Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(12): 20660–20683.
DOI: 10.3934/math.20221133
Received: 15 May 2022
Revised: 05 August 2022
Accepted: 21 August 2022
Published: 23 September 2022

Research article

Regularization of a final value problem for a linear and nonlinear biharmonic equation with observed data in L^q space

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Abstract: In this work, we focus on the final value problem of an inverse problem for both linear and nonlinear biharmonic equations. The aim of this study is to provide a regularized method for the bi-harmonic equation, once the observed data are obtained at a terminal time in $L^q(\Omega)$. We obtain an approximated solution using the Fourier series truncation method and the terminal input data in $L^q(\Omega)$ for $q \neq 2$. In comparision with previous studies, the most highlight of this study is the error between the exact and regularized solutions to be estimated in $L^q(\Omega)$; wherein an embedding between $L^q(\Omega)$ and Hilbert scale spaces $\mathcal{H}^\rho(\Omega)$ is applied.

Keywords: biharmonic equations; Fourier truncation method; inverse problem; regularization; Sobolev embedding
Mathematics Subject Classification: 35A05, 35A08, 26A33, 35B05, 35B65, 35R11

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N = 1, 2, 3)$ a sufficiently smooth boundary $\partial\Omega$. In this study, we consider a nonlinear biharmonic equation, as follows:

$$\Delta^2 u \equiv u_{tttt} + 2u_{ttxx} + u_{xxxx} = G(x, t, u(x, t)), \quad \text{in} \quad \Omega \times (0, T], \tag{1.1}$$

satisfying the following boundary conditions:

$$u(x, t) = \Delta u(x, t) = 0, \quad \partial\Omega \times (0, T), \tag{1.2}$$

and

$$\begin{cases} u(x,T) = f(x), & \dfrac{\partial u}{\partial t}(x,T) = 0, \quad \text{in} \quad \Omega, \\ \Delta u(x,T) = g(x), & \dfrac{\partial \Delta u}{\partial t}(x,T) = 0, \quad \text{in} \quad \Omega, \end{cases} \tag{1.3}$$

wherein the condition (1.2) is the Navier boundary condition; and the condition (1.3) is the mixed Dirichlet-Neumann boundary condition. The function $u = u(x,t)$ represents a concentration of contaminant at a position x and at time t. The data $f, g \in L^q(\Omega)$ and $G \in L^\infty(0,T; L^q(\Omega))$ are defined later on. However, in actual conditions, there are always included errors in the measurement methods of a physical process, so we have the following conditions:

$$\|f_\delta - f\|_{L^q(\Omega)} + \|g_\delta - g\|_{L^q(\Omega)} + \left\|G_\delta - G\right\|_{L^\infty(0,T;L^q(\Omega))} \leq \delta. \tag{1.4}$$

The biharmonic equation plays an important role in engineering and physics. It arises in the deformation of thin plates, the motion of fluids, free boundary problems and nonlinear elasticity, see [1–5]. Therefore, the biharmonic equation has a long history of research. It has been studied by many authors at early time. The most highlighted studies on numerical methods for the biharmonic equation are described in [5–10]. In particular, Smith [6] presented a numerical method for solving the biharmonic difference equation using finite difference methods. Ehrlich [7] has improved the iteration scheme to lead the Smith's result to be a special case of his study. Recently, Tuan et al. [11] have studied an approximate solution for a nonlinear biharmonic equation with discrete random data. Especially, in applications to radar imaging, Matevossian et al. [12,13] have focused on the solution of the biharmonic equation with the Dirichlet, Neumann and Cauchy boundary value problems for the Poisson equation using the scattering model.

Regarding the regularization for biharmonic equations, the authors in [14] considered a nonlinear biharmonic equation, and proved that problem (1.1) under the conditions (1.2) and (1.3) is ill-posed in the sense of Hadamard, and showed the error estimates. The corresponding regularized solutions in their study are strongly converged to the exact solution in $L^2(\Omega)$ under some priori assumptions on the solution. Besides, there are many other studies on linear homogeneous biharmonic equations; however, most of previous studies are focused on the regularization for biharmonic equations in $L^q(\Omega)$ with $q = 2$; and the convergent rate in $L^q(\Omega)$, with $q \neq 2$ is still not well implemented (Nam et al. [14]). Therefore, it can be stated that our study in this paper is one of the first results regarding the inverse problem for the biharmonic equation, once the observed data is obtained in the $L^q(\Omega)$ space with $q \neq 2$. The main objective of this study is to establish regularized solutions for problem (1.1) under the conditions (1.2) and (1.3) and showed the regularized solution is converged to the exact solution; in the linear case refered to (3.13), and in the nonlinear case referred to (3.73).

For evaluation cases in $L^q(\Omega)$ spaces, the most obstacle is unable to use Parseval's equality; therefore, we applied the embedding between $L^q(\Omega)$ and Hilbert scales spaces $\mathcal{H}^\rho(\Omega)$ to overcome this limitation; and Lemma 2.1 will be used throughout this article. The manuscript is proceeding as follows:

- The first part deals with the inverse problem with a defined source function. In this subsection, we introduce the mild solution of problem (1.1) under the conditions (1.2) and (1.3) with the observed data $f_\delta, g_\delta \in L^q(\Omega)$, and $G_\delta \in L^\infty(0,T; L^q(\Omega))$. Then, applying the Fourier series truncation method, we estimate the error between the regular and exact solution in the $L^{\frac{2N}{N-4\rho}}(\Omega)$.

- The second part of the manuscript investigates the inverse initial value problem for problem (1.1) under the conditions (1.2) and (1.3) with a nonlinear source function. In this section, the main results to be obtained are theorems: (i) The existence and the well-posedness of regularized solutions using Banach fixed point theorem; and (ii) The convergent rate between the regularized solution and the exact solution through the estimation of $\left\|\widetilde{V}^{\delta}_{\mathcal{N}_\delta}(\cdot,t) - u(\cdot,t)\right\|_{L^{\frac{2N}{N-4\rho}}(\Omega)}$.

Hence, this manuscript is organized as follows. In Section 2, some preliminaries such as definition and Lemmas are given. Section 3 introduces some results on regularization of problem (1.1) under the conditions (1.2) and (1.3) in the linear and non-linear cases. Numerical examples is described in Section 4 associate with observed data in $L^q(\Omega)$.

2. Preliminaries

We begin this section by introducing some preliminary definitions and basic lemmas that are needed for our analysis.

Definition 2.1. *Assume that $-\Delta$ has the eigenvalues $\lambda_k,\ k \in \mathbb{N}^*$:*

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots \to \infty, \tag{2.1}$$

and the corresponding eigenelements $e_k(x)$, which form an orthonormal basis in $L^2(\Omega)$.

Definition 2.2. *Let $\langle \cdot, \cdot \rangle$ be an inner product in $L^2(\Omega)$. The notation $\|\cdot\|_X$ stands for in the norm in the Banach space. We denote by $L^q(0,T;X)$, $1 \le q < \infty$, the Banach space of real-valued functions $u : (0,T) \to X$ measurable, providing that*

$$\|u\|_{L^q(0,T;X)} = \Big(\int_0^T \big\|u(t)\big\|_q dt \Big)^{\frac{1}{q}} < \infty, \ \ for\ 1 \le q < \infty, \tag{2.2}$$

while

$$\|u\|_{L^\infty(0,T;X)} = \operatorname{ess}\sup_{t\in(0,T)} \|u(t)\|_X, \quad for\ q = \infty.$$

Definition 2.3. *(see [8]) For any $\sigma \ge 0$, we also define the space*

$$\mathcal{H}^\sigma(\Omega) = \Big\{ u \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2\sigma} \big|\langle u, e_k \rangle\big|^2 < +\infty \Big\}, \tag{2.3}$$

then $\mathcal{H}^\sigma(\Omega)$ is a Hilbert space endowed with the norm

$$\|u\|_{\mathcal{H}^\sigma(\Omega)} = \Big(\sum_{k=1}^{\infty} \lambda_k^{2\sigma} \big|\langle u, e_k \rangle\big|^2 \Big)^{\frac{1}{2}}. \tag{2.4}$$

Lemma 2.1. *(see [15]) The following inclusions hold true:*

$$\begin{aligned} &L^q(\Omega) \hookrightarrow \mathcal{H}^\sigma(\Omega),\ if\ -\frac{d}{4} < \sigma \le 0,\ \ q \ge \frac{2d}{d-4\sigma}, \\ &\mathcal{H}^\sigma(\Omega) \hookrightarrow L^q(\Omega),\ if\ 0 < \sigma \le \frac{d}{4},\ \ q \le \frac{2d}{d-4\sigma}. \end{aligned} \tag{2.5}$$

3. Results on regularization of the biharmonic equation in L^q spaces

First of all, we present the formula of a mild solution of problem (1.1) as follows.

