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Research article

Analytical solutions of incommensurate fractional differential equation systems with fractional order $1 < \alpha,\beta < 2$ via bivariate Mittag-Leffler functions

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Abstract: In this paper, we derive the explicit analytical solution of incommensurate fractional differential equation systems with fractional order $1 < \alpha,\beta < 2$. The derivation is extended from a recently published paper by Huseynov et al. in [1], which is limited for incommensurate fractional order $0 < \alpha,\beta < 1$. The incommensurate fractional differential equation systems were first converted to Volterra integral equations. Then, the Mittag-Leffler function and Picard's successive approximations were used to obtain the analytical solution of incommensurate fractional order systems with $1 < \alpha,\beta < 2$. The solution will be simplified via some combinatorial concepts and bivariate Mittag-Leffler function. Some special cases will be discussed, while some examples will be given at the end of this paper.

Keywords: incommensurate fractional order system; bivariate Mittag-Leffler function; Picard's successive approximations; analytical solutions
Mathematics Subject Classification: 34A08, 34A25, 26A33

1. Introduction

System of fractional differential equations with incommensurate order derivatives have received increasing attention recently as this incommensurate order derivative is better in describing the real phenomena, such as financial system [2, 3], circuit simulation [4], eco-epidemiological model [5], HIV model [6] and modeling glucose-insulin regulatory system [7]. In this research direction, many works had been done to study stability analysis [8–11], synchronization [12] and other rich dynamical behaviour [13, 14].

Due to the emerging of cross-discipline research in this incommensurate fractional order system, finding the solution of the incommensurate fractional order system is becoming more and more important. In this case, numerical methods, such as the predictor-corrector scheme [15, 16], are always used to obtain the solution for the incommensurate fractional order system. Apart from this, some algorithms are developed to obtain the approximation solution for incommensurate fractional order systems, such as the Adomian decomposition algorithm [17], reduced-order model approximation via genetic algorithm [18]. However, not much research was done to find the analytical solution or exact solution for this incommensurate fractional order system. Until recently, Huseynov et al. in [1] successfully derive the analytical solution for the incommensurate fractional order $0 < \alpha, \beta < 1$ by converting the system into a corresponding Volterra integral equation. Besides that, Ahmadova et al. [19] found the analytical solution for this incommensurate fractional order system via trivariate Mittag-Leffer functions. However, their proposed methods are only limited to incommensurate fractional order $0 < \alpha, \beta < 1$. Hence, this motivates us to derive the analytical solution for a higher order of incommensurate fractional order system.

In this paper, we extend the work by Huseynov et al. in [1], which is limited for incommensurate fractional order system for $0 < \alpha, \beta < 1$. We intend to derive the analytical solution of higher order incommensurate fractional order system. Specifically, for $\alpha, \beta \in (1, 2)$, we consider the incommensurate fractional order system as follows:

$$\begin{aligned} {}^{C}\mathrm{D}^{\alpha} x_1(t) &= a_{11} x_1(t) + a_{12} x_2(t) + g_1(t), \\ {}^{C}\mathrm{D}^{\beta} x_2(t) &= a_{21} x_1(t) + a_{22} x_2(t) + g_2(t), \end{aligned} \tag{1.1}$$

with initial condition $x_1(0) = x_1^0$, $x_2(0) = x_2^0$, $x_1'(0) = x_1^1$ and $x_2'(0) = x_2^1$. The physical meaning of such an incommensurate fractional order system as well as the advantages of using incommensurate models over the classical one (compare to commensurate models) are shown in [2–14]. The fractional derivatives are defined with Caputo sense and the initial value problems to be solved for $x_1, x_2 \in C^1[0, \infty)$. Similar to the works in [1], we convert the system in (1.1) to Volterra integral equations and Picard's successive approximations were used to derive the analytical expression of the solution for an incommensurate fractional order system for $1 < \alpha, \beta < 2$. Similar to [1], we use Picard's successive approximations to solve the Volterra integral equations arise because this method is based on the Banach fixed point theorem. In order to obtain the fixed point of a functional operator, start with an arbitrary function (i.e. the zeroth approximation) and apply the operator repeatedly to obtain a sequence of successive approximations which should converge towards the fixed point. This method has been applied to derive the explicit analytical solution of incommensurate fractional differential equation systems with fractional order $0 < \alpha, \beta < 1$ [1]. The solution will be simplified via some combinatorial concepts and bivariate Mittag-Leffler function. In short, this paper aims to contribute to analytical method that gives new explicit solutions to a certain class of fractional differential systems. These kind of explicit solutions for solving fractional differential equations or systems have been increasingly investigated by researchers in these research areas, such as in [20–24]. In short, we hope to contribute in obtaining an explicit analytical solution for fractional calculus problems, which is relatively less investigated compared to numerical solution, such as in [25–32].

The rest of this paper is structured as follows. Section 2 is devoted to some preliminaries regarding some important definitions, concepts and notations in fractional calculus and special functions. Section 3 is devoted to presenting the derivation of analytical solutions for the incommensurate fractional order system in higher order. Moreover, some special cases will be discussed in Section 4. Sections 5 and 6 are devoted to presenting some examples and conclusion of this paper, respectively.

2. Preliminaries

In this section, we briefly explain some important definitions, concepts and notations in fractional calculus and special functions, which is important for obtaining the analytical solution for this incommensurate fractional order system.

2.1. Caputo fractional derivative

Definition 1. *Let $\alpha > 0$, $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$, $n = \alpha$ if $\alpha \in \mathbb{N}$ and $x > 0$. The left Caputo fractional derivative of a function of order α, denoted by ${}^{C}\mathrm{D}_x^{\alpha} f(x)$ is*

$$ {}^{C}\mathrm{D}_x^{\alpha}\, f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}}\, \mathrm{d}\tau, \tag{2.1} $$

with $n - 1 \leq \alpha < n$.

For Caputo fractional derivative, we have this important expression:

$$ {}^{C}\mathrm{D}_x^{\alpha}\, x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, \text{for } \beta > \alpha. \tag{2.2} $$

2.2. Mittag-Leffler function

Definition 2. *For $Re(\alpha), Re(\beta) > 0$, the classical Mittag-Leffler function (i.e. one parameter) and two-parameter Mittag-Leffler function are defined as*

$$ E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}. \tag{2.3} $$

Definition 3. *[33] For $Re(\alpha), Re(\beta), Re(\gamma) > 0$, the three-parameter version of bivariate Mittag-Leffler function can be defined as:*

$$ E_{\alpha,\beta,\gamma}(x, y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(k+l)!x^k y^l}{\Gamma(\alpha k + \beta l + \gamma)k!l!}. \tag{2.4} $$

The convergence of this bivariate Mittag-Leffler function was shown in Section 2, the new bivariate Mittag-Leffler function in [33]. The Mittag-Leffler function is used as the solution of system of fractional differential equations as this Mittag-Leffler function is the generalization of the exponential function, which exponential function is widely used to express the solution of integer order system of differential equations. The Mittag-Leffler function is a series which the terms are up to infinity. Hence, to calculate these Mittag-Leffler functions, ones can refer the numerical algorithm such as in [34–36]. With Caputo fractional derivative, we have this important expression for the fractional derivative involving Mittag-Leffler function:

$$ {}^{C}\mathrm{D}_x^{\alpha}(E_\alpha(\lambda x^\alpha)) = \lambda(E_\alpha(\lambda x^\alpha)). \tag{2.5}$$

In this paper, we will use some important integration with was introduced in [1] as follows:

$$\int_0^t (t-\tau)^{a-1}\tau^{b-1}\,\mathrm{d}\tau = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}t^{a+b-1},\ \text{for } a>0, b>0. \tag{2.6}$$

$$\int_u^t (t-\tau)^{a-1}(\tau-u)^{b-1}\,\mathrm{d}\tau = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}(t-u)^{a+b-1},\ \text{for } a>0, b>0. \tag{2.7}$$

$$\int_0^t\int_0^\tau (t-\tau)^{a-1}(\tau-u)^{b-1}f(u)\,\mathrm{d}u\mathrm{d}\tau = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\int_0^t (t-u)^{a+b-1}f(u)\,\mathrm{d}u,\ \text{for } a>0, b>0. \tag{2.8}$$

Remarks: We can also write $\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = B(a,b)$, where $B(a,b)$ is the Beta function.

For the $f(\tau) = \tau^v$, where $v > 0$, using Eq (2.6), we have the following integration involving Mittag-Leffler function.

$$\begin{aligned}\int_0^t (t-\tau)^a E_{\alpha,\beta}(\lambda(t-\tau)^b)\tau^v\,\mathrm{d}\tau &= \sum_{k=0}^\infty \frac{\int_0^t \lambda^k(t-\tau)^{bk+a}\tau^v\,\mathrm{d}\tau}{\Gamma(\alpha k+\beta)}\\ &= \sum_{k=0}^\infty \frac{\lambda^k\Gamma(bk+a+1)\Gamma(v+1)t^{bk+a+v+1}}{\Gamma(\alpha k+\beta)\Gamma(bk+a+v+2)}.\end{aligned} \tag{2.9}$$

If $a = \beta - 1$ and $b = \alpha$, from Eq (2.9), we obtain

$$\int_0^t (t-\tau)^{\beta-1}\tau^v E_{\alpha,\beta}(\lambda(t-\tau)^\alpha)\,\mathrm{d}\tau = \Gamma(v+1)t^{\beta+v}E_{\alpha,\beta+v+1}(\lambda t^\alpha).$$

The lower incomplete gamma function is defined for $Re(\alpha) > 0, Re(z) > 0$ as follows:

$$\gamma(\alpha, z) = \int_0^z e^{-t}t^{\alpha-1}\,\mathrm{d}t. \tag{2.10}$$

Definition 4. *Hypergeometric functions ${}_2F_1(a_1, a_2; b; z)$ and ${}_1F_2(a; b_1, b_2; z)$ are defined by the series*

$$\begin{aligned}{}_2F_1(a_1,a_2;b;z) &= \sum_{k=0}^\infty \frac{(a_1)_k(a_2)_k}{(b)_k}\frac{z^k}{k!}, |z|<1,\\ {}_1F_2(a;b_1,b_2;z) &= \sum_{k=0}^\infty \frac{(a)_k}{(b_1)_k(b_2)_k}\frac{z^k}{k!},\end{aligned} \tag{2.11}$$

where the pochhammer symbol, $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$.

For the sake of simplicity, throughout the writing, we use $\sum_{n_1,n_2,\cdots,n_k=0}^\infty$ to represent multiple series $\sum_{n_1=0}^\infty\sum_{n_2=0}^\infty\cdots\sum_{n_k=0}^\infty$.

3. Main result

In order to derive the analytical solutions for the incommensurate fractional differential equation systems with order $1 < \alpha, \beta < 2$ as in Eq (1.1), basically one can follow the following steps:
Step 1: Write the system in Volterra integral equations of second kind.
Step 2: Perform the Picard's successive approximations.
Step 3: Simplify the solution by using some combinatorial formulae.
Step 4: Verify the solution by using substitution.

Here, we will derive the inhomogeneous case. By setting $g_1(t) = 0$ and $g_2(t)$, the Eq (1.1) will reduce to the homogeneous case. Similarly, if we take the value of $\alpha = \beta$, the incommensurate fractional differential equation systems with fractional order $1 < \alpha, \beta < 2$ will be reduced to commensurate fractional differential equation systems with fractional order 1 to 2.

Step 1: Write the system in Volterra integral equations of second kind.

Using the result from Theorem 5.15 in [37], we obtain the single fractional differential equation for $\sigma \in (1, 2)$ in Caputo sense as follows:

$$ {}^{C}\mathrm{D}^{\sigma} y(t) = \lambda y(t) + h(t), \ y(0) = y_0, \ y'(0) = y_1. \tag{3.1} $$

We have the following solution:

$$ y(t) = y_0 E_\sigma(\lambda t^\sigma) + y_1 t E_{\sigma,2}(\lambda t^\sigma) + \int_0^t (t-\tau)^{\sigma-1} h(\tau) E_{\sigma,\sigma}(\lambda(t-\tau)^\sigma)\, \mathrm{d}\tau. \tag{3.2} $$

Using Eq (3.2), the Volterra integral equation of second kind for the equation in (1.1) can be written as

$$ \begin{aligned} x_1(t) &= x_1^0 E_\alpha(a_{11}t^\alpha) + x_1^1 t E_{\alpha,2}(a_{11}t^\alpha) + \int_0^t (t-\tau)^{\alpha-1}[a_{12}x_2(\tau) + g_1(\tau)] E_{\alpha,\alpha}(a_{11}(t-\tau)^\alpha)\, \mathrm{d}\tau, \\ x_2(t) &= x_2^0 E_\beta(a_{22}t^\beta) + x_2^1 t E_{\beta,2}(a_{22}t^\beta) + \int_0^t (t-\tau)^{\beta-1}[a_{21}x_1(\tau) + g_2(\tau)] E_{\beta,\beta}(a_{22}(t-\tau)^\beta)\, \mathrm{d}\tau. \end{aligned} \tag{3.3} $$

Substituting $x_2(t)$ into the first equation in (3.3) and $x_1(t)$ into the second equation in (3.3), we obtain the following:

$$ \begin{aligned} x_1(t) &= x_1^0 E_\alpha(a_{11}t^\alpha) + x_1^1 t E_{\alpha,2}(a_{11}t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(a_{11}(t-\tau)^\alpha) \Bigg[a_{12} \Bigg(x_2^0 E_\beta(a_{22}\tau^\beta) \\ &\quad + x_2^1 \tau E_{\beta,2}(a_{22}\tau^\beta) + \int_0^\tau (\tau-u)^{\beta-1} E_{\beta,\beta}(a_{22}(\tau-u)^\beta)(a_{21}x_1(u) + g_2(u))\, \mathrm{d}u \Bigg) + g_1(\tau) \Bigg] \mathrm{d}\tau \\ &= x_1^0 E_\alpha(a_{11}t^\alpha) + x_1^1 t E_{\alpha,2}(a_{11}t^\alpha) + a_{12}x_2^0 \int_0^t \sum_{n_1,n_2=0}^{\infty} \frac{a_{11}^{n_1}(t-\tau)^{n_1\alpha+\alpha-1}}{\Gamma(n_1\alpha+\alpha)} \frac{a_{22}^{n_2}\tau^{n_2\beta}}{\Gamma(n_2\beta+1)}\, \mathrm{d}\tau \\ &\quad + a_{12}x_2^1 \int_0^t \sum_{n_1,n_2=0}^{\infty} \frac{a_{11}^{n_1}(t-\tau)^{n_1\alpha+\alpha-1}}{\Gamma(n_1\alpha+\alpha)} \frac{a_{22}^{n_2}\tau^{n_2\beta+1}}{\Gamma(n_2\beta+2)}\, \mathrm{d}\tau + \int_0^t \sum_{n_1=0}^{\infty} \frac{a_{11}^{n_1}(t-\tau)^{n_1\alpha+\alpha-1}}{\Gamma(n_1\alpha+\alpha)} g_1(\tau)\, \mathrm{d}\tau \\ &\quad + a_{12} \int_0^t \int_0^\tau \sum_{n_1,n_2=0}^{\infty} \frac{a_{11}^{n_1}(t-\tau)^{n_1\alpha+\alpha-1}}{\Gamma(n_1\alpha+\alpha)} \frac{a_{22}^{n_2}(\tau-u)^{n_2\beta+\beta-1}}{\Gamma(n_2\beta+\beta)} (a_{21}x_1(u) + g_2(u))\, \mathrm{d}u\, \mathrm{d}\tau. \end{aligned} \tag{3.4} $$

Using identity as in Eq (2.6) to Eq (2.8) , we obtain

$$\begin{aligned} x_1(t) = {} & x_1^0 E_\alpha(a_{11}t^\alpha) + x_1^1 t E_{\alpha,2}(a_{11}t^\alpha) + a_{12}x_2^0 \sum_{n_1,n_2=0}^{\infty} \frac{a_{11}^{n_1} a_{22}^{n_2} t^{n_1\alpha+n_2\beta+\alpha}}{\Gamma(n_1\alpha+n_2\beta+\alpha+1)} \\ & + a_{12}x_2^1 \sum_{n_1,n_2=0}^{\infty} \frac{a_{11}^{n_1} a_{22}^{n_2} t^{n_1\alpha+n_2\beta+\alpha+1}}{\Gamma(n_1\alpha+n_2\beta+\alpha+2)} + \sum_{n_1=0}^{\infty} \frac{a_{11}^{n_1}}{\Gamma(n_1\alpha+\alpha)} \int_0^t (t-\tau)^{n_1\alpha+\alpha-1} g_1(\tau)\, \mathrm{d}\tau \\ & + a_{12} \sum_{n_1,n_2=0}^{\infty} \frac{a_{11}^{n_1} a_{22}^{n_2}}{\Gamma(n_1\alpha+n_2\beta+\alpha+\beta)} \int_0^t (t-\tau)^{n_1\alpha+n_2\beta+\alpha+\beta-1} (a_{21}x_1(\tau)+g_2(\tau))\, \mathrm{d}\tau. \end{aligned} \tag{3.5}$$

By using a similar approach, we obtain the expression for $x_2(t)$ as follows:

$$\begin{aligned} x_2(t) = {} & x_2^0 E_\beta(a_{22}t^\beta) + x_2^1 t E_{\beta,2}(a_{22}t^\beta) + a_{21}x_1^0 \sum_{n_1,n_2=0}^{\infty} \frac{a_{22}^{n_1} a_{11}^{n_2} t^{n_1\beta+n_2\alpha+\beta}}{\Gamma(n_1\beta+n_2\alpha+\beta+1)} \\ & + a_{21}x_1^1 \sum_{n_1,n_2=0}^{\infty} \frac{a_{22}^{n_1} a_{11}^{n_2} t^{n_1\beta+n_2\alpha+\beta+1}}{\Gamma(n_1\beta+n_2\alpha+\beta+2)} \\ & + a_{21} \sum_{n_1,n_2=0}^{\infty} \frac{a_{22}^{n_1} a_{11}^{n_2}}{\Gamma(n_1\beta+n_2\alpha+\alpha+\beta)} \int_0^t (t-\tau)^{n_1\beta+n_2\alpha+\alpha+\beta-1} (a_{12}x_2(\tau)+g_1(\tau))\, \mathrm{d}\tau \\ & + \sum_{n_1=0}^{\infty} \frac{a_{22}^{n_1}}{\Gamma(n_1\beta+\beta)} \int_0^t (t-\tau)^{n_1\beta+\beta-1} g_2(\tau)\, \mathrm{d}\tau. \end{aligned} \tag{3.6}$$

Step 2: Perform the Picard's successive approximation.

