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Research article

The linear k-arboricity of symmetric directed trees

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Abstract: A linear k-diforest is a directed forest in which every connected component is a directed path of length at most k. The linear k-arboricity of a digraph D is the minimum number of arc-disjoint linear k-diforests whose union covers all the arcs of D. In this paper, we study the linear k-arboricity for symmetric directed trees and fully determine the linear 2-arboricity for all symmetric directed trees.

Keywords: linear k-arboricity; digraphs; directed trees; symmetric directed trees; linear arboricity
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1. Introduction

In this paper, a digraph is a finite loopless directed graph without parallel arcs (arcs with the same head and the same tail) and an undirected graph is also a finite and simple graph. A *linear forest* is a forest in which every connected component is a path. The *linear arboricity* of a graph G, defined by Harary [14], is the minimum number of edge-disjoint linear forests whose union covers all the edges of G and is denoted by $la(G)$. Let $\Delta(G)$ be the maximum degree of an undirected graph G. Akiyama et al. [1] proposed the following conjecture for the linear arboricity of graphs.

Conjecture 1. *[1] For an undirected graph G,*

$$la(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil. \tag{1.1}$$

Later, Habib and Péroche [13] introduced the *linear k-arboricity* of a graph G, which is the minimum number of edge-disjoint k-linear forests (forests in which every connected component is a path of length at most k) whose union covers all the edges of G and is denoted by $la_k(G)$. Moreover, Habib and Péroche [13] proposed a conjecture about the value of linear k-arboricity which subsumes Akiyama's conjecture.

Conjecture 2. *[13] For an undirected graph G with n vertices,*

$$la_k(G) \leq \begin{cases} \left\lceil \dfrac{\Delta(G)n+1}{2\lfloor kn/(k+1)\rfloor} \right\rceil & when\ \Delta(G) < n-1, \\ \left\lceil \dfrac{\Delta(G)n}{2\lfloor kn/(k+1)\rfloor} \right\rceil & when\ \Delta(G) = n-1. \end{cases} \tag{1.2}$$

Aimed at these two conjectures, considerable studies have been done over recent years. Alon et al. [2, 3] studied the linear arboricity by using probabilistic methods; Ferber et al. [10] improved Alon's results recently. There are also several studies for special graphs: Chang et al. [8] studied the linear k-arboricity for trees; Wu [20] studied the linear arboricity for planar graphs; Fu et al. [11] studied the linear 3-arboricity for $K_{n,n}$ and K_n. Till now, these two conjectures are still open (see [4–9, 11, 12, 16, 18–22] for more related results).

It is natural to consider similar problems for digraphs. Let $D = (V(D), A(D))$ be a digraph. We denote $\Delta^+(D) = max\{d^+(v)|\ for\ all\ v \in V\}$, $\Delta^-(D) = max\{d^-(v)|\ for\ all\ v \in V\}$ and $\Delta(D) = max\{\Delta^+(D), \Delta^-(D)\}$. The underlying graph $S(D)$ of D is the undirected simple graph with the same vertex set of D by replacing each arc by an edge with the same ends. A *linear diforest* is a directed forest in which every connected component is a directed path. The *linear arboricity* of D, defined by Nakayama and Péroche [17], is the minimum number of arc-disjoint linear diforests whose union covers the arcs of D and denoted by $\overrightarrow{la}(D)$. Nakayama and Péroche [17] also conjectured that $\overrightarrow{la}(D) \leq \Delta(D) + 1$. Actually, Nakayama-Péroche conjecture is equivalent to say that the linear arboricity of a d-regular digraph D (i.e. every vertex in D has in-degree d and out-degree d) is $d + 1$. In 2017, He et al. [15] found that the symmetric complete digraphs K_3^* and K_5^* have the linear arboricity $d + 2$ ($d = 2, 4$ *respectively*) which is contrary to Nakayama-Péroche conjecture. Then they proposed a modified conjecture.

Conjecture 3. *[15] For a d-regular digraph D,*

$$\overrightarrow{la}(D) = d + 1\ \ except\ D\ is\ K_3^*\ or\ K_5^*. \tag{1.3}$$

Recently, Zhou et al. [23] studied the linear k-arboricity for digraphs. The *linear k-arboricity* of a digraph D is the minimum number of arc-disjoint linear k-diforests (diforests in which every connected component is a directed path of length at most k) whose union covers all the arcs of D and denoted by $\overrightarrow{la}_k(D)$. A linear k-diforests decomposition of D is called *minimum* if the number of linear k-diforests is equal to the linear k-arboricity. Zhou et al. [23] also determined the linear 2-arboricity and linear 3-arboricity for symmetric complete digraphs and symmetric complete bipartie digraphs, and proposed the following conjecture.

Conjecture 4. *For a digraph D with n vertices, if* $k = n - 1$, *then*

$$\overrightarrow{la}_k(D) \leq \begin{cases} \left\lceil \dfrac{\Delta(D)n}{\lfloor kn/(k+1)\rfloor} \right\rceil & when\ \Delta(D) = n-1\ and\ D\ is\ not\ K_3^*\ and\ K_5^*, \\ \left\lceil \dfrac{\Delta(D)n+1}{\lfloor kn/(k+1)\rfloor} \right\rceil & when\ \Delta(D) < n-1\ or\ D\ is\ K_3^*\ or\ K_5^*. \end{cases} \tag{1.4}$$

If $k < n-1$, then

$$\overrightarrow{la}_k(D) \leq \begin{cases} \left\lceil \frac{\Delta(D)n}{\lfloor kn/(k+1) \rfloor} \right\rceil & \text{when } \Delta(D) = n-1, \\ \left\lceil \frac{\Delta(D)n+1}{\lfloor kn/(k+1) \rfloor} \right\rceil & \text{when } \Delta(D) < n-1. \end{cases} \tag{1.5}$$

In this paper, we study the linear k-arboricity for symmetric directed trees. In Section 2, we give some useful lemmas and determine the linear 2-arboricity for symmetric directed stars and double-stars. In Section 3, we characterize two families of symmetric directed trees, whose linear 2-arboricity is equal to the maximum degree plus one. In Section 4, we fully determine the linear 2-arboricity for symmetric directed trees. The conclusion of the linear k-arboricity of symmetric directed trees for any k with further research directions form the arguments of the last section.