3.1. Mild solution of problem (1.1)

The solution of problem (1.1) can be written in the following Fourier series form:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)e_k(x), \quad \text{where} \quad u_k(t) = \langle u(\cdot,t), e_k\rangle\cdot \tag{3.1}$$

We have a particular solution of problem (1.1) in the form

$$\begin{aligned} u_k(t) = &\cosh\left(\sqrt{\lambda_k}(T-t)\right)\langle f, e_k\rangle + \frac{(T-t)\sinh\left(\sqrt{\lambda_k}(T-t)\right)}{2\sqrt{\lambda_k}}\langle g, e_k\rangle \\ &+ \int_t^T \frac{(r-t)\cosh\left(\sqrt{\lambda_k}(r-t)\right)}{2\lambda_k}\langle G(\cdot,r), e_k\rangle dr \\ &+ \int_t^T \frac{\sinh\left(\sqrt{\lambda_k}(r-t)\right)}{2\lambda_k\sqrt{\lambda_k}}\langle G(\cdot,r), e_k\rangle dr. \end{aligned} \tag{3.2}$$

Substituting the result into (3.1), we will have the formal solution of problem (1.1). The following steps, we are going to find the mild solution of problem (1.1) when the source function is linear and nonlinear.

3.2. Mild solution for a linear source function

By applying the Fourier truncation method, we provide a regularization solution as follows:

$$\begin{aligned} u_\delta^{\mathcal{N}_{tr}}(x,t) = &\sum_{k=1}^{\mathcal{N}_{tr}} \cosh\left(\sqrt{\lambda_k}(T-t)\right)\langle f_\delta, e_k\rangle e_k(x) + \sum_{k=1}^{\mathcal{N}_{tr}} \frac{(T-t)\sinh\left(\sqrt{\lambda_k}(T-t)\right)}{2\sqrt{\lambda_k}}\langle g_\delta, e_k\rangle e_k(x) \\ &+ \sum_{k=1}^{\mathcal{N}_{tr}} \left(\int_t^T \frac{(r-t)\cosh\left(\sqrt{\lambda_k}(r-t)\right)}{2\lambda_k}\langle G_\delta(\cdot,t), e_k\rangle dr\right) e_k(x) \\ &+ \sum_{k=1}^{\mathcal{N}_{tr}} \left(\int_t^T \frac{\sinh\left(\sqrt{\lambda_k}(r-t)\right)}{2\lambda_k\sqrt{\lambda_k}}\langle G_\delta(\cdot,t), e_k\rangle dr\right) e_k(x), \end{aligned} \tag{3.3}$$

whereby $\mathcal{N}_{tr}$ is a parameter regularization which will be defined later. From (3.3), it allows us to deduce that the mild solution to problem (1.1) in the following form:

$$u(x,t) = \sum_{k=1}^{\infty} \cosh\Big(\sqrt{\lambda_k}(T-t)\Big)\langle f, e_k\rangle e_k(x) + \frac{(T-t)\sinh\Big(\sqrt{\lambda_k}(T-t)\Big)}{2\sqrt{\lambda_k}}\langle g, e_k\rangle e_k(x)$$
$$+ \sum_{k=1}^{\infty}\Bigg(\int_t^T \frac{(r-t)\cosh\Big(\sqrt{\lambda_k}(r-t)\Big)}{2\lambda_k}\langle G(\cdot,r), e_k\rangle dr\Bigg)e_k(x)$$
$$+ \sum_{k=1}^{\infty}\Bigg(\int_t^T \frac{\sinh\Big(\sqrt{\lambda_k}(r-t)\Big)}{2\lambda_k\sqrt{\lambda_k}}\langle G(\cdot,r), e_k\rangle dr\Bigg)e_k(x). \tag{3.4}$$

Theorem 3.1. *Let assume that $f_\delta, g_\delta, G_\delta \in L^q(\Omega)\times L^q(\Omega)\times L^\infty(0,T;L^q(\Omega))$ are observed data such that*

$$\|f_\delta - f\|_{L^q(\Omega)} + \|g_\delta - g\|_{L^q(\Omega)} + \Big\|G_\delta - G\Big\|_{L^\infty(0,T;L^q(\Omega))} \le \delta. \tag{3.5}$$

Let $u \in L^\infty(0,T;\mathcal{H}^{\sigma+\gamma}(\Omega))$ for any $\gamma > 0$. By choosing $\mathcal{N}_{tr} = \left(\frac{1-\alpha}{T\sqrt{C_1}}\log(\delta^{-1})\right)^N$ for any $0 < \alpha < 1$, then we have $\Big\|u_\delta^{\mathcal{N}_{tr}}(\cdot,t) - u(\cdot,t)\Big\|_{L^{\frac{2N}{N-4\sigma}}(\Omega)}$ is of order

$$\max\left\{\delta^\alpha\left(\frac{1-\alpha}{T\sqrt{C_1}}\log(\delta^{-1})\right)^{2\sigma+\frac{N}{2}\left(\frac{2}{q}-1\right)}, \left(\frac{1-\alpha}{T\sqrt{C_1}}\log(\delta^{-1})\right)^{-2\gamma}\right\}. \tag{3.6}$$

Proof. Because of the Sobolev embedding $L^q(\Omega) \hookrightarrow \mathcal{H}^{\frac{N(q-2)}{4q}}(\Omega)$, we have

$$\begin{aligned}&\Big\|f_\delta - f\Big\|_{\mathcal{H}^{\frac{N(q-2)}{4q}}(\Omega)} + \Big\|g_\delta - g\Big\|_{\mathcal{H}^{\frac{N(q-2)}{4q}}(\Omega)} + \Big\|G_\delta - G\Big\|_{L^\infty(0,T;\mathcal{H}^{\frac{N(q-2)}{4q}}(\Omega))}\\ \le& C_1\Big\|f_\delta - f\Big\|_{L^q(\Omega)} + C_1\Big\|g_\delta - g\Big\|_{L^q(\Omega)} + C_1\Big\|G_\delta - G\Big\|_{L^\infty(0,T;L^q(\Omega))}\\ \le& C_1\delta,\end{aligned} \tag{3.7}$$

with C_1 depends on N, p. Our goal in this theorem is to assess the convergence error of $\Big\|u(\cdot,t) - u_\delta^{\mathcal{N}_{tr}}(\cdot,t)\Big\|_{\mathcal{H}^\sigma(\Omega)}$. Next, we first introduce the following function:

$$\widetilde{U}^{\mathcal{N}_{tr}}(x,t) = \sum_{k=1}^{\mathcal{N}_{tr}} \cosh\Big(\sqrt{\lambda_k}(T-t)\Big)\langle f, e_k\rangle e_k(x) + \sum_{k=1}^{\mathcal{N}_{tr}} \frac{(T-t)\sinh\Big(\sqrt{\lambda_k}(T-t)\Big)}{2\sqrt{\lambda_k}}\langle g, e_k\rangle e_k(x)$$
$$+ \sum_{k=1}^{\mathcal{N}_{tr}}\Bigg(\int_t^T \frac{(r-t)\cosh\Big(\sqrt{\lambda_k}(r-t)\Big)}{2\lambda_k}\langle G(\cdot,r), e_k\rangle dr\Bigg)e_k(x)$$
$$+ \sum_{k=1}^{\mathcal{N}_{tr}}\Bigg(\int_t^T \frac{\sinh\Big(\sqrt{\lambda_k}(r-t)\Big)}{2\lambda_k\sqrt{\lambda_k}}\langle G(\cdot,r), e_k\rangle dr\Bigg)e_k(x). \tag{3.8}$$

Using the triangle inequality, we receive

$$\Big\|u_\delta^{\mathcal{N}_{tr}}(\cdot,t) - u(\cdot,t)\Big\|_{\mathcal{H}^\sigma(\Omega)} \le \Big\|u_\delta^{\mathcal{N}_{tr}}(\cdot,t) - \widetilde{U}^{\mathcal{N}_{tr}}(\cdot,t)\Big\|_{\mathcal{H}^\sigma(\Omega)} + \Big\|\widetilde{U}^{\mathcal{N}_{tr}}(\cdot,t) - u(\cdot,t)\Big\|_{\mathcal{H}^\sigma(\Omega)}. \tag{3.9}$$