Using Picard's successive approximation, the solution of the Volterra integral equations as in Eqs (3.5) and (3.6) can be obtained via setting

$$\begin{aligned} x_{1,0}(t) = {} & x_1^0 E_\alpha(a_{11}t^\alpha) + x_1^1 t E_{\alpha,2}(a_{11}t^\alpha) + a_{12}x_2^0 \sum_{n_1,n_2=0}^{\infty} \frac{a_{11}^{n_1} a_{22}^{n_2} t^{n_1\alpha+n_2\beta+\alpha}}{\Gamma(n_1\alpha+n_2\beta+\alpha+1)} \\ & + a_{12}x_2^1 \sum_{n_1,n_2=0}^{\infty} \frac{a_{11}^{n_1} a_{22}^{n_2} t^{n_1\alpha+n_2\beta+\alpha+1}}{\Gamma(n_1\alpha+n_2\beta+\alpha+2)} + \sum_{n_1=0}^{\infty} \frac{a_{11}^{n_1}}{\Gamma(n_1\alpha+\alpha)} \int_0^t (t-\tau)^{n_1\alpha+\alpha-1} g_1(\tau)\, \mathrm{d}\tau \\ & + a_{12} \sum_{n_1,n_2=0}^{\infty} \frac{a_{11}^{n_1} a_{22}^{n_2}}{\Gamma(n_1\alpha+n_2\beta+\alpha+\beta)} \int_0^t (t-\tau)^{n_1\alpha+n_2\beta+\alpha+\beta-1} g_2(\tau)\, \mathrm{d}\tau, \\ x_{1,m}(t) = {} & x_1^0 E_\alpha(a_{11}t^\alpha) + x_1^1 t E_{\alpha,2}(a_{11}t^\alpha) + a_{12}x_2^0 \sum_{n_1,n_2=0}^{\infty} \frac{a_{11}^{n_1} a_{22}^{n_2} t^{n_1\alpha+n_2\beta+\alpha}}{\Gamma(n_1\alpha+n_2\beta+\alpha+1)} \\ & + a_{12}x_2^1 \sum_{n_1,n_2=0}^{\infty} \frac{a_{11}^{n_1} a_{22}^{n_2} t^{n_1\alpha+n_2\beta+\alpha+1}}{\Gamma(n_1\alpha+n_2\beta+\alpha+2)} + \sum_{n_1=0}^{\infty} \frac{a_{11}^{n_1}}{\Gamma(n_1\alpha+\alpha)} \int_0^t (t-\tau)^{n_1\alpha+\alpha-1} g_1(\tau)\, \mathrm{d}\tau \\ & + a_{12} \sum_{n_1,n_2=0}^{\infty} \frac{a_{11}^{n_1} a_{22}^{n_2}}{\Gamma(n_1\alpha+n_2\beta+\alpha+\beta)} \int_0^t (t-\tau)^{n_1\alpha+n_2\beta+\alpha+\beta-1} (a_{21}x_{1,m-1}(\tau)+g_2(\tau))\, \mathrm{d}\tau, \end{aligned} \tag{3.7}$$

and

$$x_{2,0}(t) = x_2^0 E_\beta(a_{22}t^\beta) + x_2^1 t E_{\beta,2}(a_{22}t^\beta) + a_{21}x_1^0 \sum_{n_1,n_2=0}^{\infty} \frac{a_{22}^{n_1} a_{11}^{n_2} t^{n_1\beta+n_2\alpha+\beta}}{\Gamma(n_1\beta + n_2\alpha + \beta + 1)}$$
$$+ a_{21}x_1^1 \sum_{n_1,n_2=0}^{\infty} \frac{a_{22}^{n_1} a_{11}^{n_2} t^{n_1\beta+n_2\alpha+\beta+1}}{\Gamma(n_1\beta + n_2\alpha + \beta + 2)} + \sum_{n_1=0}^{\infty} \frac{a_{22}^{n_1}}{\Gamma(n_1\beta + \beta)} \int_0^t (t-\tau)^{n_1\beta+\beta-1} g_2(\tau)\, \mathrm{d}\tau$$
$$+ a_{21} \sum_{n_1,n_2=0}^{\infty} \frac{a_{22}^{n_1} a_{11}^{n_2}}{\Gamma(n_1\beta + n_2\alpha + \alpha + \beta)} \int_0^t (t-\tau)^{n_1\beta+n_2\alpha+\alpha+\beta-1} g_1(\tau)\, \mathrm{d}\tau,$$

$$x_{2,m}(t) = x_2^0 E_\beta(a_{22}t^\beta) + x_2^1 t E_{\beta,2}(a_{22}t^\beta) + a_{21}x_1^0 \sum_{n_1,n_2=0}^{\infty} \frac{a_{22}^{n_1} a_{11}^{n_2} t^{n_1\beta+n_2\alpha+\beta}}{\Gamma(n_1\beta + n_2\alpha + \beta + 1)}$$
$$+ a_{21}x_1^1 \sum_{n_1,n_2=0}^{\infty} \frac{a_{22}^{n_1} a_{11}^{n_2} t^{n_1\beta+n_2\alpha+\beta+1}}{\Gamma(n_1\beta + n_2\alpha + \beta + 2)} + \sum_{n_1=0}^{\infty} \frac{a_{22}^{n_1}}{\Gamma(n_1\beta + \beta)} \int_0^t (t-\tau)^{n_1\beta+\beta-1} g_2(\tau)\, \mathrm{d}\tau$$
$$+ a_{21} \sum_{n_1,n_2=0}^{\infty} \frac{a_{22}^{n_1} a_{11}^{n_2}}{\Gamma(n_1\beta + n_2\alpha + \alpha + \beta)} \int_0^t (t-\tau)^{n_1\beta+n_2\alpha+\alpha+\beta-1} (a_{21}x_{2,m-1}(\tau) + g_1(\tau))\, \mathrm{d}\tau. \tag{3.8}$$

For $m = 1$, using the identities (2.6)–(2.8), we have

$$x_{1,1}(t) = x_{1,0}(t) + a_{12}a_{21}x_1^0 \sum_{n_1,n_2,n_3=0}^{\infty} \frac{a_{11}^{n_1+n_3} a_{22}^{n_2} t^{(n_1+n_3)\alpha+n_2\beta+\alpha+\beta}}{\Gamma((n_1+n_3)\alpha + n_2\beta + \alpha + \beta + 1)}$$
$$+ a_{12}a_{21}x_1^1 \sum_{n_1,n_2,n_3=0}^{\infty} \frac{a_{11}^{n_1+n_3} a_{22}^{n_2} t^{(n_1+n_3)\alpha+n_2\beta+\alpha+\beta+1}}{\Gamma((n_1+n_3)\alpha + n_2\beta + \alpha + \beta + 2)}$$
$$+ a_{12}^2 a_{21} x_2^0 \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \frac{a_{11}^{n_1+n_3} a_{22}^{n_2+n_4} t^{(n_1+n_3)\alpha+(n_2+n_4)\beta+2\alpha+\beta}}{\Gamma((n_1+n_3)\alpha + (n_2+n_4)\beta + 2\alpha + \beta + 1)}$$
$$+ a_{12}^2 a_{21} x_2^1 \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \frac{a_{11}^{n_1+n_3} a_{22}^{n_2+n_4} t^{(n_1+n_3)\alpha+(n_2+n_4)\beta+2\alpha+\beta+1}}{\Gamma((n_1+n_3)\alpha + (n_2+n_4)\beta + 2\alpha + \beta + 2)}$$
$$+ a_{12}a_{21} \sum_{n_1,n_2,n_3=0}^{\infty} \frac{a_{11}^{n_1+n_3} a_{22}^{n_2} \int_0^t (t-\tau)^{(n_1+n_3)\alpha+n_2\beta+2\alpha+\beta-1} g_1(\tau)\, \mathrm{d}\tau}{\Gamma((n_1+n_3)\alpha + n_2\beta + 2\alpha + \beta)}$$
$$+ a_{12}^2 a_{21} \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \frac{a_{11}^{n_1+n_3} a_{22}^{n_2+n_4} \int_0^t (t-\tau)^{(n_1+n_3)\alpha+(n_2+n_4)\beta+2\alpha+2\beta-1} g_2(\tau)\, \mathrm{d}\tau}{\Gamma((n_1+n_3)\alpha + (n_2+n_4)\beta + 2\alpha + 2\beta)}. \tag{3.9}$$

Meanwhile, for $m = 2$, we have

$$x_{1,2}(t) = x_{1,1}(t) + a_{12}^2 a_{21}^2 x_1^0 \sum_{n_1,\cdots,n_5=0}^{\infty} \frac{a_{11}^{n_1+n_3+n_5} a_{22}^{n_2+n_4} t^{(n_1+n_3+n_5)\alpha+(n_2+n_4)\beta+2\alpha+2\beta}}{\Gamma((n_1+n_3+n_5)\alpha + (n_2+n_4)\beta + 2\alpha + 2\beta + 1)}$$
$$+ a_{12}^2 a_{21}^2 x_1^1 \sum_{n_1,\cdots,n_5=0}^{\infty} \frac{a_{11}^{n_1+n_3+n_5} a_{22}^{n_2+n_4} t^{(n_1+n_3+n_5)\alpha+(n_2+n_4)\beta+2\alpha+2\beta+1}}{\Gamma((n_1+n_3+n_5)\alpha + (n_2+n_4)\beta + 2\alpha + 2\beta + 2)}$$
$$+ a_{12}^3 a_{21}^2 x_2^0 \sum_{n_1,\cdots,n_6=0}^{\infty} \frac{a_{11}^{n_1+n_3+n_5} a_{22}^{n_2+n_4+n_6} t^{(n_1+n_3+n_5)\alpha+(n_2+n_4+n_6)\beta+3\alpha+2\beta}}{\Gamma((n_1+n_3+n_5)\alpha + (n_2+n_4+n_6)\beta + 3\alpha + 2\beta + 1)}$$

$$+ a_{12}^3 a_{21}^2 x_2^1 \sum_{n_1,\cdots,n_6=0}^{\infty} \frac{a_{11}^{n_1+n_3+n_5} a_{22}^{n_2+n_4+n_6} t^{(n_1+n_3+n_5)\alpha+(n_2+n_4+n_6)\beta+3\alpha+2\beta+1}}{\Gamma((n_1+n_3+n_5)\alpha+(n_2+n_4+n_6)\beta+3\alpha+2\beta+2)}$$

$$+ a_{12}^2 a_{21}^2 \sum_{n_1,\cdots,n_5=0}^{\infty} \frac{a_{11}^{n_1+n_3+n_5} a_{22}^{n_2+n_4} \int_0^t (t-\tau)^{(n_1+n_3+n_5)\alpha+(n_2+n_4)\beta+3\alpha+2\beta-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma((n_1+n_3+n_5)\alpha+(n_2+n_4)\beta+3\alpha+2\beta)}$$

$$+ a_{12}^3 a_{21}^2 \sum_{n_1,\cdots,n_6=0}^{\infty} \frac{a_{11}^{n_1+n_3+n_5} a_{22}^{n_2+n_4+n_6} \int_0^t (t-\tau)^{(n_1+n_3+n_5)\alpha+(n_2+n_4+n_6)\beta+3\alpha+3\beta-1} g_2(\tau)\,\mathrm{d}\tau}{\Gamma((n_1+n_3+n_5)\alpha+(n_2+n_4+n_6)\beta+3\alpha+3\beta)}. \tag{3.10}$$

In general, after some algebraic manipulation, we obtain

$$x_{1,m}(t) = x_{1,m-1}(t)$$

$$+ a_{12}^m a_{21}^m x_1^0 \sum_{n_1,\cdots,n_{2m+1}=0}^{\infty} \frac{a_{11}^{n_1+n_3+\cdots+n_{2m+1}} a_{22}^{n_2+n_4+\cdots+2m} t^{(n_1+n_3+\cdots+n_{2m+1})\alpha+(n_2+n_4+\cdots+n_{2m})\beta+m\alpha+m\beta}}{\Gamma((n_1+n_3+\cdots+n_{2m+1})\alpha+(n_2+n_4+\cdots+n_{2m})\beta+m\alpha+m\beta+1)}$$

$$+ a_{12}^m a_{21}^m x_1^1 \sum_{n_1,\cdots,n_{2m+1}=0}^{\infty} \frac{a_{11}^{n_1+n_3+\cdots+n_{2m+1}} a_{22}^{n_2+n_4+\cdots+2m} t^{(n_1+n_3+\cdots+n_{2m+1})\alpha+(n_2+n_4+\cdots+n_{2m})\beta+m\alpha+m\beta+1}}{\Gamma((n_1+n_3+\cdots+n_{2m+1})\alpha+(n_2+n_4+\cdots+n_{2m})\beta+m\alpha+m\beta+2)}$$

$$+ a_{12}^{m+1} a_{21}^m x_2^0 \sum_{n_1,\cdots,n_{2m+2}=0}^{\infty} \frac{a_{11}^{n_1+n_3+\cdots+n_{2m+1}} a_{22}^{n_2+n_4+\cdots+n_{2m+2}} t^{(n_1+n_3+\cdots+n_{2m+1})\alpha+(n_2+n_4+\cdots+n_{2m+2})\beta+(m+1)\alpha+m\beta}}{\Gamma((n_1+n_3+\cdots+n_{2m+1})\alpha+(n_2+n_4+\cdots+n_{2m+2})\beta+(m+1)\alpha+m\beta+1)}$$

$$+ a_{12}^{m+1} a_{21}^m x_2^1 \sum_{n_1,\cdots,n_{2m+2}=0}^{\infty} \frac{a_{11}^{n_1+n_3+\cdots+n_{2m+1}} a_{22}^{n_2+n_4+\cdots+n_{2m+2}} t^{(n_1+n_3+\cdots+n_{2m+1})\alpha+(n_2+n_4+\cdots+n_{2m+2})\beta+(m+1)\alpha+m\beta+1}}{\Gamma((n_1+n_3+\cdots+n_{2m+1})\alpha+(n_2+n_4+\cdots+n_{2m+2})\beta+(m+1)\alpha+m\beta+2)}$$

$$+ a_{12}^m a_{21}^m \sum_{n_1,\cdots,n_{2m+1}=0}^{\infty} \frac{a_{11}^{n_1+\cdots+n_{2m+1}} a_{22}^{n_2+\cdots+n_{2m}} \int_0^t (t-\tau)^{(n_1+\cdots+n_{2m+1})\alpha+(n_2+\cdots+n_{2m})\beta+(m+1)\alpha+m\beta-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma((n_1+\cdots+n_{2m+1})\alpha+(n_2+\cdots+n_{2m})\beta+(m+1)\alpha+m\beta)}$$

$$+ a_{12}^{m+1} a_{21}^m \sum_{n_1,\cdots,n_{2m+2}=0}^{\infty} \frac{a_{11}^{n_1+\cdots+n_{2m+1}} a_{22}^{n_2+\cdots+n_{2m+2}} \int_0^t (t-\tau)^{(n_1+\cdots+n_{2m+1})\alpha+(n_2+\cdots+n_{2m+2})\beta+(m+1)(\alpha+\beta)-1} g_2(\tau)\,\mathrm{d}\tau}{\Gamma((n_1+\cdots+n_{2m+1})\alpha+(n_2+\cdots+n_{2m+2})\beta+(m+1)(\alpha+\beta))}, \tag{3.11}$$

where $n_1+\cdots+n_{2m+1}$ and $n_2+\cdots+n_{2m}$ denote $n_1+n_3+\cdots+n_{2m+1}$ and $n_2+n_4+\cdots+n_{2m}$, respectively.

When $m \to \infty$, we can rewrite the solution of $x_1(t)$ as follows:

$$x_1(t) = \sum_{k=0}^{\infty} a_{12}^k a_{21}^k x_1^0 \sum_{n_1,\cdots,n_{2k+1}=0}^{\infty} \frac{a_{11}^{n_1+n_3+\cdots+n_{2k+1}} a_{22}^{n_2+n_4+\cdots+2k} t^{(n_1+n_3+\cdots+n_{2k+1})\alpha+(n_2+n_4+\cdots+n_{2k})\beta+k\alpha+k\beta}}{\Gamma((n_1+n_3+\cdots+n_{2k+1})\alpha+(n_2+n_4+\cdots+n_{2k})\beta+k\alpha+k\beta+1)}$$

$$+ \sum_{k=0}^{\infty} a_{12}^k a_{21}^k x_1^1 \sum_{n_1,\cdots,n_{2k+1}=0}^{\infty} \frac{a_{11}^{n_1+n_3+\cdots+n_{2k+1}} a_{22}^{n_2+n_4+\cdots+2k} t^{(n_1+n_3+\cdots+n_{2k+1})\alpha+(n_2+n_4+\cdots+n_{2k})\beta+k\alpha+k\beta+1}}{\Gamma((n_1+n_3+\cdots+n_{2k+1})\alpha+(n_2+n_4+\cdots+n_{2k})\beta+k\alpha+k\beta+2)}$$

$$+ \sum_{k=0}^{\infty} a_{12}^{k+1} a_{21}^k x_2^0 \sum_{n_1,\cdots,n_{2k+2}=0}^{\infty} \frac{a_{11}^{n_1+n_3+\cdots+n_{2k+1}} a_{22}^{n_2+n_4+\cdots+n_{2k+2}} t^{(n_1+n_3+\cdots+n_{2k+1})\alpha+(n_2+n_4+\cdots+n_{2k+2})\beta+(k+1)\alpha+k\beta}}{\Gamma((n_1+n_3+\cdots+n_{2k+1})\alpha+(n_2+n_4+\cdots+n_{2k+2})\beta+(k+1)\alpha+k\beta+1)}$$

$$+ \sum_{k=0}^{\infty} a_{12}^{k+1} a_{21}^k x_2^1 \sum_{n_1,\cdots,n_{2k+2}=0}^{\infty} \frac{a_{11}^{n_1+n_3+\cdots+n_{2k+1}} a_{22}^{n_2+n_4+\cdots+n_{2k+2}} t^{(n_1+n_3+\cdots+n_{2k+1})\alpha+(n_2+n_4+\cdots+n_{2k+2})\beta+(k+1)\alpha+k\beta+1}}{\Gamma((n_1+n_3+\cdots+n_{2k+1})\alpha+(n_2+n_4+\cdots+n_{2k+2})\beta+(k+1)\alpha+k\beta+2)}$$

$$+ \sum_{k=0}^{\infty} a_{12}^k a_{21}^k \sum_{n_1,\cdots,n_{2k+1}=0}^{\infty} \frac{a_{11}^{n_1+\cdots+n_{2k+1}} a_{22}^{n_2+\cdots+n_{2k}} \int_0^t (t-\tau)^{(n_1+\cdots+n_{2k+1})\alpha+(n_2+\cdots+n_{2k})\beta+(k+1)\alpha+k\beta-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma((n_1+\cdots+n_{2k+1})\alpha+(n_2+\cdots+n_{2k})\beta+(k+1)\alpha+k\beta)}$$

$$+\sum_{k=0}^{\infty}a_{12}^{k+1}a_{21}^{k}\sum_{n_1,\cdots,n_{2k+2}=0}^{\infty}\frac{a_{11}^{n_1+\cdots+n_{2k+1}}a_{22}^{n_2+\cdots+n_{2k+2}}\int_0^t(t-\tau)^{(n_1+\cdots+n_{2k+1})\alpha+(n_2+\cdots+n_{2k+2})\beta+(k+1)(\alpha+\beta)-1}g_2(\tau)\,d\tau}{\Gamma((n_1+\cdots+n_{2k+1})\alpha+(n_2+\cdots+n_{2k+2})\beta+(k+1)(\alpha+\beta))}. \tag{3.12}$$

