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Research article

On rate type fluid flow induced by rectified sine pulses

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Abstract: This investigation aims to present the unsteady motion of second grade fluid in an oscillating duct induced by rectified sine pulses. Some of the most dominant means for solving problems in engineering, mathematics and physics are transform methods. The objective is to modify the domain of the present problem to a new domain which is easier for evaluation. Such modifications can be done by different ways, one such way is by using transforms. In present work Fourier sine transform and Laplace transform techniques are used. The solution thus obtained is in form of steady state, with combination of transient solution which fulfills all required initial and boundary conditions. The influence of various parameters of interest for both developing and retarding flows on the flow characteristics will also be sketched and discussed. Also, the problem is reduced to the flow model where side walls are absent by bringing the aspect ratio parameter (ratio of length to width) to zero.

Keywords: second grade fluid (SG fluid); exact solutions; non-Newtonian fluid (NN fluid); magnetohydrodynamics (MHD); porous medium; rectified sine pulses (RSP); Fourier sine transform (FST); Laplace transform (LT); initial and boundary conditions (IBC)
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1. Introduction

Subjects like non-Newtonian flow and rheology have wide range of practical applications in nature. In many applied processing chemical industries, behaviour of NN fluid can be observed. The rheological properties and characteristics are complex to define and thus, require deep study to understand. Due to the fact many researchers in different fields (e.g chemists, applied mathematics, physicists) find it interesting to simplify such complex structured models, amongst others, few of whom may have regarded the subject as central to their disciplines. In application of handling and processing of complex materials like emulsions, polymer melts, foams, slurries, and solutions e.t.c, engineers having background knowledge in field of process and chemical engineering, have dominant interest. Also, practicing engineers, scientists and theoretical mathematicians find this subject very applicable in their fields. Based on applications in different fields, fluid flow models are studied in many different ways.

Many solutions for fluid flow through different mediums and cross sections e.g rectangular, triangular and circular e.t.c region, are discussed by many researchers. The analytic solution of strokes for second grade fluid problem was proposed by Nazar et al. [20] and Fetecau and Fetecau [8] for evaluating velocity field. For evaluation of tangential stresses and velocity field regarding unsteady fluid of an Oldroyd-B, Fetecau et al. [9] used Fourier sine transform for obtaining solution. This flow was generated by the constant acceleration of plate placed between coupled perpendicular side walls. The exact solution using Fourier transform was obtained by Siddiqui [22], for the fluid flow generated by the periodic oscillation, placed between coupled parallel plates. The velocity tooth pulses regarding an Oldroyd-B fluid model causing hydromagnetics channel flow was discussed by Ghosh and Sana [10]. A study was done for determining the effect of sawtooth pulses on the unsteady MHD flow of an Oldroyd-B model and Laplace transform was used for obtaining the solution by Khan and Zeeshan [18]. Study regarding the fluid flow generated by sawtooth pulses are discussed in [10, 11].

Fluid Flows can be observed in many engineering, medical and agricultural problems e.g., dams, blood, and drains. The study of side walls in a duct and its effect on unsteady flow of a second grade fluid was proposed by Erdogan and Imrak [5]. Furthermore, the rectangular duct was allowed to move parallel to its length and also to an applied pressure gradient whose sides were at rest. The effect caused by the rectangular oscillating duct was studied by Nazar et al. [21], Khan and Anjum [17] for evaluating the Maxwell and Burger fluids respectively using the Laplace transforms and double finite Fourier sine for solution.

The fluid flow through porous space is important to study in areas like biomechanics, geomechanics and industry e.g flow of water through rocks, filtration of fluids and flow for regulation of skin. The Analytic solution can be obtained for such flow models by using the technique of Fourier sine transform for determining the variable accelerated flow as well as for the constant accelerated flow. Using such technique, Husain et al. [12] discussed the unsteady motion of fluid for an Oldroyd-B model passing through the porous medium. The effect of periodic pressure gradient on the oscillating viscous fluid was proposed by Johri and Singh [13] for the space of rectangular cross-section. The exact solution for the unsteady MHD flow was obtained using the Fourier and laplace transform for the Oldroyd-B fluid model passing through the long porous rectangular region by Sultan el al. [24]. Further studies of flows for non-Newtonian fluids through duct have been examined by Erdogan and Imrak [6], and Khan et al. [15, 16]. Interested readers can also be referred to [1–3, 7, 14, 19].

The equations for the differential fluid models are mostly nonlinear which are known as non-Newtonian fluids. The exact solution for such fluid models are difficult to obtain. But approximate solution can be obtained by using the technique of numerical solution. The importance of exact solution is that they completely describe the rheological effects as well as profile of fluid flow of a model and which can be used for validation and evaluation of convergence, consistency and stability of asymptotic and numeric prototype. In this article, we mainly emphasize on the induced rectified sine pulses on the fluid model which flows through a porous rectangular duct. The solution is obtained using Laplace transform and Fourier sine transform. Also, non-dimensional aspect ratio (ration of length to width) is obtained by transforming the equation of linear momentum to non-dimensional form. Also as a special case, we will obtain the explicit expressions for the velocity and the associated tangential stresses of the Newtonian fluid through duct.