Next, we evaluate (3.9) through two steps as follows:
Step 1: Estimate of $\left\|u_\delta^{\mathcal{N}_{tr}}(\cdot,t)-\widetilde{U}^{\mathcal{N}_{tr}}(\cdot,t)\right\|_{\mathcal{H}^\sigma(\Omega)}$, we find that

$$\begin{aligned}
u_\delta^{\mathcal{N}_{tr}}(x,t)-\widetilde{U}^{\mathcal{N}_{tr}}(x,t)=&\sum_{k=1}^{\mathcal{N}_{tr}}\cosh\Big(\sqrt{\lambda_k}(T-t)\Big)\langle f-f_\delta,e_k\rangle e_k(x)\\
&+\sum_{k=1}^{\mathcal{N}_{tr}}\frac{(T-t)\sinh\Big(\sqrt{\lambda_k}(T-t)\Big)}{2\sqrt{\lambda_k}}\langle g-g_\delta,e_k\rangle e_k(x)\\
&+\sum_{k=1}^{\mathcal{N}_{tr}}\left(\int_t^T\frac{(r-t)\cosh\Big(\sqrt{\lambda_k}(r-t)\Big)}{2\lambda_k}\langle G(\cdot,r)-G_\delta(\cdot,r),e_k\rangle dr\right)e_k(x)\\
&+\sum_{k=1}^{\mathcal{N}_{tr}}\left(\int_t^T\frac{\sinh\Big(\sqrt{\lambda_k}(r-t)\Big)}{2\lambda_k\sqrt{\lambda_k}}\langle G(\cdot,r)-G_\delta(\cdot,r),e_k\rangle dr\right)e_k(x)\\
=&A_1(x,t)+A_2(x,t),
\end{aligned}\tag{3.10}$$

whereby

$$\begin{aligned}
A_1(x,t)=&\sum_{k=1}^{\mathcal{N}_{tr}}\cosh\Big(\sqrt{\lambda_k}(T-t)\Big)\langle f-f_\delta,e_k\rangle e_k(x)\\
&+\sum_{k=1}^{\mathcal{N}_{tr}}\frac{(T-t)\sinh\Big(\sqrt{\lambda_k}(T-t)\Big)}{2\sqrt{\lambda_k}}\langle g-g_\delta,e_k\rangle e_k(x),\\
A_2(x,t)=&\sum_{k=1}^{\mathcal{N}_{tr}}\left(\int_t^T\frac{(r-t)\cosh\Big(\sqrt{\lambda_k}(r-t)\Big)}{2\lambda_k}\langle G(\cdot,r)-G_\delta(\cdot,r),e_k\rangle dr\right)e_k(x)\\
&+\sum_{k=1}^{\mathcal{N}_{tr}}\left(\int_t^T\frac{\sinh\Big(\sqrt{\lambda_k}(r-t)\Big)}{2\lambda_k\sqrt{\lambda_k}}\langle G(\cdot,r)-G_\delta(\cdot,r),e_k\rangle dr\right)e_k(x).
\end{aligned}\tag{3.11}$$

First of all, estimating the $A_1(x,t)$, it is easy to check that $\cosh(x)\le\exp(x)$ and $\sinh(x)\le\exp(x)$, $\forall x>0$, this implies that

$$\cosh\big(\sqrt{\lambda_k}(T-t)\big)\le\exp\big(\sqrt{\lambda_k}(T-t)\big),\quad \sinh\big(\sqrt{\lambda_k}(T-t)\big)\le\exp\big(\sqrt{\lambda_k}(T-t)\big).\tag{3.12}$$

Therefore, it gives

$$\begin{aligned}
\left\|A_1(\cdot,t)\right\|^2_{\mathcal{H}^\sigma(\Omega)}\le&2\sum_{k=1}^{\mathcal{N}_{tr}}\lambda_k^{2\sigma-\frac{N(q-2)}{2q}}\lambda_k^{\frac{N(q-2)}{2q}}\exp\big(2\sqrt{\lambda_k}T\big)\left|\langle f-f_\delta,e_k\rangle\right|^2\\
&+2\sum_{k=1}^{\mathcal{N}_{tr}}\lambda_k^{2\sigma-\frac{N(q-2)}{2q}}\lambda_k^{\frac{N(q-2)}{2q}}T^2\frac{\exp(2\sqrt{\lambda_k}T)}{4\lambda_k}\left|\langle g-g_\delta,e_k\rangle\right|^2.
\end{aligned}\tag{3.13}$$

In the fact that $\lambda_k \le C_2 k^{\frac{2}{N}}$, and noting that $\sigma \ge \frac{N}{4} - \frac{N}{2q}$, we can verify that for $k \le \mathcal{N}_{tr}$,

$$\begin{aligned}
&2\lambda_k^{2\sigma - \frac{N(q-2)}{2q}} \exp(2\sqrt{\lambda_k}T) + \frac{T^2}{2\lambda_k}\lambda_k^{2\sigma - \frac{N(q-2)}{2q}} \exp(2\sqrt{\lambda_k}T)\\
=&2\exp(2\sqrt{\lambda_k}T)\lambda_k^{2\sigma - \frac{N(q-2)}{2q}} + \frac{T^2}{2\lambda_1}\exp(2\sqrt{\lambda_k}T)\lambda_k^{2\sigma - \frac{N(q-2)}{2q}}\\
\le& C_3 \exp(2\sqrt{C_2}k^{\frac{1}{N}}T)k^{\frac{4\sigma}{N} - \frac{q-2}{q}} + C_4 \exp(2\sqrt{C_2}k^{\frac{1}{N}}T)k^{\frac{4\sigma}{N} - \frac{q-2}{q}},
\end{aligned} \tag{3.14}$$

in which $C_3 = 2C_2^{2\sigma - \frac{N(q-2)}{2q}}$ and $C_4 = \dfrac{T^2}{2\lambda_1}C_2^{2\sigma - \frac{N(q-2)}{2q}}$.

Combining (3.13) and (3.14), we have

$$\begin{aligned}
\|A_1(\cdot,t)\|^2_{\mathcal{H}^\sigma(\Omega)} &\le C_5 \exp(2T\sqrt{C_2}k^{\frac{1}{N}})k^{\frac{4\sigma}{N} - \frac{q-2}{q}}\Big(\sum_{k=1}^{\mathcal{N}_{tr}} \lambda_k^{\frac{N(q-2)}{2q}} \big|\langle f - f_\delta, e_k\rangle\big|^2 + \sum_{k=1}^{\mathcal{N}_{tr}} \lambda_k^{\frac{N(q-2)}{2q}} \big|\langle g - g_\delta, e_k\rangle\big|^2\Big)\\
&\le C_5 \exp(2T\sqrt{C_2}k^{\frac{1}{N}})k^{\frac{4\sigma}{N} - \frac{q-2}{q}}\Big(\big\|f - f_\delta\big\|^2_{\mathcal{H}^{\frac{N(q-2)}{4q}}(\Omega)} + \|g - g_\delta\|^2_{\mathcal{H}^{\frac{N(q-2)}{4q}}(\Omega)}\Big),
\end{aligned} \tag{3.15}$$

with $C_5 = 2\max\{C_3, C_4\}$. It follows from (3.7) that

$$\|A_1(\cdot,t)\|^2_{\mathcal{H}^\sigma(\Omega)} \le 2C_5 \exp\Big(2\sqrt{C_2}(\mathcal{N}_{tr})^{\frac{1}{N}}T\Big)(\mathcal{N}_{tr})^{\frac{4\sigma}{N} - \frac{q-2}{q}}\ \delta^2. \tag{3.16}$$

Next, considering the term $\|A_2(.,t)\|_{\mathcal{H}^\sigma(\Omega)}$, applying the Parseval's equality, we can see that

$$\begin{aligned}
\big\|A_2(.,t)\big\|^2_{\mathcal{H}^\sigma(\Omega)} &\le 2\sum_{k=1}^{\mathcal{N}_{tr}} \lambda_k^{2\sigma}\Big[\int_t^T (r-t)\frac{\exp\big(\sqrt{\lambda_k}(r-t)\big)}{2\lambda_k}\langle G(\cdot,r) - G_\delta(\cdot,r), e_k\rangle dr\Big]^2\\
&\quad + 2\sum_{k=1}^{\mathcal{N}_{tr}} \lambda_k^{2\sigma}\Big[\int_t^T \frac{\exp\big(\sqrt{\lambda_k}(r-t)\big)}{2\lambda_k\sqrt{\lambda_k}}\langle G(\cdot,r) - G_\delta(\cdot,r), e_k\rangle dr\Big]^2.
\end{aligned} \tag{3.17}$$

