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Research article

On two new contractions and discontinuity on fixed points

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Abstract: This paper deals with a well known open problem raised by Kannan (Bull. Calcutta Math. Soc., 60: 71–76, 1968) and B. E. Rhoades (Contemp. Math., 72: 233–245, 1988) on the existence of general contractions which have fixed points, but do not force the continuity at the fixed point. We propose some new affirmative solutions to this question using two new contractions called (ψ,φ)-$\mathcal{A}$-contraction and (ψ,φ)-$\mathcal{A}'$-contraction inspired by the results of H. Garai et al. (Applicable Analysis and Discrete Mathematics, 14(1): 33–54, 2020) and P. D. Proinov (J. Fixed Point Theory Appl. (2020) 22: 21). Some new fixed point and common fixed point results in compact metric spaces and also in complete metric spaces are proved in which the corresponding contractive mappings are not necessarily continuous at their fixed points. Moreover, we show that new solutions to characterize the completeness of metric spaces. Several examples are provided to verify the validity of our main results.

Keywords: fixed point; (ψ,φ)-$\mathcal{A}$-contraction; (ψ,φ)-$\mathcal{A}'$-contraction; discontinuity at the fixed point
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

In 1922, Banach [1] introduced the following theorem, which is well known as Banach Contraction Principle to establish the existence of solutions for integral equations.

Theorem 1.1. [1] Let (X,d) be a complete metric space and $T : X \mapsto X$ be a contractive mapping, that

is, there exists $L \in [0, 1)$ such that
$$d(Tx, Ty) \leq Ld(x, y),$$
for all $x, y \in X$. Then, we have the following assertions hold:

(i) T has a unique fixed point.
(ii) For each $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to the fixed point of T.

Since then, the research on fixed points of contractive mappings has continued through to the present. There are many contractive conditions to furnish existence (and uniqueness) of fixed points of different types of mappings in different settings. It is an interesting fact that, among the several kinds of contractive conditions, some ones force the corresponding mappings to be continuous on the entire domain and some ones force the corresponding mappings to be continuous on some particular points of the domain. However there are number of contractive conditions which cannot guarantee the continuity of mappings, although in most of the cases the continuity of the mapping is assumed. Kannan's work [2] can be considered as the start of the problem on continuity of contractive mappings at the fixed point. In [2], Kannan introduced a weaker contractive condition and proved a very interesting fixed point theorem which stated that every self-mapping T defined on a complete metric space (X, d) has a unique fixed point if the following contraction holds (called Kannan type contraction)
$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)],$$
where $\lambda \in (0, \frac{1}{2})$, for all $x, y \in X$. It may be observed that Kannan type contraction does not require the continuity of the mapping T for the existence of the fixed point. However, a mapping T satisfying Kannan type contractive condition turns out to be continuous at the fixed point.

In this direction, in [3], Rhoades compared 250 contractive conditions (including Kannan type contraction) and showed that though most of the contractions do not force the corresponding mappings to be continuous on the entire domain, all of them force the mapping to be continuous at their corresponding fixed points. Moreover, he re-examined the continuity of a large number of contractive mappings in detail and claimed that all of the contractive conditions assure that mappings are continuous at the fixed points although continuity is not assumed in all the cases [4]. Motivated by his observation, Rhoades proposed an exciting open problem as follows:

Open Problem 1.1 Whether there exists any contractive condition which can ensure the existence and uniqueness of a fixed point which does not force the corresponding mapping to be continuous at the underlying fixed points.

The first answer of this interesting open problem was achieved by Pant in [5], stated as the following theorem.

Theorem 1.2. [5] Let f be a self-mapping of a complete metric space (X, d) such that for any $x, y \in X$,

(i) $d(fx, fy) \leq \varphi(m(x, y))$,
(ii) for any $\epsilon > 0$, there exists a $\delta > 0$ such that
$$\epsilon < m(x, y) < \epsilon + \delta \Longrightarrow d(fx, fy) \leq \epsilon,$$

where $m(x, y) = \max\{d(x, fx), d(y, fy)\}$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\varphi(t) < t$ for all $t > 0$. Then f has a unique fixed point, say z. Moreover, f is continuous at z if and only if $\lim_{x\to z} m(x, z) = 0$.

After this, some new solutions to this problem of continuity at fixed point and applications of such results have been reported (e.g. Bisht and Pant [6, 7], Bisht and Rakočević [8, 9], Bisht and Özgür [10], Çelik and Özgür [11], Özgür and Taş [12, 13, 20], Pant et al. [14, 15, 21], Rashid et al. [16], Taş and Özgür [17], Taş et al. [18], Zheng and Wang [19]). Recently, H. Garai et al. [22] provided another solution to this open problem by introducing two new types of contractive mappings, called $\mathcal{A}$-contractive and $\mathcal{A}'$- contractive mappings which cover many well known contractions, such as, Edelstein type contraction, Kannan type contraction, Chatterjea type contraction, Hardy-Rogers type contraction and so on. They also proved some new fixed point theorems involving these two contractive mappings for which are not necessarily continuous at their fixed points. Moreover, in 2020, inspired by Wardowski and Dung [23], W.M. Alfaqih et al. introduced the new notion of F^*-weak contractions [24] and utilized the same to prove some fixed point theorems which also give some affirmative answers to Open Problem 1.1.

On the other hand, P. D. Proinov [25] studied the problem of finding (sufficient) conditions on the auxiliary functions $\psi, \varphi : (0, \infty) \mapsto \mathbb{R}$ that guarantee that T has a unique fixed point and that Picard iterative sequence $\{T^n x\}$ converges to the fixed point for every initial point x in a complete metric space (X, d). Also, he proved that recent fixed point theorems of Wardowski [26] and Jleli and Samet [27] are equivalent to a special case of the well-known fixed point theorem of Skof [28].

The aim of this paper is twofold:

1. to introduce two new notions of (ψ, φ)-$\mathcal{A}$-contraction and (ψ, φ)-$\mathcal{A}'$-contraction and utilize them to prove some fixed point theorems (resp. common fixed point theorems).
2. to present some new affirmative answers to the Open Problem 1.1 via these new kinds of contractions.

2. Preliminaries

In 2012, Wardowski [26] introduced the definition of an F-contraction mapping as follows:
Let (X, d) be a metric space. A mapping $T : X \mapsto X$ is said to be a F-contraction if there exists a real number $\tau > 0$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \quad \Rightarrow \quad \tau + F(d(Tx, Ty) \leq F(d(x, y)),$$

where $F : \mathbb{R}_+ \mapsto \mathbb{R}$ is a mapping satisfying the following conditions:
$(F1)$ F is strictly increasing, that is, for all $x, y \in \mathbb{R}_+$, $x < y$, $F(x) < F(y)$;
$(F2)$ For each sequence $\{\alpha_n\}_{n=1}^{+\infty}$ of positive numbers,

$$\lim_{n\to+\infty} \alpha_n = 0 \quad \text{if and only if} \quad \lim_{n\to+\infty} F(\alpha_n) = -\infty;$$

$(F3)$ There exists $k \in (0, 1)$ such that $\lim_{\alpha\to0^+} \alpha^k F(\alpha) = 0$.
Let $\mathcal{F}$ be the set of all functions satisfying $(F1)$–$(F3)$.
Example 2.1. [26] The following functions belong to $\mathcal{F}$.
(1) $F(\alpha) = \ln \alpha, \alpha > 0$.
(2) $F(\alpha) = \ln \alpha + \alpha, \alpha > 0$.
(3) $F(\alpha) = -\frac{1}{\sqrt{\alpha}}, \alpha > 0$.
(4) $F(\alpha) = \ln(\alpha^2 + \alpha), \alpha > 0$.

Further, Wardowski [26] stated a modified version of Banach contraction principle as follows.

Theorem 2.1. [26] Let T be a self-mapping on a complete metric space (X, d). If T forms a F-contraction, then it possesses a unique fixed point x^*. Moreover, for any $x \in X$ the sequence $\{T^n x\}$ is convergent to x^*.

Inspired by Wardowski's contribution, there is a sustained endeavor of many authors to extend and improve this concept by relaxing or excluding some of the conditions $(F1)$–$(F3)$ or generalizing the shape of the respective F-contraction. In this respect, Secelean [29] proved that $(F3)$ can be replaced by adding certain boundedness condition on the operator T. Furthermore, if F is continuous then condition $(F3)$ can be dropped without any extra assumption on T. Piri and Kumam [30] replaced $(F3)$ by the continuity of F, which was essentially motivated by the fact that most of the utilized functions in the existing literature were continuous. Vetro [31] extended the F-contraction by replacing the constant τ with a function. Secelean and Wardowski [32] introduced a new concept of ψF-contraction which strictly generalized F-contraction by weakening $(F1)$ and considering the family of a certain class of increasing functions ψ. Lukács and Kajántó [33] defined a new version of F-contraction by omitting $(F2)$ condition in b-metric spaces. For more sequent Wardowski's results, one can refer to [34–37] and so forth.

In 2020, W.M. Alfaqih et al. [24] introduced the notion of F^*-weak contraction by deleting $(F1)$, $(F3)$ and removing one way implication of $(F2)$. Also, they presented some fixed point results corresponding to this type contraction and gave an affirmative answer to Open Problem 1.1.

Let $\mathcal{F}'$ be the set of all functions $F : \mathbb{R}_+ \mapsto \mathbb{R}$ satisfying the following condition:

$(F2')$: for every sequence $\{\beta_n\} \subset (0, +\infty)$,

$$\lim_{n\to+\infty} F(\beta_n) = -\infty \Rightarrow \lim_{n\to+\infty} \beta_n = 0.$$

Obviously, $\mathcal{F} \subset \mathcal{F}'$. However, the converse inclusion is not true in general (see Example 2.1–2.2 in [24] for details).

Definition 2.1. [24] Let (X, d) be a metric space. A self-mapping T on X is said to be an F^*-weak contraction if there exist $\tau > 0$ and $F \in \mathcal{F}'$ such that

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(m(x, y)),$$

where $m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$.

Theorem 2.2. [24] Let (X, d) be a complete metric space and $T : X \mapsto X$ an F^*-weak contraction. If F is continuous, then

(1) T has a unique fixed point (say $z \in X$),

(2) $\lim\limits_{n\to\infty} T^n x = z$ for all $x \in X$.

Moreover, T is continuous at z if and only if $\lim\limits_{n\to\infty} m(x, z) = 0$.

Very recently, Proinov [25], to extend and unify many existing results, proved that the fixed point theorem of Skof [28], in the setting of metric spaces, covers many existing results, including the attractive results of Wardowski [26] and Jleli-Samet [27] by introducing the following theorem.

Theorem 2.3. [25] Let (X, d) be a metric space and $T : X \mapsto X$ be a mapping such that $\psi(d(Tx, Ty)) \leq \varphi(d(x, y))$ for all $x, y \in X$, with $d(Tx, Ty) > 0$, where the functions $\psi, \varphi : (0, \infty) \to \mathbb{R}$ satisfy the following conditions:

(i) ψ is nondecreasing;

(ii) $\varphi(t) < \psi(t)$ for $t > 0$;
(iii) $\limsup_{t\to\varepsilon^+}\varphi(t) < \psi(\varepsilon^+)$ for any $\varepsilon > 0$.
Then T has a unique fixed point $x^* \in X$ and the iterative sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Setting $\varphi(t) = \psi(t) - \tau$ in Theorem 2.3, we can obtain the following corollary.

Corollary 2.1. [25] Let (X, d) be a metric space and $T : X \mapsto X$ be a mapping satisfying the following condition:
$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \tau$ for all $x, y \in X$ with $d(Tx, Ty) > 0$,
where $\tau > 0$ and $\psi : (0, +\infty) \mapsto \mathbb{R}$ is nondecreasing. Then T has a unique fixed point ξ and the sequence $\{T^n x\}$ is convergent to ξ for every $x \in X$.

Indeed, Corollary 2.1 improves Theorem 1.1 and the results of Secelean [29], Piri and Kumam [30] and Lukács and Kajántó [33]. In fact, Corollary 2.1 shows that both conditions $(F2)$ and $(F3)$ can be omitted from Theorem 2.1. Besides, the strictness of monotonicity of F is not necessary.

Throughout the rest part of this paper, we denote by $X, \mathbb{R}_+, \mathbb{N}$ the nonempty set, the set of non-negative real numbers and the set of natural numbers, respectively.

Now, we recall the notions of $\mathcal{A}$-contractive and $\mathcal{A}'$-contractive mappings introduced by H. Garai et al. in [22].

We denote by $\mathcal{A}$ the collection of all mappings $f : \mathbb{R}_+^3 \mapsto \mathbb{R}_+$ which satisfy the following conditions:

$(\mathcal{A}_1)$ f is continuous.
$(\mathcal{A}_2)$ If $v > 0$ and $u < f(u, v, v)$ or $u < f(v, u, v)$ or $u < f(v, v, u)$, then $u < v$.
$(\mathcal{A}_3)$ $f(u, v, w) \leq u + v + w$, for all $u, v, w \in \mathbb{R}_+$.