2. Flow configuration and basic governing equations

Defining the Cauchy stress for a second grade incompressible homogeneous fluid and denoting by $\mathbb{T}$ as,

$$\mathbb{T} = -p\mathbb{I} + \mu\tau_1 + \alpha_1\tau_2 + \alpha_2\tau_1^2, \tag{2.1}$$

where hydrostatic pressure is denoted by p ,visco-elasticity is denoted by α_1, cross-viscosity is denoted by α_2, the dynamic viscosity is denoted μ and the Rivlin-Ericksen kinematic tensors are defined by τ_1 and τ_2.

$$\begin{aligned} \tau_1 &= (\nabla\mathbf{U}) + (\nabla\mathbf{U})^T, \\ \tau_2 &= \frac{d\tau_1}{dt} + \tau_1(\nabla\mathbf{U}) + (\nabla\mathbf{U})^T\tau_1 \end{aligned} \tag{2.2}$$

with $\frac{d}{dt}$ is the material time derivative, ∇ is the gradient operator, $\mathbf{U}$ is the velocity field and the transpose operation is denoted by the superposed T. It is worthwhile promising, when imposing the so-called Clausius-Duhem condition and the fluid is locally in equilibrium then by minimizing the Helmholtz free energy the stability and existence of the compatible solution for a thermodynamical models are often guaranteed as refer to [4]. These thermodynamic stability conditions impose following restrictions on normal stress moduli:

$$\begin{aligned} &\mu \geq 0, \\ &\alpha_1 \geq 0, \\ &\alpha_1 + \alpha_2 = 0. \end{aligned}$$

Here, the flow of an unsteady and an incompressible fluid through a rectangular cross-section duct possessing electrically conducting second grade fluid whose sides are at $x = 0, x = d, y = 0$ and $y = h$. At time $t = 0$, both the duct and the fluid are at rest. The velocity and stress fields are in the form mentioned in [23]

$$\mathbf{U} = \mathbf{U}(x, y, t) = w(x, y, t)\,\hat{\mathbf{k}}, \;\; \mathbf{S} = \mathbf{S}(x, y, t), \tag{2.3}$$

where the $z - directional$ unit vector is denoted by $\hat{\mathbf{k}}$.

For the second grade fluid the Darcy resistance R fulfills the following equation

$$R = -\frac{\mu\phi}{k}\Big(1 + \frac{\alpha}{\mu}\frac{\partial}{\partial t}\Big)\mathbf{U}, \tag{2.4}$$

where μ and α are material constants, ϕ is the porosity and k is the permeability of the porous medium.

For the magnetohydrodynamics flow, the governing equations for the incompressible fluid will be

$$\nabla.\mathbf{U} = 0$$
$$\rho\Big(\partial_t\mathbf{U} + (\mathbf{U}.\nabla)\mathbf{U}\Big) = \nabla.\mathbf{S} + J \times B + R. \tag{2.5}$$

Invoking (2.1)–(2.4) into the (2.5), omitting the body forces and pressure gradient, we obtain the following equation

$$\partial_t\, w(x,y,t) = (\nu + \alpha\partial_t)\Big(\partial_x^2 + \partial_y^2\Big)w(x,y,t) - \frac{\sigma\beta_0^2}{\rho}w(x,y,t) - \frac{\phi}{k}(\nu + \alpha\partial_t)w(x,y,t), \tag{2.6}$$

where $\alpha = \frac{\alpha_1}{\rho}$, kinematic viscosity of the fluid is denoted by $\nu = \frac{\mu}{\rho}$, ρ is the constant density, β_0 is the strength of applied to magnetic field and the electrical conductivity is denoted as σ.

3. Flow problem and non-dimensionalization

In this section, the boundary value problem corresponding to the aforementioned fluid flow is obtained together with relevant boundary conditions. The following non-dimensional relations are introduced in (2.6) as,

$$w^* = \frac{wd}{\nu},\ x^* = \frac{x}{d},\ y^* = \frac{y}{h},\ z^* = \frac{z}{d},\ t^* = \frac{t\nu}{d^2},$$
$$\alpha^* = \frac{\alpha}{d^2},\ T^* = \frac{T\nu}{d^2},\ \tau_1 = \frac{S_{xz}d^2}{\rho\nu^2},\ \tau_2 = \frac{S_{yz}d^2}{\rho\nu^2}. \tag{3.1}$$

After dropping the asterik for brevity and using the same notations for dimensionless quantities by abuse of notations, (2.6) in non-dimensional form becomes

$$\partial_t\, w(x,y,t) = (1 + \alpha\partial_t)\Big(\partial_x^2 + B^2\partial_y^2\Big)w(t) - \Omega w(x,y,t) - \epsilon(1 + \alpha\partial_t)w(x,y,t), \tag{3.2}$$

where $\Omega = \frac{\sigma\beta_0^2}{\rho}$ is the magnetic parameter, the non-dimensional parameter $B = \frac{d}{h}$ is the aspect ratio and $\epsilon = \frac{\nu\phi}{k}$ is the porosity parameter.