2. Preliminary

Lemma 1. *Let H be a subdigraph of a digraph D. Then $\overrightarrow{la}_k(H) \leq \overrightarrow{la}_k(D)$.*

Lemma 2. *For a digraph D with n vertices,*

$$\overrightarrow{la}_1(D) \geq \overrightarrow{la}_2(D) \geq ... \geq \overrightarrow{la}_{n-1}(D) = \overrightarrow{la}(D). \tag{2.1}$$

Lemma 3. *For a digraph $D = (V(D), A(D))$ with n vertices and m arcs,*

$$\overrightarrow{la}_k(D) \geq \max \left\{ \Delta(D), \left\lceil \frac{m}{\lfloor \frac{kn}{k+1} \rfloor} \right\rceil \right\}. \tag{2.2}$$

If D is a symmetric digraph, we just give two opposite directions to the linear forests of the minimum linear forests decomposition of $S(D)$ and get the following trivial bound for $\overrightarrow{la}_k(D)$.

Lemma 4. *Let D be a symmetric digraph. Then $\overrightarrow{la}_k(D) \leq 2la_k(S(D))$.*

Nakayama and Péroche [17] studied the linear arboricity for symmetric directed trees and obtained the following result.

Theorem 5. *[17] Let T^* be a symmetric directed tree with more than two vertices, then $\overrightarrow{la}(T^*) = \Delta(T^*)$.*

For the linear 2-arboricity of several families of special symmetric directed tree, we have the following three results.

Theorem 6. *Let S^* be a symmetric directed star with more than two vertices, then $\overrightarrow{la}_2(S^*) = \Delta(S^*)$.*

Proof. Assume that $V(S^*) = \{x, x_0, x_1, ..., x_{\Delta-1}\}$ and x is the center vertex of S^*. We can decompose the arcs of S^* into directed 2-paths $x_i x x_{i+1(mod\ \Delta)}$ ($i \in \{0, 1, ..., \Delta - 1\}$) and those directed 2-paths form $\Delta(S^*)$ linear 2-diforests. □

A double-star with $a + b + 2$ vertices and the degree sequence $(a + 1, b + 1, 1, ..., 1)$ is denoted by $S_{a,b}$. Let $S^*_{a,b}$ be a symmetric directed double-star.

Theorem 7. *(1)* $\overrightarrow{la}_2(S^*_{2,1}) = 4$*;*

(2) $\overrightarrow{la}_2(S^*_{a,a}) = a + 2$*;*

(3) $\overrightarrow{la}_2(S^*_{a,b}) = b + 1$*, here* $a \geq 1, b \geq 3, a < b$*.*

Proof. (1) It is easy to check.

(2) Assume that $V(S^*_{a,a}) = \{x, y, x_0, x_1 ..., x_{a-1}, y_0, ..., y_{a-1}\}$, where x, y have degree $a+1$ and $x_i, y_i (0 \leq i \leq a-1)$ are adjacent to x and y respectively. We construct $\{F_0, F_1, ..., F_{a+1}\}$ to be a linear 2-diforests decomposition of $S^*_{a,a}$ with $F_i = \{x_i x x_{i+1(mod\ a-1)} | 0 \leq i \leq a-2\} \cup \{y_i y y_{i+1} | 0 \leq i \leq a-2\}$, $F_{a-1} = \{x_{a-1} x y\}$, $F_a = \{y x x_{a-1}\}$ and $F_{a+1} = \{y_{a-1} y y_0\}$.

(3) We have a similar decomposition as (2). □

Theorem 8. *Let* T^* *be a symmetric directed tree with n vertices.*

(1) If $\Delta(T^*) = 1$*, then* $\overrightarrow{la}_2(T^*) = 2$*;*

(2) If $\Delta(T^*) = 2$ *and* $n \geq 4$*, then* $\overrightarrow{la}_2(T^*) = 3$*;*

(3) If $\Delta(T^*) = 3$ *and* $n \geq 5$*, then* $\overrightarrow{la}_2(T^*) = 4$*.*

Proof. It is easy to check (1) and (2). We know that each T^* with $\Delta(T^*) = 3$ has $S^*_{2,1}$ as a subdigraph, thus $\overrightarrow{la}_2(T^*) \geq 4$ by Theorem 7. On the other hand, $\overrightarrow{la}_2(T^*) \leq 4$ by Lemma 4. □

Let T^* be a symmetric directed tree and the vertex v in $S(T^*)$ be a root. The *depth* of a vertex in $S(T^*)$ is the number of edges from the vertex to v. The *tree-depth* of $S(T^*)$ with root v is the value of the maximum vertex depth based on the root v. Suppose that the tree-depth of $S(T^*)$ with root v is k, we call a vertex x is a *key* vertex in T^* if the vertex depth of x is $k-1$.