Example 2.2. [22] Here are some examples of mappings belonging to the class $\mathcal{A}$ given by the following:
(1) $f(u, v, w) = \frac{v+w}{2}$.
(2) $f(u, v, w) = \frac{u+v}{2}$.
(3) $f(u, v, w) = \frac{1}{2}\max\{u + v, v + w, w + u\}$.
(4) $f(u, v, w) = \max\{u, v, w\}$.
(5) $f(u, v, w) = \max\{v, w\}$.
(6) $f(u, v, w) = \max\{u, \alpha v + (1 - \alpha)w, (1 - \alpha)v + \alpha w\}, 0 \leq \alpha < 1$.
(7) $f(u, v, w) = xu + yv + zw$, where x, y, z are positive real numbers such that $x + y + z = 1$.
(8) $f(u, v, w) = \sqrt{vw}$.
(9) $f(u, v, w) = u$.
(10) $f(u, v, w) = (uvw)^{\frac{1}{3}}$.

We denote by $\mathcal{A}'$ the collection of all mappings $f : \mathbb{R}_+^3 \to \mathbb{R}_+$ which satisfy the following conditions:

$(\mathcal{A}'_1)$ f is continuous.
$(\mathcal{A}'_2)$ If $v > 0$ and $u < f(u, v, v)$ or $u < f(v, u, v)$ or $u < f(v, v, u)$, then $u < v$.
$(\mathcal{A}'_3)$ If $v > 0$ and $u < f(v, u + v, 0)$, then $u < v$.
$(\mathcal{A}'_4)$ If $v \leq v_1$, then $f(u, v, w) \leq f(u, v_1, w)$, for all $u, w \in \mathbb{R}_+$.
$(\mathcal{A}'_5)$ $f(u, u, u) \leq u$, for all $u \in \mathbb{R}_+$.
$(\mathcal{A}'_6)$ $f(u, v, w) \leq u + v + w$, for all $u, v, w \in \mathbb{R}_+$.

Example 2.3. [22] Some examples of mappings f belonging to $\mathcal{A}'$ are the following:
(1) $f(u, v, w) = \frac{1}{3}(u + v + w)$.

(2) $f(u,v,w) = \frac{1}{2}\max\{u,v,w\}$.
(3) $f(u,v,w) = \frac{1}{2}(v+w)$.

Motivated by the contributions of H. Garai et al. [22] and Proinov [25], we will introduce two new notions of contractions called (ψ,φ)-$\mathcal{A}$-contraction and (ψ,φ)-$\mathcal{A}'$-contraction as follows.

Definition 2.2. Let (X,d) be a metric space. A self-mapping T on X is said to be an (ψ,φ)-$\mathcal{A}$-contraction, if for every $x,y \in X$ such that $d(Tx,Ty) > 0$, the following inequality

$$\psi(d(Tx,Ty)) \leq \varphi(m(x,y)), \tag{2.1}$$

holds, where $\psi,\varphi : (0,+\infty) \mapsto \mathbb{R}$ are two functions such that $\varphi(t) < \psi(t)$, for $t > 0$ and $m(x,y)$ is defined by $m(x,y) = f(d(x,y), d(x,Tx), d(y,Ty))$, $f \in \mathcal{A}$.

Definition 2.3. Let (X,d) be a metric space. A self-mapping T on X is said to be an (ψ,φ)-$\mathcal{A}'$-contraction, if for every $x,y \in X$ such that $d(Tx,Ty) > 0$, the following inequality

$$\psi(d(Tx,Ty)) \leq \varphi(m'(x,y)), \tag{2.2}$$

holds, where $\psi,\varphi : (0,+\infty) \mapsto \mathbb{R}$ are two functions such that $\varphi(t) < \psi(t)$, for $t > 0$ and $m'(x,y)$ is defined by $m'(x,y) = f(d(x,y), d(x,Ty), d(y,Tx))$, $f \in \mathcal{A}'$.

Definition 2.4. Let (X,d) be a metric space. A pair (T,S) of self-mappings on X is said to be an (ψ,φ)-$\mathcal{A}$-contraction, if for every $x,y \in X$ such that $d(Tx,Sy) > 0$, the following inequality

$$\psi(d(Tx,Sy)) \leq \varphi(M(x,y)), \tag{2.3}$$

holds, where $\psi,\varphi : (0,+\infty) \mapsto \mathbb{R}$ are two functions such that $\varphi(t) < \psi(t)$, for $t > 0$ and $M(x,y)$ is defined by $M(x,y) = f(d(x,y), d(x,Tx), d(y,Sy))$, $f \in \mathcal{A}$.

Definition 2.5. Let (X,d) be a metric space. A pair (T,S) of self-mappings on X is said to be an (ψ,φ)-$\mathcal{A}'$-contraction, if for every $x,y \in X$ such that $d(Tx,Sy) > 0$, the following inequality

$$\psi(d(Tx,Sy)) \leq \varphi(M'(x,y)), \tag{2.4}$$

holds, where $\psi,\varphi : (0,+\infty) \mapsto \mathbb{R}$ are two functions such that $\varphi(t) < \psi(t)$, for $t > 0$ and $m'(x,y)$ is defined by $M'(x,y) = f(d(x,y), d(x,Sy), d(y,Tx))$, $f \in \mathcal{A}'$.

Definition 2.6. [38] A mapping T on a metric space (X,d) is said to be orbitally continuous if, for any sequence $\{y_n\}$ in $O_x(T)$, $y_n \to u$ implies $Ty_n \to Tu$ as $n \to +\infty$, where $O_x(T) = \{T^n x : n \geq 0\}$ is the orbit of T at x.

It is easy to observe that a continuous mapping is orbitally continuous, but not conversely.

Definition 2.7. [39] A self-mapping T of a metric space (X,d) is called k-continuous, $k = 1,2,3,\ldots$, if $T^k x_n \to Tx$ whenever $\{x_n\}$ is a sequence in X such that $T^{k-1}x_n \to x$.

It was shown in [39] that continuity of T^k and k-continuity of T are independent conditions when $k > 1$ and continuity $\Rightarrow$ 2-continuity $\Rightarrow$ 3-continuity $\Rightarrow \ldots$, but not conversely. It is also easy to see that 1-continuity is equivalent to continuity.

Definition 2.8. [40] Let (X,d) be a metric space and $T : X \mapsto X$. A mapping $f : X \mapsto \mathbb{R}$ is said to be T-orbitally lower semi-continuous at $z \in X$ if $\{x_n\}$ is a sequence in $O_x(T)$ for some $x \in X$, $\lim_{n\to\infty} x_n = z$ implies $f(z) \leq \lim_{n\to\infty} \inf f(x_n)$.

Proposition 2.1 [41] Let (X, d) be a metric space, $T : X \mapsto X$ and $z \in X$. If T is orbitally continuous at z or T is k-continuous at z for some $k \neq 1$, then the function $f(x) := d(x, Tx)$ is T-orbitally lower semi-continuous at z.

It is noted that T-orbital lower semi-continuity of $f(x) = d(x, Tx)$ is weaker than both orbital continuity and k-continuity of T (see Example 1 in [41]).

3. Main results

At the beginning of this section, we first investigate new solutions to the Open Problem 1.1 using the (ψ, φ)-$\mathcal{A}$-contraction which generates a unique fixed point (resp. common fixed point) in compact metric spaces and complete metric spaces.

3.1. New fixed point results via (ψ, φ)-$\mathcal{A}$-contractions

Theorem 3.1. Let (X, d) be a compact metric space and $T : X \mapsto X$ be a (ψ, φ)-$\mathcal{A}$-contraction such that T is orbitally continuous. Also, assume that ψ is nondecreasing. Then we have the following assertions:
(i) T has a unique fixed point $z \in X$.
(ii) If $u > f(u, 0, 0)$ for all $u > 0$, then the sequence $\{T^n x_0\}$ of iterates converges to that fixed point for each $x_0 \in X$.
(iii) Further, if $f(0, 0, u) = 0$ implies $u = 0$, then T is continuous at the fixed point z if and only if $\lim_{x \to z} m(z, x) = 0$, where

$$m(z, x) = f(d(z, x), d(z, Tz), d(x, Tx)).$$

Proof. (1) Starting with an arbitrary point $x_0 \in X$, we define a sequence $\{x_n\} \subseteq X$ by $x_{n+1} = Tx_n = T^n x_0$ for $n \in \mathbb{N} \cup \{0\}$. Let $\alpha_n = d(x_n, x_{n+1})$, for $n \in \mathbb{N} \cup \{0\}$.
Now, we will prove that $\{\alpha_n\}$ converges to 0.
It is trivial if $\alpha_n = 0$ for some $n \in \mathbb{N} \cup \{0\}$.
Suppose now that $\alpha_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$.
Using (2.1), with $x = x_n, y = x_{n+1}$, we have

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &= \psi(d(Tx_{n-1}, Tx_n)) \\ &\leq \varphi(m(x_{n-1}, x_n)) \\ &= \varphi(f(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}))) \\ &< \psi(f(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}))).\end{aligned}$$

From the monotonicity of ψ, we have

$$d(x_n, x_{n+1}) < f(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})).$$

Therefore, by $(\mathcal{A}_2)$, we have

$$\alpha_n = d(x_n, x_{n+1}) < d(x_{n-1}, x_n) = \alpha_{n-1},$$

which shows that $\{\alpha_n\}$ is a decreasing sequence of nonnegative real numbers and hence converges to some nonnegative real number $r \geq 0$.

Again, since X is compact, there exists a convergent subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ and let $\lim_{k\to+\infty} x_{n_k} = z$. Further, by the orbital continuity of T, we have

$$r = \lim_{k\to+\infty} d(x_{n_k}, x_{n_k+1}) = d(z, Tz).$$

Again, we have

$$r = \lim_{k\to+\infty} d(x_{n_k+1}, x_{n_k+2}) = d(Tz, T^2z).$$

If $r > 0$, then $z \neq Tz$, from (2.1) and ($\mathcal{A}_2$), we have

$$\begin{aligned} d(Tz, T^2z) &< f(d(z, Tz), d(z, Tz), d(Tz, T^2z)) \\ &\Rightarrow d(Tz, T^2z) < d(z, Tz) \\ &\Rightarrow r < r, \end{aligned}$$

which leads to a contradiction. So we have $r = 0$, that is, $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$, and z is a fixed point of T.

Next, we will prove the uniqueness of the fixed point. For this, let z' be another fixed point of T. Then we have

$$\begin{aligned} \psi(d(Tz, Tz')) &\leq \varphi(m(z, z')) \\ &= \varphi(f(d(z, z'), d(z, Tz), d(z', Tz'))) \\ &< \psi(f(d(z, z'), d(z, Tz), d(z', Tz'))) \\ &= \psi(f(d(z, z'), 0, 0)). \end{aligned}$$

It follows from the monotonicity of ψ that

$$d(z, z') < f(d(z, z'), 0, 0). \tag{3.1}$$

From ($\mathcal{A}_3$), we have

$$d(z, z') \geq f(d(z, z'), 0, 0),$$

which contradicts to (3.1). So $z = z'$.

(2) Next, we assume that $u > f(u, 0, 0)$ for all $u > 0$. We consider the sequence of real numbers $\{s_n\}$, where $s_n = d(z, x_n)$. Define a function $g(x) = d(z, x)$ for all $x \in X$. Clearly, g is continuous on X, and hence $g(X)$ is bounded. Thus, $\{s_n\}$ is a bounded sequence of a real numbers. Since the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to z, we get that

$$\lim_{k\to+\infty} d(z, x_{n_k}) = 0,$$

i.e., $\lim_{k\to+\infty} s_{n_k} = 0$. Thus 0 is a cluster point of the sequence $\{s_n\}$. Let c be any cluster point of $\{s_n\}$. Then there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $s_{n_i} \to c$. So $d(z, x_{n_i}) \to c$ as $i \to +\infty$. Therefore, we have

$$|s_{n_i+1} - s_{n_i}| = |d(x_{n_i+1}, z) - d(x_{n_i}, z)|$$

$$\leq d(x_{n_i+1}, x_{n_i}) \to 0,$$

as $i \to +\infty$ and hence $\lim_{i\to+\infty} s_{n_i+1} = \lim_{i\to+\infty} s_{n_i}$.
We now prove that $c = 0$. If $c > 0$, then $\lim_{i\to+\infty} d(z, x_{n_i}) > 0$ and so we may assume that $x_{n_i} \neq z$ for all $i \geq 1$. Then we have

$$\begin{aligned}\psi(d(Tx_{n_i}, Tz)) &\leq \varphi(m(x_{n_i}, z))\\ &= \varphi(f(d(x_{n_i}, z), d(x_{n_i}, Tx_{n_i}), d(z, Tz)))\\ &< \psi(f(d(x_{n_i}, z), d(x_{n_i}, Tx_{n_i}), d(z, Tz))),\end{aligned}$$

which implies that

$$s_{n_i+1} = d(Tx_{n_i}, Tz) < f(d(x_{n_i}, z), d(x_{n_i}, x_{n_i+1}), d(z, Tz)).$$

Taking limits as $i \to +\infty$ in the above inequality, we have

$$c < f(c, 0, 0),$$

which contradicts to assumption (ii). So, $c = 0$. Therefore, 0 is the only cluster point of the bounded sequence $\{s_n\}$ and so this sequence also converges to 0. Hence $\{x_n\}$ converges to z. Since x_0 is arbitrary point in X, it follows that $\{T^n x_0\}$ converges to the fixed point z for each $x_0 \in X$.