At time $t = 0^+$, the duct is generated impulsively from rest due to periodically applied rectified sine pulses. Assuming the following IBC in non-dimensional form as,

$$w(x,y,0) = 0, \quad x,y \in [0,1],$$

$$
\begin{aligned}
w(0,y,t) &= w(1,y,t) = w(x,0,t) = w(x,1,t) \\
&= 2\sum_{p=1}^{\infty}(-1)^p H_{pT}(t) \times \sin\{\frac{\pi}{T}(t-pT)\} \\
&+ \sin\{\frac{\pi}{T}\,t\}H(t) \text{for } x,y\in[0,1] \quad t\in[0,\infty),
\end{aligned} \tag{3.3}
$$

where the Heaviside unit step function is denoted as $H(\cdot)$ and time period is denoted as T,

$$
H_\phi(t) = H(t-\phi) = \begin{cases} 0, & t \le \phi \\ 1, & t > \phi\ . \end{cases}
$$

4. Expressions for velocity field and shear stress

4.1. Calculations for the velocity field

Applying Laplace transform (LT) to (3.2), and using condition in (3.3), to obtain

$$
q\,\overline{w}(x,y,q) = (1+\alpha q)\Big(\partial_x^2 + B^2\partial_y^2\Big)\overline{w}(x,y,q) - \Omega\overline{w}(x,y,q) - \epsilon(1+\alpha q)\overline{w}(x,y,q). \tag{4.1}
$$

The LT $\overline{w}(x,y,q)$ of the function $w(x,y,t)$ must satisfy the IBC

$$
\begin{aligned}
\overline{w}(0,y,q) &= \overline{w}(1,y,q) = \overline{w}(x,0,q) = \overline{w}(x,1,q) \\
&= \frac{\pi/T}{q^2+(\pi/T)^2}\Big\{2\sum_{p=1}^{\infty}(-1)^p \exp(-p\,qT) + 1\Big\} \text{ for } x,y\in[0,1].
\end{aligned} \tag{4.2}
$$

By virtue of the property

$$
\mathfrak{L}\{H(t-c)g(t-c)\} = exp(-cs)G(s).
$$

Multiplying (4.1) by $\sin(\xi_m x)\sin(\lambda_n y)$ and integrating with respect to x and y from 0 to 1, where $\lambda_n = n\pi$ and $\xi_m = m\pi$, while corresponding conditions given in (4.2) and following [23], gives

$$
\begin{aligned}
\overline{\mathrm{w}}(m,n,q) &= a_{mn}\lambda_{mn}^2\frac{\pi}{T}\frac{1+\alpha q}{\{q^2+(\pi/T)^2\}\{(1+\alpha\lambda_{mn}^2+\alpha\epsilon)q+\lambda_{mn}^2+\Omega+\epsilon\}} \\
&\times\Big\{1+2\sum_{p=1}^{\infty}(-1)^p\exp(-p\,qT)\Big\},
\end{aligned} \tag{4.3}
$$

where in (4.3)

$$
\begin{aligned}
a_{mn} &= (\{1-(-1)^m\}\{1-(-1)^n\})/(\lambda_n\xi_m), \\
\lambda_{mn}^2 &= \lambda_n^2 B^2 + \xi_m^2, \\
m,n &= 1,2,3,\cdots\ .
\end{aligned}
$$

Re-writing (4.3) as,

$$\overline{\mathtt{w}}(m,n,q) = a_{mn}\frac{\pi}{T}\Big\{\frac{1}{q^2+(\pi/T)^2} - \frac{(1+\alpha\epsilon)q+\Omega+\epsilon}{\{q^2+(\pi/T)^2\}\{\alpha_{mn}q+\beta_{mn}\}}\Big\}$$
$$\times\Big\{1+2\sum_{p=1}^{\infty}(-1)^p\exp(-p\,qT)\Big\}, \tag{4.4}$$

where $\alpha_{mn} = 1+\alpha\epsilon+\alpha\lambda_{mn}^2$ and $\beta_{_{mn}} = \lambda_{mn}^2+\Omega+\epsilon$. Following [23], to obtain