For the linear 2-arborcity of symmetric directed trees, we have the following upper bound.

Theorem 9. $\overrightarrow{la}_2(T^*) \leq \Delta(T^*) + 1$.

Proof. If $\Delta(T^*) \leq 3$, then the result holds by Theorem 8. Now we assume that $\Delta(T^*) \geq 4$. By Theorem 6, the result holds when T^* is a symmetric directed star. Thus we assume that T^* is not a symmetric directed star.

Suppose that $|V(T^*)| = n$. We prove the theorem by induction on n. First, in $S(T^*)$ with a root, we choose a key vertex x adjacent to a non-leaf vertex y and s leaves $x_1, x_2, ..., x_s$ ($s \leq \Delta(T^*) - 1$). Let $T^*_0 = T^* - \{x_1, x_2, ..., x_s\}$. By induction, $\overrightarrow{la}_2(T^*_0) \leq \Delta(T^*_0) + 1 \leq \Delta(T^*) + 1$. Let $\{F_1, F_2, ..., F_{\Delta(T^*)+1}\}$ be a linear 2-diforests decomposition of T^*_0 (some F_i may be empty) and we assume that the arcs $xy \in F_{\Delta(T^*)}$ and $yx \in F_{\Delta(T^*)+1}$ without loss of generality. Let $X = \{x, x_1, x_2, ..., x_s\}$ and the induced subdigraph $T^*[X]$ be a symmetric directed star which can be decomposed into $\Delta(T^*) - 1$ directed paths P_i ($1 \leq i \leq \Delta(T) - 1$) of lengths at most two (P_i may be empty). Then $F'_i = F_i \cup P_i$ ($1 \leq i \leq \Delta(T) - 1$) are arc-disjoint linear 2-diforests of T^*. $\{F'_1, F'_2, ..., F'_{\Delta(T^*)-1}, F_{\Delta(T^*)}, F_{\Delta(T^*)+1}\}$ is a linear 2-diforests decomposition of T^*. Thus, $\overrightarrow{la}_2(T^*) \leq \Delta(T^*) + 1$. □

3. Two families of symmetric directed trees

Let T^*_1 be a symmetric directed tree (not a directed star) and x is a key vertex adjacent to $\Delta(T^*_1) - 1$ leaves x_1, x_2,..., $x_{\Delta(T^*_1)-1}$ and a non-leaf vertex y. Thus the depth of x is $k-1$ and the depth of y is $k-2$

in a rooted tree of $S(T_1^*)$. Suppose that $v, v_1, v_2, ..., v_t$ (not containing x) are all adjacent to y, where the depth of v is $k-3$ and the depth of others are $k-1$. We obtain a new symmetric directed tree $T_1^{'}$ from $T_1^* - \{x_1, x_2, ..., x_{\Delta(T_1^*)-1}\}$ by adding a new leaf x' and the arcs $x'y, yx'$, and we call $T_1^{'}$ the *grafting graph* of T_1^*. For the linear 2-arboricity of T_1^*, we have the following lemma.

Lemma 10. *If a symmetric directed tree T_1^* satisfies one of the following conditions, then $\overrightarrow{la}_2(T_1^*) = \Delta(T_1^*) + 1$.*

(1) $\Delta(T_1^*)$ is odd, $\overrightarrow{la}_2(T_1^{'}) = \Delta(T_1^*)$ and any minimum linear 2-diforests decomposition of $T_1^{'}$ must have the 2-paths $xyx^{'}$, $x^{'}yv$, vyx in three linear 2-diforests or have $x^{'}yx$, $vyx^{'}$, xyv in three linear 2-diforests;

(2) $\Delta(T_1^*)$ is odd, $\overrightarrow{la}_2(T_1^{'}) = \Delta(T_1^*)$ and any minimum linear 2-diforests decomposition of $T_1^{'}$ must have the 2-paths $xyx^{'}$, $x^{'}yv_i$, v_iyx in three linear 2-diforests or have $x^{'}yx$, $v_iyx^{'}$, xyv_i in three linear 2-diforests for some $1 \le i \le t$ (without loss of generality, we say $i = 1$).

Proof. If $T_1^{'}$ satisfies condition (1), we assume that $\overrightarrow{la}_2(T_1^*) = \Delta(T_1^*)$. Let $\{F_1, F_2, ..., F_{\Delta(T_1^*)}\}$ be a linear 2-diforests decomposition of T_1^*. We denote $U = \{y, x, x_1, ..., x_{\Delta(T_1^*)-1}\}$. Then the arcs of the induced subdigraph $T_1^*[U]$ must be decomposed into $\Delta(T_1^*)$ directed 2-paths, which are in different linear 2-diforests. Assume that the 2-path yxx_1 in F_1 and the 2-path $x_{\Delta(T_1^*)-1}xy$ in F_2. Let $F_1' = F_1 - xx_1 + x'y$ and $F_2' = F_2 - x_{\Delta(T_1^*)-1}x + yx'$. We delete vertices $x_1, x_2..., x_{\Delta(T_1^*)-1}$ and the arcs incident with them in F_i $(3 \le i \le \Delta(T_1^*))$, and denote the resulting linear 2-diforests by F_i' $(3 \le i \le \Delta(T_1^*))$ respectively. Thus $\{F_1', F_2', ..., F'_{\Delta(T_1^*)}\}$ is a new linear 2-diforests decomposition of $T_1^{'}$, which conflicts with condition (1). Therefore $\overrightarrow{la}_2(T_1^*) = \Delta(T_1^*) + 1$.