(3) Next, we assume that $f(0, 0, u) = 0$ implies $u = 0$. Let T be continuous at the fixed point z. To show $\lim_{x\to z} m(z, x) = 0$, let $\{y_n\}$ be a sequence in X converging to z. Then

$$\begin{aligned}\lim_{n\to+\infty} m(z, y_n) &= \lim_{n\to+\infty} f(d(z, y_n), d(z, Tz), d(y_n, Ty_n))\\ &= f(0, 0, d(z, Tz)) = 0.\end{aligned}$$

Therefore, $\lim_{x\to z} m(z, x) = 0$.
Conversely, let $\lim_{x\to z} m(z, x) = 0$. To prove T is continuous at the fixed point z, let $\{y_n\}$ be a sequence in X converging to z. Therefore, we have

$$\begin{aligned}&\lim_{n\to+\infty} m(z, y_n) = 0\\ \Longrightarrow\ &\lim_{n\to+\infty} f(d(z, y_n), d(z, Tz), d(y_n, Ty_n)) = 0\\ \Longrightarrow\ &f(0, 0, \lim_{n\to+\infty} d(y_n, Ty_n)) = 0\\ \Longrightarrow\ &\lim_{n\to+\infty} d(y_n, Ty_n) = 0\\ \Longrightarrow\ &\lim_{n\to+\infty} Ty_n = \lim_{n\to+\infty} y_n = z = Tz.\end{aligned}$$

So T is continuous at the fixed point z.

Now, we give some illustrative examples of Theorem 3.1.

Example 3.1. Let $X = [0, 2]$ and d be the usual metric on X. Consider the self-mapping $T : X \to X$ defined by

$$Tx = \begin{cases} \frac{1}{4}x & ; \quad x \geq 1 \\ 0 & ; \quad x < 1 \end{cases}$$

It is easy to see that T satisfies the conditions of Theorem 3.1 with the functions $f(u,v,w) = \frac{v+w}{2}$ defined for all $u,v,w \in \mathbb{R}^+$, $\psi(t) = t$ and $\varphi(t) = \frac{2}{3}t$, for $t > 0$. Notice that 0 is the unique fixed point of the orbitally continuous (ψ,φ)-$\mathcal{A}$-contractive mapping T. Also, we have $u > f(u,0,0)$ for all $u > 0$ and hence, the sequence $\{T^n x_0\}$ of iterates converges to the fixed point 0 for each $x_0 \in X$. Since $f(0,0,u) = 0$ implies $u = 0$, we can check the continuity of T by calculating the limit

$$\begin{aligned}\lim_{x\to 0} m(0,x) &= \lim_{x\to 0} f(d(0,x), d(0,T0), d(x,Tx)) \\ &= \lim_{x\to 0} f(|x|, 0, |x|) = \lim_{x\to 0}\frac{|x|}{2} = 0.\end{aligned}$$

This shows that T is continuous at the fixed point 0.

Example 3.2. Let $X = [0,2]$ and d be the usual metric on X. Consider the self-mapping $T : X \to X$ defined by

$$Tx = \begin{cases} 1 & ; \quad 0 \le x \le 1 \\ 0 & ; \quad 1 < x \le 2 \end{cases}.$$

It is easy to see that T satisfies the conditions of Theorem 3.1 with the functions $f(u,v,w) = \max\{v,w\}$ defined for all $u,v,w \in \mathbb{R}^+$, $\psi(t) = t$ and

$$\varphi(t) = \begin{cases} \frac{t}{2} & ; \quad 0 < t \le 1 \\ \sqrt{t} & ; \quad 1 < t \end{cases},$$

for $t > 0$. Notice that 1 is the unique fixed point of the orbitally continuous (ψ,φ)-$\mathcal{A}$-contractive mapping T. Also, we have $u > f(u,0,0)$ for all $u > 0$ and hence, the sequence $\{T^n x_0\}$ of iterates converges to the fixed point 0 for each $x_0 \in X$. Since $f(0,0,u) = 0$ implies $u = 0$, we can check the continuity of T by calculating the limit

$$\begin{aligned}\lim_{x\to 1} m(1,x) &= \lim_{x\to 1} f(d(1,x), d(1,T1), d(x,Tx)) \\ &= \lim_{x\to 1} \max\{d(1,T1), d(x,Tx)\} \\ &\neq 0. \quad \text{(not exist)}\end{aligned}$$

This shows that T is discontinuous at the fixed point 1.

Theorem 3.2. Let (X,d) be a compact metric space and a pair (T,S) of self-mappings on X be an (ψ,φ)-$\mathcal{A}$-contraction such that T and S are orbitally continuous. Also, assume that ψ is nondecreasing. Then we have the following assertions:

(i) T and S have a unique common fixed point $z \in X$.

(ii) If $u > f(u,0,0)$ for all $u > 0$, then the sequences $\{T^n x_0\}$ and $\{S^n x_0\}$ of iterates converge to that fixed point for each $x_0 \in X$.

(iii) Further, if $f(0,u,0) = 0$ implies $u = 0$, then T is continuous at the fixed point z if and only if $\lim_{x\to z} M(x,z) = 0$. Also, if $f(0,0,u) = 0$ implies $u = 0$, then S is continuous at the fixed point z if and only if $\lim_{y\to z} M(z,y) = 0$, where

$M(x,z) = f(d(x,z), d(x,Tx), d(z,Sz))$ and $M(z,y) = f(d(z,y), d(z,Tz), d(y,Sy))$.

Proof. (1) Let $x_0 \in X$ be an arbitrary point. We define a sequence $\{x_n\} \subseteq X$ such that $x_{2n+1} = Tx_{2n}$, $x_{2n+2} = Sx_{2n+1}$, for $n \in \mathbb{N} \cup \{0\}$. Let $\alpha_n = d(x_n, x_{n+1})$, for $n \in \mathbb{N} \cup \{0\}$.

Now, we will prove that $\{\alpha_n\}$ converges to 0.
It is trivial if $\alpha_n = 0$ for some $n \in \mathbb{N} \cup \{0\}$. Suppose now that $\alpha_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$.
Using (2.3), with $x = x_{2n}, y = x_{2n+1}$, we have

$$\begin{aligned}\psi(d(x_{2n+1}, x_{2n+2})) &= \psi(d(Tx_{2n}, Sx_{2n+1}))\\ &\leq \varphi(M(x_{2n}, x_{2n+1}))\\ &= \varphi(f(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})))\\ &< \psi(f(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}))).\end{aligned}$$

From the monotonicity of ψ, we have

$$d(x_{2n+1}, x_{2n+2}) < f(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})).$$

By $(\mathcal{A}_2)$, we have

$$\alpha_{2n+1} = d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1}) = \alpha_{2n}.$$

Using similar arguments, we can also obtain that $\alpha_{2n} < \alpha_{2n-1}$.
Thus, $\{\alpha_n\}$ is a decreasing sequence of nonnegative real numbers and hence converges to some nonnegative real number $r \geq 0$.
Again, since X is compact, there exists a convergent subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ and let $\lim_{k\to+\infty} x_{n_k} = z$.
Further, by the orbital continuity of T, we have

$$r = \lim_{k\to+\infty} d(x_{n_k}, Tx_{n_k}) = d(z, Tz),$$

where $n_k = 2j$, $j \in \mathbb{N}$. If $r > 0$, then $z \neq Tz$, from (2.3) and $(\mathcal{A}_2)$, we have

$$\begin{aligned}\psi(d(Tz, Sx_{2j+1})) &\leq \varphi(M(z, x_{2j+1}))\\ &= \varphi(f(d(z, x_{2j+1}), d(z, Tz), d(x_{2j+1}, Sx_{2j+1})))\\ &< \psi(f(d(z, x_{2j+1}), d(z, Tz), d(x_{2j+1}, Sx_{2j+1}))).\end{aligned}$$

From the monotonicity of ψ, we have

$$d(Tz, Sx_{2j+1}) < f(d(z, x_{2j+1}), d(z, Tz), d(x_{2j+1}, Sx_{2j+1})).$$

Taking limits as $j \to +\infty$ in the above inequality, we have

$$d(Tz, z) < f(0, d(z, Tz), 0)),$$

which implies that $d(Tz, z) < 0$, a contradiction. Hence, $r = 0$ and z is a fixed point of S.
Using the same manner in the case that T is orbitally continuous, we can conclude that z is a fixed point of T. Therefore, z is a common fixed point of T and S.
Next, we will prove the uniqueness of the common fixed point. For this, let z' be another common fixed point of T and S, that is, $z' = Tz' = Sz'$. Then we have

$$\psi(d(Tz, Sz')) \leq \varphi(M(z, z'))$$

$$
\begin{aligned}
&= \varphi(f(d(z,z'), d(z,Tz), d(z',Sz'))) \\
&< \psi(f(d(z,z'), d(z,Tz), d(z',Tz'))) \\
&= \psi(f(d(z,z'), 0, 0)).
\end{aligned}
$$

It follows from the monotonicity of ψ that

$$d(z,z') < f(d(z,z'), 0, 0).$$

By $(\mathcal{A}_3)$, we have

$$d(z,z') \geq f(d(z,z'), 0, 0),$$

which contradicts to the above inequality. So $z = z'$.

(2) Next, we assume that $u > f(u, 0, 0)$ for all $u > 0$. We consider the sequence of real numbers $\{s_n\}$ where $s_n = d(z, x_n)$. Define a function $g(x) = d(z, x)$ for all $x \in X$. Clearly, g is continuous on X, and hence $g(X)$ is bounded. Thus, $\{s_n\}$ is a bounded sequence of a real numbers. Since the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to z, we get that

$$\lim_{k\to+\infty} d(z, x_{n_k}) = 0,$$

i.e., $\lim_{k\to+\infty} s_{n_k} = 0$. Thus 0 is a cluster point of the sequence $\{s_n\}$. Let c be any cluster point of $\{s_n\}$. Then there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $s_{n_i} \to c$. So $d(z, x_{n_i}) \to c$ as $i \to +\infty$. Therefore, we have

$$
\begin{aligned}
|s_{n_i+1} - s_{n_i}| &= |d(x_{n_i+1}, z) - d(x_{n_i}, z)| \\
&\leq d(x_{n_i+1}, x_{n_i}) \to 0,
\end{aligned}
$$

as $i \to +\infty$ and hence $\lim_{i\to+\infty} s_{n_i+1} = \lim_{i\to+\infty} s_{n_i}$.
We now prove that $c = 0$. If $c > 0$, then $\lim_{i\to+\infty} d(z, x_{n_i}) > 0$ and so we may assume that $x_{n_i} \neq z$ for all $i \geq 1$. Then, for all $n_i = 2j, i \geq 1, j \in \mathbb{N}$, we have

$$
\begin{aligned}
\psi(d(Tx_{2j}, Sz)) &\leq \varphi(m(x_{2j}, z)) \\
&= \varphi(f(d(x_{2j}, z), d(x_{2j}, Tx_{2j}), d(z, Sz))) \\
&< \psi(f(d(x_{2j}, z), d(x_{2j}, Tx_{2j}), d(z, Sz))),
\end{aligned}
$$

which implies that

$$s_{2j+1} = d(Tx_{2j}, Sz) < f(d(x_{2j}, z), d(x_{2j}, x_{2j+1}), d(z, Sz)).$$

Taking limits as $j \to +\infty$ in the above inequality, we have

$$c < f(c, 0, 0),$$

which contradicts to assumption (ii). So, $c = 0$. Therefore, 0 is the only cluster point of the bounded sequence $\{s_n\}$ and so this sequence also converges to 0. Hence $\{x_n\}$ converges to z. Since x_0 is arbitrary

point in X, it follows that $\{T^n x_0\}$ converges to the fixed point z for each $x_0 \in X$.
Using the similar arguments as mentioned above, we can also obtain that $\{S^n x_0\}$ converges to the fixed point z for each $x_0 \in X$.

(3) Next, we assume that $f(0,u,0) = 0$ implies $u = 0$. Let T be continuous at the fixed point z. To show $\lim\limits_{x\to z} M(x,z) = 0$, let $\{t_n\}$ be a sequence in X converging to z. Then

$$\begin{aligned}\lim_{n\to+\infty} M(t_n,z) &= \lim_{n\to+\infty} f(d(t_n,z), d(t_n,Tt_n), d(z,Sz))\\ &= f(0, d(z,Tz), 0) = 0.\end{aligned}$$

Therefore, $\lim\limits_{x\to z} M(x,z) = 0$.
Conversely, let $\lim\limits_{x\to z} M(x,z) = 0$. To prove T is continuous at the fixed point z, let $\{t_n\}$ be a sequence in X converging to z. Therefore, we have

$$\begin{aligned}&\lim_{n\to+\infty} M(t_n,z) = 0\\ \Longrightarrow\ &\lim_{n\to+\infty} f(d(t_n,z), d(t_n,Tt_n), d(z,Sz)) = 0\\ \Longrightarrow\ &f(0, \lim_{n\to+\infty} d(t_n,Tt_n), 0) = 0\\ \Longrightarrow\ &\lim_{n\to+\infty} d(t_n,Tt_n) = 0\\ \Longrightarrow\ &\lim_{n\to+\infty} Tt_n = \lim_{n\to+\infty} t_n = z = Tz.\end{aligned}$$

So T is continuous at the fixed point z. The same conclusion can be drawn for S by using similar arguments.