From (3.17), using the Hölder's inequality, we receive

$$\begin{aligned}
\big\|A_2(.,t)\big\|^2_{\mathcal{H}^\sigma(\Omega)} \le& 2\sum_{k=1}^{\mathcal{N}_{tr}} \lambda_k^{2\sigma}\int_t^T (r-t)^2\frac{\exp\big(2\sqrt{\lambda_k}(r-t)\big)}{4\lambda_k^2}\big|\langle G(\cdot,r) - G_\delta(\cdot,r), e_k\rangle\big|^2 dr\\
&+ 2\sum_{k=1}^{\mathcal{N}_{tr}} \lambda_k^{2\sigma}\int_t^T \frac{\exp\big(2\sqrt{\lambda_k}(r-t)\big)}{4\lambda_k^3}\big|\langle G(.,r) - G_\delta(.,r), e_k\rangle\big|^2 dr\\
\le&(2T^2+1)\sum_{k=1}^{\mathcal{N}_{tr}} \exp\big(2\sqrt{\lambda_k}T\big)\lambda_k^{2\sigma - \frac{N(q-2)}{2q}}\lambda_k^{\frac{N(q-2)}{2q}}\int_t^T \big|\langle G(.,r) - G_\delta(.,r), e_k\rangle\big|^2 dr.
\end{aligned} \tag{3.18}$$

Because of $\lambda_k \le C_2 k^{\frac{2}{N}}$, and noting that $\sigma \ge \dfrac{N}{4} - \dfrac{N}{2q}$, we can verify that for $k \le \mathcal{N}_{tr}$,

$$\lambda_k^{2\sigma - \frac{N(q-2)}{2q}} \exp\big(2\sqrt{\lambda_k}(r-t)\big) \le C_3 \exp\Big(2\sqrt{C_2}k^{\frac{1}{N}}T\Big)k^{\frac{4\sigma}{N} - \frac{q-2}{q}} \le C_3 \exp\Big(2\sqrt{C_2}(\mathcal{N}_{tr})^{\frac{1}{N}}T\Big)(\mathcal{N}_{tr})^{\frac{4\sigma}{N} - \frac{q-2}{q}}. \tag{3.19}$$

From (3.18), we noticed that

$$\begin{aligned}\|A_2(.,t)\|^2_{\mathcal{H}^\sigma(\Omega)} \le & C_3 \exp\Big(2\sqrt{C_2}(\mathcal{N}_{tr})^{\frac{1}{N}}T\Big)(\mathcal{N}_{tr})^{\frac{4\sigma}{N}-\frac{q-2}{q}}\ (2T^2+1)\sum_{k=1}^{\mathcal{N}_{tr}}\lambda_k^{\frac{N(q-2)}{2q}}\int_t^T \Big|\langle G(.,r)-G_\delta(.,r),e_k\rangle\Big|^2 dr\\ \le & C_3 \exp\Big(2\sqrt{C_2}(\mathcal{N}_{tr})^{\frac{1}{N}}T\Big)(\mathcal{N}_{tr})^{\frac{4\sigma}{N}-\frac{q-2}{q}}\ (2T^2+1)\int_t^T \Big\|G(\cdot,r)-G_\delta(\cdot,r)\Big\|^2_{\mathcal{H}^{\frac{N(q-2)}{4q}}(\Omega)}dr. \end{aligned} \quad (3.20)$$

Due to condition (3.7) and from the estimation in (3.20), one has

$$\|A_2(.,t)\|^2_{\mathcal{H}^\sigma(\Omega)} \le C_3 \exp\Big(2\sqrt{C_2}(\mathcal{N}_{tr})^{\frac{1}{N}}T\Big)(\mathcal{N}_{tr})^{\frac{4\sigma}{N}-\frac{q-2}{q}}\ (2T^3+T)\delta^2. \quad (3.21)$$

From all the estimation above, we received

$$\begin{aligned}&\Big\|u_\delta^{\mathcal{N}_{tr}}(\cdot,t)-\widetilde{U}^{\mathcal{N}_{tr}}(\cdot,t)\Big\|^2_{\mathcal{H}^\sigma(\Omega)}\\ \le & 2\|A_1(.,t)\|^2_{\mathcal{H}^\sigma(\Omega)}+2\|A_2(.,t)\|^2_{\mathcal{H}^\sigma(\Omega)}\\ \le & 2C_5\exp\Big(2\sqrt{C_2}(\mathcal{N}_{tr})^{\frac{1}{N}}T\Big)(\mathcal{N}_{tr})^{\frac{4\sigma}{N}-\frac{q-2}{q}}\ \delta^2\\ & +2C_3C_1\exp\Big(2\sqrt{C_2}(\mathcal{N}_{tr})^{\frac{1}{N}}T\Big)(\mathcal{N}_{tr})^{\frac{4\sigma}{N}-\frac{q-2}{q}}\ (2T^3+T)\delta^2. \end{aligned} \quad (3.22)$$

Step 2: Estimate of $\Big\|\widetilde{U}^{\mathcal{N}_{tr}}(\cdot,t)-u(\cdot,t)\Big\|_{\mathcal{H}^\sigma(\Omega)}$, from (3.4) and (3.8), and using the Parseval's inequality, we deduce that

$$\begin{aligned}\widetilde{U}^{\mathcal{N}_{tr}}(x,t)-u(x,t) = & \sum_{k=\mathcal{N}_{tr}+1}^{\infty}\cosh\Big(\sqrt{\lambda_k}(T-t)\Big)\langle f,e_k\rangle e_k(x)\\ & +\sum_{k=1}^{\mathcal{N}_{tr}}\frac{(T-t)\sinh\Big(\sqrt{\lambda_k}(T-t)\Big)}{2\sqrt{\lambda_k}}\langle g,e_k\rangle e_k(x)\\ & +\sum_{k=\mathcal{N}_{tr}+1}^{\infty}\Bigg(\int_t^T\frac{(r-t)\cosh\Big(\sqrt{\lambda_k}(r-t)\Big)}{2\lambda_k}\langle G(\cdot,r),e_k\rangle dr\Bigg)e_k(x)\\ & +\sum_{k=\mathcal{N}_{tr}+1}^{\infty}\Bigg(\int_t^T\frac{\sinh\Big(\sqrt{\lambda_k}(r-t)\Big)}{2\lambda_k\sqrt{\lambda_k}}\langle G(\cdot,r),e_k\rangle dr\Bigg)e_k(x). \end{aligned} \quad (3.23)$$

From (3.23), for any $\gamma>0$, we received

$$\Big\|\widetilde{U}^{\mathcal{N}_{tr}}(\cdot,t)-u(\cdot,t)\Big\|^2_{\mathcal{H}^\sigma(\Omega)} = \sum_{k=\mathcal{N}_{tr}+1}^{\infty}\lambda_k^{2\sigma}\Big|\langle u(\cdot,t),e_k\rangle\Big|^2 = \sum_{k=\mathcal{N}_{tr}+1}^{\infty}\lambda_k^{-2\gamma}\lambda_k^{2\sigma+2\gamma}\Big|\langle u(\cdot,t),e_k\rangle\Big|^2. \quad (3.24)$$

In case $k>\mathcal{N}_{tr}$, there exists a postive constant $C_6>0$ such that $\lambda_k^{-2\gamma}\le C_6k^{-\frac{4\gamma}{N}}$, we have

$$\Big\|\widetilde{U}^{\mathcal{N}_{tr}}(\cdot,t)-u(\cdot,t)\Big\|^2_{\mathcal{H}^\sigma(\Omega)} \le C_6(\mathcal{N}_{tr})^{-\frac{4\gamma}{N}}\Big\|u(\cdot,t)\Big\|^2_{\mathcal{H}^{\sigma+\gamma}(\Omega)} \le C_6(\mathcal{N}_{tr})^{-\frac{4\gamma}{N}}\|u\|^2_{L^\infty(0,T;\mathcal{H}^{\sigma+\gamma}(\Omega))}. \quad (3.25)$$