It is similar to prove the result when $T_1^{'}$ satisfies condition (2). We omit the proof here. □

We denote the family of symmetric directed trees which satisfy the condition (1) (resp. condition (2)) in Lemma 10 by Φ (resp. Ψ). For a symmetric directed tree $T_1^* \in \Phi \cup \Psi$, the structure of the grafting graph of T_1^* is shown in Figure 1.

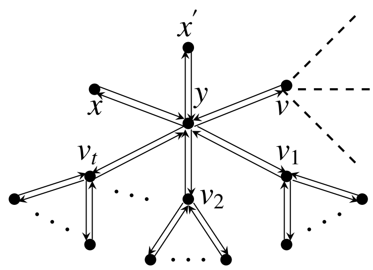

Figure 1. The structure of the grafting graph.

Lemma 11. *Suppose that a symmetric directed tree $T_1^* \in \Phi$. Then the followings hold:*

(1) $d_{T_1^*}(v_i) = \Delta(T_1^*)$, $1 \le i \le t$;

(2) $\Delta(T_1^*) = 2t + 3$.

Proof. Since $\overrightarrow{la}_2(T_1^{'}) = \Delta(T_1^*)$, let $\{F_1, F_2, ..., F_{\Delta(T_1^*)}\}$ be a linear 2-diforests decomposition of $T_1^{'}$.

Firstly, we claim that each linear 2-diforest F_i $(1 \leq i \leq \Delta(T_1^*))$ must have at least one arc incident with the vertex y. Otherwise, suppose that F_j $(j \neq 1)$ has no arc incident with y. Without loss of generality, we assume that the 2-path vyx' in F_1. Then we take away yx' from F_1 and put it into F_j. We have a new linear 2-diforest decomposition of T_1', where vyx' is not a 2-path in any linear 2-diforest. It is contrary to the definition of graphs in Φ.

We assume that $d_{T_1^*}(v_1) \leq \Delta(T_1^*) - 1$. Since $d_{T_1^*}(v_1) \leq \Delta(T_1^*) - 1$, without loss of generality, we assume that $xyv \in F_1$ and v_1 is an isolated vertex in F_1. Then we have three cases to discuss.

Case 1. Assume that $yv_1 \in F_2$, i.e., yv_1 a 1-path in F_2. Let $F_1' = F_1 - xy$ and $F_2' = F_2 + xy$. It is easy to check $\{F_1', F_2', F_3, ..., F_{\Delta(T_1^*)}\}$ is a new linear 2-diforests decomposition of T_1', a contradiction to the definition of graphs in Φ.

Case 2. Assume that $yv_1u \in F_2$, where u is a leaf adjacent to v_1. Let $F_1' = F_1 - xy + v_1u$ and $F_2' = F_2 - v_1u + xy$. $\{F_1', F_2', F_3, ..., F_{\Delta(T_1^*)}\}$ is a new linear 2-diforests decomposition of T_1', a contradiction to the definition of graphs in Φ.

Case 3. Assume that $v_2yv_1 \in F_2$. Let T_v' be the component of $T_1' - y$ containing v. The subsets of F_i $(i = 1, 2)$ containing all the linear diforests of T_v' are denoted by F_i^v $(i = 1, 2)$. Let $F_1' = F_1 + yv_1 - yv - F_1^v + F_2^v$, $F_2' = F_1 - yv_1 + yv + F_1^v - F_2^v$. Thus $\{F_1', F_2', F_3, ..., F_{\Delta(T_1^*)}\}$ is a new linear 2-diforests decomposition of T_1', a contradiction to the definition of graphs in Φ.

So $d_{T_1^*}(v_1) = \Delta(T_1^*)$ and (1) also holds for the other v_i $(2 \leq i \leq t)$.

Since every v_i $(1 \leq i \leq t)$ reaches the maximum degree and $\overrightarrow{la}_2(T_1') = \Delta(T_1^*)$, the 2-path v_iyv_j $(1 \leq i, j \leq t)$ should not be contained in any linear 2-diforest. Thus yv_i and v_iy $(1 \leq i \leq t)$ should be in different linear 2-diforests. By the definition of graphs in Φ, there are totally $2t + 3$ linear diforest in which y is not an isolated vertex. Since each linear 2-diforest F_i $(1 \leq i \leq \Delta(T_1^*))$ must have at least one arc incident with the vertex y, $\Delta(T_1^*) = 2t + 3$ holds. □

Lemma 12. *Suppose that a symmetric directed tree $T_1^* \in \Psi$. Then the followings hold:*

(1) $d_{T_1^*}(v_i) = \Delta(T_1^*)$, $2 \leq i \leq t$;

(2) $\Delta(T_1^*) = 2t + 3$;

(3) $d_{T_1'}(v) \geq 2$, $d_{T_1'}(v_1) \leq \Delta(T_1^*) - 1$.

Proof. The proof of (1) and (2) is similar as in Lemma 11, we omit it here. It is easy to get $d_{T_1'}(v_1) \leq \Delta(T_1^*) - 1$. Let $\{F_1, F_2, ..., F_{\Delta(T_1^*)}\}$ be a linear 2-diforests decomposition of T_1'. Suppose that $d_{T_1'}(v) = 1$. Without loss of generality, we assume that $x'yx \in F_1$ and $vy \in F_2$. Let $F_1' = F_1 - yx$ and $F_2' = F_2 + yx$. Then we obtain a new linear 2-diforests decomposition $\{F_1', F_2', F_3, ..., F_{\Delta(T_1^*)}\}$ of T_1', a contradiction to the definition of graphs in Ψ. □

4. The linear 2-arboricity of symmetric directed trees

In the following lemma, we generalize the definition of grafting graphs and study the linear 2-arboricity of symmetric directed trees not belonging to Φ and Ψ.