The following example illustrates Theorem 3.2.

Example 3.3. Let us consider the compact metric space (X,d) and the self-mapping T considered in Example 3.1. Define the self-mappings $S : X \to X$ as

$$Sx = \begin{cases} 0 & ; \quad x \geq 1 \\ \frac{1}{4}x & ; \quad x < 1 \end{cases}$$

It is easy to see that the pair (T,S) satisfies the conditions of Theorem 3.2 with the functions $f(u,v,w) = \frac{v+w}{2}$ defined for all $u,v,w \in \mathbb{R}^+$, $\psi(t) = t$ and $\varphi(t) = \frac{2}{3}t$, for $t > 0$. Clearly, 0 is the unique common fixed point of the orbitally continuous (ψ,φ)-$\mathcal{A}$-contractive mappings T and S. Since we have $u > f(u,0,0)$ for all $u > 0$, the sequences $\{T^n x_0\}$ and $\{S^n x_0\}$ of iterates converge to the common fixed point 0 for each $x_0 \in X$. Notice that $f(0,u,0) = 0$ implies $u = 0$ and $f(0,0,u) = 0$ implies $u = 0$. So, we can check the continuity of T and S at the fixed point 0 by means of the limits

$$\begin{aligned}\lim_{x\to 0} M(x,0) &= \lim_{x\to 0} f(d(x,0), d(x,Tx), d(0,S0))\\ &= \lim_{x\to 0} f(|x|, |x|, 0) = \lim_{x\to 0}\frac{|x|}{2} = 0\end{aligned}$$

and

$$\lim_{y\to 0} M(0,y) = \lim_{y\to 0} f(d(0,y), d(0,T0), d(y,Sy))$$

$$= \lim_{y\to 0} f(|y|, 0, \left|y - \frac{y}{4}\right|) = \lim_{y\to 0} \frac{3|y|}{8} = 0.$$

This shows that both of the self-mappings T and S are continuous at the common fixed point 0.

Example 3.4. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ and define a metric $d : X \times X \mapsto X$ on X by

$$d(x,y) = \begin{cases} 0 & ; \quad x = y \\ e^{x+y} & ; \quad x \neq y \end{cases}$$

It is easy to check that (X, d) is a complete but not-compact metric space.

Define a function $T : X \mapsto X$ by $T(\frac{1}{n}) = \frac{1}{4n}$ for all $n \in \mathbb{N}$. Also, we take $f \in \mathcal{A}$, where $f(u, v, w) = u$, for $u, v, w \in \mathbb{R}^+$ and define

$$\psi(t) = \begin{cases} \frac{1}{2}\ln t & ; \quad t > 1 \\ \frac{1}{4}\ln(\frac{t}{2}) & ; \quad 0 < t \leq 1 \end{cases}$$

and

$$\varphi(t) = \begin{cases} \frac{1}{3}\ln t & ; \quad t > 1 \\ \frac{1}{2}\ln(\frac{t}{2}) & ; \quad 0 < t \leq 1 \end{cases}$$

Let $x, y \in X$ be arbitrary with $x \neq y$ and take $x = \frac{1}{n}$, $y = \frac{1}{m}$ with $n \neq m$. Therefore, we have

$$\psi(d(Tx,Ty)) = \frac{1}{2}\ln[e^{\frac{1}{4n}+\frac{1}{4m}}] = \frac{1}{2}(\frac{1}{4n} + \frac{1}{4m}) = \frac{1}{8n} + \frac{1}{8m},$$

and

$$\begin{aligned} \varphi(f(d(x,y), d(x,Tx), d(y,Ty))) &= \varphi(e^{\frac{1}{n}+\frac{1}{m}}) \\ &= \frac{1}{3}\ln(e^{\frac{1}{n}+\frac{1}{m}}) \\ &= \frac{1}{3}(\frac{1}{n} + \frac{1}{m}) \\ &= \frac{1}{3n} + \frac{1}{3m}. \end{aligned}$$

Thus, it is easy to check that $\psi(d(Tx,Ty)) \leq \varphi(f(d(x,y), d(x,Tx), d(y,Ty)))$. Hence, T is a (ψ,φ)-$\mathcal{A}$-contractive mapping and also T is orbitally continuous, but T is fixed point free.

If we take the underlying space as complete, we need additional conditions on $f \in \mathcal{A}$ and/or T and S to validate the conclusions of the above theorems.

Theorem 3.3. Let (X, d) be a complete metric space and $T : X \mapsto X$ be a (ψ,φ)-$\mathcal{A}$-contraction such that either T is orbitally continuous or k-continuous or T^k is continuous for some $k \in \mathbb{N}$. Also, assume that ψ is nondecreasing and f satisfies the following conditions:

(i) for any $\epsilon > 0$, there exists an $\delta > 0$ such that

$$f(d(x,y), d(x,Tx), d(y,Ty)) < \epsilon + \delta \Longrightarrow d(Tx,Ty) \leq \frac{\epsilon}{3},$$

for all $x, y \in X$.

(ii) $f(0, 0, u) = 0$ implies $u = 0$.

Then we have the following assertions:

(1) T has a unique fixed point z.
(2) The sequence $\{T^n x_0\}$ of iterates converges to z for each $x_0 \in X$.
(3) Moreover, T is continuous at z if and only if $\lim_{x\to z} m(z,x) = 0$, where

$$m(z,x) = f(d(z,x), d(z,Tz), d(x,Tx)).$$

Proof.(1) Let $x_0 \in X$ be an arbitrary point and define a sequence $\{x_n\} \subseteq X$ by $x_{n+1} = Tx_n = T^n x_0$ for all natural numbers $n \geq 1$. Let $\alpha_n = d(x_n, x_{n+1})$, for all natural numbers $n \geq 1$. Analysis similar to the proof in Theorem 3.1 shows that $\{\alpha_n\}$ is a decreasing sequence of nonnegative real numbers and hence converges to some nonnegative real number $r \geq 0$.
We claim that $r = 0$. If not, by assumption (i), there exists an δ such that

$$f(d(x,y), d(x,Tx), d(y,Ty)) < 3r + \delta \Rightarrow d(Tx,Ty) \leq r.$$

Since $\{\alpha_n\}$ converges to r, for the above δ, there exists an $N \in \mathbb{N}$ such that for $n \geq N$,

$$\alpha_n < r + \frac{\delta}{3},$$

that is,

$$d(x_n, x_{n+1}) < r + \frac{\delta}{3}.$$

So, together with $(\mathcal{A}_3)$, we have

$$\begin{aligned} & f(d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_{n+1}, x_{n+2})) \\ & \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+1}, x_{n+2}) \\ & = \alpha_n + \alpha_{n+1} + \alpha_{n+1} \\ & < 3\alpha_n \\ & < 3r + \delta. \end{aligned}$$

Therefore, $d(x_{n+1}, x_{n+2}) \leq r$, that is $\alpha_{n+1} \leq r$. But this contradicts to the fact that $\{\alpha_n\}$ converges to r and we must have $r = 0$, that is, $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$.

Next, we will show that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$ be arbitrary. It follows from the condition (i) that there exits an $\delta > 0$ such that

$$f(d(x,y), d(x,Tx), d(y,Ty)) < \epsilon + \delta \Rightarrow d(Tx,Ty) \leq \frac{\epsilon}{3},$$

for all $x, y \in X$. Without loss of generality, we assume that $\delta < \epsilon$. Since $\lim_{n\to+\infty} d(x_{n+1}, x_n) = 0$, there exists an $N \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) < \frac{\delta}{3} < \frac{\epsilon}{3} < \epsilon,$$

for all $n \geq N$. By induction on p, we will show that

$$d(x_N, x_{N+p}) < \epsilon, \tag{3.2}$$

for all $p \in \mathbb{N}$. Clearly, (3.2) holds true for $p = 1$. Suppose that (3.2) is true for p, that is, $d(x_N, x_{N+p}) < \epsilon$. Then we have

$$\begin{aligned} & f(d(x_N, x_{N+p}), d(x_N, x_{N+1}), d(x_{N+p}, x_{N+p+1})) \\ & \leq d(x_N, x_{N+p}) + d(x_N, x_{N+1}) + d(x_{N+p}, x_{N+p+1}) \\ & < \epsilon + \frac{\delta}{3} + \frac{\delta}{3} \\ & < \epsilon + \delta. \end{aligned}$$

Therefore, $d(x_{N+1}, x_{N+p+1}) \leq \frac{\epsilon}{3}$. So we have

$$\begin{aligned} & d(x_N, x_{N+p+1}) \\ & \leq d(x_N, x_{N+1}) + d(x_{N+1}, x_{N+p+1}) \\ & < \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ & < \epsilon. \end{aligned}$$

Hence, (3.2) is true for $p + 1$, Thus (3.2) holds for all $p \geq 1$. In a similar manner we can obtain that

$$d(x_n, x_{n+p}) < \epsilon,$$

for all $n \geq N$ and $p \geq 1$. Therefore, $\{x_n\}$ is a Cauchy sequence in a complete metric space (X, d) and hence converges to some $z \in X$.

Suppose that T admits the following types of continuity, respectively.

Case 1. T is orbitally continuous. Since $\{x_n\}$ converges to z, orbital continuity implies that $Tx_n \to Tz$. This yields $Tz = z$, since $T^n x_0 \to z$. Therefore, z is a fixed point of T.

Case 2. T is k-continuous for some $k \in \mathbb{N}$. Since $T^{k-1}x_n \to z$, k-continuity of T implies that $T^k x_n \to Tz$. Hence $z = Tz$ as $T^k x_n \to z$. Therefore, z is a fixed point of T.

Case 3. T^k is continuous for some $k \in \mathbb{N}$. We have that $\lim_{n\to+\infty} T^k x_n = T^k z$ which yields $T^k z = z$ as $x_n \to T^k z$. If $Tz \neq z$, then $T^{k-1}z \neq z$. So we have

$$\begin{aligned} \psi(d(T^k x_n, Tz)) &= \psi(d(TT^{k-1}x_n, z)) \\ &\leq \varphi(f(d(T^{k-1}x_n, z), d(T^{k-1}x_n, T^k x_n), d(z, Tz))) \\ &< \psi(f(d(T^{k-1}x_n, z), d(T^{k-1}x_n, T^k x_n), d(z, Tz))). \end{aligned}$$

Using the monotonicity of ψ and $(\mathcal{A}_2)$ we also get

$$d(T^k x_n, Tz) < f(d(T^{k-1}x_n, z), d(T^{k-1}x_n, T^k x_n), d(z, Tz)),$$

which leads to $d(Tz, z) < d(Tz, z)$ by taking limits as $k \to +\infty$, a contradiction. So we must have $Tz = z$, i.e., z is a fixed point of T.

For the uniqueness of the common fixed point, let z' be another common fixed point of T and S, that is, $z' = Tz' = Sz'$. Then we have

$$\psi(d(Tz, Sz')) \leq \varphi(m(z, z'))$$

$$= \varphi(f(d(z,z'), d(z,Tz), d(z',Sz')))$$
$$< \psi(f(d(z,z'), d(z,Tz), d(z',Tz')))$$
$$= \psi(f(d(z,z'), 0, 0)).$$

It follows from the monotonicity of ψ that

$$d(z,z') < f(d(z,z'), 0, 0).$$

By $(\mathcal{A}_3)$, we have

$$d(z,z') \geq f(d(z,z'), 0, 0),$$

which contradicts to above inequality. So $z = z'$.
The remaining parts of the proof of this theorem is similar to that of Theorem 3.1 and so is omitted.

Theorem 3.4. Let (X, d) be a complete metric space and a pair (T, S) of self-mappings be an (ψ, φ)-$\mathcal{A}$-contraction such that either T and S are orbitally continuous or k-continuous or T^k is continuous for some $k \in \mathbb{N}$. Also, assume that ψ is nondecreasing and f satisfies the following conditions:
(i) for any $\epsilon > 0$, there exists an $\delta > 0$ such that

$$f(d(x,y), d(x,Tx), d(y,Sy)) < \epsilon + \delta \Longrightarrow d(Tx,Sy) \leq \frac{\epsilon}{3},$$

for all $x, y \in X$.
(ii) $f(0,0,u) = 0$ implies $u = 0$.
Then we have the followings assertions:
(1) T and S has a unique common fixed point z.
(2) The sequences $\{T^n x_0\}$ and $\{S^n x_0\}$ of iterates converge to z for each $x_0 \in X$.
(3) Moreover, T is continuous at z if and only if $\lim_{x\to z} M(x,z) = 0$ and S is continuous at z if and only if $\lim_{x\to z} M(z,y) = 0$, where $M(x,z) = f(d(x,z), d(x,Tx), d(z,Sz))$, and $M(z,y) = f(d(z,y), d(z,Tz), d(y,Sy))$.