Combining (3.22) to (3.25), we conclude that

$$\begin{aligned}\left\|u_\delta^{\mathcal{N}_{tr}}(\cdot,t)-u(\cdot,t)\right\|^2_{\mathcal{H}^\sigma(\Omega)} \le & 2\left\|u_\delta^{\mathcal{N}_{tr}}(\cdot,t)-\widetilde{U}^{\mathcal{N}_{tr}}(\cdot,t)\right\|^2_{\mathcal{H}^\sigma(\Omega)}+2\left\|\widetilde{U}^{\mathcal{N}_{tr}}(\cdot,t)-u(\cdot,t)\right\|^2_{\mathcal{H}^\sigma(\Omega)}\\ \le & 8C_6\exp\Big(2T\sqrt{C_2}(\mathcal{N}_{tr})^{\frac{1}{N}}\Big)(\mathcal{N}_{tr})^{\frac{4\sigma}{N}-\frac{q-2}{q}}\ \delta^2\\ & +4C_3C_1\exp\Big(2T\sqrt{C_2}(\mathcal{N}_{tr})^{\frac{1}{N}}\Big)(\mathcal{N}_{tr})^{\frac{4\sigma}{N}-\frac{q-2}{q}}\ (2T^3+T)\delta^2\\ & +2C_6(\mathcal{N}_{tr})^{-\frac{4\gamma}{N}}\|u\|^2_{L^\infty(0,T;\mathcal{H}^{\sigma+\gamma}(\Omega))}. \end{aligned} \tag{3.26}$$

By choosing $\mathcal{N}_{tr}=\Big(\dfrac{1-\alpha}{T\sqrt{C_2}}\log\big(\dfrac{1}{\delta}\big)\Big)^N$, we need the following results:

$$\exp\big(2T\sqrt{C_2}(\mathcal{N}_{tr})^{\frac{1}{N}}\big)(\mathcal{N}_{tr})^{\frac{4\sigma}{N}-\frac{q-2}{q}}\ \delta^2=\delta^{2\alpha}\Big(\frac{1-\alpha}{T\sqrt{C_2}}\log\big(\frac{1}{\delta}\big)\Big)^{4\sigma+N\left(\frac{2}{q}-1\right)}. \tag{3.27}$$

The provision of this theorem is completed. □

3.3. Mild solution for a nonlinear source function

In this section, we will study the initial inverse problem for nonlinear of source term.

$$\begin{cases}\Delta^2 u\equiv u_{tttt}+2u_{ttxx}+u_{xxxx}=G(x,t,u(x,t)), & \text{in}\quad \Omega\times(0,T],\\ u(x,t)=\Delta u(x,t)=0, & \partial\Omega\times(0,T),\end{cases} \tag{3.28}$$

with the final condition

$$\begin{cases}u(x,T)=f(x), & \dfrac{\partial u}{\partial t}(x,T)=0, \qquad \text{in}\quad \Omega,\\ \Delta u(x,T)=g(x), & \dfrac{\partial \Delta u}{\partial t}(x,T)=0, \qquad \text{in}\quad \Omega.\end{cases} \tag{3.29}$$

We assume that $v\in L^2(\Omega)$, let $\mathcal{P}_i(z-t),\ i=\{1,2,3,4\}$ is an operator defined as follows:

$$\begin{aligned}\mathcal{P}_1(z-t)v&=\sum_{k=1}^{\infty}\cosh\big(\sqrt{\lambda_k}(z-t)\big)\langle v,e_k\rangle e_k(x),\\ \mathcal{P}_2(z-t)v&=\sum_{k=1}^{\infty}\frac{(z-t)\sinh\Big(\sqrt{\lambda_k}(z-t)\Big)}{2\sqrt{\lambda_k}}\langle v,e_k\rangle e_k(x),\\ \mathcal{P}_3(z-t)v&=\sum_{k=1}^{\infty}\frac{(z-t)\cosh\Big(\sqrt{\lambda_k}(z-t)\Big)}{2\lambda_k}\langle v,e_k\rangle e_k(x),\\ \mathcal{P}_4(z-t)v&=\sum_{k=1}^{\infty}\frac{\sinh\Big(\sqrt{\lambda_k}(z-t)\Big)}{2\lambda_k\sqrt{\lambda_k}}\langle v,e_k\rangle e_k(x).\end{aligned} \tag{3.30}$$

From the way to set the operator (3.30), the mild solution of the problem (3.28) under the condition (3.29) is as follows:

$$u(t)=\mathcal{P}_1(T-t)f+\mathcal{P}_2(T-t)g+\int_t^T\mathcal{P}_3(r-t)G(u(r))dr+\int_t^T\mathcal{P}_4(r-t)G(u(r))dr. \tag{3.31}$$

We regularized the mild solution (3.31) by Fourier method. Assume that we have $\ell \in L^2(\Omega)$, then we define $\ell_{\delta,\mathcal{N}_{tr}}$ as follows:

$$\mathbb{T}_{\mathcal{N}_\delta}\ell = \sum_{k=1}^{\mathcal{N}_\delta} \langle \ell, e_k\rangle e_k(x), \tag{3.32}$$

without loss of generality, we can completely assume that $f_\delta, g_\delta \in L^q(\Omega)$ such that $\|f - f_\delta\|_{L^q(\Omega)} + \|g - g_\delta\|_{L^q(\Omega)} \le \delta$. Therefore, we build the structure of regularized solution and symbols it is $\widetilde{V}^\delta_{\mathcal{N}_\delta}$.

$$\begin{aligned}\widetilde{V}^\delta_{\mathcal{N}_\delta}(t) =& \mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta} f_\delta + \mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta} g_\delta \\ &+ \int_t^T \mathcal{P}_3(r-t)\mathbb{T}_{\mathcal{N}_\delta} G(\widetilde{V}^\delta_{\mathcal{N}_\delta}(r))dr + \int_t^T \mathcal{P}_4(r-t)\mathbb{T}_{\mathcal{N}_\delta} G(\widetilde{V}^\delta_{\mathcal{N}_\delta}(r))dr. \end{aligned} \tag{3.33}$$

The next theorem will provide details about the existence and the well-posedness of regularized solutions.

Theorem 3.2. *Let the terminal data $\ell \in L^q(\Omega)$, then the nonlinear integral equation* (3.33) *has a unique solution $\widetilde{V}^\delta_{\mathcal{N}_\delta}(x,t) \in L^\infty(0,T; L^{\frac{2N}{N-4\rho}}(\Omega))$, then we have the following estimate:*

$$\left\|\widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t)\right\|_{L^{\frac{2N}{N-4\rho}}(\Omega)} \le \sqrt{2}(T-t)(\mathcal{N}_\delta)^{\rho-\frac{N(2-q)}{4q}} \exp\Big((\sqrt{\mathcal{N}_\delta}+m)(T-t)\Big)\|g_\delta\|_{L^q(\Omega)}, \tag{3.34}$$

where $m \ge \sqrt{\mathcal{N}_\delta} + 2\mathcal{L}_f(\mathcal{N}_\delta)^\rho \max\{T,1\}\sqrt{\mathcal{N}_\delta}$.