Lemma 13. *Let T^* be a symmetric directed tree which is not in Φ and Ψ. Suppose that in a rooted tree of T^*, x is a key vertex adjacent to a non-leaf vertex y and s leaves x_1, x_2,..., x_s $(s \geq 1)$. We define the generalized grafting graphs T' as follows:*

- *If $s \le \Delta(T^*) - 2$, let $T^{'} = T^* - \{x_1, x_2, ..., x_s\}$;*
- *If $s = \Delta(T^*) - 1$, let $V(T^{'}) = V(T^*) - \{x_1, x_2, ..., x_s\} + \{x'\}$ where x' is a new vertex, and the arcs of T' contain $x'y, yx'$ and all the arcs of T^* except those incident with $x_1, x_2, ..., x_s$.*

Then, $\overrightarrow{la}_2(T^*) \le \Delta(T^*)$ if and only if $\overrightarrow{la}_2(T^{'}) \le \Delta(T^*)$.

Proof. Suppose that $\overrightarrow{la}_2(T^*) \le \Delta(T^*)$. If $s \le \Delta(T^*) - 2$, then $T^{'}$ is a subdigraph of T^*. Thus $\overrightarrow{la}_2(T^{'}) \le \overrightarrow{la}_2(T^*) \le \Delta(T^*)$ by Lemma 1. If $s = \Delta(T^*) - 1$. We use the 2-paths $xyx^{'}$ and $x^{'}yx$ to replace the 2-paths yxx_i and x_jxy $(i, j \in \{1, 2, ..., \Delta(T^*) - 1\})$ respectively and delete all the other paths incident with x and x_i $(1 \le i \le s)$ in the minimum linear 2-diforests decomposition of T^*. As a result, we get a linear 2-diforests decomposition of $T^{'}$ with at most $\Delta(T^*)$ linear 2-diforests.

Now we suppose that $\overrightarrow{la}_2(T^{'}) \le \Delta(T^*)$.

We assume $s \le \Delta(T^*) - 2$. If $\Delta(T^*) = 3$ and $|V(T^*)| = 5$, then T^* is isomorphic to $S^*_{2,1} \in \Phi$. If $\Delta(T^*) = 3$ and $|V(T^*)| \ge 6$, $S^*_{2,1}$ is an induced subdigraph of $T^{'}$. But $\overrightarrow{la}_2(T^{'}) = 4 > 3 = \Delta(T^*)$ by Theorem 8.

If $\Delta(T^*) \ge 4$, let $\{F_1, F_2, ..., F_{\Delta(T^*)}\}$ be a linear 2-diforests decomposition of $T^{'}$. We assume that the arcs $yx \in F_1$ and $xy \in F_2$ without loss of generality. Then, let $X = \{x, x_1, x_2, ..., x_s\}$ and the induced subdigraph $T^*[X]$ which is a symmetric directed star can form $\Delta(T^*) - 2$ directed path P_i $(3 \le i \le \Delta(T^*))$ of lengths at most two. Let $F^{'}_i = F_i \cup P_i$ $(3 \le i \le \Delta(T^*))$. Thus $\{F_1, F_2, F^{'}_3..., F^{'}_{\Delta(T^*)}\}$ is a linear 2-diforest decomposition of T^* .

Then, we consider $s = \Delta(T^*) - 1$. We make a discussion on the degree of y in $T^{'}$.

Suppose that $d_{T'}(y) = 3$ and $z(\ne x, x')$ is a vertex adjacent to y in $T^{'}$. If z is a leaf, then we claim that $d_{T^*}(x) \ge 4$. Otherwise, T^* is isomorphic to $S^*_{2,1}$ or a directed 3-path with $\overrightarrow{la}_2(T^{'}) = \Delta(T^*) + 1$. Thus $\overrightarrow{la}_2(T^*) \le \Delta(T^*)$ by Theorem 7. If z is not a leaf, then there must exist a vertex v adjacent to z in $T^{'}$. Let $\{F_1, F_2, ..., F_{\Delta(T^*)}\}$ be a linear 2-diforests decomposition of $T^{'}$. Assume the 2-paths $x^{'}yx$, xyz and $zyx^{'}$ are in F_1, F_2, F_3 respectively.

In the following, we construct $\Delta(T^*)$ linear 2-diforests in T^* by the $\Delta(T^*)$ linear 2-diforests of T'. We only need to consider three cases.

Case 1.1. Suppose that the 1-path zv is in some F_i, without loss of generality, we say F_4. We move yz from F_2 into F_4. Next, we add $x_{\Delta(T^*)-1}x$ into F_2 to form a 2-path $x_{\Delta(T^*)-1}xy$, and delete $yx^{'}$ from F_3. Thus we obtain some new linear 2-diforests denoted by $F^{'}_2$, $F^{'}_3$ and $F^{'}_4$, corresponding to F_2, F_3, F_4 resectively. We construct $F^{'}_1 = F_1 - x'yx + yxx_{\Delta(T^*)-1}$. Then let $X = \{x, x_1, x_2, ..., x_{\Delta(T^*)-2}\}$ and the induced subdigraph $T^*[X]$ which is a symmetric directed star can form $\Delta(T^*) - 2$ directed paths P_i $(3 \le i \le \Delta(T^*))$ of lengths at most two. Let $F^{''}_i = F^{'}_i \cup P_i (i = 3, 4)$, $F^{'}_i = F_i \cup P_i (i \in \{5, ..., \Delta(T^*)\})$. Thus$\{F^{'}_1, F^{'}_2, F^{''}_3, F^{''}_4, F^{'}_5, ..., F^{'}_{\Delta(T^*)}\}$ is a linear 2-diforest decomposition of T^*.