Proof.(1) Let $x_0 \in X$ be an arbitrary point and define a sequence $\{x_n\} \subseteq X$ by $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for $n \in \mathbb{N} \cup \{0\}$. Let $\alpha_n = d(x_n, x_{n+1})$, for $n \in \mathbb{N} \cup \{0\}$. Analysis similar to the proof in Theorem 3.2 shows that $\{\alpha_n\}$ is a decreasing sequence of nonnegative real numbers and hence converges to some nonnegative real number $r \geq 0$.
We claim that $r = 0$. If not, by assumption (i), there exists an δ such that

$$f(d(x,y), d(x,Tx), d(y,Ty)) < 3r + \delta \Rightarrow d(Tx,Ty) \leq r,$$

for all $x, y \in X$. Since $\{\alpha_n\}$ converges to r, so does $\{\alpha_{2n}\}$. For the above δ, there exists an $N \in \mathbb{N}$ such that for $n \geq N$,

$$\alpha_{2n} < r + \frac{\delta}{3},$$

that is,

$$d(x_{2n}, x_{2n+1}) < r + \frac{\delta}{3}.$$

So, together with $(\mathcal{A}_3)$, we have

$$\begin{aligned}
&f(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}))\\
&\leq d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\\
&= \alpha_{2n} + \alpha_{2n} + \alpha_{2n+1}\\
&< 3\alpha_{2n}\\
&< 3r + \delta.
\end{aligned}$$

Therefore, $d(x_{2n+1}, x_{2n+2}) \leq r$, that is $\alpha_{2n+1} \leq r$. But this contradicts to the fact that $\{\alpha_n\}$ converges to r and we must have $r = 0$, that is, $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$.
Next, we will show that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$ be arbitrary. It follows from the condition (i) that there exits an $\delta > 0$ such that

$$f(d(x, y), d(x, Tx), d(y, Sy)) < \epsilon + \delta \Rightarrow d(Tx, Sy) \leq \frac{\epsilon}{3},$$

for all $x, y \in X$. Without loss of generality, we assume that $\delta < \epsilon$. Since $\{\alpha_n\}$ converges to 0, so does $\{\alpha_{2n}\}$. Then there exists an $N \in \mathbb{N}$ such that

$$d(x_{2n}, x_{2n+1}) < \frac{\delta}{3} < \frac{\epsilon}{3} < \epsilon,$$

for all $2n \geq N$. By induction on p, we will show that

$$d(x_{2N}, x_{2N+p}) < \epsilon, \tag{3.3}$$

for all $p \in \mathbb{N}$. Clearly, (3.3) holds true for $p = 1$. Suppose that (3.3) is true for p, that is, $d(x_{2N}, x_{2N+p}) < \epsilon$. Then we have

$$\begin{aligned}
&f(d(x_{2N}, x_{2N+p}), d(x_{2N}, x_{2N+1}), d(x_{2N+p}, x_{2N+p+1}))\\
&\leq d(x_{2N}, x_{2N+p}) + d(x_{2N}, x_{2N+1}) + d(x_{2N+p}, x_{2N+p+1})\\
&< \epsilon + \frac{\delta}{3} + \frac{\delta}{3}\\
&< \epsilon + \delta.
\end{aligned}$$

Therefore, $d(x_{2N+1}, x_{2N+p+1}) \leq \frac{\epsilon}{3}$. So we have

$$\begin{aligned}
&d(x_{2N}, x_{2N+p+1})\\
&\leq d(x_{2N}, x_{2N+1}) + d(x_{2N+1}, x_{2N+p+1})\\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3}\\
&< \epsilon.
\end{aligned}$$

Hence, (3.3) is true for $p + 1$, Thus (3.3) holds for all $p \geq 1$. Moreover, we can obtain that

$$d(x_{2n}, x_{2n+p}) < \epsilon,$$

for all $2n \geq N$ and $p \geq 1$.
Using the same argument, we also have

$$d(x_{2n+1}, x_{2n+1+p}) < \epsilon,$$

for all $2n + 1 \geq N$ and $p \geq 1$.
Therefore, $\{x_n\}$ is a Cauchy sequence in a complete metric space (X, d) and hence converges to some $z \in X$.
Suppose that T admits the following types of continuity, respectively.
Case 1. T is orbitally continuous. Since $\{x_n\}$ converges to z, orbital continuity implies that $Tx_n \to Tz$. This yields $Tz = z$, since $T^n x_0 \to z$. Therefore, z is a fixed point of T.
Case 2. T is k-continuous for some $k \in \mathbb{N}$. Since $T^{k-1}x_n \to z$, k-continuity of T implies that $T^k x_n \to Tz$. Hence $z = Tz$ as $T^k x_n \to z$. Therefore, z is a fixed point of T.
Case 3. T^k is continuous for some $k \in \mathbb{N}$, then $\lim_{n\to+\infty} T^k x_n = T^k z$. This yields $T^k z = z$ as $T^k x_n \to z$. If $Tz \neq z$, then $T^{k-1}z \neq z$. So we have

$$\begin{aligned}
\psi(d(T^k x_n, Sz)) &= \psi(d(TT^{k-1}x_n, Sz)) \\
&\leq \varphi(f(d(T^{k-1}x_n, z), d(T^{k-1}x_n, T^k x_n), d(z, Sz))) \\
&< \psi(f(d(T^{k-1}x_n, z), d(T^{k-1}x_n, T^k x_n), d(z, Sz))).
\end{aligned}$$

Using the monotonicity of ψ, we also get

$$d(T^k x_n, Sz) < f(d(T^{k-1}x_n, z), d(T^{k-1}x_n, T^k x_n), d(z, Sz)).$$

Taking limit in the above inequality as $n \to +\infty$, we have

$$d(z, Sz) < f(0, 0, d(z, Sz)).$$

which leads to $d(z, Sz) < d(z, Sz)$ by taking limits as $k \to +\infty$, a contradiction. So we must have $Sz = z$, i.e., z is a fixed point of S.
Using the same manner, we can obtain that z is a fixed point of T.
Hence, z is a common fixed point of T and S.
The remaining parts of the proof of this theorem is similar to that of Theorem 3.2 and so is omitted.
Remark 3.1. From Proposition 2.1, we can obtain that the T-orbital lower semi-continuity of $f(x) = d(x, Tx)$ can be deduced from the orbital continuity or k-continuity of T for all $k \neq 1$. In such a case, if $\{x_n\} \subset O_x(T)$ and $x_n \to z$ satisfying $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, by the T-orbital lower semi-continuity of $f(x) = d(x, Tx)$, one has

$$d(z, Tz) \leq \liminf_{n\to\infty} d(x_n, Tx_n) = 0,$$

which implies that $Tz = z$, that is z is a fixed point of T.
Corollary 3.1. Replacing the orbital continuity of T (or both T and S) by the T-orbital lower semi-continuity of $f(x) = d(x, Tx)$ (or both $f(x) = d(x, Tx)$ and $g(x) = d(x, Sx)$) in Theorem 3.1 (or Theorem 3.2), the conclusion remains true.

Corollary 3.2. Replacing the orbital continuity or k-continuity of T (or both T and S) by the T-orbital lower semi-continuity of $f(x) = d(x, Tx)$ (or both $f(x) = d(x, Tx)$ and $g(x) = d(x, Sx)$) in Theorem 3.3 (or Theorem 3.4), the conclusion remains true.
Define $m_i(x, y), i = 1, 2, 3, 4$ as follows:

$$m_1(x, y) = \max\{d(x, Tx), d(y, Ty)\}. \quad m_2(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$
$$m_3(x, y) = \max\{d(x, Tx), d(y, Sy)\}. \quad m_4(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy)\}.$$

Define $\varphi(t) = \psi(t) - \tau, \tau > 0$ or $\psi(t) = t$ for $t > 0$. We obtain the following corollaries.
Corollary 3.3. Let (X, d) be a complete metric space and $T : X \mapsto X$ either T be orbitally continuous or k-continuous or T^k continuous for some $k \in \mathbb{N}$ and satisfy the following conditions:
(i) for any $\epsilon > 0$, there exists an $\delta > 0$ such that

$$m_i(x, y) < \epsilon + \delta \Longrightarrow d(Tx, Ty) \leq \frac{\epsilon}{3}, i = 1, 2$$

for all $x, y \in X$.
(ii) $d(Tx, Ty) \leq \varphi(m_i(x, y)), i = 1, 2$ for all $x, y \in X$ with $d(Tx, Ty) > 0$, where $\varphi(t) < t$, for $t > 0$.
Then T admits a unique fixed point z and the sequence $\{T^n x_0\}$ is convergent to z for every $x_0 \in X$. Moreover, T is continuous at z if and only if $\lim_{x \to z} m_i(x, z) = 0$, $i = 1, 2$.
Corollary 3.4. Let (X, d) be a complete metric space and a pair (T, S) of self-mappings either T and S be orbitally continuous or k-continuous or T^k is continuous for some $k \in \mathbb{N}$ and satisfy the following assumptions:
(i) for any $\epsilon > 0$, there exists an $\delta > 0$ such that

$$m_i(x, y) < \epsilon + \delta \Longrightarrow d(Tx, Sy) \leq \frac{\epsilon}{3}, i = 3, 4$$

for all $x, y \in X$.
(ii) $\psi(d(Tx, Sy)) \leq \psi(m_i(x, y)) - \tau, i = 3, 4$ for all $x, y \in X$ with $d(Tx, Sy) > 0$, where $\varphi(t) < t$, for $t > 0$.
Then T and S have a unique common fixed point z and the sequences $\{T^n x_0\}$ and $\{S^n x_0\}$ are convergent to z for every $x_0 \in X$. Moreover, T and S are continuous at z if and only if $\lim_{x \to z} m_i(x, z) = 0$ and $\lim_{x \to z} m_i(z, y) = 0$, $i = 3, 4$, respectively.

3.2. New fixed point results via (ψ, φ)-$\mathcal{A}'$-contractions

We now obtain some fixed point and common fixed theorems concerning the (ψ, φ)-$\mathcal{A}'$-contraction in compact metric spaces and complete metric spaces.
Theorem 3.5. Let (X, d) be a compact metric space and $T : X \mapsto X$ be a (ψ, φ)-$\mathcal{A}'$-contraction such that T is orbitally continuous. Also, assume that ψ is nondecreasing. Then we have the following assertions:
(i) T has a unique fixed point z.
(ii) If $u > f(u, 0, 0)$ for all $u > 0$, the sequence $\{T^n x_0\}$ converges to the fixed point z for every $x_0 \in X$.
(iii) Further, if $f(0, u, 0) = 0$ implies $u = 0$, then T is continuous at the fixed point z if and only if $\lim_{x \to z} m'(z, x) = 0$, where

$$m'(z, x) = f(d(z, x), d(z, Tx), d(x, Tz)).$$

Proof. (1) Let $x_0 \in X$ be an arbitrary point and define a sequence $\{x_n\} \subseteq X$ by $x_{n+1} = Tx_n = T^n x_0$ for all $n \in \mathbb{N} \cup \{0\}$. Taking $\alpha_n = d(x_n, x_{n+1})$, for all $n \in \mathbb{N} \cup \{0\}$.
Now, we prove that $\{\alpha_n\}$ converges to 0.
It is trivial, if $\alpha_n = 0$ for some $n \in \mathbb{N} \cup \{0\}$. Suppose that $\alpha_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$.
Using (2.3), with $x = x_n, y = x_{n+1}$, we have

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &= \psi(d(Tx_{n-1}, Tx_n))\\ &\leq \varphi(m'(x_{n-1}, x_n))\\ &= \varphi(f(d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1}), d(x_n, x_n)))\\ &< \psi(f(d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1}), 0)).\end{aligned}$$

From the monotonicity of ψ and $(\mathcal{A}'_3)$, $(\mathcal{A}'_4)$, we have that

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &< \psi(f(d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1}), 0))\\ &\leq \psi(f(d(x_{n-1}, x_n), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0))\\ &\Rightarrow d(x_n, x_{n+1}) < f(d(x_{n-1}, x_n), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0)\\ &\Rightarrow d(x_n, x_{n+1}) < d(x_{n-1}, x_n).\end{aligned}$$

That is

$$\alpha_n = d(x_n, x_{n+1}) < d(x_{n-1}, x_n) = \alpha_{n-1},$$

which shows that $\{\alpha_n\}$ is a decreasing sequence of nonnegative real numbers and hence converges to some nonnegative real number $r \geq 0$.
Again, since X is compact, there exists a convergent subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ and let $\lim_{k\to+\infty} x_{n_k} = z$.
Further, by the orbital continuity of T, we have

$$r = \lim_{k\to+\infty} d(x_{n_k}, x_{n_k+1}) = d(z, Tz).$$

Again, we have

$$r = \lim_{k\to+\infty} d(x_{n_k+1}, x_{n_k+2}) = d(Tz, T^2 z).$$

If $r > 0$, then $z \neq Tz$, from (2.3) and $(\mathcal{A}'_4)$, we have

$$\begin{aligned}d(Tz, T^2 z) &< f(d(z, Tz), d(z, T^2 z), d(Tz, Tz))\\ &\leq f(d(z, Tz), d(z, Tz) + d(Tz, T^2 z), 0)\\ &\Rightarrow d(Tz, T^2 z) < d(z, Tz)\\ &\Rightarrow r < r,\end{aligned}$$

which is a contradiction. So we have $r = 0$ and z is a fixed point of T.
Next, we will prove the uniqueness of the fixed point. For this, let z' be another fixed point of T. Then we have

$$\psi(d(Tz, Tz')) \leq \varphi(m'(z, z'))$$

$$= \varphi(f(d(z,z'),d(z,Tz'),d(z',Tz)))$$
$$< \psi(f(d(z,z'),d(z,Tz'),d(z',Tz))),$$

which implies that

$$d(z,z') = d(Tz,Tz') < f(d(z,z'),d(z,Tz'),d(z',Tz)). \tag{3.4}$$

By $(\mathcal{A}'_5)$, we also have

$$d(z,z') \geq f(d(z,z'),d(z,z'),d(z,z')),$$

which contradicts (3.4). So $z = z'$.