Proof. Let any $\ell \in \mathcal{H}^{\sigma_1}(\Omega)$, suppose that $\sigma_1 \ge \sigma$, we get

$$\begin{aligned} &\left\|\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}\ell + \mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}\ell\right\|^2_{\mathcal{H}^{\sigma_1}(\Omega)} \\ \le& 2\left\|\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}\ell\right\|^2_{\mathcal{H}^{\sigma_1}(\Omega)} + 2\left\|\mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}\ell\right\|^2_{\mathcal{H}^{\sigma_1}(\Omega)} \\ \le& 2\sum_{k=1}^{\mathcal{N}_\delta} \lambda_k^{2\sigma_1-2\sigma}\cosh^2\Big(\sqrt{\lambda_k}(T-t)\Big)\lambda_k^{2\sigma}\left|\langle \ell, e_k\rangle\right|^2 \\ &+ 2\sum_{k=1}^{\mathcal{N}_\delta} \lambda_k^{2\sigma_1-2\sigma}\frac{(T-t)^2\sinh^2(\sqrt{\lambda_k}(T-t))}{4\lambda_k}\lambda_k^{2\sigma}\left|\langle \ell, e_k\rangle\right|^2. \end{aligned} \tag{3.35}$$

From (3.35) and in view of (3.12), we receive

$$\begin{aligned} &\left\|\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}\ell + \mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}\ell\right\|^2_{\mathcal{H}^{\sigma_1}(\Omega)} \\ \le& 2(\mathcal{N}_\delta)^{2\sigma_1-2\sigma}\exp\Big(2\sqrt{\mathcal{N}_\delta}(T-t)\Big)\|\ell\|^2_{\mathcal{H}^\sigma(\Omega)} \\ &+ 2(T-t)^2(\mathcal{N}_\delta)^{2\sigma_1-2\sigma}\exp\big(2\sqrt{\mathcal{N}_\delta}(T-t)\big)\|\ell\|^2_{\mathcal{H}^\sigma(\Omega)}. \end{aligned} \tag{3.36}$$

By a similar argument above (3.36), we can find also that

$$\begin{aligned}\left\|\mathcal{P}_3(r-t)\mathbb{T}_{\mathcal{N}_\delta}\ell\right\|^2_{\mathcal{H}^{\sigma_1}(\Omega)} &= \sum_{k=1}^{\mathcal{N}_\delta}\lambda_k^{2\sigma_1-2\sigma}\frac{(r-t)^2\cosh^2\left(\sqrt{\lambda_k}(r-t)\right)}{4\lambda_k^2}\lambda_k^{2\sigma}\left|\langle\ell,e_k\rangle\right|^2\\ &\le(\mathcal{N}_\delta)^{2\sigma_1-2\sigma}(r-t)^2\exp\left(2\sqrt{\mathcal{N}_\delta}(r-t)\right)\left\|\ell\right\|_{\mathcal{H}^\sigma(\Omega)},\end{aligned} \tag{3.37}$$

and we can see that

$$\begin{aligned}\|\mathcal{P}_4(r-t)\mathbb{T}_{\mathcal{N}_\delta}\ell\|^2_{\mathcal{H}^{\sigma_1}(\Omega)} &= \sum_{k=1}^{\mathcal{N}_\delta}\lambda_k^{2\sigma_1-2\sigma}\frac{\sinh^2\left(\sqrt{\lambda_k}(r-t)\right)}{4\lambda_k^3}\lambda_k^{2\sigma}\left|\langle\ell,e_k\rangle\right|^2\\ &\le(\mathcal{N}_\delta)^{2\sigma_1-2\sigma}\exp\left(2\sqrt{\mathcal{N}_\delta}(r-t)\right)\|\ell\|_{\mathcal{H}^\sigma(\Omega)}.\end{aligned} \tag{3.38}$$

For $m>0$, we denote by $L^\infty_m(0,T;L^{\frac{2N}{N-4\rho}}(\Omega))$ the function space $L^\infty(0,T;L^{\frac{2N}{N-4\rho}}(\Omega))$ with the following norm:

$$\left\|v\right\|_m := ess\sup_{0\le t\le T}\left\|\exp\left(m(t-T)\right)v(\cdot,t)\right\|_{L^{\frac{2N}{N-4\rho}}(\Omega)},\quad v\in L^{\frac{2N}{N-4\rho}}(\Omega). \tag{3.39}$$

Next, we define a nonlinear map $\mathcal{M}: L^\infty_m(0,T;L^{\frac{2N}{N-4\rho}}(\Omega))\hookrightarrow L^\infty_m(0,T;L^{\frac{2N}{N-4\rho}}(\Omega))$ by

$$\begin{aligned}\mathcal{M}u(t) =&\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}f_\delta+\mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}g_\delta\\ &+\int_t^T\mathcal{P}_3(r-t)\mathbb{T}_{\mathcal{N}_\delta}G(u(r))dr+\int_t^T\mathcal{P}_4(r-t)\mathbb{T}_{\mathcal{N}_\delta}G(u(r))dr.\end{aligned} \tag{3.40}$$

- Case 1: $u=0$, we have $\mathcal{M}(u=0)=\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}f_\delta+\mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}g_\delta$. From Lemma 2.1 and $0\le\rho\le\frac{N}{4}$, with the Sobolev embedding $\mathcal{H}^\rho(\Omega)\hookrightarrow L^{\frac{2N}{N-4\rho}}(\Omega)$, there exists a constant C_1 depends on N,ρ such that

$$\begin{aligned}&\left\|\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}f_\delta+\mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}g_\delta\right\|_{L^{\frac{2N}{N-4\rho}}(\Omega)}\\ \le&C_7(N,\rho)\left\|\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}f_\delta+\mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}\ell\right\|_{\mathcal{H}^\rho(\Omega)}.\end{aligned} \tag{3.41}$$

From (3.41), use evaluation results in (3.36), taking square root on the both side, we choose $\sigma_1=\rho$ and $\sigma=\frac{N(2-q)}{4q}$, it gives

$$\begin{aligned}&\left\|\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}f_\delta+\mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}g_\delta\right\|_{\mathcal{H}^\rho(\Omega)}\\ \le&\sqrt{2}(\mathcal{N}_\delta)^{\rho-\frac{N(2-q)}{4q}}\exp\left(\sqrt{\mathcal{N}_\delta}(T-t)\right)\left\|f_\delta\right\|_{\mathcal{H}^{\frac{N(2-q)}{4q}}(\Omega)}\\ &+\sqrt{2}(T-t)(\mathcal{N}_\delta)^{\rho-\frac{N(2-q)}{4q}}\exp\left(\sqrt{\mathcal{N}_\delta}(T-t)\right)\left\|g_\delta\right\|_{\mathcal{H}^{\frac{N(2-q)}{4q}}(\Omega)}.\end{aligned} \tag{3.42}$$

For $1<q<2$, using Lemma 2.1, we find that $L^q(\Omega)\hookrightarrow\mathcal{H}^{\frac{N(2-q)}{4q}}(\Omega)$. Therefore, there exist a constant $C_8(N,\rho)$ such that

$$\begin{aligned}&\Big\|\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}f_\delta+\mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}g_\delta\Big\|_{\mathcal{H}^{\frac{N(2-q)}{4q}}(\Omega)}\\&\le C_8(N,\rho)\Big(\sqrt{2}(\mathcal{N}_\delta)^{\rho-\frac{N(2-q)}{4q}}\exp\Big(\sqrt{\mathcal{N}_\delta}(T-t)\Big)\Big\|f_\delta\Big\|_{L^q(\Omega)}\\&\quad+\sqrt{2}(T-t)(\mathcal{N}_\delta)^{\rho-\frac{N(2-q)}{4q}}\exp\big(\sqrt{\mathcal{N}_\delta}(T-t)\big)\Big\|g_\delta\Big\|_{L^q(\Omega)}\Big).\end{aligned}\tag{3.43}$$

Combining (3.41) to (3.43), this leads to

$$\begin{aligned}&\Big\|\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}f_\delta+\mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}g_\delta\Big\|_{L^{\frac{2N}{N-4\rho}}(\Omega)}\\&\le C_9(N,\rho)\Big(\sqrt{2}(\mathcal{N}_\delta)^{\rho-\frac{N(2-q)}{4q}}\exp\Big(\sqrt{\mathcal{N}_\delta}(T-t)\Big)\Big\|f_\delta\Big\|_{L^q(\Omega)}\\&\quad+\sqrt{2}(T-t)(\mathcal{N}_\delta)^{\rho-\frac{N(2-q)}{4q}}\exp\big(\sqrt{\mathcal{N}_\delta}(T-t)\big)\Big\|g_\delta\Big\|_{L^q(\Omega)}\Big),\end{aligned}\tag{3.44}$$

whereby $C_9(N,\rho)=C_8(N,\rho)C_7(N,\rho)$. Combining with above arguments, we deduce that

$$\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}f_\delta+\mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}g_\delta\in L^\infty_m(0,T;L^{\frac{2N}{N-4\rho}}(\Omega)).\tag{3.45}$$

- Case 2: In this case, we take two function $u_1,u_2\in L^\infty_m(0,T;L^{\frac{2N}{N-4\rho}}(\Omega))$, from (3.40), it is easy to see that

$$\begin{aligned}\mathcal{M}u_1(t)-\mathcal{M}u_2(t)=&\int_t^T\mathcal{P}_3(r-t)\Big(\mathbb{T}_{\mathcal{N}_\delta}G(u_1(r))-\mathbb{T}_{\mathcal{N}_\delta}G(u_2(r))\Big)dr\\&+\int_t^T\mathcal{P}_4(r-t)\Big(\mathbb{T}_{\mathcal{N}_\delta}G(u_1(r))-\mathbb{T}_{\mathcal{N}_\delta}G(u_2(r))\Big)dr.\end{aligned}\tag{3.46}$$