Similarly, by symmetry, we have successive approximations for $x_2(t)=\lim_{m\to\infty}x_{2,m}(t)$ as follows:

$$\begin{aligned}x_2(t)=&\sum_{k=0}^{\infty}a_{12}^{k}a_{21}^{k}x_2^0\sum_{n_1,\cdots,n_{2k+1}=0}^{\infty}\frac{a_{22}^{n_1+n_3+\cdots+n_{2k+1}}a_{11}^{n_2+n_4+\cdots+2k}t^{(n_1+n_3+\cdots+n_{2k+1})\beta+(n_2+n_4+\cdots+n_{2k})\alpha+k\alpha+k\beta}}{\Gamma((n_1+n_3+\cdots+n_{2k+1})\beta+(n_2+n_4+\cdots+n_{2k})\alpha+k\alpha+k\beta+1)}\\&+\sum_{k=0}^{\infty}a_{12}^{k}a_{21}^{k}x_2^1\sum_{n_1,\cdots,n_{2k+1}=0}^{\infty}\frac{a_{22}^{n_1+n_3+\cdots+n_{2k+1}}a_{11}^{n_2+n_4+\cdots+2k}t^{(n_1+n_3+\cdots+n_{2k+1})\beta+(n_2+n_4+\cdots+n_{2k})\alpha+k\alpha+k\beta+1}}{\Gamma((n_1+n_3+\cdots+n_{2k+1})\beta+(n_2+n_4+\cdots+n_{2k})\alpha+k\alpha+k\beta+2)}\\&+\sum_{k=0}^{\infty}a_{12}^{k}a_{21}^{k+1}x_1^0\sum_{n_1,\cdots,n_{2k+2}=0}^{\infty}\frac{a_{22}^{n_1+n_3+\cdots+n_{2k+1}}a_{11}^{n_2+n_4+\cdots+n_{2k+2}}t^{(n_1+n_3+\cdots+n_{2k+1})\beta+(n_2+n_4+\cdots+n_{2k+2})\alpha+k\alpha+(k+1)\beta}}{\Gamma((n_1+n_3+\cdots+n_{2k+1})\beta+(n_2+n_4+\cdots+n_{2k+2})\alpha+k\alpha+(k+1)\beta+1)}\\&+\sum_{k=0}^{\infty}a_{12}^{k}a_{21}^{k+1}x_1^1\sum_{n_1,\cdots,n_{2k+2}=0}^{\infty}\frac{a_{22}^{n_1+n_3+\cdots+n_{2k+1}}a_{11}^{n_2+n_4+\cdots+n_{2k+2}}t^{(n_1+n_3+\cdots+n_{2k+1})\beta+(n_2+n_4+\cdots+n_{2k+2})\alpha+k\alpha+(k+1)\beta+1}}{\Gamma((n_1+n_3+\cdots+n_{2k+1})\beta+(n_2+n_4+\cdots+n_{2k+2})\alpha+k\alpha+(k+1)\beta+2)}\\&+\sum_{k=0}^{\infty}a_{12}^{k}a_{21}^{k+1}\sum_{n_1,\cdots,n_{2k+2}=0}^{\infty}\frac{a_{22}^{n_1+\cdots+n_{2k+1}}a_{11}^{n_2+\cdots+n_{2k+2}}\int_0^t(t-\tau)^{(n_1+\cdots+n_{2k+1})\beta+(n_2+\cdots+n_{2k+2})\alpha+(k+1)(\alpha+\beta)-1}g_1(\tau)\,d\tau}{\Gamma((n_1+\cdots+n_{2k+1})\beta+(n_2+\cdots+n_{2k+2})\alpha+(k+1)(\alpha+\beta))}\\&+\sum_{k=0}^{\infty}a_{12}^{k}a_{21}^{k}\sum_{n_1,\cdots,n_{2k+1}=0}^{\infty}\frac{a_{22}^{n_1+\cdots+n_{2k+1}}a_{11}^{n_2+\cdots+n_{2k}}\int_0^t(t-\tau)^{(n_1+\cdots+n_{2k+1})\beta+(n_2+\cdots+n_{2k})\alpha+k\alpha+(k+1)\beta-1}g_2(\tau)\,d\tau}{\Gamma((n_1+\cdots+n_{2k+1})\beta+(n_2+\cdots+n_{2k})\alpha+k\alpha+(k+1)\beta)}.\end{aligned} \tag{3.13}$$

Step 3: Simplify the solution by using some combinatorial formulae.

We write j as all the odd-indexed terms together and m as all the even-indexed appear together, i.e. $j=n_1+n_3+\cdots+n_{2k+1}$, $m=n_2+n_4+\cdots+n_{2k}$ or $m=n_2+n_4+\cdots+n_{2k+2}$, we obtain,

$$\begin{aligned}x_1(t)=&x_1^0\sum_{n_1=0}^{\infty}\frac{a_{11}^{n_1}t^{n_1\alpha}}{\Gamma(n_1\alpha+1)}+x_1^0\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\substack{n_1,n_2,\cdots,n_{2k+1}:\\n_1+n_3+\cdots+n_{2k+1}=j,\\n_2+n_4+\cdots+n_{2k}=m}}\frac{a_{12}^{k}a_{21}^{k}a_{11}^{j}a_{22}^{m}t^{(j+k)\alpha+(m+k)\beta}}{\Gamma((j+k)\alpha+(m+k)\beta+1)}\\&+x_1^1\sum_{n_1=0}^{\infty}\frac{a_{11}^{n_1}t^{n_1\alpha+1}}{\Gamma(n_1\alpha+2)}+x_1^1\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\substack{n_1,n_2,\cdots,n_{2k+1}:\\n_1+n_3+\cdots+n_{2k+1}=j,\\n_2+n_4+\cdots+n_{2k}=m}}\frac{a_{12}^{k}a_{21}^{k}a_{11}^{j}a_{22}^{m}t^{(j+k)\alpha+(m+k)\beta+1}}{\Gamma((j+k)\alpha+(m+k)\beta+2)}\\&+x_2^0\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\substack{n_1,n_2,\cdots,n_{2k+2}:\\n_1+n_3+\cdots+n_{2k+1}=j,\\n_2+n_4+\cdots+n_{2k+2}=m}}\frac{a_{12}^{k+1}a_{21}^{k}a_{11}^{j}a_{22}^{m}t^{(j+k+1)\alpha+(m+k)\beta}}{\Gamma((j+k+1)\alpha+(m+k)\beta+1)}\\&+x_2^1\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\substack{n_1,n_2,\cdots,n_{2k+2}:\\n_1+n_3+\cdots+n_{2k+1}=j,\\n_2+n_4+\cdots+n_{2k+2}=m}}\frac{a_{12}^{k+1}a_{21}^{k}a_{11}^{j}a_{22}^{m}t^{(j+k+1)\alpha+(m+k)\beta+1}}{\Gamma((j+k+1)\alpha+(m+k)\beta+2)}+\sum_{n_1=0}^{\infty}\frac{a_{11}^{n_1}\int_0^t(t-\tau)^{n_1\alpha+\alpha-1}g_1(\tau)\,d\tau}{\Gamma(n_1\alpha+\alpha)}\end{aligned}$$

$$+\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\substack{n_1,n_2,\cdots,n_{2k+1}:\\ n_1+n_3+\cdots+n_{2k+1}=j,\\ n_2+n_4+\cdots+n_{2k}=m}}\frac{a_{12}^{k}a_{21}^{k}a_{11}^{j}a_{22}^{m}\int_0^t(t-\tau)^{(j+k+1)\alpha+(m+k)\beta-1}g_1(\tau)\,\mathrm{d}\tau}{\Gamma((j+k+1)\alpha+(m+k)\beta)}$$

$$+\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\substack{n_1,n_2,\cdots,n_{2k+2}:\\ n_1+n_3+\cdots+n_{2k+1}=j,\\ n_2+n_4+\cdots+n_{2k+2}=m}}\frac{a_{12}^{k+1}a_{21}^{k}a_{11}^{j}a_{22}^{m}\int_0^t(t-\tau)^{(j+k+1)\alpha+(m+k+1)\beta-1}g_2(\tau)\,\mathrm{d}\tau}{\Gamma((j+k+1)\alpha+(m+k+1)\beta)}. \tag{3.14}$$

Then, we have the simple combinatorial identity as follows for any k, j and m:

$$\begin{aligned}\sum_{\substack{n_1,n_2,\cdots,n_{2k+1}:\\ n_1+n_3+\cdots+n_{2k+1}=j,\\ n_2+n_4+\cdots+n_{2k}=m}}(1)&=\left|\left\{(n_1,n_3,\cdots,n_{2k+1}):\sum=j\right\}\right|\left|\left\{n_2,n_4,\cdots,n_{2k}):\sum=m\right\}\right|\\ &=\frac{(k+j)!}{k!j!}\frac{(k+m-1)!}{(k-1)!m!}=\binom{k+j}{k}\binom{k+m-1}{k-1}\end{aligned} \tag{3.15}$$

$$\begin{aligned}\sum_{\substack{n_1,n_2,\cdots,n_{2k+2}:\\ n_1+n_3+\cdots+n_{2k+1}=j,\\ n_2+n_4+\cdots+n_{2k+2}=m}}(1)&=\left|\left\{(n_1,n_3,\cdots,n_{2k+1}):\sum=j\right\}\right|\left|\left\{n_2,n_4,\cdots,n_{2k+2}):\sum=m\right\}\right|\\ &=\frac{(k+j)!}{k!j!}\frac{(k+m)!}{k!m!}=\binom{k+j}{k}\binom{k+m}{k}\end{aligned} \tag{3.16}$$

Applying Eqs (3.15) and (3.16) to Eq (3.14) yields

$$\begin{aligned}x_1(t)&=x_1^0\sum_{n_1=0}^{\infty}\frac{a_{11}^{n_1}t^{n_1\alpha}}{\Gamma(n_1\alpha+1)}+x_1^0\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_{12}^{k}a_{21}^{k}a_{11}^{j}a_{22}^{m}t^{(j+k)\alpha+(m+k)\beta}\binom{k+j}{k}\binom{k+m-1}{k-1}}{\Gamma((j+k)\alpha+(m+k)\beta+1)}\\ &+x_1^1\sum_{n_1=0}^{\infty}\frac{a_{11}^{n_1}t^{n_1\alpha+1}}{\Gamma(n_1\alpha+2)}+x_1^1\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_{12}^{k}a_{21}^{k}a_{11}^{j}a_{22}^{m}t^{(j+k)\alpha+(m+k)\beta+1}\binom{k+j}{k}\binom{k+m-1}{k-1}}{\Gamma((j+k)\alpha+(m+k)\beta+2)}\\ &+x_2^0\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_{12}^{k+1}a_{21}^{k}a_{11}^{j}a_{22}^{m}t^{(j+k+1)\alpha+(m+k)\beta}\binom{k+j}{k}\binom{k+m}{k}}{\Gamma((j+k+1)\alpha+(m+k)\beta+1)}\\ &+x_2^1\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_{12}^{k+1}a_{21}^{k}a_{11}^{j}a_{22}^{m}t^{(j+k+1)\alpha+(m+k)\beta+1}\binom{k+j}{k}\binom{k+m}{k}}{\Gamma((j+k+1)\alpha+(m+k)\beta+2)}+\sum_{n_1=0}^{\infty}\frac{a_{11}^{n_1}\int_0^t(t-\tau)^{n_1\alpha+\alpha-1}g_1(\tau)\,\mathrm{d}\tau}{\Gamma(n_1\alpha+\alpha)}\\ &+\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_{12}^{k}a_{21}^{k}a_{11}^{j}a_{22}^{m}\int_0^t(t-\tau)^{(j+k+1)\alpha+(m+k)\beta-1}g_1(\tau)\,\mathrm{d}\tau\binom{k+j}{k}\binom{k+m-1}{k-1}}{\Gamma((j+k+1)\alpha+(m+k)\beta)}\\ &+\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_{12}^{k+1}a_{21}^{k}a_{11}^{j}a_{22}^{m}\int_0^t(t-\tau)^{(j+k+1)\alpha+(m+k+1)\beta-1}g_2(\tau)\,\mathrm{d}\tau\binom{k+j}{k}\binom{k+m}{k}}{\Gamma((j+k+1)\alpha+(m+k+1)\beta)}.\end{aligned} \tag{3.17}$$

By writing some of the terms in Mittag-Leffler function and let all the summations start from 0, we have

$$
\begin{aligned}
x_1(t) = {} & x_1^0 E_\alpha(a_{11}t^\alpha) + x_1^0 \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_{12}^{k+1}a_{21}^{k+1}a_{11}^{j}a_{22}^{m}t^{(j+k+1)\alpha+(m+k+1)\beta}\binom{k+j+1}{k+1}\binom{k+m}{k}}{\Gamma((j+k+1)\alpha+(m+k+1)\beta+1)} \\
& + x_1^1 tE_{\alpha,2}(a_{11}t^\alpha) + x_1^1 \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_{12}^{k+1}a_{21}^{k+1}a_{11}^{j}a_{22}^{m}t^{(j+k+1)\alpha+(m+k+1)\beta+1}\binom{k+j+1}{k+1}\binom{k+m}{k}}{\Gamma((j+k+1)\alpha+(m+k+1)\beta+2)} \\
& + x_2^0 \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_{12}^{k+1}a_{21}^{k}a_{11}^{j}a_{22}^{m}t^{(j+k+1)\alpha+(m+k)\beta}\binom{k+j}{k}\binom{k+m}{k}}{\Gamma((j+k+1)\alpha+(m+k)\beta+1)} \\
& + x_2^1 \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_{12}^{k+1}a_{21}^{k}a_{11}^{j}a_{22}^{m}t^{(j+k+1)\alpha+(m+k)\beta+1}\binom{k+j}{k}\binom{k+m}{k}}{\Gamma((j+k+1)\alpha+(m+k)\beta+2)} \\
& + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(a_{11}(t-\tau)^\alpha)g_1(\tau)\,\mathrm{d}\tau \\
& + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_{12}^{k+1}a_{21}^{k+1}a_{11}^{j}a_{22}^{m}\int_0^t (t-\tau)^{(j+k+2)\alpha+(m+k+1)\beta-1}g_1(\tau)\,\mathrm{d}\tau\,\binom{k+j+1}{k+1}\binom{k+m}{k}}{\Gamma((j+k+2)\alpha+(m+k+1)\beta)} \\
& + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_{12}^{k+1}a_{21}^{k}a_{11}^{j}a_{22}^{m}\int_0^t (t-\tau)^{(j+k+1)\alpha+(m+k+1)\beta-1}g_2(\tau)\,\mathrm{d}\tau\,\binom{k+j}{k}\binom{k+m}{k}}{\Gamma((j+k+1)\alpha+(m+k+1)\beta)}.
\end{aligned}
\tag{3.18}
$$

In the same manner, for $x_2(t)$, we obtain the following expression

$$
\begin{aligned}
x_2(t) = {} & x_2^0 E_\beta(a_{22}t^\beta) + x_2^0 \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_{12}^{k+1}a_{21}^{k+1}a_{11}^{j}a_{22}^{m}t^{(m+k+1)\alpha+(j+k+1)\beta}\binom{k+m+1}{k+1}\binom{k+j}{k}}{\Gamma((m+k+1)\alpha+(j+k+1)\beta+1)} \\
& + x_2^1 tE_{\beta,2}(a_{22}t^\beta) + x_2^1 \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_{12}^{k+1}a_{21}^{k+1}a_{11}^{j}a_{22}^{m}t^{(m+k+1)\alpha+(j+k+1)\beta+1}\binom{k+m+1}{k+1}\binom{k+j}{k}}{\Gamma((m+k+1)\alpha+(j+k+1)\beta+2)} \\
& + x_1^0 \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_{12}^{k}a_{21}^{k+1}a_{11}^{j}a_{22}^{m}t^{(m+k+1)\alpha+(j+k)\beta}\binom{k+m}{k}\binom{k+j}{k}}{\Gamma((m+k+1)\alpha+(j+k)\beta+1)} \\
& + x_1^1 \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_{12}^{k}a_{21}^{k+1}a_{11}^{j}a_{22}^{m}t^{(m+k+1)\alpha+(j+k)\beta+1}\binom{k+m}{k}\binom{k+j}{k}}{\Gamma((m+k+1)\alpha+(j+k)\beta+2)} + \int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}(a_{22}(t-\tau)^\beta)g_2(\tau)\,\mathrm{d}\tau \\
& + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_{12}^{k+1}a_{21}^{k+1}a_{11}^{j}a_{22}^{m}\int_0^t (t-\tau)^{(m+k+2)\alpha+(j+k+1)\beta-1}g_2(\tau)\,\mathrm{d}\tau\,\binom{k+m+1}{k+1}\binom{k+j}{k}}{\Gamma((m+k+2)\alpha+(j+k+1)\beta)} \\
& + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_{12}^{k}a_{21}^{k+1}a_{11}^{j}a_{22}^{m}\int_0^t (t-\tau)^{(m+k+1)\alpha+(j+k+1)\beta-1}g_1(\tau)\,\mathrm{d}\tau\,\binom{k+m}{k}\binom{k+j}{k}}{\Gamma((m+k+1)\alpha+(j+k+1)\beta)}.
\end{aligned}
\tag{3.19}
$$

Using $\binom{k+j+1}{k+1}\binom{k+m}{k} = \frac{(k+j+1)!}{(k+1)!j!}\frac{(k+m)!}{k!m!}$ and $\binom{k+j}{k}\binom{k+m}{k} = \frac{(k+j)!}{k!j!}\frac{(k+m)!}{k!m!}$ and assuming $p = j+k$ and $q = m+k$

and $a_{11},\ a_{22} \neq 0$, we obtain

$$
\begin{aligned}
x_1(t) = {} & x_1^0 E_\alpha(a_{11}t^\alpha) + x_1^1 t E_{\alpha,2}(a_{11}t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(a_{11}(t-\tau)^\alpha) g_1(\tau)\,\mathrm{d}\tau \\
& + x_1^0 a_{12}a_{21} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha+\beta}}{\Gamma(p\alpha+q\beta+\alpha+\beta+1)} \sum_{k=0}^{\min(p,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{(p+1)!}{(k+1)!(p-k)!} \frac{q!}{k!(q-k)!} \\
& + x_1^1 a_{12}a_{21} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha+\beta+1}}{\Gamma(p\alpha+q\beta+\alpha+\beta+2)} \sum_{k=0}^{\min(p,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{(p+1)!}{(k+1)!(p-k)!} \frac{q!}{k!(q-k)!} \\
& + x_2^0 a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha}}{\Gamma(p\alpha+q\beta+\alpha+1)} \sum_{k=0}^{\min(p,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{p!}{k!(p-k)!} \frac{q!}{k!(q-k)!} \\
& + x_2^1 a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha+1}}{\Gamma(p\alpha+q\beta+\alpha+2)} \sum_{k=0}^{\min(p,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{p!}{k!(p-k)!} \frac{q!}{k!(q-k)!} \\
& + a_{12}a_{21} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+2\alpha+\beta-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+2\alpha+\beta)} \sum_{k=0}^{\min(p,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{(p+1)!}{(k+1)!(p-k)!} \frac{q!}{k!(q-k)!} \\
& + a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+\alpha+\beta-1} g_2(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+\alpha+\beta)} \sum_{k=0}^{\min(p,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{p!}{k!(p-k)!} \frac{q!}{k!(q-k)!}. \qquad (3.20)
\end{aligned}
$$

The above equation can also be rewritten as follows:

$$
\begin{aligned}
x_1(t) = {} & x_1^0 E_\alpha(a_{11}t^\alpha) + x_1^1 t E_{\alpha,2}(a_{11}t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(a_{11}(t-\tau)^\alpha) g_1(\tau)\,\mathrm{d}\tau \\
& + x_1^0 \frac{a_{12}a_{21}}{a_{22}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha}}{\Gamma(p\alpha+q\beta+\alpha+1)} \sum_{k=0}^{\min(p,q-1)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{(p+1)!}{(k+1)!(p-k)!} \frac{(q-1)!}{k!(q-k-1)!} \\
& + x_1^1 \frac{a_{12}a_{21}}{a_{22}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha+1}}{\Gamma(p\alpha+q\beta+\alpha+2)} \sum_{k=0}^{\min(p,q-1)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{(p+1)!}{(k+1)!(p-k)!} \frac{(q-1)!}{k!(q-k-1)!} \\
& + x_2^0 a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha}}{\Gamma(p\alpha+q\beta+\alpha+1)} \sum_{k=0}^{\min(p,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{p!}{k!(p-k)!} \frac{q!}{k!(q-k)!} \\
& + x_2^1 a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha+1}}{\Gamma(p\alpha+q\beta+\alpha+2)} \sum_{k=0}^{\min(p,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{p!}{k!(p-k)!} \frac{q!}{k!(q-k)!} \\
& + \frac{a_{12}a_{21}}{a_{22}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+2\alpha-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+2\alpha)} \\
& \times \sum_{k=0}^{\min(p,q-1)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{(p+1)!}{(k+1)!(p-k)!} \frac{(q-1)!}{k!(q-k-1)!} \\
& + a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+\alpha+\beta-1} g_2(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+\alpha+\beta)}
\end{aligned}
$$

$$\times \sum_{k=0}^{\min(p,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{p!}{k!(p-k)!} \frac{q!}{k!(q-k)!}. \tag{3.21}$$