(2) Next, we consider the sequence of real numbers $\{s_n\}$ where $s_n = d(z, x_n)$. We will show that $\{s_n\}$ converges to 0. If $x_n = z$ for some $n \in \mathbb{N}$, then $\{s_n\}$ converges to 0. So, we assume that $x_n \neq z$ for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned}\psi(s_{n+1}) &= \psi(d(Tx_n,Tz)\\ &\leq \varphi(m'(x_n,z))\\ &= \varphi(f(d(x_n,z),d(x_n,Tz),d(z,Tx_n)))\\ &< \psi(f(d(x_n,z),d(x_n,Tz),d(z,Tx_n)))\\ &= \psi(f(s_n,s_n,s_{n+1})).\end{aligned}$$

Hence, we have $s_{n+1} < s_n$, and this is true for all natural numbers n. Then $\{s_n\}$ is a decreasing sequence of real numbers. Also, since $\{x_{n_k}\}$ converges to z, it follows that $\{s_{n_k}\}$ converges to 0. Therefore, $\{s_n\}$ must converges to 0, that is, $\{x_n\}$ converges to the fixed point z.

(3) Next, we assume that $f(0,u,0) = 0$ implies $u = 0$. Let T be continuous at the fixed point z. To show that $\lim\limits_{x\to z} m'(x,z) = 0$, let $\{y_n\}$ be a sequence in X converging to z. Then

$$\begin{aligned}\lim_{n\to+\infty} m'(z,y_n) &= \lim_{n\to+\infty} f(d(z,y_n),d(z,Ty_n),d(y_n,Tz))\\ &= f(0,0,0) \leq 0.\end{aligned}$$

Therefore, $\lim\limits_{x\to z} m'(z,x) = 0$.
Conversely, let $\lim\limits_{x\to z} m'(z,x) = 0$. To prove T is continuous at the fixed point z, let $\{y_n\}$ be a sequence in X converging to z. Therefore, we have

$$\begin{aligned}&\lim_{n\to+\infty} m'(z,y_n) = 0\\ \Longrightarrow\ &\lim_{n\to+\infty} f(d(z,y_n),d(z,Ty_n),d(y_n,z)) = 0\\ \Longrightarrow\ &f(0,\lim_{n\to+\infty} d(y_n,Ty_n),0) = 0\\ \Longrightarrow\ &\lim_{n\to+\infty} d(y_n,Ty_n) = 0\\ \Longrightarrow\ &\lim_{n\to+\infty} Ty_n = \lim_{n\to+\infty} y_n = z = Tz.\end{aligned}$$

So T is continuous at the fixed point z.

The following example illustrates Theorem 3.5.

Example 3.5. Let $X = [-1, 1]$ and d be the usual metric on X. Consider the self-mapping $T : X \to X$ defined by

$$Tx = \begin{cases} -\frac{1}{4}x & ; \quad 0 < x \leq 1 \\ 0 & ; \quad -1 \leq x \leq 0 \end{cases}$$

T satisfies the conditions of Theorem 3.5 with the functions $f(u, v, w) = \frac{v+w}{2}$ defined for all $u, v, w \in \mathbb{R}$, $\psi(t) = \frac{t}{2}$ and $\varphi(t) = \frac{t}{4}$ $(t > 0)$. The point 0 is the unique fixed point of the orbitally continuous (ψ, φ)-$\mathcal{A}'$-contractive mapping T. Since $f(0, u, 0) = 0$ implies $u = 0$, we can check the continuity of T by calculating the limit $\lim\limits_{x\to 0} m'(0, x)$. We have

$$\begin{aligned} \lim_{x\to 0^+} m'(0, x) &= \lim_{x\to 0^+} f(d(0, x), d(0, Tx), d(x, T0)) \\ &= \lim_{x\to 0^+} f(|x|, \frac{|x|}{4}, |x|) = \lim_{x\to 0^+} \frac{5|x|}{8} = 0 \end{aligned}$$

and

$$\lim_{x\to 0^-} m'(0, x) = \lim_{x\to 0^-} f(|x|, 0, |x|) = \lim_{x\to 0^+} \frac{|x|}{2} = 0.$$

Thus, we obtain $\lim\limits_{x\to 0^+} m'(0, x) = 0$ and this shows that T is continuous at the fixed point 0.

Theorem 3.6. Let (X, d) be a compact metric space and a pair (T, S) of self-mappings on X be an (ψ, φ)-$\mathcal{A}'$-contraction such that T and S are orbitally continuous. Also, assume that ψ is nondecreasing. Then we have the following assertions:

(i) T and S have a unique common fixed point $z \in X$.

(ii) The sequences $\{T^n x_0\}$ and $\{S^n x_0\}$ of iterates converge to that fixed point for each $x_0 \in X$.

(iii) Further, if $f(0, 0, u) = 0$ implies $u = 0$, then T is continuous at the fixed point z if and only if $\lim\limits_{x\to z} M'(x, z) = 0$. Also, if $f(0, 0, u) = 0$ implies $u = 0$, then S is continuous at the fixed point z if and only if $\lim\limits_{y\to z} M'(z, y) = 0$, where

$M'(x, z) = f(d(x, z), d(x, Sz), d(z, Tx))$ and $M'(z, y) = f(d(z, y), d(z, Sy), d(y, Tz))$.

Proof. (1) Let $x_0 \in X$ be an arbitrary point. We define a sequence $\{x_n\} \subseteq X$ such that $x_{2n+1} = Tx_{2n}$, $x_{2n+2} = Sx_{2n+1}$, for $n \in \mathbb{N} \cup \{0\}$. Let $\alpha_n = d(x_n, x_{n+1})$, for $n \in \mathbb{N} \cup \{0\}$.

Now, we will prove that $\{\alpha_n\}$ converges to 0.

It is trivial if $\alpha_n = 0$ for some $n \in \mathbb{N} \cup \{0\}$. Suppose now that $\alpha_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$.

Using (2.4), with $x = x_{2n}, y = x_{2n+1}$, we have

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n+2})) &= \psi(d(Tx_{2n}, Sx_{2n+1})) \\ &\leq \varphi(M'(x_{2n}, x_{2n+1})) \\ &= \varphi(f(d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n+1}), d(x_{2n+1}, Tx_{2n}))) \\ &= \varphi(f(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1}))) \\ &= \varphi(f(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+2}), 0)) \\ &< \psi(f(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+2}), 0)). \end{aligned}$$

From the monotonicity of ψ and $(\mathcal{A}'_4)$, we have

$$d(x_{2n+1}, x_{2n+2}) < f(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}), 0).$$

By $(\mathcal{A}'_3)$, we have

$$\alpha_{2n+1} = d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1}) = \alpha_{2n}.$$

Using the similar arguments, we can also obtain that $\alpha_{2n} < \alpha_{2n-1}$.
Thus, $\{\alpha_n\}$ is a decreasing sequence of nonnegative real numbers and hence converges to some nonnegative real number $r \geq 0$.
Again, since X is compact, there exists a convergent subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ and let $\lim_{k\to+\infty} x_{n_k} = z$.
Further, by the orbital continuity of T, we have

$$r = \lim_{k\to+\infty} d(x_{n_k}, Sx_{n_k}) = d(z, Tz),$$

where $n_k = 2j$, $j \in \mathbb{N}$. If $r > 0$, then $z \neq Tz$, from (2.4), we have

$$\begin{aligned}
\psi(d(Tz, Sx_{2j+1})) &< \varphi(M'(z, x_{2j+1})) \\
&= \varphi(f(d(z, x_{2j+1}), d(z, Sx_{2j+1}), d(x_{2j+1}, Tz))) \\
&< \psi(f(d(z, x_{2j+1}), d(z, x_{2j+2}), d(x_{2j+1}, Tz))).
\end{aligned}$$

From the monotonicity of ψ, we have

$$d(Tz, Sx_{2j+1}) < f(d(z, x_{2j+1}), d(z, x_{2j+2}), d(x_{2j+1}, Tz)).$$

Taking limits as $j \to +\infty$ in the above inequality, we have

$$d(Tz, z) < f(0, 0, d(z, Tz)),$$

which implies that $d(Tz, z) < 0$, a contradiction. Hence, $r = 0$ and z is a fixed point of T.
Using the same manner in the case that S is orbitally continuous, we can conclude that z is a fixed point of S. Therefore, z is a common fixed point of T and S.
Next, we will prove the uniqueness of the common fixed point. For this, let z' be another common fixed point of T and S, that is, $z' = Tz' = Sz'$. Then we have

$$\begin{aligned}
\psi(d(Tz, Sz')) &\leq \varphi(M'(z, z')) \\
&= \varphi(f(d(z, z'), d(z, Sz'), d(z', Tz))) \\
&< \psi(f(d(z, z'), d(z, Sz'), d(z', Tz))) \\
&= \psi(f(d(z, z'), d(z, z'), d(z, z'))).
\end{aligned}$$

It follows from the monotonicity of ψ that

$$d(z, z') < f(d(z, z'), d(z, z'), d(z, z')).$$

By $(\mathcal{A}'_5)$, we have

$$f(d(z, z'), d(z, z'), d(z, z') \leq d(z, z'),$$

which contradicts to above inequality. So $z = z'$.

(2) We consider the sequence of real numbers $\{s_n\}$ where $s_n = d(z, x_n)$. Define a function $g(x) = d(z, x)$ for all $x \in X$. Clearly, g is continuous on X, and hence $g(X)$ is bounded. Thus, $\{s_n\}$ is a bounded sequence of a real numbers. Since the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to z, we get that

$$\lim_{k\to+\infty} d(z, x_{n_k}) = 0,$$

i.e., $\lim_{k\to+\infty} s_{n_k} = 0$. Thus 0 is a cluster point of the sequence $\{s_n\}$. Let c be any cluster point of $\{s_n\}$. Then there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $s_{n_i} \to c$. So $d(z, x_{n_i}) \to c$ as $i \to +\infty$. Therefore, we have

$$\begin{aligned} |s_{n_i+1} - s_{n_i}| &= |d(x_{n_i+1}, z) - d(x_{n_i}, z)| \\ &\leq d(x_{n_i+1}, x_{n_i}) \to 0, \end{aligned}$$

as $i \to +\infty$ and hence $\lim_{i\to+\infty} s_{n_i+1} = \lim_{i\to+\infty} s_{n_i}$.
We now prove that $c = 0$. If $c > 0$, then $\lim_{i\to+\infty} d(z, x_{n_i}) > 0$ and so we may assume that $x_{n_i} \neq z$ for all $i \geq 1$. Then, for all $n_i = 2j, i \geq 1, j \in \mathbb{N}$, we have

$$\begin{aligned} \psi(d(Tx_{2j}, Sz)) &\leq \varphi(M'(x_{2j}, z)) \\ &= \varphi(f(d(x_{2j}, z), d(x_{2j}, Sz), d(z, Tx_{2j}))) \\ &< \psi(f(d(x_{2j}, z), d(x_{2j}, Sz), d(z, Tx_{2j}))), \end{aligned}$$

which implies that

$$s_{2j+1} = d(Tx_{2j}, Sz) < f(d(x_{2j}, z), d(x_{2j}, Sz), d(z, Tx_{2j})).$$

Taking limits as $j \to +\infty$ in the above inequality, we have

$$c < f(c, c, c),$$

which contradicts to $(\mathcal{A}'_5)$. So, $c = 0$. Therefore, 0 is the only cluster point of the bounded sequence $\{s_n\}$ and so this sequence also converges to 0. Hence $\{x_n\}$ converges to z. Since x_0 is arbitrary point in X, it follows that $\{T^n x_0\}$ converges to the fixed point z for each $x_0 \in X$.
Using the similar arguments as mentioned above, we can also obtain that $\{S^n x_0\}$ converges to the fixed point z for each $x_0 \in X$.