Taking any $m>0$, this implies that

$$\begin{aligned}&\Big\|\exp\big(m(t-T)\big)\Big(\mathcal{M}u_1(t)-\mathcal{M}u_2(t)\Big)\Big\|_{\mathcal{H}^\rho(\Omega)}\\&\le\int_t^T\exp\big(m(t-T)\big)\underbrace{\Big\|\mathcal{P}_3(r-t)\Big(\mathbb{T}_{\mathcal{N}_\delta}G(u_1(r))-\mathbb{T}_{\mathcal{N}_\delta}G(u_2(r))\Big)\Big\|_{\mathcal{H}^\rho(\Omega)}}_{\mathcal{D}_1(r,t)}dr\\&\quad+\int_t^T\exp\big(m(t-T)\big)\underbrace{\Big\|\mathcal{P}_4(r-t)\Big(\mathbb{T}_{\mathcal{N}_\delta}G(u_1(r))-\mathbb{T}_{\mathcal{N}_\delta}G(u_2(r))\Big)\Big\|_{\mathcal{H}^\rho(\Omega)}}_{\mathcal{D}_2(r,t)}dr.\end{aligned}\tag{3.47}$$

If we reuse estimates of (3.37) and (3.38) with $\sigma_1=\rho$ and $\sigma=0$, and in the two reviews below, we have used the Sobolev embedding $L^{\frac{2N}{N-4\rho}}(\Omega)\hookrightarrow L^2(\Omega)$. So, $\mathcal{D}_1(r,t)$ and $\mathcal{D}_2(r,t)$ can be bounded as follows:

$$\begin{aligned}\mathcal{D}_1(r,t)&\le(\mathcal{N}_\delta)^\rho(r-t)\exp\big(\sqrt{\mathcal{N}_\delta}(r-t)\big)\Big\|G(u_1(r))-G(u_2(r))\Big\|_{L^2(\Omega)}\\&\le\mathcal{L}_f(\mathcal{N}_\delta)^\rho(r-t)\exp\big(\sqrt{\mathcal{N}_\delta}(r-t)\big)\Big\|u_1(r)-u_2(r)\Big\|_{L^2(\Omega)}\\&\le\mathcal{L}_f(\mathcal{N}_\delta)^\rho(r-t)\exp\big(\sqrt{\mathcal{N}_\delta}(r-t)\big)\Big\|u_1(r)-u_2(r)\Big\|_{L^{\frac{2N}{N-4\rho}}(\Omega)},\end{aligned}\tag{3.48}$$

and by prove similarly, we obtain

$$\begin{aligned}\mathcal{D}_2(r,t) &\leq (\mathcal{N}_\delta)^\rho \exp\Big(\sqrt{\mathcal{N}_\delta}(r-t)\Big)\Big\|G(u_1(r))-G(u_2(r))\Big\|_{L^2(\Omega)}\\ &\leq \mathcal{L}_f(\mathcal{N}_\delta)^\rho \exp\Big(\sqrt{\mathcal{N}_\delta}(r-t)\Big)\Big\|u_1(r)-u_2(r)\Big\|_{L^{\frac{2N}{N-4\rho}}(\Omega)}.\end{aligned} \tag{3.49}$$

Combining (3.40) to (3.49), we have

$$\begin{aligned}&\Big\|\exp(m(t-T))\Big(\mathcal{M}u_1(t)-\mathcal{M}u_2(t)\Big)\Big\|_{\mathcal{H}^\rho(\Omega)}\\ \leq&\mathcal{L}_f(\mathcal{N}_\delta)^\rho\int_t^T \exp(m(t-T))(r-t)^2\exp\Big(2\sqrt{\mathcal{N}_\delta}(r-t)\Big)\Big\|u_1(r)-u_2(r)\Big\|_{L^{\frac{2N}{N-4\rho}}(\Omega)}dr\\ &+\mathcal{L}_f(\mathcal{N}_\delta)^\rho\int_t^T \exp(m(t-T))\exp\Big(2\sqrt{\mathcal{N}_\delta}(r-t)\Big)\Big\|u_1(r)-u_2(r)\Big\|_{L^{\frac{2N}{N-4\rho}}(\Omega)}.\end{aligned} \tag{3.50}$$

From (3.50), make a simple transformation, we get

$$\begin{aligned}&\Big\|\exp(m(t-T))\Big(\mathcal{M}u_1(t)-\mathcal{M}u_2(t)\Big)\Big\|_{\mathcal{H}^\rho(\Omega)}\\ \leq&\mathcal{L}_f(\mathcal{N}_\delta)^\rho\int_t^T \exp(m(t-r))(r-t)\exp\Big(\sqrt{\mathcal{N}_\delta}(r-t)\Big)\exp(m(r-T))\Big\|u_1(r)-u_2(r)\Big\|_{L^{\frac{2N}{N-4\rho}}(\Omega)}dr\\ &+\mathcal{L}_f(\mathcal{N}_\delta)^\rho\int_t^T \exp(m(t-r))\exp\Big(\sqrt{\mathcal{N}_\delta}(r-t)\Big)\exp(m(r-T))\Big\|u_1(r)-u_2(r)\Big\|_{L^{\frac{2N}{N-4\rho}}(\Omega)}.\end{aligned} \tag{3.51}$$

Using the fact that, we get

$$\Big\|u_1-u_2\Big\|_m := ess\sup_{0\leq r\leq T}\exp(m(r-T))\Big\|u_1(r)-u_2(r)\Big\|_{L^{\frac{2N}{N-4\rho}}(\Omega)}. \tag{3.52}$$

From (3.47) and (3.52), we follow that

$$\begin{aligned}&\Big\|\exp(m(t-T))\Big(\mathcal{M}u_1(t)-\mathcal{M}u_2(t)\Big)\Big\|_{\mathcal{H}^\rho(\Omega)}\\ \leq&\mathcal{L}_f(\mathcal{N}_\delta)^\rho\Big(\int_t^T \exp(m(t-r))(r-t)\exp\Big(\sqrt{\mathcal{N}_\delta}(r-t)\Big)dr\Big)\Big\|u_1-u_2\Big\|_m\\ &+\mathcal{L}_f(\mathcal{N}_\delta)^\rho\Big(\int_t^T \exp(m(t-r))\exp\Big(\sqrt{\mathcal{N}_\delta}(r-t)\Big)dr\Big)\Big\|u_1-u_2\Big\|_m.\end{aligned} \tag{3.53}$$

Form (3.53), we can see that

$$\Big(\int_t^T \exp(m(t-r))(r-t)\exp\Big(\sqrt{\mathcal{N}_\delta}(r-t)\Big)dr\Big) \leq T\int_t^T \exp\Big((\sqrt{\mathcal{N}_\delta}-m)(r-t)\Big)dr \leq \frac{T}{m-\sqrt{\mathcal{N}_\delta}},$$

$$\left(\int_t^T \exp\left(m(t-r)\right)\exp\Big(\sqrt{\mathcal{N}_\delta}(r-t)\Big)dr\right) \le \frac{1}{m-\sqrt{\mathcal{N}_\delta}}. \tag{3.54}$$

We combine estimation from (3.53) to (3.54), and Sobolev embedding as in $\mathcal{H}^\rho(\Omega) \hookrightarrow L^{\frac{2N}{N-4\rho}}(\Omega)$, one obtains

$$\begin{aligned}
&\Big\|\exp\left(m(t-T)\right)\Big(\mathcal{M}u_1(t)-\mathcal{M}u_2(t)\Big)\Big\|_{L^{\frac{2N}{N-4\rho}}(\Omega)} \\
\lesssim &\Big\|\exp\left(m(t-T)\right)\Big(\mathcal{M}u_1(t)-\mathcal{M}u_2(t)\Big)\Big\|_{\mathcal{H}^\rho(\Omega)} \le \frac{\max\{T,1\}\mathcal{L}_f(\mathcal{N}_\delta)^\rho}{m-\sqrt{\mathcal{N}_\delta}}\big\|u_1-u_2\big\|_m.
\end{aligned} \tag{3.55}$$