$$\overline{\mathtt{w}}(m,n,q) = a_{mn}\frac{(\pi/T)}{q^2+(\pi/T)^2}\Big\{2\sum_{p=1}^{\infty}(-1)^p\exp(-p\,qT)+1\Big\} - \frac{a_{mn}}{(\pi/T)^2\alpha_{mn}^2+\beta_{mn}^2}$$
$$\times\Big[(\pi/T)\Big\{(\Omega+\epsilon)\alpha_{mn}-\beta_{mn}(1+\alpha\epsilon)\Big\}\Big\{\frac{1}{q+(\beta_{mn}/\alpha_{mn})} - \frac{q}{q^2+(\pi/T)^2}\Big\}$$
$$+\Big\{(\pi/T)^2\alpha_{mn}(1+\alpha\epsilon)+\beta_{mn}(\Omega+\epsilon)\Big\}\frac{(\pi/T)}{q^2+(\pi/T)^2}\Big]$$
$$\times\Big\{2\sum_{p=1}^{\infty}(-1)^p\exp(-p\,qT)+1\Big\}. \tag{4.5}$$

Taking inverse Fourier sine of (4.5), and then inverse Laplace transform of the resulting equation, leads to

$$w(x,y,t) = \sin\{\frac{\pi}{T}t\}H(t) + 2\sum_{p=1}^{\infty}(-1)^pH_{pT}(t)\sin\{\frac{\pi}{T}(t-pT)\} - 16\sum_{m,n=0}^{\infty}\frac{\sin(\xi_s x)}{\xi_s}$$
$$\times\frac{\sin(\lambda_l y)}{\lambda_l}\frac{1}{(\pi/T)^2\alpha_{mn}^2+\beta_{mn}^2}\Big[(\pi/T)\Big\{\alpha_{mn}(\Omega+\epsilon)-\beta_{mn}(1+\alpha\epsilon)\Big\}$$
$$\times\Big\{\exp\{-\frac{\beta_{mn}}{\alpha_{mn}}t\} - \cos\{\frac{\pi}{T}t\}\Big\}H(t) + \Big\{(\pi/T)^2\alpha_{mn}(1+\alpha\epsilon)+\beta_{mn}(\Omega+\epsilon)\Big\}$$
$$\times\sin\{\frac{\pi}{T}t\}H(t) + 2\sum_{p=1}^{\infty}(-1)^pH_{pT}(t)\Big\{(\pi/T)\{\alpha_{mn}(\Omega+\epsilon)-\beta_{mn}(1+\alpha\epsilon)\}$$
$$\times\Big\{\exp\{-\frac{\beta_{mn}}{\alpha_{mn}}(t-pT)\} - \cos\{\frac{\pi}{T}(t-pT)\}\Big\}\Big], \tag{4.6}$$

where $s = 2m-1$, $l = 2n-1$, $\xi_s = \xi_{2m-1} = (2m-1)\pi$ and $\xi_1 = \xi_{2\mathrm{n}-1} = (2\mathrm{n}-1)\pi$.

4.2. Calculation for tangential stresses

In the considered problem $S_{xx} = S_{xy} = S_{yy} = 0$, and the meaningful equations as,

$$S_{xz}(x,y,t) = (\mu+\alpha_1\partial_t)\partial_x w(x,y,t)\,;$$
$$S_{yz}(x,y,t) = (\mu+\alpha_1\partial_t)\partial_y w(x,y,t). \tag{4.7}$$

Applying the non-dimensional scheme (3.1) to (4.7), to obtain

$$\tau_1(x,y,t) = (1+\alpha\partial_t)\partial_x w(x,y,t)\,;$$

$$\tau_2(x,y,t) = (1+\alpha\partial_t)\partial_y w(x,y,t). \tag{4.8}$$

Now applying the LT to (4.8), to obtain

$$\begin{aligned}\overline{\tau}_1(x,y,q) &= (1+\alpha q)\partial_x \overline{w}(x,y,q)\,;\\ \overline{\tau}_2(x,y,q) &= (1+\alpha q)\partial_y \overline{w}(x,y,q).\end{aligned} \tag{4.9}$$

Taking the inverse Fourier Sine transform of (4.5), then plug in the resulting value of $\overline{w}(x,y,q)$ into (4.9), to obtain

$$\begin{aligned}\overline{\tau}_1(x,y,q) = &-16\frac{\pi}{T}\sum_{m,n=0}^{\infty}\cos(\xi_s x)\frac{\sin(\lambda_l y)}{\lambda_l}\frac{\alpha(1+\alpha\epsilon)q^2+(1+2\alpha\epsilon+\alpha\Omega)q+\Omega+\epsilon}{(q^2+(\pi/T)^2)(\alpha_{mn}q+\beta_{mn})}\\ &\times\left(2\sum_{p=1}^{\infty}(-1)^p\exp(-p\,qT)+1\right),\end{aligned} \tag{4.10}$$

$$\begin{aligned}\overline{\tau}_2(x,y,q) = &-16\,\frac{\pi}{T}\sum_{m,n=0}^{\infty}\cos(\lambda_l y)\frac{\sin(\xi_s x)}{\xi_s}\frac{\alpha(1+\alpha\epsilon)q^2+(1+2\alpha\epsilon+\alpha\Omega)q+\Omega+\epsilon}{(q^2+(\pi/T)^2)(\alpha_{mn}q+\beta_{mn})}\\ &\times\left(2\sum_{p=1}^{\infty}(-1)^p\exp(-p\,qT)+1\right).\end{aligned} \tag{4.11}$$