Similarly, by symmetric, we obtain

$$\begin{aligned}
x_2(t) &= x_2^0 E_\beta(a_{22}t^\beta) + x_2^1 t E_{\beta,2}(a_{22}t^\beta) + \int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}(a_{22}(t-\tau)^\beta) g_2(\tau)\,\mathrm{d}\tau \\
&+ x_2^0 \frac{a_{12}a_{21}}{a_{11}} \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\beta}}{\Gamma(p\alpha+q\beta+\beta+1)} \sum_{k=0}^{\min(p-1,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{(p-1)!}{k!(p-k-1)!} \frac{(q+1)!}{(k+1)!(q-k)!} \\
&+ x_2^1 \frac{a_{12}a_{21}}{a_{11}} \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\beta+1}}{\Gamma(p\alpha+q\beta+\beta+2)} \sum_{k=0}^{\min(p-1,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{(p-1)!}{k!(p-k-1)!} \frac{(q+1)!}{(k+1)!(q-k)!} \\
&+ x_1^0 a_{21} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\beta}}{\Gamma(p\alpha+q\beta+\beta+1)} \sum_{k=0}^{\min(p,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{p!}{k!(p-k)!} \frac{q!}{k!(q-k)!} \\
&+ x_1^1 a_{21} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\beta+1}}{\Gamma(p\alpha+q\beta+\beta+2)} \sum_{k=0}^{\min(p,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{p!}{k!(p-k)!} \frac{q!}{k!(q-k)!} \\
&+ a_{21} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+\alpha+\beta-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+\alpha+\beta)} \sum_{k=0}^{\min(p,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{p!}{k!(p-k)!} \frac{q!}{k!(q-k)!} \\
&+ \frac{a_{12}a_{21}}{a_{11}} \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+2\beta-1} g_2(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+2\beta)} \\
&\times \sum_{k=0}^{\min(p-1,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{(p-1)!}{k!(p-k-1)!} \frac{(q+1)!}{(k+1)!(q-k)!}.
\end{aligned} \tag{3.22}$$

Letting $A = \frac{a_{12}a_{21}}{a_{11}a_{22}}$ with $a_{11},\ a_{22} \neq 0$, we can simplify the inner series in (3.21) into hypergeometric function expression using the following identities.

$$\begin{aligned}
&\sum_{k=0}^{\min(p,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{p!}{k!(p-k)!} \frac{q!}{k!(q-k)!} = {}_2F_1(-p,-q;1;A), \\
&\sum_{k=0}^{\min(p,q-1)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{(p+1)!}{(k+1)!(p-k)!} \frac{(q-1)!}{k!(q-k-1)!} = (p+1){}_2F_1(-p,1-q;2;A).
\end{aligned} \tag{3.23}$$

Hence, we have $x_1(t)$ as follows:

$$\begin{aligned}
x_1(t) &= x_1^0 E_\alpha(a_{11}t^\alpha) + x_1^1 t E_{\alpha,2}(a_{11}t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(a_{11}(t-\tau)^\alpha) g_1(\tau)\,\mathrm{d}\tau \\
&+ x_1^0 a_{11} A \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha}}{\Gamma(p\alpha+q\beta+\alpha+1)} (p+1){}_2F_1(-p,1-q;2;A)
\end{aligned}$$

$$+ x_1^1 a_{11} A \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha+1}}{\Gamma(p\alpha+q\beta+\alpha+2)} (p+1)_2F_1(-p,1-q;2;A)$$
$$+ x_2^0 a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha}}{\Gamma(p\alpha+q\beta+\alpha+1)} \, {}_2F_1(-p,-q;1;A)$$
$$+ x_2^1 a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha+1}}{\Gamma(p\alpha+q\beta+\alpha+2)} \, {}_2F_1(-p,-q;1;A)$$
$$+ a_{11} A \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+2\alpha-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+2\alpha)} (p+1)_2F_1(-p,1-q;2;A)$$
$$+ a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+\alpha+\beta-1} g_2(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+\alpha+\beta)} \, {}_2F_1(-p,-q;1;A). \tag{3.24}$$

The same is applied for $x_2(t)$, where we can simplify the inner series in (3.22) into a hypergeometric function expression using the following identities.

$$\sum_{k=0}^{\min(p,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{p!}{k!(p-k)!}\frac{q!}{k!(q-k)!} = {}_2F_1(-p,-q;1;A),$$
$$\sum_{k=0}^{\min(p-1,q)} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^k \frac{(p-1)!}{k!(p-k-1)!}\frac{(q+1)!}{(k+1)!(q-k)!} = (q+1)_2F_1(1-p,-q;2;A). \tag{3.25}$$

Hence, we obtain $x_2(t)$ as follows:

$$x_2(t) = x_2^0 E_\beta(a_{22}t^\beta) + x_2^1 t E_{\beta,2}(a_{22}t^\beta) + \int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}(a_{22}(t-\tau)^\beta) g_2(\tau)\,\mathrm{d}\tau$$
$$+ x_2^0 a_{22} A \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\beta}}{\Gamma(p\alpha+q\beta+\beta+1)} (q+1)_2F_1(1-p,-q;2;A)$$
$$+ x_2^1 a_{22} A \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\beta+1}}{\Gamma(p\alpha+q\beta+\beta+2)} (q+1)_2F_1(1-p,-q;2;A)$$
$$+ x_1^0 a_{21} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\beta}}{\Gamma(p\alpha+q\beta+\beta+1)} \, {}_2F_1(-p,-q;1;A)$$
$$+ x_1^1 a_{21} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\beta+1}}{\Gamma(p\alpha+q\beta+\beta+2)} \, {}_2F_1(-p,-q;1;A)$$
$$+ a_{22} A \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+2\beta-1} g_2(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+2\beta)} (q+1)_2F_1(1-p,-q;2;A)$$
$$+ a_{21} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+\alpha+\beta-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+\alpha+\beta)} \, {}_2F_1(-p,-q;1;A). \tag{3.26}$$

Substituting $q = 0$ in the double series with $q = 1$ in Eq (3.24) (i.e. the 4th, 5th and 8th terms in the RHS of Eq (3.24)), and using ${}_2F_1(-p, 1; 2; A) = \frac{1-(1-A)^{p+1}}{A(p+1)}$, we obtain the following new expression for the 4th, 5th and 8th terms in the RHS of Eq (3.24).

$$
\begin{aligned}
&x_1^0 a_{11} A \sum_{p=0}^{\infty} \frac{a_{11}^p t^{(p+1)\alpha}}{\Gamma((p+1)\alpha+1)}(p+1)\,{}_2F_1(-p,1;2;A) = x_1^0 E_\alpha(a_{11}t^\alpha) - x_1^0 E_\alpha(a_{11}(1-A)t^\alpha),\\
&x_1^1 a_{11} A \sum_{p=0}^{\infty} \frac{a_{11}^p t^{(p+1)\alpha+1}}{\Gamma((p+1)\alpha+2)}(p+1)\,{}_2F_1(-p,1;2;A) = x_1^1 t E_{\alpha,2}(a_{11}t^\alpha) - x_1^1 t E_{\alpha,2}(a_{11}(1-A)t^\alpha),\\
&a_{11} A \sum_{p=0}^{\infty} \frac{a_{11}^p \int_0^t (t-\tau)^{(p+1)\alpha+\alpha-1} g_1(\tau) d\tau}{\Gamma((p+1)\alpha+\alpha)}(p+1)\,{}_2F_1(-p,1;2;A)\\
&= \int_0^t (t-\tau)^{\alpha-1}\left[E_{\alpha,\alpha}(a_{11}(t-\tau)^\alpha) - E_{\alpha,\alpha}(a_{11}(1-A)(t-\tau)^\alpha)\right] g_1(\tau)\,\mathrm{d}\tau.
\end{aligned}
\tag{3.27}
$$

Hence, we have the final solutions for the $x_1(t)$ to the system (1.1) as follows:

$$
\begin{aligned}
x_1(t) = {} & x_1^0 E_\alpha(a_{11}(1-A)t^\alpha) + x_1^1 t E_{\alpha,2}(a_{11}(1-A)t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(a_{11}(1-A)(t-\tau)^\alpha) g_1(\tau)\,\mathrm{d}\tau\\
& + x_1^0 a_{11} A \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha}}{\Gamma(p\alpha+q\beta+\alpha+1)}(p+1)\,{}_2F_1(-p,1-q;2;A)\\
& + x_1^1 a_{11} A \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha+1}}{\Gamma(p\alpha+q\beta+\alpha+2)}(p+1)\,{}_2F_1(-p,1-q;2;A)\\
& + x_2^0 a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha}}{\Gamma(p\alpha+q\beta+\alpha+1)}\,{}_2F_1(-p,-q;1;A)\\
& + x_2^1 a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha+1}}{\Gamma(p\alpha+q\beta+\alpha+2)}\,{}_2F_1(-p,-q;1;A)\\
& + a_{11} A \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+2\alpha-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+2\alpha)}(p+1)\,{}_2F_1(-p,1-q;2;A)\\
& + a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+\alpha+\beta-1} g_2(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+\alpha+\beta)}\,{}_2F_1(-p,-q;1;A).
\end{aligned}
\tag{3.28}
$$

Using similar approach, the final solutions for the $x_2(t)$ to the system (1.1) is given as follows:

$$
\begin{aligned}
x_2(t) = {} & x_2^0 E_\beta(a_{22}(1-A)t^\beta) + x_2^1 t E_{\beta,2}(a_{22}(1-A)t^\beta) + \int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}(a_{22}(1-A)(t-\tau)^\beta) g_2(\tau)\,\mathrm{d}\tau\\
& + x_2^0 a_{22} A \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\beta}}{\Gamma(p\alpha+q\beta+\beta+1)}(q+1)\,{}_2F_1(1-p,-q;2;A)
\end{aligned}
$$

$$+ x_2^1 a_{22} A \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\beta+1}}{\Gamma(p\alpha+q\beta+\beta+2)}(q+1)_2F_1(1-p,-q;2;A)$$
$$+ x_1^0 a_{21} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\beta}}{\Gamma(p\alpha+q\beta+\beta+1)}\ {}_2F_1(-p,-q;1;A)$$
$$+ x_1^1 a_{21} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\beta+1}}{\Gamma(p\alpha+q\beta+\beta+2)}\ {}_2F_1(-p,-q;1;A)$$
$$+ a_{21} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+\alpha+\beta-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+\alpha+\beta)}\ {}_2F_1(-p,-q;1;A)$$
$$+ a_{22} A \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+2\beta-1} g_2(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+2\beta)}(q+1)_2F_1(1-p,-q;2;A). \tag{3.29}$$

Step 4: Verify the solution by using substitution.

Finally, we can verify the solutions by substituting (3.28) (i.e. $x_1(t)$) and (3.29) (i.e. $x_2(t)$) into ${}^C\mathrm{D}^\alpha x_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + g_1(t)$, which is the first equation of incommensurate fractional order system (1.1). Hence, the right-hand-side of the first equation of (1.1) is given by

$$\begin{aligned}
& a_{11}x_1(t) + a_{12}x_2(t) + g_1(t) \\
={}& x_1^0 a_{11} E_\alpha(a_{11}t^\alpha) + x_1^1 a_{11} t E_{\alpha,2}(a_{11}t^\alpha) + \left[a_{11}\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(a_{11}(t-\tau)^\alpha) g_1(\tau)\,\mathrm{d}\tau + g_1(t)\right] \\
& + x_2^0 a_{12} E_\beta(a_{22}t^\beta) + x_2^1 a_{12} t E_{\beta,2}(a_{22}t^\beta) + a_{12}\int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}(a_{22}(t-\tau)^\beta) g_2(\tau)\,\mathrm{d}\tau \\
& + x_1^0 a_{11} A \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta}}{\Gamma(p\alpha+q\beta+1)} \left(p\ {}_2F_1(1-p,1-q;2;A) + {}_2F_1(-p,1-q;1;A)\right) \\
& + x_1^1 a_{11} A \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+1}}{\Gamma(p\alpha+q\beta+2)} \left(p\ {}_2F_1(1-p,1-q;2;A) + {}_2F_1(-p,1-q;1;A)\right) \\
& + x_2^0 a_{12} \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta}}{\Gamma(p\alpha+q\beta+1)} \left({}_2F_1(1-p,-q;1;A) + Aq\ {}_2F_1(1-p,1-q;2;A)\right) \\
& + x_2^1 a_{12} \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+1}}{\Gamma(p\alpha+q\beta+2)} \left({}_2F_1(1-p,-q;1;A) + Aq\ {}_2F_1(1-p,1-q;2;A)\right) \\
& + a_{11} A \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+\alpha-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+\alpha)} \left(p\ {}_2F_1(1-p,1-q;2;A) + {}_2F_1(-p,1-q;1;A)\right) \\
& + a_{12} \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+\beta-1} g_2(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+\beta)} \left({}_2F_1(1-p,-q;1;A) + Aq\ {}_2F_1(1-p,1-q;2;A)\right),
\end{aligned} \tag{3.30}$$

while the left-hand-side of the equation is given by

$$
\begin{aligned}
{}^{C}\mathrm{D}^{\alpha}x_1(t) = \; & a_{11}x_1^0E_{\alpha}(a_{11}t^{\alpha}) + a_{11}x_1^1tE_{\alpha,2}(a_{11}t^{\alpha}) + \; {}^{C}\mathrm{D}^{\alpha}\left(\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(a_{11}(t-\tau)^{\alpha})g_1(\tau)\,\mathrm{d}\tau\right) \\
& + x_1^0a_{11}A\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\frac{a_{11}^pa_{22}^qt^{p\alpha+q\beta}}{\Gamma(p\alpha+q\beta+1)}(p+1)\,{}_2F_1(-p,1-q;2;A) \\
& + x_1^1a_{11}A\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\frac{a_{11}^pa_{22}^qt^{p\alpha+q\beta+1}}{\Gamma(p\alpha+q\beta+2)}(p+1)\,{}_2F_1(-p,1-q;2;A) \\
& + x_2^0a_{12}E_{\beta}(a_{22}t^{\beta}) + x_2^0a_{12}\sum_{p=1}^{\infty}\sum_{q=0}^{\infty}\frac{a_{11}^pa_{22}^qt^{p\alpha+q\beta}}{\Gamma(p\alpha+q\beta+1)}\;{}_2F_1(-p,-q;1;A) \\
& + x_2^1a_{12}tE_{\beta,2}(a_{22}t^{\beta}) + x_2^1a_{12}\sum_{p=1}^{\infty}\sum_{q=0}^{\infty}\frac{a_{11}^pa_{22}^qt^{p\alpha+q\beta+1}}{\Gamma(p\alpha+q\beta+2)}\;{}_2F_1(-p,-q;1;A) \\
& + a_{11}A\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\frac{a_{11}^pa_{22}^q\int_0^t(t-\tau)^{p\alpha+q\beta+\alpha-1}g_1(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+\alpha)}(p+1)\,{}_2F_1(-p,1-q;2;A) \\
& + a_{12}\int_0^t(t-\tau)^{\beta-1}E_{\beta,\beta}(a_{22}(t-\tau)^{\beta})g_2(\tau)\,\mathrm{d}\tau \\
& + a_{12}\sum_{p=1}^{\infty}\sum_{q=0}^{\infty}\frac{a_{11}^pa_{22}^q\int_0^t(t-\tau)^{p\alpha+q\beta+\beta-1}g_2(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+\beta)}\;{}_2F_1(-p,-q;1;A).
\end{aligned}
\tag{3.31}
$$

Using the Lemma 2.1 in [1], all the terms in Eqs (3.31) and (3.30) are equivalent, except for the third term of Eqs (3.31) and (3.30). Hence, we show here using some algebraic manipulation that the third term of Eq (3.31) is indeed equivalent to the third term in Eq (3.30). We have

$$
\begin{aligned}
& {}^{C}\mathrm{D}^{\alpha}\left(\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(a_{11}(t-\tau)^{\alpha})g_1(\tau)\,\mathrm{d}\tau\right) \\
&= {}^{C}\mathrm{D}^{\alpha}\left(\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}g_1(\tau)\,\mathrm{d}\tau + \sum_{k=1}^{\infty}\int_0^t\frac{a_{11}^k(t-\tau)^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}g_1(\tau)\,\mathrm{d}\tau\right) \\
&= {}^{C}\mathrm{D}^{\alpha}\left(\mathrm{I}^{\alpha}g_1(t)\right) + {}^{C}\mathrm{D}^{\alpha}\left(\sum_{k=1}^{\infty}\int_0^t\frac{a_{11}^k(t-\tau)^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}g_1(\tau)\,\mathrm{d}\tau\right) \\
&= g_1(t) + \sum_{k=1}^{\infty}\int_0^t\frac{a_{11}^k(t-\tau)^{(k-1)\alpha+\alpha-1}}{\Gamma((k-1)\alpha+\alpha)}g_1(\tau)\,\mathrm{d}\tau \\
&= g_1(t) + a_{11}\sum_{k=0}^{\infty}\int_0^t\frac{a_{11}^k(t-\tau)^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}g_1(\tau)\,\mathrm{d}\tau \\
&= g_1(t) + a_{11}\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(a_{11}(t-\tau)^{\alpha})g_1(\tau)\,\mathrm{d}\tau.
\end{aligned}
\tag{3.32}
$$

By applying Lemma 2.1 in [1] and Eq (3.32), the first equation in (1.1) holds true, while the second equation in (1.1) can be verified using a similar approach. Hence, we have the theorem as follows:

Theorem 1. *The incommensurate fractional differential equation systems with fractional order* $1 < \alpha, \beta < 2$ *are given by:*

$$\begin{aligned} {}^{C}\mathrm{D}^{\alpha} x_1(t) &= a_{11} x_1(t) + a_{12} x_2(t) + g_1(t), \\ {}^{C}\mathrm{D}^{\beta} x_2(t) &= a_{21} x_1(t) + a_{22} x_2(t) + g_2(t), \end{aligned} \tag{3.33}$$

with initial conditions $x_1(0) = x_1^0,\ x_2(0) = x_2^0,\ x_1'(0) = x_1^1,\ x_2'(0) = x_2^1$ *and constant* $A = \frac{a_{12}a_{21}}{a_{11}a_{22}} (\neq 1)$, $a_{11}, a_{22} \neq 0$ *have the solutions as follows:*

$$\begin{aligned} x_1(t) = {} & x_1^0 E_{\alpha}(a_{11}(1-A)t^{\alpha}) + x_1^1 t E_{\alpha,2}(a_{11}(1-A)t^{\alpha}) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(a_{11}(1-A)(t-\tau)^{\alpha}) g_1(\tau)\,\mathrm{d}\tau \\ & + x_1^0 a_{11} A \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha}}{\Gamma(p\alpha+q\beta+\alpha+1)} (p+1)\,{}_2F_1(-p, 1-q; 2; A) \\ & + x_1^1 a_{11} A \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha+1}}{\Gamma(p\alpha+q\beta+\alpha+2)} (p+1)\,{}_2F_1(-p, 1-q; 2; A) \\ & + x_2^0 a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha}}{\Gamma(p\alpha+q\beta+\alpha+1)}\ {}_2F_1(-p, -q; 1; A) \\ & + x_2^1 a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha+1}}{\Gamma(p\alpha+q\beta+\alpha+2)}\ {}_2F_1(-p, -q; 1; A) \\ & + a_{11} A \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+2\alpha-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+2\alpha)} (p+1)\,{}_2F_1(-p, 1-q; 2; A) \\ & + a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+\alpha+\beta-1} g_2(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+\alpha+\beta)}\ {}_2F_1(-p, -q; 1; A), \end{aligned} \tag{3.34}$$

$$\begin{aligned} x_2(t) = {} & x_2^0 E_{\beta}(a_{22}(1-A)t^{\beta}) + x_2^1 t E_{\beta,2}(a_{22}(1-A)t^{\beta}) + \int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}(a_{22}(1-A)(t-\tau)^{\beta}) g_2(\tau)\,\mathrm{d}\tau \\ & + x_2^0 a_{22} A \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\beta}}{\Gamma(p\alpha+q\beta+\beta+1)} (q+1)\,{}_2F_1(1-p, -q; 2; A) \\ & + x_2^1 a_{22} A \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\beta+1}}{\Gamma(p\alpha+q\beta+\beta+2)} (q+1)\,{}_2F_1(1-p, -q; 2; A) \\ & + x_1^0 a_{21} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\beta}}{\Gamma(p\alpha+q\beta+\beta+1)}\ {}_2F_1(-p, -q; 1; A) \\ & + x_1^1 a_{21} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\beta+1}}{\Gamma(p\alpha+q\beta+\beta+2)}\ {}_2F_1(-p, -q; 1; A) \\ & + a_{21} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+\alpha+\beta-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+\alpha+\beta)}\ {}_2F_1(-p, -q; 1; A) \end{aligned}$$

$$+ a_{22}A\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{a_{11}^{p}a_{22}^{q}\int_0^t(t-\tau)^{p\alpha+q\beta+2\beta-1}g_2(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+2\beta)}(q+1)_2F_1(1-p,-q;2;A). \tag{3.35}$$

4. Some special cases

In this section, we will present some special cases of Theorem 1, which including the case when $A = 1$, $a_{11} = 0$, or $a_{22} = 0$, respectively. In order to achieve these, we need the following lemmas involving bivariate Mittag-Leffler function:

Lemma 1. *[1] For $\alpha, \beta > 0$ and $\gamma - 1 > \lfloor\alpha\rfloor$, we have*

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\left[t^{\gamma-1}E_{\alpha,\beta,\gamma}(\lambda_1 t^{\alpha}, \lambda_2 t^{\beta})\right] = t^{\gamma-\alpha-1}E_{\alpha,\beta,\gamma-\alpha}(\lambda_1 t^{\alpha}, \lambda_2 t^{\beta}), \tag{4.1}$$

for any $t, \alpha, \beta, \gamma, \lambda_1, \lambda_2 \in \mathbb{R}$.

Proof: See Lemma 2.2 in [1].

Lemma 2. *[1] For $\alpha, \beta > 0$, we have*

$$\begin{aligned}
1 + a_{11}t^{\alpha}E_{\alpha,\beta,\alpha+1}(a_{11}t^{\alpha}, a_{22}t^{\beta}) + a_{22}t^{\beta}E_{\alpha,\beta,\beta+1}(a_{11}t^{\alpha}, a_{22}t^{\beta}) &= E_{\alpha,\beta,1}(a_{11}t^{\alpha}, a_{22}t^{\beta}),\\
\frac{t^{\alpha-1}}{\Gamma(\alpha)} + a_{11}t^{2\alpha-1}E_{\alpha,\beta,2\alpha}(a_{11}t^{\alpha}, a_{22}t^{\beta}) + a_{22}t^{\alpha+\beta-1}E_{\alpha,\beta,\alpha+\beta}(a_{11}t^{\alpha}, a_{22}t^{\beta}) &= t^{\alpha-1}E_{\alpha,\beta,\alpha}(a_{11}t^{\alpha}, a_{22}t^{\beta}),\\
\frac{t^{\beta-1}}{\Gamma(\beta)} + a_{11}t^{\alpha+\beta-1}E_{\alpha,\beta,\alpha+\beta}(a_{11}t^{\alpha}, a_{22}t^{\beta}) + a_{22}t^{2\beta-1}E_{\alpha,\beta,2\beta}(a_{11}t^{\alpha}, a_{22}t^{\beta}) &= t^{\beta-1}E_{\alpha,\beta,\beta}(a_{11}t^{\alpha}, a_{22}t^{\beta}),
\end{aligned} \tag{4.2}$$

for any $t, \alpha, \beta \in \mathbb{R}$.

Proof: See [1].

4.1. The $A = 1$ case

In this case, we have the hypergeometric function with $A = 1$, i.e. $a_{11}a_{22} = a_{12}a_{21}$. The following identities are important for finding the explicit analytical solution of system (1.1).

$$\begin{aligned}
{}_2F_1(-p,-q;1;1) &= \frac{\Gamma(1)\Gamma(p+q+1)}{\Gamma(p+1)\Gamma(q+1)} = \frac{\Gamma(p+q+1)}{\Gamma(p+1)\Gamma(q+1)} = \binom{p+q}{q}\\
(p+1)_2F_1(-p,1-q;2;1) &= (p+1)\frac{\Gamma(2)\Gamma(p+q+1)}{\Gamma(p+2)\Gamma(q+1)} = \frac{\Gamma(p+q+1)}{\Gamma(p+1)\Gamma(q+1)} = \binom{p+q}{q}.
\end{aligned} \tag{4.3}$$

Using Eqs (3.24), (3.26), $A = 1$ and the identities in Eq (4.3), we can express $x_1(t)$ as follows:

$$\begin{aligned}
x_1(t) = {} & x_1^0E_{\alpha}(a_{11}t^{\alpha}) + x_1^1tE_{\alpha,2}(a_{11}t^{\alpha}) + \int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(a_{11}(t-\tau)^{\alpha})g_1(\tau)\,\mathrm{d}\tau\\
& + x_1^0a_{11}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\frac{a_{11}^{p}a_{22}^{q}t^{p\alpha+q\beta+\alpha}}{\Gamma(p\alpha+q\beta+\alpha+1)}\binom{p+q}{q} + x_1^1a_{11}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\frac{a_{11}^{p}a_{22}^{q}t^{p\alpha+q\beta+\alpha+1}}{\Gamma(p\alpha+q\beta+\alpha+2)}\binom{p+q}{q}
\end{aligned}$$

$$+ x_2^0 a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha}}{\Gamma(p\alpha+q\beta+\alpha+1)}\binom{p+q}{q} + x_2^1 a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha+1}}{\Gamma(p\alpha+q\beta+\alpha+2)}\binom{p+q}{q}$$
$$+ a_{11} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+2\alpha-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+2\alpha)}\binom{p+q}{q} \tag{4.4}$$
$$+ a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+\alpha+\beta-1} g_2(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+\alpha+\beta)}\binom{p+q}{q}.$$

Expanding the Mittag-Leffler function and bivariate Mittag-Leffler function in the first three terms of Eq (4.4) and rearranging the terms in RHS of (4.4) yields

$$\begin{aligned}
x_1(t) &= x_1^0 + x_1^0 \sum_{p=0}^{\infty} \frac{a_{11}^{p+1} t^{p\alpha+\alpha}}{\Gamma(p\alpha+\alpha+1)} + x_1^0 a_{11} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha}}{\Gamma(p\alpha+q\beta+\alpha+1)}\binom{p+q}{q} \\
&\quad + x_2^0 a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha}}{\Gamma(p\alpha+q\beta+\alpha+1)}\binom{p+q}{q} \\
&\quad + x_1^1 t + x_1^1 t \sum_{p=0}^{\infty} \frac{a_{11}^{p+1} t^{p\alpha+\alpha}}{\Gamma(p\alpha+\alpha+2)} + x_1^1 a_{11} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha+1}}{\Gamma(p\alpha+q\beta+\alpha+2)}\binom{p+q}{q} \\
&\quad + x_2^1 a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha+1}}{\Gamma(p\alpha+q\beta+\alpha+2)}\binom{p+q}{q} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g_1(\tau)\,\mathrm{d}\tau + \sum_{p=0}^{\infty} \frac{a_{11}^{p+1} \int_0^t (t-\tau)^{p\alpha+2\alpha-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+2\alpha)} \\
&\quad + a_{11} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+2\alpha-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+2\alpha)}\binom{p+q}{q} \\
&\quad + a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+\alpha+\beta-1} g_2(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+\alpha+\beta)}\binom{p+q}{q} \\
&= x_1^0 + \left(x_1^0 a_{11} + x_2^0 a_{12}\right) \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha}}{\Gamma(p\alpha+q\beta+\alpha+1)}\binom{p+q}{q} \\
&\quad + x_1^1 t + \left(x_1^1 a_{11} + x_2^1 a_{12}\right) \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta+\alpha+1}}{\Gamma(p\alpha+q\beta+\alpha+2)}\binom{p+q}{q} \\
&\quad + \frac{\int_0^t (t-\tau)^{\alpha-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma(\alpha)} + a_{11} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+2\alpha-1} g_1(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+2\alpha)}\binom{p+q}{q} \\
&\quad + a_{12} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^p a_{22}^q \int_0^t (t-\tau)^{p\alpha+q\beta+\alpha+\beta-1} g_2(\tau)\,\mathrm{d}\tau}{\Gamma(p\alpha+q\beta+\alpha+\beta)}\binom{p+q}{q}.
\end{aligned} \tag{4.5}$$

Rewriting some terms in the above equation using bivariate Mittag-Leffler function yields

$$
\begin{aligned}
x_1(t) &= x_1^0 + \left(x_1^0 a_{11} + x_2^0 a_{12}\right) t^{\alpha} E_{\alpha,\beta,\alpha+1}\left(a_{11}t^{\alpha}, a_{22}t^{\beta}\right) \\
&+ x_1^1 t + \left(x_1^1 a_{11} + x_2^1 a_{12}\right) t^{\alpha+1} E_{\alpha,\beta,\alpha+2}\left(a_{11}t^{\alpha}, a_{22}t^{\beta}\right) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g_1(\tau)\,\mathrm{d}\tau \\
&+ a_{11} \int_0^t (t-\tau)^{2\alpha-1} E_{\alpha,\beta,2\alpha}\left(a_{11}(t-\tau)^{\alpha}, a_{22}(t-\tau)^{\beta}\right) g_1(\tau)\,\mathrm{d}\tau \\
&+ a_{12} \int_0^t (t-\tau)^{\alpha+\beta-1} E_{\alpha,\beta,\alpha+\beta}\left(a_{11}(t-\tau)^{\alpha}, a_{22}(t-\tau)^{\beta}\right) g_2(\tau)\,\mathrm{d}\tau. \qquad (4.6)
\end{aligned}
$$

Using the similar approach for $x_2(t)$, we then have the following theorem:

Theorem 2. *The incommensurate fractional differential equation systems with fractional order* $1 < \alpha, \beta < 2$

$$
\begin{aligned}
{}^{C}\mathrm{D}^{\alpha} x_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + g_1(t), \\
{}^{C}\mathrm{D}^{\beta} x_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + g_2(t),
\end{aligned}
$$

with initial conditions $x_1(0) = x_1^0,\ x_2(0) = x_2^0,\ x_1'(0) = x_1^1,\ x_2'(0) = x_2^1$ *and constant* $A = \frac{a_{12}a_{21}}{a_{11}a_{22}} = 1$ *has the following solutions given by:*

$$
\begin{aligned}
x_1(t) &= x_1^0 + \left(x_1^0 a_{11} + x_2^0 a_{12}\right) t^{\alpha} E_{\alpha,\beta,\alpha+1}\left(a_{11}t^{\alpha}, a_{22}t^{\beta}\right) \\
&+ x_1^1 t + \left(x_1^1 a_{11} + x_2^1 a_{12}\right) t^{\alpha+1} E_{\alpha,\beta,\alpha+2}\left(a_{11}t^{\alpha}, a_{22}t^{\beta}\right) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g_1(\tau)\,\mathrm{d}\tau \\
&+ a_{11} \int_0^t (t-\tau)^{2\alpha-1} E_{\alpha,\beta,2\alpha}\left(a_{11}(t-\tau)^{\alpha}, a_{22}(t-\tau)^{\beta}\right) g_1(\tau)\,\mathrm{d}\tau \\
&+ a_{12} \int_0^t (t-\tau)^{\alpha+\beta-1} E_{\alpha,\beta,\alpha+\beta}\left(a_{11}(t-\tau)^{\alpha}, a_{22}(t-\tau)^{\beta}\right) g_2(\tau)\,\mathrm{d}\tau,
\end{aligned} \qquad (4.7)
$$

$$
\begin{aligned}
x_2(t) &= x_2^0 + \left(x_1^0 a_{21} + x_2^0 a_{22}\right) t^{\beta} E_{\alpha,\beta,\beta+1}\left(a_{11}t^{\alpha}, a_{22}t^{\beta}\right) \\
&+ x_2^1 t + \left(x_1^1 a_{21} + x_2^1 a_{22}\right) t^{\beta+1} E_{\alpha,\beta,\beta+2}\left(a_{11}t^{\alpha}, a_{22}t^{\beta}\right) + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} g_2(\tau)\,\mathrm{d}\tau \\
&+ a_{21} \int_0^t (t-\tau)^{\alpha+\beta-1} E_{\alpha,\beta,\alpha+\beta}\left(a_{11}(t-\tau)^{\alpha}, a_{22}(t-\tau)^{\beta}\right) g_1(\tau)\,\mathrm{d}\tau \\
&+ a_{22} \int_0^t (t-\tau)^{2\beta-1} E_{\alpha,\beta,2\beta}\left(a_{11}(t-\tau)^{\alpha}, a_{22}(t-\tau)^{\beta}\right) g_2(\tau)\,\mathrm{d}\tau.
\end{aligned} \qquad (4.8)
$$

Proof: The solution is proved when the first equation of (1.1) is satisfied. Hence, using Eqs (4.7) and (4.8), the LHS of (1.1) (after taking the fractional derivative for Eq (4.7) with Lemma 1) and RHS of (1.1) are shown as follows:

$$
\begin{aligned}
{}^{C}\mathrm{D}^{\alpha} x_1(t) = {} & \left(x_1^0 a_{11} + x_2^0 a_{12}\right) E_{\alpha,\beta,1}\left(a_{11}t^{\alpha}, a_{22}t^{\beta}\right) + \left(x_1^1 a_{11} + x_2^1 a_{12}\right) t E_{\alpha,\beta,2}\left(a_{11}t^{\alpha}, a_{22}t^{\beta}\right) \\
& + g_1(t) + a_{11}\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\beta,\alpha}\left(a_{11}(t-\tau)^{\alpha}, a_{22}(t-\tau)^{\beta}\right) g_1(\tau)\,\mathrm{d}\tau \\
& + a_{12}\int_0^t (t-\tau)^{\beta-1} E_{\alpha,\beta,\beta}\left(a_{11}(t-\tau)^{\alpha}, a_{22}(t-\tau)^{\beta}\right) g_2(\tau)\,\mathrm{d}\tau, \qquad (4.9)
\end{aligned}
$$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + g_1(t) = {} & a_{11}x_1^0 + a_{11}\left(x_1^0 a_{11} + x_2^0 a_{12}\right) t^{\alpha} E_{\alpha,\beta,\alpha+1}\left(a_{11}t^{\alpha}, a_{22}t^{\beta}\right) \\
& + x_1^1 a_{11} t + a_{11}\left(x_1^1 a_{11} + x_2^1 a_{12}\right) t^{\alpha+1} E_{\alpha,\beta,\alpha+2}\left(a_{11}t^{\alpha}, a_{22}t^{\beta}\right) \\
& + \frac{a_{11}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} g_1(\tau)\,\mathrm{d}\tau \\
& + a_{11}a_{11}\int_0^t (t-\tau)^{2\alpha-1} E_{\alpha,\beta,2\alpha}\left(a_{11}(t-\tau)^{\alpha}, a_{22}(t-\tau)^{\beta}\right) g_1(\tau)\,\mathrm{d}\tau \\
& + a_{11}a_{12}\int_0^t (t-\tau)^{\alpha+\beta-1} E_{\alpha,\beta,\alpha+\beta}\left(a_{11}(t-\tau)^{\alpha}, a_{22}(t-\tau)^{\beta}\right) g_2(\tau)\,\mathrm{d}\tau \\
& + a_{12}x_2^0 + a_{12}\left(x_1^0 a_{21} + x_2^0 a_{22}\right) t^{\beta} E_{\alpha,\beta,\beta+1}\left(a_{11}t^{\alpha}, a_{22}t^{\beta}\right) \\
& + x_2^1 a_{12} t + a_{12}\left(x_1^1 a_{21} + x_2^1 a_{22}\right) t^{\beta+1} E_{\alpha,\beta,\beta+2}\left(a_{11}t^{\alpha}, a_{22}t^{\beta}\right) \\
& + a_{12}a_{21}\int_0^t (t-\tau)^{\alpha+\beta-1} E_{\alpha,\beta,\alpha+\beta}\left(a_{11}(t-\tau)^{\alpha}, a_{22}(t-\tau)^{\beta}\right) g_1(\tau)\,\mathrm{d}\tau \\
& + \frac{a_{12}}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} g_2(\tau)\,\mathrm{d}\tau \\
& + a_{12}a_{22}\int_0^t (t-\tau)^{2\beta-1} E_{\alpha,\beta,2\beta}\left(a_{11}(t-\tau)^{\alpha}, a_{22}(t-\tau)^{\beta}\right) g_2(\tau)\,\mathrm{d}\tau + g_1(t). \qquad (4.10)
\end{aligned}
$$

In order to verify the LHS (i.e. Eq (4.9)) is equal to RHS (i.e. Eq (4.10)) for the first equation in (1.1), we will compare the terms containing x_1^0, x_2^0, x_1^1, x_2^1, $g_1(\tau)$ and $g_2(\tau)$. First, we take part of (4.10) involving x_1^0 to prove its equivalence with the corresponding x_1^0 term in (4.9). Since $A = 1$, then $a_{11}a_{22} = a_{12}a_{21}$ which yields

$$
\begin{aligned}
& x_1^0\Big(a_{11} + a_{11}^2 t^{\alpha} E_{\alpha,\beta,\alpha+1}\left(a_{11}t^{\alpha}, a_{22}t^{\beta}\right) + a_{12}a_{21}t^{\beta} E_{\alpha,\beta,\beta+1}\left(a_{11}t^{\alpha}, a_{22}t^{\beta}\right)\Big) \\
& = x_1^0 a_{11}\Big(1 + a_{11}t^{\alpha} E_{\alpha,\beta,\alpha+1}\left(a_{11}t^{\alpha}, a_{22}t^{\beta}\right) + a_{22}t^{\beta} E_{\alpha,\beta,\beta+1}\left(a_{11}t^{\alpha}, a_{22}t^{\beta}\right)\Big) \\
& = x_1^0 a_{11}\left(1 + \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^{p+1} a_{22}^{q} t^{p\alpha+q\beta+\alpha}}{\Gamma(p\alpha+q\beta+\alpha+1)}\binom{p+q}{q} + \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^{p} a_{22}^{q+1} t^{p\alpha+q\beta+\beta}}{\Gamma(p\alpha+q\beta+\beta+1)}\binom{p+q}{q}\right) \\
& = x_1^0 a_{11}\left(1 + \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} \frac{a_{11}^{p} a_{22}^{q} t^{p\alpha+q\beta}}{\Gamma(p\alpha+q\beta+1)}\binom{p+q-1}{q} + \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{a_{11}^{p} a_{22}^{q} t^{p\alpha+q\beta}}{\Gamma(p\alpha+q\beta+1)}\binom{p+q-1}{q-1}\right) \\
& = x_1^0 a_{11}\left(1 + \sum_{p=1}^{\infty} \frac{a_{11}^{p} t^{p\alpha}}{\Gamma(p\alpha+1)} + \sum_{q=1}^{\infty} \frac{a_{22}^{q} t^{q\beta}}{\Gamma(q\beta+1)} + \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \frac{a_{11}^{p} a_{22}^{q} t^{p\alpha+q\beta}}{\Gamma(p\alpha+q\beta+1)}\left[\binom{p+q-1}{q} + \binom{p+q-1}{q-1}\right]\right)
\end{aligned}
$$

$$= x_1^0 a_{11}\Bigg(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{a_{11}^p a_{22}^q t^{p\alpha+q\beta}}{\Gamma(p\alpha+q\beta+1)}\binom{p+q}{q}\Bigg)$$
$$= x_1^0 a_{11} E_{\alpha,\beta,1}\left(a_{11}t^\alpha, a_{22}t^\beta\right). \tag{4.11}$$