Note that, in Case 1.1, the approach of constructing a linear 2-diforests decomposition of T^* is firstly adjusting the original linear 2-diforests decomposition of T', then partitioning the arcs of $T^*[X]$ into directed paths of lengths at most two and finally combining these paths with the adjusted linear 2-diforests of T'. And the approach in the following cases is similar, thus we will show the details of the adjusting step and omit the details of the other two steps.

Case 1.2. Suppose that h is another vertex adjacent to z and without loss of generality, the 2-path $vzh \in F_4$. Let $T^{'}_v$ be the component of $T^{'} - z$ containing v. The subsets of $F_i (i = 2, 4)$ containing all the linear diforests of $T_v^{'}$ are denoted by $F_i^v(i = 2,4)$. Let $F_2' = F_2 - yz + vz + F_4^v - F_2^v$ and $F_4' = F_4 + yz - vz + F_2^v - F_4^v$. Then similarly as in Case 1.1, we decompose $T^*[X]$ into $\Delta(T^*) - 2$ directed paths of lengths at most two and can construct a linear 2-diforest decomposition of T^*.

Case 1.3. Suppose that the vertex h is adjacent to v and without loss of generality, the 2-path $zvh \in F_4$. Let $F_2' = F_2 - xy$ and $F_4' = F_4 + xy$. Then we can construct a linear 2-diforest decomposition of T^* similarly as in Case 1.1.

Then we assume that $d_{T'}(y) \geq 4$. Let $v_1, v_2, ..., v_t$ be the vertices which are adjacent to y (except x, x'). We know that $t \geq 2$. Let $\{F_1, F_2, ..., F_{\Delta(T^*)}\}$ be a linear 2-diforest decomposition of $T^{'}$. We only need to consider the following three cases and other cases are similar to proof.

Case 2.1. Suppose that the 2-paths $x^{'}yx$, xyv_1, $v_2yx^{'}$ are in F_1, F_2, F_3 respectively. Let $T_{v_1}^{'}$ be the component of $T^{'} - y$ that contains v_1. The subsets of $F_i(i = 2,3)$ containing all the linear diforests of $T_{v_1}^{'}$ are denoted by $F_i^{v_1}(i = 2,3)$. Let $F_2^{'} = F_2 - yv_1 + yx' - F_2^{v_1} + F_3^{v_1}$ and $F_3^{'} = F_3 + yv_1 - yx' + F_2^{v_1} - F_3^{v_1}$. Then similarly as in Case 1.1, we can construct a linear 2-diforest decomposition of T^*.

Case 2.2. Suppose that the 2-paths $x^{'}yx$, xyv_1, $v_1yx^{'}$ are in F_1, F_2, F_3 respectively and the 2-path $v_2yv_3 \in F_4$. Let $T_{v_3}^{'}$ be the directed subtree of $T^{'} - y$ that contains v_3. The subsets of $F_i(i = 3,4)$ containing all the linear diforests of $T_{v_3}^{'}$ are denoted by $F_i^{v_3}$. Let $F_3^{'} = F_3 - yx' + yv_3 - F_3^{v_3} + F_4^{v_3}$ and $F_4^{'} = F_4 + yx' - yv_3 + F_3^{v_3} - F_4^{v_3}$. Then similarly as in Case 1.1, we can construct a linear 2-diforest decomposition of T^*. The proof also works when the 1-path $v_2y \in F_4$.

Case 2.3. Suppose that the 2-paths $x^{'}yx$, xyv_1, $v_1yx^{'}$ are in F_1, F_2, F_3 respectively. We also suppose that there is no 2-path v_ayv_b $(2 \leq a, b \leq t)$ and no 1-path v_ay or yv_a $(2 \leq a \leq t)$ in any linear 2-diforest F_i $(1 \leq i \leq \Delta(T^*))$. It is easy to check that the number of linear 2-diforests, in which y is not an isolated vertex, is odd. When $\Delta(T^*)$ is even, we can find a linear 2-diforest in which y is an isolated vertex, denoted by F_4. We then take away xy from F_2 and put it into F_4 and similarly as in Case 1.1 we can construct a linear 2-diforest decomposition of T^*. When $\Delta(T^*)$ is odd, let S_3^* be an induced symmetric subtree of $T^{'}$ where $V(S_3^*) = \{x, y, v_1\}$. Due to $T^* \notin \Phi \cup \Psi$, we can find a linear 2-diforest decomposition of T' such that the arcs of S_3^* are in four different linear 2-diforests. Then we can construct a linear 2-diforest decomposition of T^* similarly as in Case 1.1.

Since we obtain a linear 2-diforest decomposition of T^* with size $\Delta(T^*)$ for each case, $\overrightarrow{la}_2(T^*) \leq \Delta(T^*)$ holds. □

We delete all leaves of a symmetric directed tree T^* and denote the remaining graph by $I(T^*)$. A symmetric directed tree T^* with maximum degree $\Delta(T^*)$ and having more than two vertices is called $\Delta(T^*)$-*critical* if $d_{T^*}(x) + d_{I(T^*)}(x) = \Delta(T^*) + 1$ for any vertex x in $I(T^*)$.