Rewriting (4.10) and (4.11) in the following equivalent form

$$\begin{aligned}\overline{\tau}_1(x,y,q) = &-16\sum_{m,n=0}^{\infty}\cos(\xi_s x)\frac{\sin(\lambda_l y)}{\lambda_l}\frac{1}{(\pi/T)^2\alpha_{mn}^2+\beta_{mn}^2}\bigg\{\frac{\pi}{T}\frac{a\beta_{mn}^2-b\alpha_{mn}\beta_{mn}+c\alpha_{mn}^2}{\alpha_{mn}}\\ &\times\frac{1}{q+(\beta_{mn}/\alpha_{mn})}+(c\beta_{mn}-a^3\beta_{mn}+a^3b\alpha_{mn})\frac{\pi/T}{q^2+(\pi/T)^2}\\ &+\frac{\pi}{T}(a^3\alpha_{mn}+b\beta_{mn}-c\alpha_{mn})\frac{q}{q^2+(\pi/T)^2}\bigg\}\bigg\{1+2\sum_{p=1}^{\infty}(-1)^p\exp(-p\,qT)\bigg\},\end{aligned} \tag{4.12}$$

$$\begin{aligned}\overline{\tau}_2(x,y,q) = &-16\sum_{m,n=0}^{\infty}\cos(\lambda_l y)\frac{\sin(\xi_s x)}{\xi_s}\frac{1}{(\pi/T)^2\alpha_{mn}^2+\beta_{mn}^2}\bigg\{\frac{\pi}{T}\frac{a\beta_{mn}^2-b\alpha_{mn}\beta_{mn}+c\alpha_{mn}^2}{\alpha_{mn}}\\ &\times\frac{1}{q+(\beta_{mn}/\alpha_{mn})}+(c\beta_{mn}-a^3\beta_{mn}+a^3b\alpha_{mn})\frac{\pi/T}{q^2+(\pi/T)^2}\\ &+\frac{\pi}{T}(a^3\alpha_{mn}+b\beta_{mn}-c\alpha_{mn})\frac{q}{q^2+(\pi/T)^2}\bigg\}\bigg\{1+2\sum_{p=1}^{\infty}(-1)^p\exp(-p\,qT)\bigg\},\end{aligned} \tag{4.13}$$

where

$$a=\alpha(1+\alpha\epsilon),\;\; b=1+2\alpha\epsilon+\alpha\Omega, c=\Omega+\epsilon.$$

Applying the inverse Laplace transform to (4.12) and (4.13), to obtain

$$\tau_1(x,y,t) = -16\sum_{m,n=0}^{\infty}\cos(\xi_s x)\frac{\sin(\lambda_l y)}{\lambda_l}\frac{1}{(\pi/T)^2\alpha_{mn}^2+\beta_{mn}^2}\Bigg[\Bigg\{\frac{\pi}{T}\frac{a\beta_{mn}^2-b\alpha_{mn}\beta_{mn}+c\alpha_{mn}^2}{\alpha_{mn}}$$
$$\times\exp\{-\frac{\beta_{mn}}{\alpha_{mn}}t\}+(c\beta_{mn}-a^3\beta_{mn}+a^3b\alpha_{mn})\sin\{\frac{\pi}{T}t\}$$
$$+\frac{\pi}{T}(a^3\alpha_{mn}+b\beta_{mn}-c\alpha_{mn})\cos\{\frac{\pi}{T}t\}\Bigg\}H(t)+2\sum_{p=1}^{\infty}(-1)^pH_{pT}(t)$$
$$\times\Bigg\{\frac{\pi}{T}\frac{a\beta_{mn}^2-b\alpha_{mn}\beta_{mn}+c\alpha_{mn}^2}{\alpha_{mn}}\exp\{-\frac{\beta_{mn}}{\alpha_{mn}}(t-pT)\}+(c\beta_{mn}-a^3\beta_{mn}+a^3b\alpha_{mn})$$
$$\times\sin\{\frac{\pi}{T}(t-pT)\}+\frac{\pi}{T}(a^3\alpha_{mn}+b\beta_{mn}-c\alpha_{mn})\cos\{\frac{\pi}{T}(t-pT)\}\Bigg\}\Bigg], \tag{4.14}$$