(3) Next, we assume that $f(0, 0, u) = 0$ implies $u = 0$. Let T be continuous at the fixed point z. To show $\lim_{x\to z} M'(x, z) = 0$, let $\{t_n\}$ be a sequence in X converging to z. Then

$$\begin{aligned} \lim_{n\to+\infty} M'(t_n, z) &= \lim_{n\to+\infty} f(d(t_n, z), d(t_n, Sz), d(z, Tt_n)) \\ &= f(0, 0, d(z, Tz)) = 0. \end{aligned}$$

Therefore, $\lim_{x\to z} M'(x, z) = 0$.
Conversely, let $\lim_{x\to z} M'(x, z) = 0$. To prove T is continuous at the fixed point z, let $\{t_n\}$ be a sequence in X converging to z. Therefore, we have

$$\lim_{n\to+\infty} M'(t_n, z) = 0$$

$$\Longrightarrow \lim_{n\to+\infty} f(d(t_n, z), d(t_n, Sz), d(z, Tt_n)) = 0$$
$$\Longrightarrow f(0, 0, \lim_{n\to+\infty} d(z, Tt_n)) = 0$$
$$\Longrightarrow \lim_{n\to+\infty} d(z, Tt_n) = 0$$
$$\Longrightarrow \lim_{n\to+\infty} Tt_n = \lim_{n\to+\infty} t_n = z = Tz.$$

So T is continuous at the fixed point z. The same conclusion can be drawn for S by using similar argument.

Example 3.6. Let $X = [-1, 1]$ and d be the usual metric on X and the self-mapping T considered in Example 3.5. Define the self-mapping $S : X \to X$ as

$$Sx = \begin{cases} 0 & ; \quad 0 < x \leq 1 \\ -\frac{1}{4}x & ; \quad -1 \leq x \leq 0 \end{cases}$$

Then the pair (T, S) satisfies the conditions of Theorem 3.6 with the functions $f(u, v, w) = \frac{v+w}{2}$ defined for all $u, v, w \in \mathbb{R}$, $\psi(t) = \frac{1}{2}t$ and $\varphi(t) = \frac{1}{4}t$ $(t > 0)$. Clearly, 0 is the unique common fixed point of the orbitally continuous (ψ, φ)-$\mathcal{A}'$-contractive mappings T and S. Notice that both of the self-mappings T and S are continuous at the common fixed point 0.

Now, we show that the compactness hypothesis of X in Theorem 3.5 can not be replaced by completeness. The following example illustrates this fact.

Example 3.7. Let us consider $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ and define a metric $d : X \times X \mapsto X$ on X by

$$d(x, y) = \begin{cases} 0 & ; \quad x = y \\ e^{xy} & ; \quad x \neq y \end{cases}.$$

It is easy to check that (X, d) is a complete but not-compact metric space.

Define the self-mapping $T : X \mapsto X$ by $T(\frac{1}{n}) = \frac{1}{4n}$, for $n \in \mathbb{N}$. Consider the functions $f \in \mathcal{A}'$ where $f(u, v, w) = \frac{1}{2}\max\{u, v, w\}$, for all $u, v, w \in \mathbb{R}^+$, and define

$$\psi(t) = \begin{cases} \ln t & ; \quad t > 1 \\ \frac{1}{2}\ln t & ; \quad 0 < t \leq 1 \end{cases}$$

and

$$\varphi(t) = \begin{cases} \frac{1}{2}\ln t & ; \quad t > 1 \\ \ln t & ; \quad 0 < t \leq 1 \end{cases}$$

Let $x, y \in X$ be arbitrary with $x \neq y$ and take $x = \frac{1}{n}$, $y = \frac{1}{m}$ with $m \neq n$. Therefore, we have

$$\psi(d(Tx, Ty)) = \ln[e^{\frac{1}{4n}\times\frac{1}{4m}})] = \frac{1}{16mn},$$

and

$$\begin{aligned}\varphi(f(d(x, y), d(x, Ty), d(y, Tx))) &= \ln[2 \times \frac{1}{2}\max\{e^{\frac{1}{mn}}, e^{\frac{1}{4nm}}, e^{\frac{1}{4mn}}\}] \\ &= \frac{1}{mn}.\end{aligned}$$

Thus, it is easy to check that $\psi(d(Tx, Ty)) \leq \varphi(m'(x, y))$.
Then it is easy to verify that T is an orbitally continuous (ψ, φ)-$\mathcal{A}'$-contractive mapping. Clearly, T is fixed point free.

Next, we add some conditions on $f \in \mathcal{A}'$ and/or T and S to obtain some fixed point and common fixed point theorems in the setting of complete metric spaces as follows.

Theorem 3.7. Let (X, d) be a complete metric space and $T : X \mapsto X$ be a (ψ, φ)-$\mathcal{A}'$-contraction. Also, assume that ψ is nondecreasing and f satisfies the following conditions:
(i) for any $\epsilon > 0$, there exists an $\delta > 0$ such that

$$f(d(x, y), d(x, Ty), d(y, Tx)) < \epsilon + \delta \Longrightarrow d(Tx, Ty) \leq \frac{\epsilon}{4},$$

for all $x, y \in X$.
(ii) $f(0, u, 0) = 0$ implies $u = 0$.
Then we have the following assertions:
(1) T has a unique fixed point z.
(2) The sequence $\{T^n x_0\}$ of iterates converges to z for each $x_0 \in X$.
(3) Moreover, T is continuous at z if and only if $\lim_{x \to z} m'(z, x) = 0$, where

$$m'(z, x) = f(d(z, x), d(z, Tx), d(x, Tz)).$$

Proof.(1) Let $x_0 \in X$ be an arbitrary point and define a sequence $\{x_n\} \subseteq X$ by $x_{n+1} = Tx_n = T^n x_0$ for $n \in \mathbb{N} \cup \{0\}$. Let $\alpha_n = d(x_n, x_{n+1})$, for $n \in \mathbb{N} \cup \{0\}$.
Now, we will prove that $\{\alpha_n\}$ converges to 0.
It is trivial, if $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$. Suppose that $\alpha_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$.
Analysis similar to that in the proof of Theorem 3.5, we can show that $\{\alpha_n\}$ is a decreasing sequence of nonnegative real numbers and hence converges to some nonnegative real number $r \geq 0$. If $r > 0$, then by assumption (i), there exists an δ such that

$$f(d(x, y), d(x, Ty), d(y, Tx)) < 3r + \delta \Rightarrow d(Tx, Ty) \leq \frac{3r}{4}.$$

Since $\{\alpha_n\}$ converges to r, for the above δ, there exists an $n \in \mathbb{N}$ such that

$$\alpha_n < r + \frac{\delta}{3},$$

that is,

$$d(x_n, x_{n+1}) < r + \frac{\delta}{3}.$$

Then, together with $(\mathcal{A}'_6)$, we have

$$\begin{aligned}
& f(d(x_n, x_{n+1}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})) \\
& \leq d(x_n, x_{n+1}) + d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1}) \\
& \leq d(x_n, x_{n+1}) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\
& = 2\alpha_n + \alpha_{n+1}
\end{aligned}$$

$$< 3\alpha_n$$
$$< 3r + \delta.$$

Therefore, $d(x_{n+1}, x_{n+2}) \leq \frac{3r}{4}$, that is $\alpha_{n+1} \leq \frac{3r}{4} < r$. But this contradicts the fact that $\{\alpha_n\}$ converges to r and we must have $r = 0$, that is, $\lim\limits_{n\to+\infty} d(x_n, x_{n+1}) = 0$.

Next, we will show that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$ be arbitrary. Then by the assumption (i), there exits an $\delta > 0$ such that

$$f(d(x,y), d(x,Ty), d(y,Tx)) < 3\epsilon + \delta \Rightarrow d(Tx,Ty) \leq \frac{3\epsilon}{4},$$

for all $x, y \in X$. Without loss of generality, we assume that $\delta < \epsilon$. Since $\lim\limits_{n\to+\infty} d(x_{n+1}, x_n) = 0$, for above δ, there exists an $N \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) < \frac{\delta}{4} < \frac{\epsilon}{4} < \epsilon,$$

for all $n \geq N$. By induction on p, we will show that

$$d(x_N, x_{N+p}) < \epsilon, \tag{3.5}$$

for all $p \in \mathbb{N}$. Clearly, (3.5) holds true for $p = 1$. Suppose that (3.5) is true for p, i.e. $d(x_N, x_{N+p}) < \epsilon$. Then we have

$$\begin{aligned}
& f(d(x_N, x_{N+p}), d(x_N, x_{N+p+1}), d(x_{N+p}, x_{N+1})\\
& \leq d(x_N, x_{N+p}) + d(x_N, x_{N+p+1}) + d(x_{N+p}, x_{N+1})\\
& \leq d(x_N, x_{N+p}) + d(x_N, x_{N+p}) + d(x_{N+p}, x_{N+p+1}) + d(x_{N+p}, x_N) + d(x_N, x_{N+1})\\
& < 3\epsilon + \frac{\delta}{4} + \frac{\delta}{4}\\
& < 3\epsilon + \delta.
\end{aligned}$$

Therefore, $d(x_{N+1}, x_{N+p+1}) \leq \frac{3\epsilon}{4}$. So we have

$$\begin{aligned}
& d(x_N, x_{N+p+1})\\
& \leq d(x_N, x_{N+1}) + d(x_{N+1}, x_{N+p+1})\\
& < \frac{\epsilon}{4} + \frac{3\epsilon}{4} = \epsilon.
\end{aligned}$$

Hence, (3.5) is true for $p + 1$. Thus (3.5) holds for all $p \geq 1$. In a similar manner we can obtain that

$$d(x_n, x_{n+p}) < \epsilon,$$

for all $n \geq N$ and $p \geq 1$. Therefore, $\{x_n\}$ is a Cauchy sequence in a complete metric space (X, d) and hence converges to some $z \in X$.
If $d(Tx_n, Tz) = 0$ for infinitely many values of n, then we have

$$d(z, Tz) \leq d(z, Tx_n) + d(Tx_n, Tz) = d(z, Tx_n) = d(z, x_{n+1}),$$

for these values of n. Taking limits as $n \to +\infty$, we have $d(z, Tz) \leq 0$, which implies that $d(z, Tz) = 0$. This means that $z = Tz$, that is z is a fixed point of T.
If $d(Tx_n, Tz) > 0$ holds for infinitely many values of n, applying (2.3) with $x = x_n, y = z$, we conclude that

$$\begin{aligned}\psi(d(Tx_n, Tz)) &\leq \varphi(m'(x_n, z)) \\ &= \varphi(f(d(x_n, z), d(x_n, Tz), d(z, Tx_n))) \\ &< \psi(f(d(x_n, z), d(x_n, Tz), d(z, Tx_n))),\end{aligned}$$

which together with the monotonicity of ψ implies that

$$0 < d(Tx_n, Tz) = d(x_{n+1}, Tz) < f(d(x_n, z), d(x_n, z), d(z, x_{n+1})).$$

Taking limits as $n \to +\infty$ in above inequality, we have

$$0 < d(z, Tz) < f(0, 0, 0) < 0,$$

which is a contradiction.
Hence, z is a fixed point of T.
The remaining parts of the proof of this theorem is similar to that of Theorem 3.5 and so is omitted.
Theorem 3.8. Let (X, d) be a complete metric space and a pair (T, S) of self-mappings be an (ψ, φ)-$\mathcal{A}'$-contraction such that either T and S are orbitally continuous or k-continuous or T^k is continuous for some $k \in \mathbb{N}$. Also, assume that ψ is nondecreasing and f satisfies the following conditions:
(i) for any $\epsilon > 0$, there exists an $\delta > 0$ such that

$$f(d(x, y), d(x, Sy), d(y, Tx)) < \epsilon + \delta \Longrightarrow d(Tx, Sy) \leq \frac{\epsilon}{4},$$

for all $x, y \in X$.
(ii) $f(0, 0, u) = 0$ implies $u = 0$.
Then we have the following assertions:
(1) T and S has a unique common fixed point z.
(2) The sequences $\{T^n x_0\}$ and $\{S^n x_0\}$ of iterates converge to z for each $x_0 \in X$.
(3) Moreover, T is continuous at z if and only if $\lim_{x\to z} M'(x, z) = 0$ and S is continuous at z if and only if $\lim_{x\to z} M'(z, y) = 0$, where $M'(x, z) = f(d(x, z), d(x, Sz), d(z, Tx))$ and $M'(z, y) = f(d(z, y), d(z, Sy), d(z, Tx))$.