Indeed, the above expression can be obtained via Lemma 2. Hence, using a similar approach, we have the proof for the terms with x_2^0 as follows:

$$\begin{aligned} &x_2^0 a_{12}\Big(1 + a_{11}t^\alpha E_{\alpha,\beta,\alpha+1}\left(a_{11}t^\alpha, a_{22}t^\beta\right) + a_{22}t^\beta E_{\alpha,\beta,\beta+1}\left(a_{11}t^\alpha, a_{22}t^\beta\right)\Big)\\ &= x_2^0 a_{12} E_{\alpha,\beta,1}\left(a_{11}t^\alpha, a_{22}t^\beta\right). \end{aligned} \tag{4.12}$$

Similarity, for the terms with x_1^1 and x_2^1, the LHS is equal to the RHS since we have

$$\begin{aligned} &x_1^1 a_{11} t + a_{11}\left(x_1^1 a_{11} + x_2^1 a_{12}\right) t^{\alpha+1} E_{\alpha,\beta,\alpha+2}\left(a_{11}t^\alpha, a_{22}t^\beta\right) + x_2^1 a_{12} t\\ &+ a_{12}\left(x_1^1 a_{21} + x_2^1 a_{22}\right) t^{\beta+1} E_{\alpha,\beta,\beta+2}\left(a_{11}t^\alpha, a_{22}t^\beta\right)\\ &= \left(x_1^1 a_{11} + x_2^1 a_{12}\right) t E_{\alpha,\beta,2}\left(a_{11}t^\alpha, a_{22}t^\beta\right). \end{aligned} \tag{4.13}$$

For the terms containing $g_1(\tau)$ and $g_2(\tau)$, using Lemma 2 and $A = 1$, we show that it is equivalent via the following equations.

$$\begin{aligned} &\frac{a_{11}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} g_1(\tau)\,\mathrm{d}\tau\\ &+ a_{11}^2\int_0^t (t-\tau)^{2\alpha-1} E_{\alpha,\beta,2\alpha}\left(a_{11}(t-\tau)^\alpha, a_{22}(t-\tau)^\beta\right) g_1(\tau)\,\mathrm{d}\tau\\ &+ a_{12}a_{21}\int_0^t (t-\tau)^{\alpha+\beta-1} E_{\alpha,\beta,\alpha+\beta}\left(a_{11}(t-\tau)^\alpha, a_{22}(t-\tau)^\beta\right) g_1(\tau)\,\mathrm{d}\tau\\ &= a_{11}\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\beta,\alpha}\left(a_{11}(t-\tau)^\alpha, a_{22}(t-\tau)^\beta\right) g_1(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} &a_{12}\Bigg(\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} g_2(\tau)\,\mathrm{d}\tau\\ &+ a_{11}\int_0^t (t-\tau)^{\alpha+\beta-1} E_{\alpha,\beta,\alpha+\beta}\left(a_{11}(t-\tau)^\alpha, a_{22}(t-\tau)^\beta\right) g_2(\tau)\,\mathrm{d}\tau\\ &+ a_{22}\int_0^t (t-\tau)^{2\beta-1} E_{\alpha,\beta,2\beta}\left(a_{11}(t-\tau)^\alpha, a_{22}(t-\tau)^\beta\right) g_2(\tau)\,\mathrm{d}\tau\Bigg)\\ &= a_{12}\int_0^t (t-\tau)^{\beta-1} E_{\alpha,\beta,\beta}\left(a_{11}(t-\tau)^\alpha, a_{22}(t-\tau)^\beta\right) g_2(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{4.15}$$

Theorem 2 is verified since the Eq (4.9) is equivalent to Eq (4.10) for each of the terms containing x_1^0, x_2^0, x_1^1, x_2^1, $g_1(\tau)$ and $g_2(\tau)$.

4.2. The $a_{11} = 0$ case

We have emphasized that a_{11} and a_{22} are not equal to zero in Theorem 1. However, we can still make assumption for these special cases. For the case $a_{11} = 0$, we consider $\alpha, \beta \in (1, 2)$. The incommensurate fractional differential equation system (1.1) is now given by

$$\begin{aligned} {}^{C}\mathrm{D}^{\alpha} x_1(t) &= a_{12} x_2(t) + g_1(t), \\ {}^{C}\mathrm{D}^{\beta} x_2(t) &= a_{21} x_1(t) + a_{22} x_2(t) + g_2(t), \end{aligned} \tag{4.16}$$

with the same initial conditions as (1.1).

Since $a_{11} = 0$, we use Eq (3.18) which makes it double series when $j = 0$, yielding

$$\begin{aligned} x_1(t) = x_1^0 &+ x_1^0 \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_{12}^{k+1} a_{21}^{k+1} a_{22}^{m} t^{(k+1)\alpha+(m+k+1)\beta}}{\Gamma((k+1)\alpha + (m+k+1)\beta + 1)} \frac{(k+m)!}{k!m!} \\ &+ x_1^1 t + x_1^1 \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_{12}^{k+1} a_{21}^{k+1} a_{22}^{m} t^{(k+1)\alpha+(m+k+1)\beta+1}}{\Gamma((k+1)\alpha + (m+k+1)\beta + 2)} \frac{(k+m)!}{k!m!} \\ &+ x_2^0 \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_{12}^{k+1} a_{21}^{k} a_{22}^{m} t^{(k+1)\alpha+(m+k)\beta}}{\Gamma((k+1)\alpha + (m+k)\beta + 1)} \frac{(k+m)!}{k!m!} \\ &+ x_2^1 \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_{12}^{k+1} a_{21}^{k} a_{22}^{m} t^{(k+1)\alpha+(m+k)\beta+1}}{\Gamma((k+1)\alpha + (m+k)\beta + 2)} \frac{(k+m)!}{k!m!} \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g_1(\tau)\, \mathrm{d}\tau \\ &+ \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_{12}^{k+1} a_{21}^{k+1} a_{22}^{m} \int_0^t (t-\tau)^{(k+2)\alpha+(m+k+1)\beta-1} g_1(\tau)\, \mathrm{d}\tau}{\Gamma((k+2)\alpha + (m+k+1)\beta)} \frac{(k+m)!}{k!m!} \\ &+ \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_{12}^{k+1} a_{21}^{k} a_{22}^{m} \int_0^t (t-\tau)^{(k+1)\alpha+(m+k+1)\beta-1} g_2(\tau)\, \mathrm{d}\tau}{\Gamma((k+1)\alpha + (m+k+1)\beta)} \frac{(k+m)!}{k!m!}. \end{aligned} \tag{4.17}$$

Rewriting the above equation using bivariate Mittag-Leffler function yields

$$\begin{aligned} x_1(t) = x_1^0 &+ x_1^0 a_{12} a_{21} t^{\alpha+\beta} E_{\alpha+\beta,\beta,\alpha+\beta+1}\left(a_{12} a_{21} t^{\alpha+\beta}, a_{22} t^{\beta}\right) \\ &+ x_1^1 t + x_1^1 a_{12} a_{21} t^{\alpha+\beta+1} E_{\alpha+\beta,\beta,\alpha+\beta+2}\left(a_{12} a_{21} t^{\alpha+\beta}, a_{22} t^{\beta}\right) \\ &+ x_2^0 a_{12} t^{\alpha} E_{\alpha+\beta,\beta,\alpha+1}\left(a_{12} a_{21} t^{\alpha+\beta}, a_{22} t^{\beta}\right) + x_2^1 a_{12} t^{\alpha+1} E_{\alpha+\beta,\beta,\alpha+2}\left(a_{12} a_{21} t^{\alpha+\beta}, a_{22} t^{\beta}\right) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g_1(\tau)\, \mathrm{d}\tau \\ &+ a_{12} a_{21} \int_0^t (t-\tau)^{2\alpha+\beta-1} E_{\alpha+\beta,\beta,2\alpha+\beta}\left(a_{12} a_{21} (t-\tau)^{\alpha+\beta}, a_{22} (t-\tau)^{\beta}\right) g_1(\tau)\, \mathrm{d}\tau \\ &+ a_{12} \int_0^t (t-\tau)^{\alpha+\beta-1} E_{\alpha+\beta,\beta,\alpha+\beta}\left(a_{12} a_{21} (t-\tau)^{\alpha+\beta}, a_{22} (t-\tau)^{\beta}\right) g_2(\tau)\, \mathrm{d}\tau. \end{aligned} \tag{4.18}$$

Since $a_{11} = 0$, the $x_2(t)$ solution can be obtained directly from the first equation of (4.16). By rearranging the first equation of (4.16), we obtain

$$\begin{aligned}x_2(t) &= \frac{{}^{C}\mathrm{D}^{\alpha}x_1(t)}{a_{12}} - \frac{g_1(t)}{a_{12}}\\ &= \frac{1}{a_{12}}\bigg(x_1^0 a_{12}a_{21}t^{\beta}E_{\alpha+\beta,\beta,\beta+1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right) + x_1^1 a_{12}a_{21}t^{\beta+1}E_{\alpha+\beta,\beta,\beta+2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)\\ &\quad + x_2^0 a_{12}E_{\alpha+\beta,\beta,1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right) + x_2^1 a_{12}tE_{\alpha+\beta,\beta,2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right) + g_1(t)\\ &\quad + a_{12}a_{21}\int_0^t (t-\tau)^{\alpha+\beta-1}E_{\alpha+\beta,\beta,\alpha+\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{22}(t-\tau)^{\beta}\right)g_1(\tau)\,\mathrm{d}\tau\\ &\quad + a_{12}\int_0^t (t-\tau)^{\beta-1}E_{\alpha+\beta,\beta,\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{22}(t-\tau)^{\beta}\right)g_2(\tau)\,\mathrm{d}\tau\bigg) - \frac{g_1(t)}{a_{12}}. \qquad (4.19)\end{aligned}$$

Using Lemma 1, and since our $\alpha, \beta > 1$, we obtain the $x_2(t)$ as follows:

$$\begin{aligned}x_2(t) &= x_1^0 a_{21}t^{\beta}E_{\alpha+\beta,\beta,\beta+1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right) + x_1^1 a_{21}t^{\beta+1}E_{\alpha+\beta,\beta,\beta+2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)\\ &\quad + x_2^0 E_{\alpha+\beta,\beta,1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right) + x_2^1 tE_{\alpha+\beta,\beta,2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)\\ &\quad + a_{21}\int_0^t (t-\tau)^{\alpha+\beta-1}E_{\alpha+\beta,\beta,\alpha+\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{22}(t-\tau)^{\beta}\right)g_1(\tau)\,\mathrm{d}\tau\\ &\quad + \int_0^t (t-\tau)^{\beta-1}E_{\alpha+\beta,\beta,\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{22}(t-\tau)^{\beta}\right)g_2(\tau)\,\mathrm{d}\tau. \qquad (4.20)\end{aligned}$$

Hence, we can obtain the following theorem.

Theorem 3. *For special case $a_{11} = 0$, the system* (1.1) *can be written as*

$$\begin{aligned}{}^{C}\mathrm{D}^{\alpha}x_1(t) &= a_{12}x_2(t) + g_1(t),\\ {}^{C}\mathrm{D}^{\beta}x_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + g_2(t),\end{aligned}$$

and the explicit analytical solution of the above system with initial conditions $x_1(0) = x_1^0$, $x_2(0) = x_2^0$, $x_1'(0) = x_1^1$, $x_2'(0) = x_2^1$ is given by:

$$\begin{aligned}x_1(t) &= x_1^0 + x_1^0 a_{12}a_{21}t^{\alpha+\beta}E_{\alpha+\beta,\beta,\alpha+\beta+1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)\\ &\quad + x_1^1 t + x_1^1 a_{12}a_{21}t^{\alpha+\beta+1}E_{\alpha+\beta,\beta,\alpha+\beta+2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)\\ &\quad + x_2^0 a_{12}t^{\alpha}E_{\alpha+\beta,\beta,\alpha+1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)\\ &\quad + x_2^1 a_{12}t^{\alpha+1}E_{\alpha+\beta,\beta,\alpha+2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}g_1(\tau)\,\mathrm{d}\tau \qquad (4.21)\\ &\quad + a_{12}a_{21}\int_0^t (t-\tau)^{2\alpha+\beta-1}E_{\alpha+\beta,\beta,2\alpha+\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{22}(t-\tau)^{\beta}\right)g_1(\tau)\,\mathrm{d}\tau\\ &\quad + a_{12}\int_0^t (t-\tau)^{\alpha+\beta-1}E_{\alpha+\beta,\beta,\alpha+\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{22}(t-\tau)^{\beta}\right)g_2(\tau)\,\mathrm{d}\tau,\end{aligned}$$

$$x_2(t) = x_1^0 a_{21} t^{\beta} E_{\alpha+\beta,\beta,\beta+1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right) + x_1^1 a_{21} t^{\beta+1} E_{\alpha+\beta,\beta,\beta+2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)$$
$$+ x_2^0 E_{\alpha+\beta,\beta,1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right) + x_2^1 t E_{\alpha+\beta,\beta,2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)$$
$$+ a_{21}\int_0^t (t-\tau)^{\alpha+\beta-1} E_{\alpha+\beta,\beta,\alpha+\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{22}(t-\tau)^{\beta}\right) g_1(\tau)\,\mathrm{d}\tau \tag{4.22}$$
$$+ \int_0^t (t-\tau)^{\beta-1} E_{\alpha+\beta,\beta,\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{22}(t-\tau)^{\beta}\right) g_2(\tau)\,\mathrm{d}\tau.$$

Proof: The theorem can be checked by substituting the solutions into the second equation of (4.16). For the LHS, take the fractional derivative for Eq (4.22) and use Lemma 1, which yields

$${}^{C}\mathrm{D}^{\beta} x_2(t) = x_1^0 a_{21} E_{\alpha+\beta,\beta,1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right) + x_1^1 a_{21} t E_{\alpha+\beta,\beta,2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)$$
$$+ x_2^0\, {}^{C}\mathrm{D}^{\beta}\left[E_{\alpha+\beta,\beta,1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)\right] + x_2^1\, {}^{C}\mathrm{D}^{\beta}\left[t E_{\alpha+\beta,\beta,2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)\right]$$
$$+ a_{21}\int_0^t (t-\tau)^{\alpha-1} E_{\alpha+\beta,\beta,\alpha}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{22}(t-\tau)^{\beta}\right) g_1(\tau)\,\mathrm{d}\tau \tag{4.23}$$
$$+ {}^{C}\mathrm{D}^{\beta}\left[\int_0^t (t-\tau)^{\beta-1} E_{\alpha+\beta,\beta,\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{22}(t-\tau)^{\beta}\right) g_2(\tau)\,\mathrm{d}\tau\right].$$

For the RHS, substitute Eqs (4.21) and (4.22) into the second equation of (4.16) yields

$$a_{21}x_1(t) + a_{22}x_2(t) + g_2(t)$$
$$= x_1^0 a_{21}\left(1 + a_{12}a_{21}t^{\alpha+\beta} E_{\alpha+\beta,\beta,\alpha+\beta+1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)\right)$$
$$+ x_1^1 a_{21} t\left(1 + a_{12}a_{21}t^{\alpha+\beta} E_{\alpha+\beta,\beta,\alpha+\beta+2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)\right)$$
$$+ x_2^0 a_{12}a_{21} t^{\alpha} E_{\alpha+\beta,\beta,\alpha+1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)$$
$$+ x_2^1 a_{12}a_{21} t^{\alpha+1} E_{\alpha+\beta,\beta,\alpha+2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right) + \frac{a_{21}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} g_1(\tau)\,\mathrm{d}\tau$$
$$+ a_{12}a_{21}^2 \int_0^t (t-\tau)^{2\alpha+\beta-1} E_{\alpha+\beta,\beta,2\alpha+\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{22}(t-\tau)^{\beta}\right) g_1(\tau)\,\mathrm{d}\tau$$
$$+ a_{12}a_{21}\int_0^t (t-\tau)^{\alpha+\beta-1} E_{\alpha+\beta,\beta,\alpha+\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{22}(t-\tau)^{\beta}\right) g_2(\tau)\,\mathrm{d}\tau \tag{4.24}$$
$$+ x_1^0 a_{21}a_{22} t^{\beta} E_{\alpha+\beta,\beta,\beta+1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right) + x_1^1 a_{21}a_{22} t^{\beta+1} E_{\alpha+\beta,\beta,\beta+2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)$$
$$+ x_2^0 a_{22} E_{\alpha+\beta,\beta,1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right) + x_2^1 a_{22} t E_{\alpha+\beta,\beta,2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)$$
$$+ a_{21}a_{22}\int_0^t (t-\tau)^{\alpha+\beta-1} E_{\alpha+\beta,\beta,\alpha+\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{22}(t-\tau)^{\beta}\right) g_1(\tau)\,\mathrm{d}\tau$$
$$+ a_{22}\int_0^t (t-\tau)^{\beta-1} E_{\alpha+\beta,\beta,\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{22}(t-\tau)^{\beta}\right) g_2(\tau)\,\mathrm{d}\tau.$$