$$\tau_2(x,y,t) = -16\sum_{m,n=0}^{\infty}\cos(\lambda_l y)\frac{\sin(\xi_s x)}{\xi_s}\frac{1}{(\pi/T)^2\alpha_{mn}^2+\beta_{mn}^2}\Bigg[\Bigg\{\frac{\pi}{T}\frac{a\beta_{mn}^2-b\alpha_{mn}\beta_{mn}+c\alpha_{mn}^2}{\alpha_{mn}}$$
$$\times\exp\{-\frac{\beta_{mn}}{\alpha_{mn}}t\}+(c\beta_{mn}-a^3\beta_{mn}+a^3b\alpha_{mn})\sin\{\frac{\pi}{T}t\}$$
$$+\frac{\pi}{T}(a^3\alpha_{mn}+b\beta_{mn}-c\alpha_{mn})\cos\{\frac{\pi}{T}t\}\Bigg\}H(t)+2\sum_{p=1}^{\infty}(-1)^pH_{pT}(t)$$
$$\times\Bigg\{\frac{\pi}{T}\frac{a\beta_{mn}^2-b\alpha_{mn}\beta_{mn}+c\alpha_{mn}^2}{\alpha_{mn}}\exp\{-\frac{\beta_{mn}}{\alpha_{mn}}(t-pT)\}$$
$$+(c\beta_{mn}-a^3\beta_{mn}+a^3b\alpha_{mn})\times\sin\{\frac{\pi}{T}(t-pT)\}$$
$$+\frac{\pi}{T}(a^3\alpha_{mn}+b\beta_{mn}-c\alpha_{mn})\cos\{\frac{\pi}{T}(t-pT)\}\Bigg\}\Bigg]. \tag{4.15}$$

4.3. Special case

For velocity field, letting $\alpha = 0$ into (4.6), (4.14) and (4.15), to get

$$w(x,y,t) = \sin\{\frac{\pi}{T}t\}H(t)+2\sum_{p=1}^{\infty}(-1)^pH_{pT}(t)\sin\{\frac{\pi}{T}(t-pT)\}-16\sum_{m,n=0}^{\infty}\frac{\sin(\xi_s x)}{\xi_s}$$
$$\times\frac{\sin(\lambda_l y)}{\lambda_l}\frac{1}{(\pi/T)^2+\beta_{mn}^2}\Bigg[\Bigg\{(\pi/T)\lambda_{mn}^2\Big\{\cos\{\frac{\pi}{T}t\}-\exp(-\beta_{mn}t)\Big\}$$
$$+\Big\{(\pi/T)^2+\beta_{mn}(\Omega+\epsilon)\Big\}\sin\{\frac{\pi}{T}t\}\Bigg\}H(t)+2\sum_{p=1}^{\infty}(-1)^pH_{pT}(t)$$
$$\times\Bigg\{(\pi/T)\lambda_{mn}^2\Big\{\cos\{\frac{\pi}{T}(t-pT)\}-\exp\{-\beta_{mn}(t-pT)\}\Big\}$$
$$+\{(\pi/T)^2+\beta_{mn}(\Omega+\epsilon)\}\sin\{\frac{\pi}{T}(t-pT)\}\Bigg\}\Bigg],$$

and the tangential stresses

$$\tau_1(x,y,q) = -16\sum_{m,n=0}^{\infty}\cos(\xi_s x)\frac{\sin(\lambda_l y)}{\lambda_l}\frac{1}{(\pi/T)^2+\beta_{mn}^2}\Bigg[\Bigg\{\frac{\pi}{T}\lambda_{mn}^2\Bigg\{\cos\{\frac{\pi}{T}t\}-\exp(-\beta_{mn}t)\Bigg\}$$
$$+\beta_{mn}(\Omega+\epsilon)\sin\{\frac{\pi}{T}t\}\Bigg\}H(t)+2\sum_{p=1}^{\infty}(-1)^p H_{pT}(t)\Bigg\{\frac{\pi}{T}\lambda_{mn}^2\Bigg\{\cos\{\frac{\pi}{T}(t-pT)\}$$
$$-\exp\{-\beta_{sl}(t-pT)\}\Bigg\}+\beta_{mn}(\Omega+\epsilon)\sin\{\frac{\pi}{T}(t-pT)\}\Bigg\}\Bigg],$$

$$\tau_2(x,y,q) = -16\sum_{m,n=0}^{\infty}\cos(\lambda_l y)\frac{\sin(\xi_s x)}{\xi_s}\frac{1}{(\pi/T)^2+\beta_{mn}^2}\Bigg[\Bigg\{\frac{\pi}{T}\lambda_{mn}^2\Bigg\{\cos\{\frac{\pi}{T}t\}-\exp(-\beta_{mn}t)\Bigg\}$$
$$+\beta_{mn}(\Omega+\epsilon)\sin\{\frac{\pi}{T}t\}\Bigg\}H(t)+2\sum_{p=1}^{\infty}(-1)^p H_{pT}(t)\Bigg\{\frac{\pi}{T}\lambda_{mn}^2\Bigg\{\cos\{\frac{\pi}{T}(t-pT)\}$$
$$-\exp\{-\beta_{mn}(t-pT)\}\Bigg\}+\beta_{mn}(\Omega+\epsilon)\sin\{\frac{\pi}{T}(t-pT)\}\Bigg\}\Bigg],$$

for Newtonian fluid flowing through oscillating duct.