Lemma 14. *If T^* (not belonging to Φ and Ψ) is a $\Delta(T^*)$-critical symmetric directed tree, then* $\overrightarrow{la}_2(T^*) = \Delta(T^*) + 1$.

Proof. Let $|I(T^*)| = n$. We make an induction on n to prove $\overrightarrow{la}_2(T^*) \geq \Delta(T^*) + 1$. For $n = 2$, T^* is a symmetric directed double-star, and $\overrightarrow{la}_2(T^*) = \Delta(T^*) + 1$ by Theorem 7.

Suppose that T^* is an $\Delta(T^*)$-critical tree with $n \geq 3$. We choose a key vertex x. By the definition of $\Delta(T^*)$-critical tree, it is adjacent to $\Delta(T^*) - 1$ leaves and a non-leaf y. Let $T^{'}$ be the grafting graph of T^*. Observe that $x \in V(I(T^*))$ and $x \notin V(I(T^{'}))$. Moveover, for any vertex $v \in I(T^{'})$, we have $d_{T'}(v) =$

$d_{T^*}(v)$ and $d_{I(T')}(v) = d_{I(T^*)}(v)$, except y which satisfies $d_{T'}(y) = d_{T^*}(y) + 1$, $d_{I(T')}(y) + 1 = d_{I(T^*)}(y)$. Thus, for any vertex $v \in I(T')$, we have $d_{T'}(v) + d_{I(T')}(v) = d_{T^*}(v) + d_{I(T^*)}(v) = \Delta(T^*) + 1$.

Next we show that $\Delta(T') = \Delta(T^*)$. If $d_{T'(y)} \geq \Delta(T^*) + 1$, then $d_{T^*}(y) = \Delta(T^*)$ and $n = 2$, which conflicts with $n \geq 3$. Thus $\Delta(T') \leq \Delta(T^*)$. There must be a vertex z in T' with $\Delta(T^*) - 1$ leaves, then $d_{T'}(z) = \Delta(T^*)$. Hence $\Delta(T') = \Delta(T^*)$ and T' is a $\Delta(T')$-critical tree.

According to the induction, $\overrightarrow{la}_2(T') \geq \Delta(T')+1 = \Delta(T^*)+1$. Therefore, we have $\overrightarrow{la}_2(T^*) \geq \Delta(T^*)+1$ by Lemma 13. □

Lemma 15. *Suppose that a symmetric directed tree $T_1^* \in \Phi$ or Ψ. Then T_1^* contains no $\Delta(T_1^*)$-critical tree as its induced symmetric subdigraph.*

Proof. Let T_1' be the grafting graph of T_1^*. And it is easy to check that $T_1' \notin \Phi, \Psi$.

Assume $\Delta(T_1^*) = 3$. Suppose that $|T_1'| \geq 5$, then $S_{2,1}^*$ is an induced subdigraph of T_1'. So $\overrightarrow{la}_2(T_1') \geq 4 = \Delta(T_1^*) + 1$, a contradiction to the definition of T_1^*. If $|T_1'| = 4$, then T_1^* is isomorphic to $S_{2,1}^*$. Thus the induced subdigraph of T_1^* includes no $\Delta(T_1^*)$-critical tree.

Since $\Delta(T_1^*)$ is odd, now we assume that $\Delta(T_1^*) \geq 5$. We claim that the induced subdigraph of T_1' includes no $\Delta(T_1^*)$-critical tree. Otherwise, by Lemma 14, $\overrightarrow{la}_2(T_1') = \Delta(T_1^*) + 1$, which is a contradiction to the definition of T_1^*. To the contrary, we assume that T_1^* has a $\Delta(T_1^*)$-critical subtree T_s^*. Since the induced subdigraph of T_1' includes no $\Delta(T_1^*)$-critical tree, T_s^* must contain the vertices $x, x_1, x_2, ..., x_{\Delta(T_1^*)-1}$. We then obtain a grafting graph T_s' from T_s^* by deleting $x_1, x_2, ..., x_{\Delta(T_1^*)-1}$ and adding a new vertex x' adjacent to y in both directions. Clearly, T_s' is a $\Delta(T_1^*)$-critical subtree in T_1', a contradiction. □

Theorem 16. *Suppose that T^* is a symmetric directed tree not belonging to Φ and Ψ, and $\Delta(T^*) \geq 4$. $\overrightarrow{la}_2(T^*) = \Delta(T^*)$ if and only if T^* contains no $\Delta(T^*)$-critical tree as its induced symmetric subdigraph.*

Proof. Suppose that $\overrightarrow{la}_2(T^*) = \Delta(T^*)$. If T^* contains a $\Delta(T^*)$-critical tree T_0^*, then $\overrightarrow{la}_2(T^*) \geq \overrightarrow{la}_2(T_0^*) = \Delta(T^*) + 1$ by Lemma 14 and Theorem 1, which is a contradiction. Therefore there is no $\Delta(T^*)$-critical tree in T^* as an induced symmetric subdigraph.

Now we assume that the induced symmetric subdigraph of T^* includes no $\Delta(T^*)$-critical tree and $\overrightarrow{la}_2(T^*) = \Delta(T^*) + 1$. Let T_{min}^* be a minimal symmetric subtree of T^* with $\overrightarrow{la}_2(T_{min}^*) = \Delta(T^*) + 1$. Recall that, $I(T_{min}^*)$ is a subtree of T_{min}^* by deleting all leaves. We have $|V(I(T_{min}^*))| \geq 2$. Otherwise, if $|V(I(T_{min}^*))| = 1$, then $\Delta(T_{min}^*) > \Delta(T^*)$, a contradiction.

