \delta_0 \,\rangle = \{id, \delta_0\}.$$

It is easy to know that H and K are subgroups, and $H \lhd G$. Since

$$\omega_i\delta_0 = \delta_i, \quad \varphi_z\delta_0 = \tau_z$$

for any $i, z \in \mathbb{Z}$, hence $G = HK$, and $H \cap K = \{id\}$, thus we have

$$G = H \rtimes K.$$

Let

$$H_1 = \langle\, \omega_1 \,\rangle = \{\omega_i \mid i \in \mathbb{Z}\},$$

one can get that $H_1 \lhd H, Z(G) \lhd H$, where $Z(G) = \langle\, \varphi_0 \,\rangle = \{id, \varphi_0\}$ is the centre of G. Furthermore

$$H_1 \cap Z(G) = \{id\},$$

and $\omega_i\varphi_0 = \varphi_i$ for any $i \in \mathbb{Z}$. Hence $H = H_1Z(G)$, and

$$H = H_1 \times Z(G).$$

H_1 is isomorphic to $\mathbb{Z}$, determined by the map $\omega_i \to i$, for any $i \in \mathbb{Z}$. K is isomorphic to $\mathbb{Z}_2$, determined by the map

$$id \to 0, \delta_0 \to 1.$$

Therefore

$$G = (H_1 \times Z(G)) \rtimes K = (\langle\, \omega_1 \,\rangle \times \langle\, \varphi_0 \,\rangle) \rtimes \langle\, \delta_0 \,\rangle \cong (\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2.$$

The proof is finished. □

By the Theorem 4.2, the following results are easily to get.

Corollary 4.3. *G is a solvable and non-nilpotent group.*

Proof. By the Proposition 4.1 and Theorem 4.2, we have

$$G = (\langle\, \omega_1 \,\rangle \times \langle\, \varphi_0 \,\rangle) \rtimes \langle\, \delta_0 \,\rangle \cong (\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2, \quad Z(G) = \langle\, \varphi_0 \,\rangle \cong \mathbb{Z}_2.$$

Since $\mathbb{Z}$ and $\mathbb{Z}_2$ are solvable groups, then $\mathbb{Z} \times \mathbb{Z}_2$ and $(\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2/\mathbb{Z} \times \mathbb{Z}_2$ are solvable groups, hence $G \cong (\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ is solvable .

It is easy to see $G/Z(G) = \langle\, \omega_1 \,\rangle \rtimes \langle\, \delta_0 \,\rangle$ and $Z\left(G/Z(G)\right) = \{id\}$, hence, $G/Z(G)$ and G are non-nilpotent groups. □

5. Conclusions

In this paper, we investigate the automorphism groups $\mathrm{Aut}(r(\mathfrak{w}^s_{2,2}))$ of representation rings $r(\mathfrak{w}^s_{2,2})$ of two classes of weak Sweedler Hopf algebras $\mathfrak{w}^s_{2,2}(s = 0, 1)$ and discuss some properties of $\mathrm{Aut}(r(\mathfrak{w}^s_{2,2}))$. We obtain that $\mathrm{Aut}(r(\mathfrak{w}^0_{2,2}))$ is isomorphic to the Klein four-group. It is shown that $\mathrm{Aut}(r(\mathfrak{w}^1_{2,2}))$ is a non-commutative infinite group, it is solvable and non-nilpotent. In addition, we prove that $\mathrm{Aut}(r(\mathfrak{w}^1_{2,2}))$ is isomorphic to $(\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, and its centre is isomorphic to $\mathbb{Z}_2$.

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Conflict of interest

The authors declared that they have no conflict of interest.

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