Similar approach will be employed in proving Theorem 2, where we will be comparing one by one of the terms containing x_1^0, x_2^0, x_1^1, x_2^1, $g_1(\tau)$ and $g_2(\tau)$. First, for the parts involving x_1^0, x_1^1 and $g_1(t)$, by using Lemma 2, we prove Eq (4.24) to be equivalent as those corresponding parts in Eq (4.23) as follows:

$$
\begin{aligned}
&x_1^0 a_{21}\left(1 + a_{12}a_{21}t^{\alpha+\beta}E_{\alpha+\beta,\beta,\alpha+\beta+1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right) + a_{22}t^{\beta}E_{\alpha+\beta,\beta,\beta+1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)\right)\\
&= x_1^0 a_{21}\left[1 + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{a_{12}^m a_{21}^m a_{22}^n t^{m\alpha+m\beta+n\beta}\binom{m+n-1}{n}}{\Gamma(m\alpha+m\beta+n\beta+1)} + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{a_{12}^m a_{21}^m a_{22}^n t^{m\alpha+m\beta+n\beta}\binom{m+n-1}{n-1}}{\Gamma(m\alpha+m\beta+n\beta+1)}\right]\\
&= x_1^0 a_{21}E_{\alpha+\beta,\beta,1}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right),
\end{aligned} \tag{4.25}
$$

$$
\begin{aligned}
&x_1^1 a_{21}t\left(1 + a_{12}a_{21}t^{\alpha+\beta}E_{\alpha+\beta,\beta,\alpha+\beta+2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right) + a_{22}t^{\beta}E_{\alpha+\beta,\beta,\beta+2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right)\right)\\
&= x_1^1 a_{21}t\left[1 + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{a_{12}^m a_{21}^m a_{22}^n t^{m\alpha+m\beta+n\beta}\binom{m+n-1}{n}}{\Gamma(m\alpha+m\beta+n\beta+2)} + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{a_{12}^m a_{21}^m a_{22}^n t^{m\alpha+m\beta+n\beta}\binom{m+n-1}{n-1}}{\Gamma(m\alpha+m\beta+n\beta+2)}\right]\\
&= x_1^1 a_{21}tE_{\alpha+\beta,\beta,2}\left(a_{12}a_{21}t^{\alpha+\beta}, a_{22}t^{\beta}\right),
\end{aligned} \tag{4.26}
$$

$$
\begin{aligned}
&a_{21}\Bigg[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}g_1(\tau)\,\mathrm{d}\tau\\
&\quad + a_{12}a_{21}\int_0^t (t-\tau)^{2\alpha+\beta-1}E_{\alpha+\beta,\beta,2\alpha+\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{22}(t-\tau)^{\beta}\right)g_1(\tau)\,\mathrm{d}\tau\\
&\quad + a_{22}\int_0^t (t-\tau)^{\alpha+\beta-1}E_{\alpha+\beta,\beta,\alpha+\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{22}(t-\tau)^{\beta}\right)g_1(\tau)\,\mathrm{d}\tau\Bigg]\\
&= a_{21}\Bigg[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}g_1(\tau)\,\mathrm{d}\tau\\
&\quad + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_{12}^{m+1}a_{21}^{m+1}a_{22}^n\int_0^t (t-\tau)^{m\alpha+m\beta+n\beta+2\alpha+\beta-1}g_1(\tau)\,\mathrm{d}\tau\binom{m+n}{n}}{\Gamma(m\alpha+m\beta+n\beta+2\alpha+\beta)}\\
&\quad + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_{12}^m a_{21}^m a_{22}^{n+1}\int_0^t (t-\tau)^{m\alpha+m\beta+n\beta+\alpha+\beta-1}g_1(\tau)\,\mathrm{d}\tau\binom{m+n}{n}}{\Gamma(m\alpha+m\beta+n\beta+\alpha+\beta)}\Bigg]\\
&= a_{21}\Bigg[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}g_1(\tau)\,\mathrm{d}\tau\\
&\quad + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{a_{12}^m a_{21}^m a_{22}^n\int_0^t (t-\tau)^{m\alpha+m\beta+n\beta+\alpha-1}g_1(\tau)\,\mathrm{d}\tau\binom{m+n-1}{n}}{\Gamma(m\alpha+m\beta+n\beta+\alpha)}\\
&\quad + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{a_{12}^m a_{21}^m a_{22}^n\int_0^t (t-\tau)^{m\alpha+m\beta+n\beta+\alpha-1}g_1(\tau)\,\mathrm{d}\tau\binom{m+n-1}{n-1}}{\Gamma(m\alpha+m\beta+n\beta+\alpha)}\Bigg]\\
&= a_{21}\Bigg[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}g_1(\tau)\,\mathrm{d}\tau\\
&\quad + \sum_{m=1}^{\infty}\frac{a_{12}^m a_{21}^m\int_0^t (t-\tau)^{m\alpha+m\beta+\alpha-1}g_1(\tau)\,\mathrm{d}\tau}{\Gamma(m\alpha+m\beta+\alpha)} + \sum_{n=1}^{\infty}\frac{a_{22}^n\int_0^t (t-\tau)^{n\beta+\alpha-1}g_1(\tau)\,\mathrm{d}\tau}{\Gamma(n\beta+\alpha)}
\end{aligned}
$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{12}^{m}a_{21}^{m}a_{22}^{n}\int_0^t(t-\tau)^{m\alpha+m\beta+n\beta+\alpha-1}g_1(\tau)\,\mathrm{d}\tau\ \left[\binom{m+n-1}{n}+\binom{m+n-1}{n-1}\right]}{\Gamma(m\alpha+m\beta+n\beta+\alpha)}\Bigg]$$

$$=a_{21}\int_0^t(t-\tau)^{\alpha-1}E_{\alpha+\beta,\beta,\alpha}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta},a_{22}(t-\tau)^{\beta}\right)g_1(\tau)\,\mathrm{d}\tau. \tag{4.27}$$

Meanwhile, for x_2^0, x_2^1 and $g_2(t)$ parts, we proceed the proving from (4.23) as follows:

$$x_2^0\ {}^C\mathrm{D}^{\beta}\left[E_{\alpha+\beta,\beta,1}\left(a_{12}a_{21}t^{\alpha+\beta},a_{22}t^{\beta}\right)\right]$$

$$=x_2^0\sum_{\substack{m,n=0\\(m,n)\neq(0,0)}}^{\infty}\frac{a_{12}^{m}a_{21}^{m}a_{22}^{n}t^{m\alpha+m\beta+n\beta-\beta}\binom{m+n}{n}}{\Gamma(m\alpha+m\beta+n\beta-\beta+1)}$$

$$=x_2^0\left[\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{a_{12}^{m}a_{21}^{m}a_{22}^{n}t^{m\alpha+m\beta+n\beta-\beta}\binom{m+n-1}{n}}{\Gamma(m\alpha+m\beta+n\beta-\beta+1)}+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{a_{12}^{m}a_{21}^{m}a_{22}^{n}t^{m\alpha+m\beta+n\beta-\beta}\binom{m+n-1}{n-1}}{\Gamma(m\alpha+m\beta+n\beta-\beta+1)}\right]$$

$$=x_2^0\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_{12}^{m+1}a_{21}^{m+1}a_{22}^{n}t^{(m+1)\alpha+m\beta+n\beta}\binom{m+n}{n}}{\Gamma((m+1)\alpha+m\beta+n\beta+1)}+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_{12}^{m}a_{21}^{m}a_{22}^{n+1}t^{m\alpha+m\beta+n\beta}\binom{m+n}{n}}{\Gamma(m\alpha+m\beta+n\beta+1)}\right]$$

$$=x_2^0\left(a_{12}a_{21}t^{\alpha}E_{\alpha+\beta,\beta,\alpha+1}\left(a_{12}a_{21}t^{\alpha+\beta},a_{22}t^{\beta}\right)+a_{22}E_{\alpha+\beta,\beta,1}\left(a_{12}a_{21}t^{\alpha+\beta},a_{22}t^{\beta}\right)\right), \tag{4.28}$$

$$x_2^1\ {}^C\mathrm{D}^{\beta}\left[tE_{\alpha+\beta,\beta,2}\left(a_{12}a_{21}t^{\alpha+\beta},a_{22}t^{\beta}\right)\right]$$

$$=x_2^1\sum_{\substack{m,n=0\\(m,n)\neq(0,0)}}^{\infty}\frac{a_{12}^{m}a_{21}^{m}a_{22}^{n}t^{m\alpha+m\beta+n\beta-\beta+1}\binom{m+n}{n}}{\Gamma(m\alpha+m\beta+n\beta-\beta+2)}$$

$$=x_2^1\left[\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{a_{12}^{m}a_{21}^{m}a_{22}^{n}t^{m\alpha+m\beta+n\beta-\beta+1}\binom{m+n-1}{n}}{\Gamma(m\alpha+m\beta+n\beta-\beta+2)}+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{a_{12}^{m}a_{21}^{m}a_{22}^{n}t^{m\alpha+m\beta+n\beta-\beta+1}\binom{m+n-1}{n-1}}{\Gamma(m\alpha+m\beta+n\beta-\beta+2)}\right]$$

$$=x_2^1\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_{12}^{m+1}a_{21}^{m+1}a_{22}^{n}t^{(m+1)\alpha+m\beta+n\beta+1}\binom{m+n}{n}}{\Gamma((m+1)\alpha+m\beta+n\beta+2)}+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_{12}^{m}a_{21}^{m}a_{22}^{n+1}t^{m\alpha+m\beta+n\beta+1}\binom{m+n}{n}}{\Gamma(m\alpha+m\beta+n\beta+2)}\right]$$

$$=x_2^1\left(a_{12}a_{21}t^{\alpha+1}E_{\alpha+\beta,\beta,\alpha+2}\left(a_{12}a_{21}t^{\alpha+\beta},a_{22}t^{\beta}\right)+a_{22}tE_{\alpha+\beta,\beta,2}\left(a_{12}a_{21}t^{\alpha+\beta},a_{22}t^{\beta}\right)\right), \tag{4.29}$$

$${}^C\mathrm{D}^{\beta}\left[\int_0^t(t-\tau)^{\beta-1}E_{\alpha+\beta,\beta,\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta},a_{22}(t-\tau)^{\beta}\right)g_2(\tau)\,\mathrm{d}\tau\right]$$

$$=\sum_{\substack{m,n=0\\(m,n)\neq(0,0)}}^{\infty}\frac{a_{12}^{m}a_{21}^{m}a_{22}^{n}\int_0^t(t-\tau)^{m\alpha+m\beta+n\beta-1}g_2(\tau)\,\mathrm{d}\tau\ \binom{m+n}{n}}{\Gamma(m\alpha+m\beta+n\beta)}$$

$$=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{a_{12}^{m}a_{21}^{m}a_{22}^{n}\int_0^t(t-\tau)^{m\alpha+m\beta+n\beta-1}g_2(\tau)\,\mathrm{d}\tau\ \binom{m+n-1}{n}}{\Gamma(m\alpha+m\beta+n\beta)}$$

$$+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{a_{12}^{m}a_{21}^{m}a_{22}^{n}\int_0^t(t-\tau)^{m\alpha+m\beta+n\beta-1}g_2(\tau)\,\mathrm{d}\tau\binom{m+n-1}{n-1}}{\Gamma(m\alpha+m\beta+n\beta)}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_{12}^{m+1}a_{21}^{m+1}a_{22}^{n}\int_0^t(t-\tau)^{m\alpha+m\beta+n\beta+\alpha+\beta-1}g_2(\tau)\,\mathrm{d}\tau\binom{m+n}{n}}{\Gamma((m\alpha+m\beta+n\beta+\alpha+\beta)}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_{12}^{m}a_{21}^{m}a_{22}^{n+1}\int_0^t(t-\tau)^{m\alpha+m\beta+n\beta+\beta-1}g_2(\tau)\,\mathrm{d}\tau\binom{m+n}{n}}{\Gamma(m\alpha+m\beta+n\beta+\beta)}$$

$$=a_{12}a_{21}\int_0^t(t-\tau)^{\alpha+\beta-1}E_{\alpha+\beta,\beta,\alpha+\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta},a_{22}(t-\tau)^{\beta}\right)g_2(\tau)\,\mathrm{d}\tau$$

$$+a_{22}\int_0^t(t-\tau)^{\beta-1}E_{\alpha+\beta,\beta,\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta},a_{22}(t-\tau)^{\beta}\right)g_2(\tau)\,\mathrm{d}\tau. \tag{4.30}$$

Since all the terms in Eqs (4.23) and (4.24) are equivalent, the solution of the system (4.16) is verified.

4.3. The $a_{22}=0$ case

For the case $a_{22}=0$, we consider $\alpha,\beta\in(1,2)$. The incommensurate fractional differential equation system (1.1) is now given by

$$\begin{aligned}{}^{C}\mathrm{D}^{\alpha}x_1(t)&=a_{11}x_1(t)+a_{12}x_2(t)+g_1(t),\\ {}^{C}\mathrm{D}^{\beta}x_2(t)&=a_{21}x_1(t)+g_2(t),\end{aligned}\tag{4.31}$$

with the same initial conditions as in Eq (1.1).

For the case $a_{22}=0$, using similar approach from the previous subsection, we obtain the following solution

$$\begin{aligned}x_1(t)=&\,x_1^0E_{\alpha+\beta,\alpha,1}\left(a_{12}a_{21}t^{\alpha+\beta},a_{11}t^{\alpha}\right)+x_1^1tE_{\alpha+\beta,\alpha,2}\left(a_{12}a_{21}t^{\alpha+\beta},a_{11}t^{\alpha}\right)\\&+x_2^0a_{12}t^{\alpha}E_{\alpha+\beta,\alpha,\alpha+1}\left(a_{12}a_{21}t^{\alpha+\beta},a_{11}t^{\alpha}\right)+x_2^1a_{12}t^{\alpha+1}E_{\alpha+\beta,\alpha,\alpha+2}\left(a_{12}a_{21}t^{\alpha+\beta},a_{11}t^{\alpha}\right)\\&+\int_0^t(t-\tau)^{\alpha-1}g_1(\tau)E_{\alpha+\beta,\alpha,\alpha}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta},a_{11}(t-\tau)^{\alpha}\right)\,\mathrm{d}\tau\\&+a_{12}\int_0^t(t-\tau)^{\alpha+\beta-1}g_2(\tau)E_{\alpha+\beta,\alpha,\alpha+\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta},a_{11}(t-\tau)^{\alpha}\right)\,\mathrm{d}\tau,\end{aligned}\tag{4.32}$$

$$\begin{aligned}x_2(t)=&\,x_1^0a_{21}t^{\beta}E_{\alpha+\beta,\alpha,\beta+1}\left(a_{12}a_{21}t^{\alpha+\beta},a_{11}t^{\alpha}\right)+x_1^1a_{21}t^{\beta+1}E_{\alpha+\beta,\alpha,\beta+2}\left(a_{12}a_{21}t^{\alpha+\beta},a_{11}t^{\alpha}\right)\\&+x_2^0+x_2^0a_{12}a_{21}t^{\alpha+\beta}E_{\alpha+\beta,\alpha,\alpha+\beta+1}\left(a_{12}a_{21}t^{\alpha+\beta},a_{11}t^{\alpha}\right)\\&+x_2^1t+x_2^1a_{12}a_{21}t^{\alpha+\beta+1}E_{\alpha+\beta,\alpha,\alpha+\beta+2}\left(a_{12}a_{21}t^{\alpha+\beta},a_{11}t^{\alpha}\right)\\&+a_{21}\int_0^t(t-\tau)^{\alpha+\beta-1}g_1(\tau)E_{\alpha+\beta,\alpha,\alpha+\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta},a_{11}(t-\tau)^{\alpha}\right)\,\mathrm{d}\tau\end{aligned}$$

$$+\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} g_2(\tau)\,\mathrm{d}\tau$$
$$+a_{12}a_{21}\int_0^t (t-\tau)^{\alpha+2\beta-1} g_2(\tau) E_{\alpha+\beta,\alpha,\alpha+2\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}, a_{11}(t-\tau)^{\alpha}\right)\,\mathrm{d}\tau. \quad (4.33)$$

4.4. *The $a_{11}=0$ and $a_{22}=0$ case*

For the case $a_{11}=0$ and $a_{22}=0$, we consider $\alpha,\beta\in(1,2)$. The incommensurate fractional differential equation system (1.1) is now given by

$$\begin{aligned} {}^{C}\mathrm{D}^{\alpha}x_1(t) &= a_{12}x_2(t)+g_1(t),\\ {}^{C}\mathrm{D}^{\beta}x_2(t) &= a_{21}x_1(t)+g_2(t), \end{aligned} \quad (4.34)$$

with the same initial conditions as in Eq (1.1).

For the case $a_{11}=0$ and $a_{22}=0$, using similar approach from the previous subsection, we obtain the following solution,

$$\begin{aligned} x_1(t) &= x_1^0 E_{\alpha+\beta}\left(a_{12}a_{21}t^{\alpha+\beta}\right) + x_1^1 t E_{\alpha+\beta,2}\left(a_{12}a_{21}t^{\alpha+\beta}\right) + x_2^0 a_{12} t^{\alpha} E_{\alpha+\beta,\alpha+1}\left(a_{12}a_{21}t^{\alpha+\beta}\right)\\ &+ x_2^1 a_{12} t^{\alpha+1} E_{\alpha+\beta,\alpha+2}\left(a_{12}a_{21}t^{\alpha+\beta}\right) + \int_0^t (t-\tau)^{\alpha-1} g_1(\tau) E_{\alpha+\beta,\alpha}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}\right)\,\mathrm{d}\tau\\ &+ a_{12}\int_0^t (t-\tau)^{\alpha+\beta-1} g_2(\tau) E_{\alpha+\beta,\alpha+\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}\right)\,\mathrm{d}\tau, \end{aligned} \quad (4.35)$$

$$\begin{aligned} x_2(t) &= x_1^0 a_{21} t^{\beta} E_{\alpha+\beta,\beta+1}\left(a_{12}a_{21}t^{\alpha+\beta}\right) + x_1^1 a_{21} t^{\beta+1} E_{\alpha+\beta,\beta+2}\left(a_{12}a_{21}t^{\alpha+\beta}\right) + x_2^0 E_{\alpha+\beta}\left(a_{12}a_{21}t^{\alpha+\beta}\right)\\ &+ x_2^1 t E_{\alpha+\beta,2}\left(a_{12}a_{21}t^{\alpha+\beta}\right) + a_{21}\int_0^t (t-\tau)^{\alpha+\beta-1} g_1(\tau) E_{\alpha+\beta,\alpha+\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}\right)\,\mathrm{d}\tau\\ &+ \int_0^t (t-\tau)^{\beta-1} g_2(\tau) E_{\alpha+\beta,\beta}\left(a_{12}a_{21}(t-\tau)^{\alpha+\beta}\right)\,\mathrm{d}\tau. \end{aligned} \quad (4.36)$$

5. Examples

This section illustrates an explicit analytical solution for the incommensurate fractional differential system for order $\alpha,\beta\in(1,2)$ using the theorems that we had derived in previous sections. Four examples will be presented using Theorems 1–3, respectively, for the case of $a_{11}=0$ and $a_{22}=0$.

Example 1. *Consider the following incommensurate linear fractional differential equation system*

$$\begin{pmatrix} \frac{\mathrm{d}^{3/2}x_1}{\mathrm{d}t^{3/2}} \\ \frac{\mathrm{d}^{4/3}x_2}{\mathrm{d}t^{4/3}} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2t \\ -4t \end{pmatrix}, \quad (5.1)$$

with respect to the initial conditions $x_1(0)=3/2$, $x_1'(0)=2$, $x_2(0)=2$ and $x_2'(0)=3$.