5. Graphical illustration

Graphs are presented for describing the flow of developing as well as retarding fluids. Time $t = 0.2$ corresponds to the developing flow and $t = 1$ corresponds to the retarding flow. Figure 1 is prepared to notice the variation in the velocity profile of developing as well as the retarding fluids flow in the absence and in the presence of side walls. Figure 1(a) displays the variation of magnitude of amplitude of oscillation of velocity profile with respect to α for developing flow in the side walls presence. It is evaluated in velocity profile of an oscillating amplitude that the magnitude decreases as α increases. The value $\alpha = 0$ corresponds to the Newtonian fluid. It is evident that the curves regarding the flow of second grade model changes to that of Newtonian model as α is minimized. Figure 1(b) shows that the magnitude of amplitude of oscillation of velocity decreases as α increases for retarding flow. Figure 1(c,d) predicts that the profile of velocity is a decreasing function of α in the absence of side walls for both flows. Moreover, it is also evident from these figures that the velocity profile is smaller in the absence of side walls while greater in the presence of side walls. Figure 2, showing the influence of the spatial variable x for both flows on the amplitude of velocity versus y. The graphs for various values of x have been plotted. It is seen for the velocity profile that the amplitude of oscillation is an increasing function of x. The influence of x is more prominent for smaller values. The influence of magnetic parameter Ω and time period T on the transient velocity profile is underlined in Figure 3. Figure 3(a) shows that the time required to reach the steady state decreases as Ω increases in the presence of side walls. But the time increases as T increases. Thus T reduces the magnetic effect on the velocity profile. The case $\Omega = 0$ corresponds to hydrodynamic flow. It can also be noted that the required time to reach the steady state for hydrodynamic flow is greater as compared to that for hydro-magnetic flow. Figure 3(b) shows the effects of Ω and T on the velocity profile in side walls absence. The result is similar as in the case for flow in the presence of side walls. Figure 4(a,b) displays that the required time decreases as ϵ increases for both conditions.

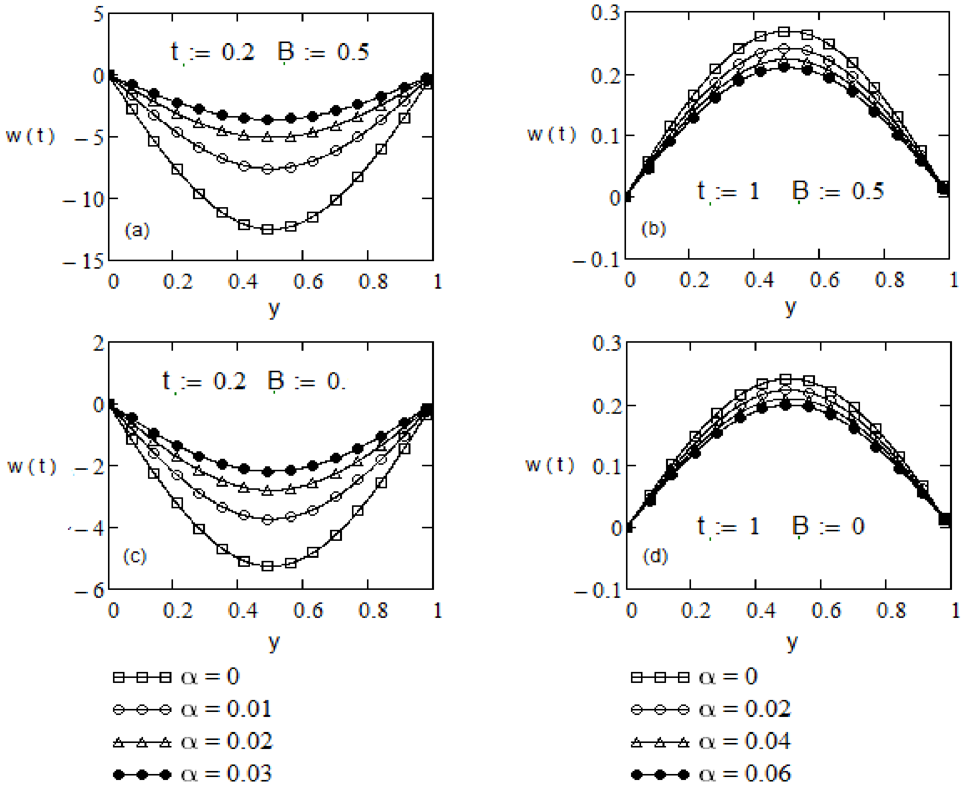

Figure 1. Velocity profiles for second grade fluid for different values of α.