Assume that a vertex x adjacent to $x_1, x_2, ..., x_s$ in T_{min}^* satisfies $d_{T_{min}^*}(x) + d_{I(T_{min}^*)}(x) \leq \Delta(T^*)$. Among all $x_1, x_2, ..., x_s$ in T_{min}^*, we assume that $x_1, x_2, ...x_r$ are in $I(T_{min}^*)$. Obviously, $1 \leq r \leq s$ and $s+r \leq \Delta(T^*)$.

We assume that $s - r \geq 2$. For each $1 \leq i \leq r$, we construct a symmetric directed tree $T_{min_i}^*$ which is the component of $T_{min}^* - x$ containing x_i and includes the arcs xx_i and x_ix. We know that $\overrightarrow{la}_2(T_{min_i}^*) \leq \Delta(T^*)$ according to the minimality of T_{min}^*. Assume that $\{DF_{i,1}, DF_{i,2}, ..., DF_{i,\Delta(T^*)}\}$ is a linear 2-diforests decomposition of $T_{min_i}^*$, where xx_i is in $DF_{i,i}$ and x_ix is in $DF_{i,i+r}$ for $1 \leq i \leq r$. Let $X = \{x, x_{r+1}, x_{r+2}, ..., x_s\}$. Since $s - r \leq \Delta(T^*) - 2r$, $T^*[X]$ can be decomposed into $\Delta(T^*) - 2r$ directed paths $P_i(i \in \{2r+1, 2r+2, ..., \Delta(T^*)\})$ of lengths at most two. Let $DF_j = \cup_{1\leq i\leq r} DF_{i,j}$ for $1 \leq j \leq 2r$ and $DF_j = \cup_{1\leq i\leq r} DF_{i,j} \cup \{P_j\}$ for $2r + 1 \leq j \leq \Delta(T^*)$. Thus $\{DF_1, DF_2, ..., DF_{\Delta(T^*)}\}$ is a linear 2-diforests decomposition of T_{min}^*. Therefore, $\overrightarrow{la}_2(G) = \Delta(T^*)$, a contradiction.

Assume that $s - r = 1$. Clearly, $r = 1$, $s = 2$. We can repeat the construction above by $\Delta(T^*) \geq 4$ and get a contradiction.

Hence, for any vertex $x \in T^*_{min}$, $d_{T^*_{min}}(x) + d_{I(T^*_{min})}(x) \geq \Delta(T^*) + 1$.

Suppose that D^* is a minimal symmetric subtree of $I(T^*_{min})$ such that $d_{D^*}(v) + d_{I(D^*)}(v) \geq \Delta(T^*) + 1$ for any vertex v in $I(D^*)$. We assume there is a vertex x of D^* satisfying $d_{D^*}(x) + d_{I(D^*)}(x) \geq \Delta(T^*) + 2$. If y is a leaf of D^* adjacent to x, then we delete y to obtain a smaller symmetric directed tree $D^* - y$ such that $d_{D^*-y}(v) + d_{I(D^*-y)}(v) \geq \Delta(T^*) + 1$ for any vertex v in $I(D^* - y)$, a contradiction. If y is not a leaf adjacent to x, then we obtain a smaller symmetric directed tree D^*_x by deleting all components of $D^* - y$ containing no vertex x. And $d_{D^*_x}(v) + d_{I(D^*_x)}(v) \geq \Delta(T^*) + 1$ for any vertex $v \in I(D^*_x)$, a contradiction. Therefore, for any vertex $v \in I(D^*)$, $d_{D^*}(v) + d_{I(D^*)}(v) = \Delta(T^*) + 1$. Thus D^* is a $\Delta(T^*)$-critical tree in T^*, a contradiction. Therefore, $\overrightarrow{la}_2(T^*) = \Delta(T^*)$. □

Based on the results above, we have the following fully characterization for the linear 2-arboricity of symmetric directed trees.

Theorem 17. *Suppose that T^* is a symmetric directed tree.*

(1) If $\Delta(T^*)$ is even, then $\overrightarrow{la}_2(T^*) = \Delta(T^*)$ if and only if T^* contains no $\Delta(T^*)$-critical tree as its induced symmetric subdigraph;

(2) If $\Delta(T^*)$ is odd and bigger than four, then $\overrightarrow{la}_2(T^*) = \Delta(T^*)$ if and only if T^* contains no $\Delta(T^*)$-critical tree as its induced symmetric subdigraph and $T^* \notin \Phi \cup \Psi$.

5. Conclusions

In this paper, we fully characterize the symmetric directed trees whose linear 2-arboricity is equal to the maximum degree or the maximum degree plus one. By Lemma 2, we know that for the symmetric directed tree T^*, if $\overrightarrow{la}_2(T^*) = \Delta(T^*)$, then $\overrightarrow{la}_k(T^*) = \Delta(T^*)$ for any $k \geq 3$. And it is easy to check, if $T^* \in \Phi \cup \Psi$, then $\overrightarrow{la}_k(T^*) = \Delta(T^*)$ when $k \geq 3$. Therefore, to fully determine the linear k-arboricity of symmetric directed trees for any $k \geq 3$, we only need to consider the symmetric directed tree T^* which contains at least one $\Delta(T^*)$-critical tree as its induced symmetric subdigraph and to figure out for which k, $\overrightarrow{la}_k(T^*) = \Delta(T^*)$.

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Conflict of interest

The authors declare that they have no competing interests.

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