Proof.(1) Let $x_0 \in X$ be an arbitrary point and define a sequence $\{x_n\} \subseteq X$ by $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Let $\alpha_n = d(x_n, x_{n+1})$, for all $n \in \mathbb{N} \cup \{0\}$. Analysis similar to the proof in Theorem 3.2 shows that $\{\alpha_n\}$ is a decreasing sequence of non-negative real numbers and hence converges to some nonnegative real number $r \geq 0$.
We claim that $r = 0$. If not, by assumption (i), there exists an δ such that

$$f(d(x, y), d(x, Sy), d(y, Tx)) < 3r + \delta \Rightarrow d(Tx, Sy) \leq \frac{3r}{4},$$

for all $x, y \in X$. Since $\{\alpha_n\}$ converges to r, so does $\{\alpha_{2n}\}$. For the above δ, there exists an $N \in \mathbb{N}$ such that for $n \geq N$,

$$\alpha_{2n} < r + \frac{\delta}{3},$$

that is,

$$d(x_{2n}, x_{2n+1}) < r + \frac{\delta}{3}.$$

So, together with $(\mathcal{A}'_3)$, we have

$$\begin{aligned}
& f(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})) \\
& = f(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+2}), 0) \\
& \leq f(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}), 0) \\
& \leq \alpha_{2n} + \alpha_{2n} + \alpha_{2n+1} \\
& < 3\alpha_{2n} \\
& < 3r + \delta.
\end{aligned}$$

Therefore, $d(x_{2n+1}, x_{2n+2}) \leq r$, that is $\alpha_{2n+1} \leq r$. But this contradicts the fact that $\{\alpha_n\}$ converges to r and we must have $r = 0$, that is, $\lim\limits_{n\to+\infty} d(x_n, x_{n+1}) = 0$.

Next, we will show that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$ be arbitrary. It follows from the condition (i) that there exits an $\delta > 0$ such that

$$f(d(x, y), d(x, Sy), d(y, Tx)) < \epsilon + \delta \Rightarrow d(Tx, Sy) \leq \frac{\epsilon}{4},$$

for all $x, y \in X$. Without loss of generality, we assume that $\delta < \epsilon$. Since $\{\alpha_n\}$ converges to 0, so does $\{\alpha_{2n}\}$. Then there exists an $N \in \mathbb{N}$ such that

$$d(x_{2n}, x_{2n+1}) < \frac{\delta}{3} < \frac{\epsilon}{3} < \epsilon,$$

for all $2n \geq N$. By induction on p, we will show that

$$d(x_{2N}, x_{2N+p}) < \epsilon, \tag{3.6}$$

for all $p \in \mathbb{N}$. Clearly, (3.6) holds true for $p = 1$. Suppose that (3.6) is true for p, that is, $d(x_{2N}, x_{2N+p}) < \epsilon$. Then we have

$$\begin{aligned}
& f(d(x_{2N}, x_{2N+p}), d(x_{2N}, x_{2N+1}), d(x_{2N+p}, x_{2N+p+1}) \\
& \leq d(x_{2N}, x_{2N+p}) + d(x_{2N}, x_{2N+1}) + d(x_{2N+p}, x_{2N+p+1}) \\
& < \epsilon + \frac{\delta}{3} + \frac{\delta}{3} \\
& < \epsilon + \delta.
\end{aligned}$$

Therefore, $d(x_{2N+1}, x_{2N+p+1}) \leq \frac{\epsilon}{4}$. So we have

$$\begin{aligned}
& d(x_{2N}, x_{2N+p+1}) \\
& \leq d(x_{2N}, x_{2N+1}) + d(x_{2N+1}, x_{2N+p+1}) \\
& < \frac{\epsilon}{3} + \frac{\epsilon}{4}
\end{aligned}$$

$$< \epsilon.$$

Hence, (3.6) is true for $p+1$, Thus (3.6) holds for all $p \geq 1$. Moreover, we can obtain that

$$d(x_{2n}, x_{2n+p}) < \epsilon,$$

for all $2n \geq N$ and $p \geq 1$.
Using the same argument, we also have

$$d(x_{2n+1}, x_{2n+1+p}) < \epsilon,$$

for all $2n+1 \geq N$ and $p \geq 1$.
Therefore, $\{x_n\}$ is a Cauchy sequence in a complete metric space (X, d) and hence converges to some $z \in X$.
Suppose that T admits the following types continuity, respectively.
Case 1. T is orbitally continuous. Since $\{x_n\}$ converges to z, orbital continuity implies that $Tx_n \to Tz$. This yields $Tz = z$, since $T^n x_0 \to z$. Therefore, z is a fixed point of T.
Case 2. T is k-continuous for some $k \in \mathbb{N}$. Since $T^{k-1}x_n \to z$, k-continuity of T implies that $T^k x_n \to Tz$. Hence $z = Tz$ as $T^k x_n \to z$. Therefore, z is a fixed point of T.
Case 3. T^k is continuous for some $k \in \mathbb{N}$, then $\lim_{n\to+\infty} T^k x_n = T^k z$. This yields $T^k z = z$ as $T^k x_n \to z$. If $Tz \neq z$, then $T^{k-1}z \neq z$. So we have

$$\begin{aligned}
\psi(d(T^k x_n, Sz)) &= \psi(d(TT^{k-1}x_n, Sz)) \\
&\leq \varphi(f(d(T^{k-1}x_n, z), d(T^{k-1}x_n, Sz), d(z, T^k x_n))) \\
&< \psi(f(d(T^{k-1}x_n, z), d(T^{k-1}x_n, Sz), d(z, T^k x_n))).
\end{aligned}$$

Using the monotonicity of ψ, we also get

$$d(T^k x_n, Sz) < f(d(T^{k-1}x_n, z), d(T^{k-1}x_n, Sz), d(z, T^k x_n)).$$

Taking limits in above inequality as $n \to +\infty$, we have

$$d(z, Sz) < f(0, d(z, Sz), 0).$$

which leads to $d(z, Sz) < 0$, a contradiction. So we must have $Sz = z$, i.e., z is a fixed point of S.
Using the same manner, we can obtain that z is a fixed point of T.
Hence, z is a common fixed point of T and S.
The remaining parts of the proof of this theorem is similar to that of Theorem 3.6 and so is omitted.
Corollary 3.5. Replacing the orbital continuity of T (or both T and S) by the T-orbital lower semi-continuity of $f(x) = d(x, Tx)$ (or both $f(x) = d(x, Tx)$ and $g(x) = d(x, Sx)$) in Theorem 3.5 (or Theorem 3.6), the conclusion remains true.
Corollary 3.6. Replacing the orbital continuity or k-continuity of T (or both T and S) by the T-orbital lower semi-continuity of $f(x) = d(x, Tx)$ (or both $f(x) = d(x, Tx)$ and $g(x) = d(x, Sx)$) in Theorem 3.7 (or Theorem 3.8), the conclusion remains true.

Define $m_i'(x, y), i = 1, 2, 3, 4$ as follows:

$$m_1'(x, y) = \frac{d(x, y) + d(x, Ty) + d(y, Tx)}{3}, \quad m_2'(x, y) = \frac{1}{2}\max\{d(x, y), d(x, Ty), d(y, Tx)\},$$

$$m_3'(x,y) = \frac{d(x,y)+d(x,Sy)+d(y,Tx)}{3}, \quad m_4'(x,y) = \frac{1}{2}\max\{d(x,y),d(x,Sy),d(y,Tx)\}.$$

Define $\varphi(t) = \psi(t) - \tau, \tau > 0$ or $\psi(t) = t$ for $t > 0$. We obtain the following corollaries.

Corollary 3.7. Let (X,d) be a complete metric space and $T : X \mapsto X$ be a mapping satisfying the following conditions:

(i) for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$m_i'(x,y) < \epsilon + \delta \Longrightarrow d(Tx,Ty) \le \frac{\epsilon}{4}, i = 1,2$$

for all $x, y \in X$.

(ii) $\psi(d(Tx,Ty)) \le \psi(m_i'(x,y)) - \tau, i = 1,2$ for all $x, y \in X$ with $d(Tx,Ty) > 0$, where $\tau > 0$ and $\psi : (0,+\infty) \mapsto \mathbb{R}$ is nondecreasing.

Then T admits a unique fixed point z and the sequence $\{T^n x_0\}$ is convergent to z for every $x_0 \in X$. Moreover, T is continuous at z if and only if $\lim\limits_{x\to z} m_i'(x,z) = 0$, $i = 1,2$.

Corollary 3.8. Let (X,d) be a complete metric space and $T : X \mapsto X$ be a mapping satisfying the following conditions:

(i) for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$m_i'(x,y) < \epsilon + \delta \Longrightarrow d(Tx,Sy) \le \frac{\epsilon}{4}, i = 3,4$$

for all $x, y \in X$.

(ii) $\psi(d(Tx,Sy)) \le \psi(m_i'(x,y)) - \tau, i = 3,4$ for all $x, y \in X$ with $d(Tx,Sy) > 0$, where $\varphi(t) < t$, for $t > 0$.

Then T and S have a unique common fixed point z and the sequences $\{T^n x_0\}$ and $\{S^n x_0\}$ are convergent to z for every $x_0 \in X$. Moreover, T and S are continuous at z if and only if $\lim\limits_{x\to z} m_i(x,z) = 0$ and $\lim\limits_{x\to z} m_i(z,y) = 0$, $i = 3,4$, respectively.

Finally, we will show that fixed point property for every self-mapping of X satisfying conditions of Theorem 3.3 or Theorem 3.7 implies completeness of X. There is, however an markable difference between the next theorems and similar theorems (e.g. Kirk [42], Brahmanical [43], Saluki [44]) giving expression of completeness in terms of fixed point property for contractive mappings. In [42–44], the contractive condition implies continuity at the fixed point. However, the next theorems establish that completeness of the space is equivalent to fixed point property for a large class of mappings including continuous as well as discontinuous mappings.

Theorem 3.9. Let (X,d) be a metric space. Suppose that every orbitally continuous or k-continuous or T^k continuous self-mapping T of X being an (ψ,φ)-$\mathcal{A}$-contraction with $\varphi(t) \le \psi(\frac{t}{3}), t > 0$ and ψ is nondecreasing as well as T satisfying assumption (i) of Theorem 3.3 has a fixed point. Then X is complete.

Proof. Suppose that all assumptions of Theorem 3.9 hold true. We will show that X is a complete metric space.

If X is not complete, then there exists a Cauchy sequence $S = \{u_n\} \subseteq X$, consisting distinct points which is not convergent.

Let $x \in X$ be given. Since x is not a limit of the sequence S, we have $d(x, S - \{x\}) > 0$ and there exists a least integer $N(x) \in \mathbb{N}$ such that $x \neq u_{N(x)}$.

Let define $T : X \mapsto X$ by $Tx = u_{N(x)}$. Then $T(x) \neq x$ for all $x \in X$.
From the definition of (ψ,φ)-$\mathcal{A}$-contraction and monotonicity of ψ, we have

$$\begin{aligned}\psi(d(Tx,Ty)) &< \varphi(m(x,y))\\ &= \varphi(f(d(x,y),d(x,Tx),d(y,Ty)))\\ &\leq \psi(\frac{f(d(x,y),d(x,Tx),d(y,Ty))}{3})\\ &\Rightarrow d(Tx,Ty) < \frac{f(d(x,y),d(x,Tx),d(y,Ty))}{3},\end{aligned}$$

which, in other words, shows that T satisfies assumption (i) of Theorem 3.3.
Since the range of T is contained in the non-convergent sequence S, there is no sequence $\{x_n\}$ in X violating the definitions of orbital continuity, 2-continuity and T^2 continuity. Thus, T satisfies all assumptions of Theorem 3.9, which does not admit a fixed point. This contradicts to the assumption that T has a fixed point. Hence, X is complete.

Theorem 3.10. Let (X,d) be a metric space. Suppose that every orbitally continuous or k- continuous or T^k continuous self-mapping T of X being an (ψ,φ)-$\mathcal{A}'$-contraction with $\varphi(t) \leq \psi(\frac{t}{4}), t > 0$ and ψ is nondecreasing as well as T satisfying assumption (i) of Theorem 3.7 has a fixed point. Then X is complete.

Proof. The same conclusion follows by the same method as in Theorem 3.9.

4. Conclusions

Some new solutions were given to the well known open problem raised by Kansan and B.E. Rhodes on the existence of general contractions which have fixed points, but do not force the continuity at the fixed point by introducing two new contractions called (ψ,φ)-$\mathcal{A}$-contraction and (ψ,φ)-$\mathcal{A}'$-contraction. By means of these notions, new fixed point (resp. common fixed point) theorems were proved. In all of the obtained results, the uniqueness of the fixed point (resp. common fixed point) was arisen. On the other hand, there are a lot of studies on the non-unique fixed points in the literature (for example see [45] and the references therein). Let (X,d) be a metric space, T be a self-mapping of X and $Fix(T) = \{x \in X : Tx = x\}$ be the fixed point set of T. A circle contained in the set $Fix(T)$ is called the fixed-circle of T (see [12] and [13] for more details). In [11], considering the geometric properties of non-unique fixed points, an extended version of Open Problem 1.1 have been stated as follows:

Is there a contractive condition which is strong enough to generate a fixed circle but which does not force the map to be continuous on its fixed circle?

A solution to this extended version was obtained in [11] with the help of some auxiliary numbers. At this point, some future directions of our study appear as the following:

By means of the notion of (ψ,φ)-$\mathcal{A}$-contractive mapping (resp. (ψ,φ)-$\mathcal{A}'$-contractive mapping);

1) New solutions to the above extended version of Open Problem 1.1 can be investigated.

2) New common fixed point (resp. coincidence point) results can be examined for the cases where the set $Fix(T)$ is not a singleton.

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Conflicts of interests

The authors declare that they have no competing interests.

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