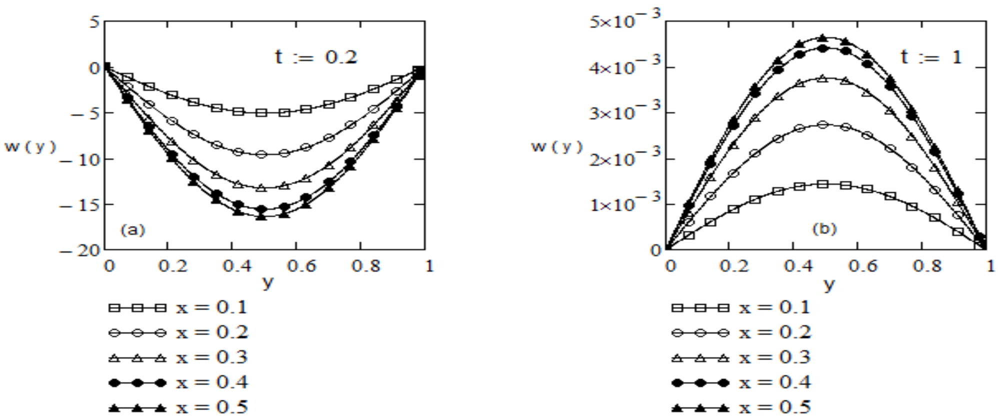

Figure 2. Velocity profiles for second grade fluid for different values of x.

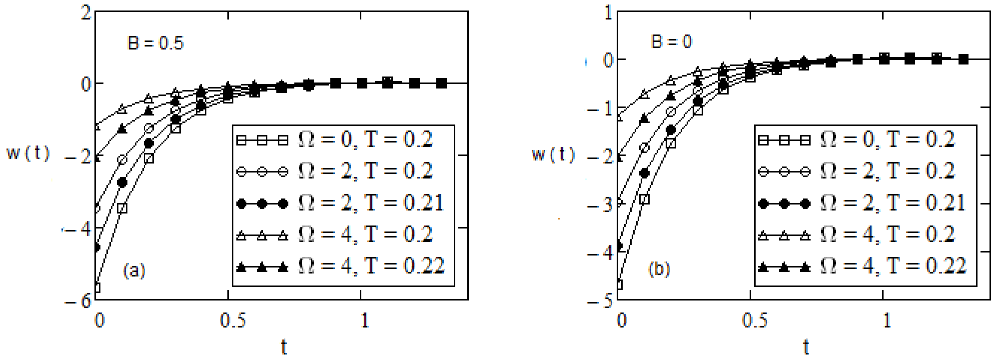

Figure 3. Velocity profiles for second grade fluid for different values of Ω and T.

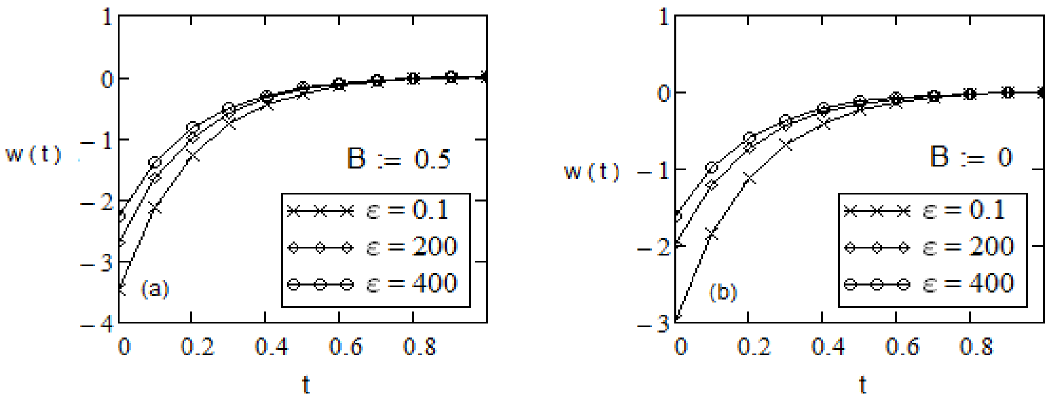

Figure 4. Velocity profiles for second grade fluid for different values of ϵ.

6. Conclusions

The exact analytic solutions for the flow of second grade fluid model through an area of a porous rectangular cross-section induced by rectified sine pulses is established. To obtain the solution, Laplace and double finite Fourier Sine transform are used for present mathematical model. In the absence of side walls *i.e*; for $B = 0$, the problem reduces to the flow between two plates.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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