Solution: Using Theorem 1, the explicit analytical solution is obtained as follows:

$$\begin{aligned}
x_1(t) = &\frac{3}{2}E_{\frac{3}{2}}\left(2t^{\frac{3}{2}}\right) + 2tE_{\frac{3}{2},2}\left(2t^{\frac{3}{2}}\right) + 2t^{\frac{5}{2}}E_{\frac{3}{2},\frac{7}{2}}\left(2t^{\frac{3}{2}}\right)\\
&+ \frac{3}{2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{3^p2^qt^{\frac{3p}{2}+\frac{4q}{3}+\frac{3}{2}}}{\Gamma\left(\frac{3p}{2}+\frac{4q}{3}+\frac{5}{2}\right)}(p+1)_2F_1\left(-p,1-q;2;\frac{1}{3}\right)\\
&+ 2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{3^p2^qt^{\frac{3p}{2}+\frac{4q}{3}+\frac{5}{2}}}{\Gamma\left(\frac{3p}{2}+\frac{4q}{3}+\frac{7}{2}\right)}(p+1)_2F_1\left(-p,1-q;2;\frac{1}{3}\right)\\
&+ 4\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{3^p2^qt^{\frac{3p}{2}+\frac{4q}{3}+\frac{3}{2}}}{\Gamma\left(\frac{3p}{2}+\frac{4q}{3}+\frac{5}{2}\right)}\,{}_2F_1\left(-p,-q;1;\frac{1}{3}\right)\\
&+ 6\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{3^p2^qt^{\frac{3p}{2}+\frac{4q}{3}+\frac{5}{2}}}{\Gamma\left(\frac{3p}{2}+\frac{4q}{3}+\frac{7}{2}\right)}\,{}_2F_1\left(-p,-q;1;\frac{1}{3}\right)\\
&+ 2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{3^p2^qt^{\frac{3p}{2}+\frac{4q}{3}+4}}{\Gamma\left(\frac{3p}{2}+\frac{4q}{3}+5\right)}(p+1)_2F_1\left(-p,1-q;2;\frac{1}{3}\right)\\
&- 8\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{3^p2^qt^{\frac{3p}{2}+\frac{4q}{3}+\frac{23}{6}}}{\Gamma\left(\frac{3p}{2}+\frac{4q}{3}+\frac{29}{6}\right)}\,{}_2F_1\left(-p,-q;1;\frac{1}{3}\right),
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
x_2(t) = &2E_{\frac{4}{3}}\left(\frac{4}{3}t^{\frac{4}{3}}\right) + 3tE_{\frac{4}{3},2}\left(\frac{4}{3}t^{\frac{4}{3}}\right) - 4t^{\frac{7}{3}}E_{\frac{4}{3},\frac{10}{3}}\left(\frac{4}{3}t^{\frac{4}{3}}\right)\\
&+ \frac{4}{3}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{3^p2^qt^{\frac{3p}{2}+\frac{4q}{3}+\frac{4}{3}}}{\Gamma\left(\frac{3p}{2}+\frac{4q}{3}+\frac{7}{3}\right)}(q+1)_2F_1\left(1-p,-q;2;\frac{1}{3}\right)\\
&+ 2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{3^p2^qt^{\frac{3p}{2}+\frac{4q}{3}+\frac{7}{3}}}{\Gamma(\frac{3p}{2}+\frac{4q}{3}+\frac{10}{3})}(q+1)_2F_1\left(1-p,-q;2;\frac{1}{3}\right)\\
&+ \frac{3}{2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{3^p2^qt^{\frac{3p}{2}+\frac{4q}{3}+\frac{4}{3}}}{\Gamma(\frac{3p}{2}+\frac{4q}{3}+\frac{7}{3})}\,{}_2F_1\left(-p,-q;1;\frac{1}{3}\right)\\
&+ 2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{3^p2^qt^{\frac{3p}{2}+\frac{4q}{3}+\frac{7}{3}}}{\Gamma(\frac{3p}{2}+\frac{4q}{3}+\frac{10}{3})}\,{}_2F_1\left(-p,-q;1;\frac{1}{3}\right)\\
&+ 2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{3^p2^qt^{\frac{3p}{2}+\frac{4q}{3}+\frac{23}{6}}}{\Gamma(\frac{3p}{2}+\frac{4q}{3}+\frac{29}{6})}\,{}_2F_1\left(-p,-q;1;\frac{1}{3}\right)\\
&- \frac{8}{3}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{3^p2^qt^{\frac{3p}{2}+\frac{4q}{3}+\frac{11}{3}}}{\Gamma(\frac{3p}{2}+\frac{4q}{3}+\frac{14}{3})}(q+1)_2F_1\left(1-p,-q;2;\frac{1}{3}\right).
\end{aligned} \tag{5.3}$$

Example 2. *Consider the following incommensurate linear fractional differential equation system*

$$\begin{pmatrix}\frac{d^{3/2}x_1}{dt^{3/2}}\\ \frac{d^{4/3}x_2}{dt^{4/3}}\end{pmatrix} = \begin{pmatrix}3/2 & 3\\ 1 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}\cos(t)\\ e^t\end{pmatrix}, \tag{5.4}$$

with respect to the initial conditions $x_1(0) = 3$, $x_1'(0) = -1$, $x_2(0) = 1$ *and* $x_2'(0) = 2$.

Solution: This example is the case when $A = 1$. Hence, using Theorem 2, we have the following explicit analytical solution

$$\begin{aligned}
x_1(t) &= 3 - t + \frac{15}{2} t^{\frac{3}{2}} E_{\frac{3}{2},\frac{4}{3},\frac{5}{2}}\left(\frac{3}{2}t^{\frac{3}{2}}, 2t^{\frac{4}{3}}\right) + \frac{9}{2} t^{\frac{5}{2}} E_{\frac{3}{2},\frac{4}{3},\frac{7}{2}}\left(\frac{3}{2}t^{\frac{3}{2}}, 2t^{\frac{4}{3}}\right) + \frac{1}{\Gamma(\frac{3}{2})}\int_0^t (t-\tau)^{\frac{1}{2}} \cos(\tau)\, \mathrm{d}\tau \\
&\quad + \frac{3}{2}\int_0^t (t-\tau)^2 E_{\frac{3}{2},\frac{4}{3},3}\left(\frac{3}{2}(t-\tau)^{\frac{3}{2}}, 2(t-\tau)^{\frac{4}{3}}\right)\cos(\tau)\, \mathrm{d}\tau \\
&\quad + 3\int_0^t (t-\tau)^{\frac{11}{6}} E_{\frac{3}{2},\frac{4}{3},\frac{17}{6}}\left(\frac{3}{2}(t-\tau)^{\frac{3}{2}}, 2(t-\tau)^{\frac{4}{3}}\right) e^{\tau}\, \mathrm{d}\tau \\
&= 3 - t + \frac{15}{2} t^{\frac{3}{2}} E_{\frac{3}{2},\frac{4}{3},\frac{5}{2}}\left(\frac{3}{2}t^{\frac{3}{2}}, 2t^{\frac{4}{3}}\right) + \frac{9}{2} t^{\frac{5}{2}} E_{\frac{3}{2},\frac{4}{3},\frac{7}{2}}\left(\frac{3}{2}t^{\frac{3}{2}}, 2t^{\frac{4}{3}}\right) \\
&\quad + \frac{\sqrt{t}}{3\Gamma(\frac{3}{2})}\left[3\sin(t)\ {}_1F_2\left(\frac{1}{4};\frac{1}{2},\frac{5}{4};\frac{-t^2}{4}\right) - t\cos(t)\ {}_1F_2\left(\frac{3}{4};\frac{3}{2},\frac{7}{4};\frac{-t^2}{4}\right)\right] \\
&\quad + \frac{3}{2}\sum_{p,q=0}^{\infty} \frac{\left(\frac{3}{2}\right)^p (2)^q}{\Gamma\left(\frac{3p}{2}+\frac{4q}{3}+4\right)} \frac{(p+q)!}{p!q!} t^{\frac{3p}{2}+\frac{4q}{3}+3}\ {}_1F_2\left(1;\frac{3p}{4}+\frac{2q}{3}+2, \frac{3p}{4}+\frac{2q}{3}+\frac{5}{2};\frac{-t^2}{4}\right) \\
&\quad + 3e^t \sum_{p,q=0}^{\infty} \frac{\left(\frac{3}{2}\right)^p (2)^q}{\Gamma\left(\frac{3p}{2}+\frac{4q}{3}+\frac{17}{6}\right)}\, \gamma\left(\frac{3p}{2}+\frac{4q}{3}+\frac{17}{6}, t\right),
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
x_2(t) &= 1 + 2t + 5t^{\frac{4}{3}} E_{\frac{3}{2},\frac{4}{3},\frac{7}{3}}\left(\frac{3}{2}t^{\frac{3}{2}}, 2t^{\frac{4}{3}}\right) + 3t^{\frac{7}{3}} E_{\frac{3}{2},\frac{4}{3},\frac{10}{3}}\left(\frac{3}{2}t^{\frac{3}{2}}, 2t^{\frac{4}{3}}\right) + \frac{1}{\Gamma(\frac{4}{3})}\int_0^t (t-\tau)^{\frac{1}{3}} e^{\tau}\, \mathrm{d}\tau \\
&\quad + \int_0^t (t-\tau)^{\frac{11}{6}} E_{\frac{3}{2},\frac{4}{3},\frac{17}{6}}\left(\frac{3}{2}(t-\tau)^{\frac{3}{2}}, 2(t-\tau)^{\frac{4}{3}}\right)\cos(\tau)\, \mathrm{d}\tau \\
&\quad + 2\int_0^t (t-\tau)^{\frac{5}{3}} E_{\frac{3}{2},\frac{4}{3},\frac{8}{3}}\left(\frac{3}{2}(t-\tau)^{\frac{3}{2}}, 2(t-\tau)^{\frac{4}{3}}\right) e^{\tau}\, \mathrm{d}\tau \\
&= 1 + 2t + 5t^{\frac{4}{3}} E_{\frac{3}{2},\frac{4}{3},\frac{7}{3}}\left(\frac{3}{2}t^{\frac{3}{2}}, 2t^{\frac{4}{3}}\right) + 3t^{\frac{7}{3}} E_{\frac{3}{2},\frac{4}{3},\frac{10}{3}}\left(\frac{3}{2}t^{\frac{3}{2}}, 2t^{\frac{4}{3}}\right) + \frac{e^t}{\Gamma(\frac{4}{3})}\, \gamma\left(\frac{4}{3}, t\right) \\
&\quad + \sum_{p,q=0}^{\infty} (1.5)^p (2)^q\, t^{\frac{9p+8q+17}{6}} \frac{(p+q)!}{p!q!}\left[\frac{1}{\Gamma\left(\frac{9p+8q+23}{6}\right)} - \frac{t^2\ {}_1F_2\left(1;\frac{9p+8q+35}{12},\frac{9p+8q+41}{12};\frac{-t^2}{4}\right)}{\Gamma\left(\frac{9p+8q+35}{6}\right)}\right] \\
&\quad + 2e^t \sum_{p,q=0}^{\infty} \frac{\left(\frac{3}{2}\right)^p (2)^q}{\Gamma\left(\frac{3p}{2}+\frac{4q}{3}+\frac{8}{3}\right)}\, \gamma\left(\frac{3p}{2}+\frac{4q}{3}+\frac{8}{3}, t\right).
\end{aligned} \tag{5.6}$$

Example 3. *Consider the following incommensurate linear fractional differential equation system*

$$\begin{pmatrix} \frac{\mathrm{d}^{1.2}x_1}{\mathrm{d}t^{1.2}} \\ \frac{\mathrm{d}^{1.5}x_2}{\mathrm{d}t^{1.5}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} t + e^t \\ e^{2t} \end{pmatrix}, \tag{5.7}$$

with respect to the initial conditions $x_1(0) = 1$, $x_1'(0) = \Gamma(1.8)$, $x_2(0) = 1$ *and* $x_2'(0) = 3$.

Solution: Since $a_{11} = 0$, Theorem 3 will be applied. For sake of simplicity, we present this solution using decimal numbers. This example have the following explicit analytical solution

$$\begin{aligned}
x_1(t) &= 1 + t\Gamma(1.8) + 5t^{2.7}E_{2.7,1.5,3.7}\left(5t^{2.7}, t^{1.5}\right) + 5\Gamma(1.8)t^{3.7}E_{2.7,1.5,4.7}\left(5t^{2.7}, t^{1.5}\right) \\
&\quad + t^{1.2}E_{2.7,1.5,2.2}\left(5t^{2.7}, t^{1.5}\right) + 3t^{2.2}E_{2.7,1.5,3.2}\left(5t^{2.7}, t^{1.5}\right) + \frac{1}{\Gamma(1.2)}\int_0^t (t-\tau)^{0.2}\left(\tau + e^{\tau}\right)\, d\tau \\
&\quad + 5\int_0^t (t-\tau)^{2.9}E_{2.7,1.5,3.9}\left(5(t-\tau)^{2.7}, (t-\tau)^{1.5}\right)\left(\tau + e^{\tau}\right)\, d\tau \\
&\quad + \int_0^t (t-\tau)^{1.7}E_{2.7,1.5,2.7}\left(5(t-\tau)^{2.7}, (t-\tau)^{1.5}\right)e^{2\tau}\, d\tau \\
&= 1 + t\Gamma(1.8) + 5t^{2.7}E_{2.7,1.5,3.7}\left(5t^{2.7}, t^{1.5}\right) + 5\Gamma(1.8)t^{3.7}E_{2.7,1.5,4.7}\left(5t^{2.7}, t^{1.5}\right) \\
&\quad + t^{1.2}E_{2.7,1.5,2.2}\left(5t^{2.7}, t^{1.5}\right) + 3t^{2.2}E_{2.7,1.5,3.2}\left(5t^{2.7}, t^{1.5}\right) \\
&\quad + \frac{t^{2.2}}{\Gamma(3.2)} + \frac{e^t\gamma(1.2, t)}{\Gamma(1.2)} + 5t^{4.9}E_{2.7,1.5,5.9}\left(5t^{2.7}, t^{1.5}\right) \\
&\quad + 5e^t\sum_{p,q=0}^{\infty}\frac{5^p\,\gamma(2.7p + 1.5q + 3.9, t)}{\Gamma(2.7p + 1.5q + 3.9)} + e^{2t}\sum_{p,q=0}^{\infty}\frac{5^p\gamma(2.7p + 1.5q + 2.7, 2t)}{2^{2.7p+1.5q+2.7}\Gamma(2.7p + 1.5q + 2.7)},
\end{aligned} \tag{5.8}$$

$$\begin{aligned}
x_2(t) &= 5t^{1.5}E_{2.7,1.5,2.5}\left(5t^{2.7}, t^{1.5}\right) + 5\Gamma(1.8)t^{2.5}E_{2.7,1.5,3.5}\left(5t^{2.7}, t^{1.5}\right) \\
&\quad + E_{2.7,1.5,1}\left(5t^{2.7}, t^{1.5}\right) + 3tE_{2.7,1.5,2}\left(5t^{2.7}, t^{1.5}\right) \\
&\quad + 5\int_0^t (t-\tau)^{1.7}E_{2.7,1.5,2.7}\left(5(t-\tau)^{2.7}, (t-\tau)^{1.5}\right)\left(\tau + e^{\tau}\right)\, d\tau \\
&\quad + \int_0^t (t-\tau)^{0.5}E_{2.7,1.5,1.5}\left(5(t-\tau)^{2.7}, (t-\tau)^{1.5}\right)e^{2\tau}\, d\tau \\
&= 5t^{1.5}E_{2.7,1.5,2.5}\left(5t^{2.7}, t^{1.5}\right) + 5\Gamma(1.8)t^{2.5}E_{2.7,1.5,3.5}\left(5t^{2.7}, t^{1.5}\right) \\
&\quad + E_{2.7,1.5,1}\left(5t^{2.7}, t^{1.5}\right) + 3tE_{2.7,1.5,2}\left(5t^{2.7}, t^{1.5}\right) + 5t^{3.7}E_{2.7,1.5,4.7}\left(5t^{2.7}, t^{1.5}\right) \\
&\quad + 5e^t\sum_{p,q=0}^{\infty}\frac{5^p\,\gamma(2.7p + 1.5q + 2.7, t)}{\Gamma(2.7p + 1.5q + 2.7)} + e^{2t}\sum_{p,q=0}^{\infty}\frac{5^p\gamma(2.7p + 1.5q + 1.5, 2t)}{2^{2.7p+1.5q+1.5}\Gamma(2.7p + 1.5q + 1.5)}.
\end{aligned} \tag{5.9}$$

Example 4. *Consider the following incommensurate linear fractional differential equation system*

$$\begin{pmatrix} \frac{d^{1.2}x_1}{dt^{1.2}} \\ \frac{d^{1.6}x_2}{dt^{1.6}} \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2.5t^{0.8} - 9t \\ \Gamma(0.8)\, t^2 \end{pmatrix}, \tag{5.10}$$

with respect to the initial conditions $x_1(0) = 2$*,* $x_1'(0) = 0$*,* $x_2(0) = -6$ *and* $x_2'(0) = 3$*.*

Solution: Since $a_{11} = a_{22} = 0$, applying the result presented in subsection 4.4 and with the help from Eq (2.9), we have the following explicit analytical solution

$$\begin{aligned}
x_1(t) &= 2E_{2.8}\left(-3t^{2.8}\right) - 18t^{1.2}E_{2.8,2.2}\left(-3t^{2.8}\right) + 9t^{2.2}E_{2.8,3.2}\left(-3t^{2.8}\right) \\
&\quad + 2.5\Gamma(1.8)t^2E_{2.8,3}\left(-3t^{2.8}\right) - 9t^{2.2}E_{2.8,3.2}\left(-3t^{2.8}\right) + 3\Gamma(0.8)t^{4.8}E_{2.8,5.8}\left(-3t^{2.8}\right),
\end{aligned} \tag{5.11}$$

$$
\begin{aligned}
x_2(t) = &- 2t^{1.6}E_{2.8,2.6}\left(-3t^{2.8}\right) - 6E_{2.8}\left(-3t^{2.8}\right) + 3tE_{2.8,2}\left(-3t^{2.8}\right) \\
&- 2.5\Gamma(1.8)t^{3.6}E_{2.8,4.6}\left(-3t^{2.8}\right) + 9t^{3.8}E_{2.8,4.8}\left(-3t^{2.8}\right) + \Gamma(0.8)t^{3.6}E_{2.8,4.6}\left(-3t^{2.8}\right).
\end{aligned}
\tag{5.12}
$$

The solution of this example is shown in Figure 1. For the purpose of validate the solution (i.e. LHS equal to RHS of the problem), we can find the LHS via fractional derivative of these $x_1(t)$ and $x_2(t)$ for the desired order (i.e. in this example, are 1.2 and 1.6 respectively). Meanwhile for the RHS, substitute the solution $x_1(t)$ and $x_2(t)$ in the RHS of problem. If the analytical expression is too lengthy, we suggest to plot the both sides up to desired power. We use Maple to perform all the computation.

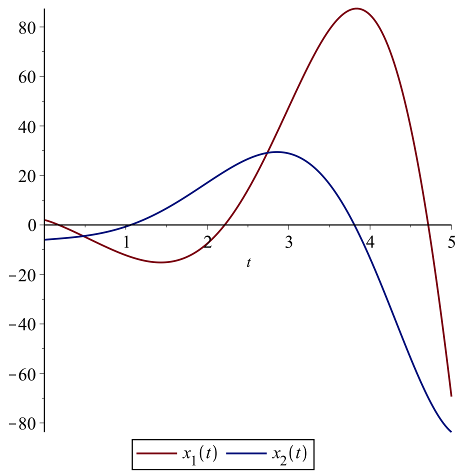

Figure 1. Solution $x_1(t)$ and $x_2(t)$ for Example 4.

6. Conclusions

This paper has successfully derived the explicit analytical solution of linear incommensurate fractional differential equation systems with fractional order $1 < \alpha, \beta < 2$. Using the new theorems, analytical solutions are obtained, and we presented them via some examples. This paper serves as an extension of the similar result recently achieved in [1, 19], which limited to fractional order $0 < \alpha, \beta < 1$. Moreover, the analytical solution obtained in this paper may enable us to investigate more rigorously the stability analysis and asymptotic stability for incommensurate fractional differential equation systems with fractional order $1 < \alpha, \beta < 2$, especially when this kind of incommensurate system may be more suitable to represent the real-world applications such as COVID-19 [38], cancer modelling, fluid flows problems. It may also be extended to higher order in the future. Explicit analytical solution for higher order (i.e. $\alpha, \beta > 2$) incommensurate fractional differential equation systems may be obtained using a similar approach.

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Conflict of interest

Authors declare that there is no conflict of interests regarding the publication of the paper.

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