Therefore, we have

$$ess\sup_{0\le t\le T}\Big\|\exp\left(m(t-T)\right)\Big(\mathcal{M}u_1(t)-\mathcal{M}u_2(t)\Big)\Big\|_{L^{\frac{2N}{N-4\rho}}(\Omega)} \le \frac{\max\{T,1\}\mathcal{L}_f(\mathcal{N}_\delta)^\rho}{m-\sqrt{\mathcal{N}_\delta}}\big\|u_1-u_2\big\|_m. \tag{3.56}$$

For any $u_1,u_2 \in L^\infty_m(0,T;L^{\frac{2N}{N-4\rho}}(\Omega))$, we conclude that

$$\big\|\mathcal{M}u_1-\mathcal{M}u_2\big\|_m \le \frac{\max\{T,1\}\mathcal{L}_f(\mathcal{N}_\delta)^\rho}{m-\sqrt{\mathcal{N}_\delta}}\big\|u_1-u_2\big\|_m. \tag{3.57}$$

From (3.57), if we choose $m \ge \sqrt{\mathcal{N}_\delta} + 2\mathcal{L}_f(\mathcal{N}_\delta)^\rho \max\{T,1\}\sqrt{\mathcal{N}_\delta}$, we can see that $\mathcal{M}$ is a contraction mapping from $L^\infty_m(0,T;L^{\frac{2N}{N-4\rho}}(\Omega)) \hookrightarrow L^\infty_m(0,T;L^{\frac{2N}{N-4\rho}}(\Omega))$, we conclude that $\mathcal{M}$ has a fixed point $\widetilde{V}^\delta_{\mathcal{N}_\delta} \in L^\infty_m(0,T;L^{\frac{2N}{N-4\rho}}(\Omega))$ throught the Banach fixed point. For the estimation (3.57), let assume that $u_1 = \widetilde{V}^\delta_{\mathcal{N}_\delta}$ and $u_2 = 0$, and we denote $\widetilde{V}^\delta_{\mathcal{N}_\delta} = \mathcal{M}(\widetilde{V}^\delta_{\mathcal{N}_\delta})$, and $\mathcal{M}u_2 = \mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}f_\delta + \mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}g_\delta$, we get

$$\begin{aligned}
&\big\|\widetilde{V}^\delta_{\mathcal{N}_\delta}\big\|_m = \big\|\mathcal{M}(\widetilde{V}^\delta_{\mathcal{N}_\delta})\big\|_m \\
\le&\Big\|\mathcal{M}(\widetilde{V}^\delta_{\mathcal{N}_\delta}) - \mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}f_\delta - \mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}g_\delta\Big\|_m + \Big\|\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}f_\delta + \mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}g_\delta\Big\|_m \\
\lesssim&\frac{1}{2}\Big\|\widetilde{V}^\delta_{\mathcal{N}_\delta}\Big\|_m + \bigg(\sqrt{2}(\mathcal{N}_\delta)^{\rho-\frac{N(2-q)}{4q}}\exp\Big(\sqrt{\mathcal{N}_\delta}(T-t)\Big)\big\|f_\delta\big\|_{L^q(\Omega)} \\
&+\sqrt{2}(T-t)(\mathcal{N}_\delta)^{\rho-\frac{N(2-q)}{4q}}\exp\big(\sqrt{\mathcal{N}_\delta}(T-t)\big)\big\|g_\delta\big\|_{L^q(\Omega)}\bigg).
\end{aligned} \tag{3.58}$$

Using the estimation (3.44), thus, we obtain that

$$\begin{aligned}
\big\|\widetilde{V}^\delta_{\mathcal{N}_\delta}\big\|_m \lesssim &2\bigg(\sqrt{2}(\mathcal{N}_\delta)^{\rho-\frac{N(2-q)}{4q}}\exp\Big(\sqrt{\mathcal{N}_\delta}(T-t)\Big)\big\|f_\delta\big\|_{L^q(\Omega)} \\
&+\sqrt{2}(T-t)(\mathcal{N}_\delta)^{\rho-\frac{N(2-q)}{4q}}\exp\big(\sqrt{\mathcal{N}_\delta}(T-t)\big)\big\|g_\delta\big\|_{L^q(\Omega)}\bigg).
\end{aligned} \tag{3.59}$$

□

In the theory below, we are going to show the error estimate between the regularized and exact solutions in the space of $L^q(\Omega)$ type.

Theorem 3.3. *Assume that problem* (1.1) *under the condition* (1.2) *has a unique solution* $u \in L^\infty(0,T;\mathcal{H}^{n+\rho}(\Omega))$ *for any* $n > 0$ *and* $0 < \rho < \frac{N}{4}$*. In addition, we assume that there exists a positive constant M such that*

$$M = ess \sup_{0\le t\le T} \left(\exp(2t\sqrt{\lambda_k})\lambda_k^{2\varsigma}\left|\langle u(t), e_k\rangle\right|^2 \right)^{\frac{1}{2}}, \tag{3.60}$$

where $\varsigma > 0$*, taking the noisy data* $f_\delta, g_\delta \in L^q(\Omega)$ *such that*

$$\left\|f_\delta - f\right\|_{L^q(\Omega)} + \left\|g_\delta - g\right\|_{L^q(\Omega)} \le \delta, \quad 1 < q < \infty. \tag{3.61}$$

By choosing $\mathcal{N}_\delta$ *such that*

$$\lim_{\delta\to 0} \mathcal{N}_\delta = \infty, \ \lim_{\epsilon\to 0} (\mathcal{N}_\delta)^{\rho - \frac{N(2-q)}{4q}} \exp(\sqrt{\mathcal{N}_\delta} T)\delta = 0. \tag{3.62}$$

Specifically in this section, we choose $\mathcal{N}_\delta$ *as follows:*

$$\mathcal{N}_\delta = \Big(\frac{T}{1-\gamma}\Big)^{-2}\big[\log(\delta^{-1})\big]^2 \ \textit{for any}\ 0 < \gamma < 1, \tag{3.63}$$

then we have

$$\left\|\widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t) - u(\cdot,t)\right\|_{L^{\frac{2N}{N-4\rho}}(\Omega)} \ \textit{is of order}. \tag{3.64}$$

$$\max\left\{\Big(\log(\delta^{-1})\Big)^{-2n}, \Big(\log(\delta^{-1})\Big)^{2\rho-2\varsigma}, \Big(\log(\delta)^{-1}\Big)^{2\rho-\frac{N(2-q)}{2q}}\delta^\gamma\right\}. \tag{3.65}$$

Proof. In this provision, we provide the upper bound for the term $\left\|\widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t) - u(\cdot,)\right\|_{\mathcal{H}^\rho(\Omega)}$, using the triangle inequality, then we get

$$\left\|\widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t) - u(\cdot,t)\right\|_{L^2(\Omega)} \le \left\|\widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t) - \widetilde{V}_{\mathcal{N}_\delta}(\cdot,t)\right\|_{L^2(\Omega)} + \left\|\widetilde{V}_{\mathcal{N}_\delta}(\cdot,t) - u(\cdot,t)\right\|_{L^2(\Omega)}. \tag{3.66}$$

From definition (3.33), we can know that

$$\begin{aligned}\widetilde{V}_{\mathcal{N}_\delta}(t) =& \mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta} f + \mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta} g \\ &+ \int_t^T \mathcal{P}_3(r-t)\mathbb{T}_{\mathcal{N}_\delta} G(\widetilde{V}_{\mathcal{N}_\delta}(r))dr + \int_t^T \mathcal{P}_4(r-t)\mathbb{T}_{\mathcal{N}_\delta} G(\widetilde{V}_{\mathcal{N}_\delta}(r))dr.\end{aligned} \tag{3.67}$$

From (3.33) and (3.67), we see that

$$\begin{aligned}\widetilde{V}_{\mathcal{N}_\delta}(t) - \widetilde{V}^\delta_{\mathcal{N}_\delta}(t) =& \mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}(f_\delta - f) + \mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}(g_\delta - g) \\ &+ \int_t^T \mathcal{P}_3(r-t)\Big[\mathbb{T}_{\mathcal{N}_\delta} G(\widetilde{V}^\delta_{\mathcal{N}_\delta}(r)) - \mathbb{T}_{\mathcal{N}_\delta} G(\widetilde{V}_{\mathcal{N}_\delta}(r))\Big]dr \\ &+ \int_t^T \mathcal{P}_4(r-t)\Big[\mathbb{T}_{\mathcal{N}_\delta} G(\widetilde{V}^\delta_{\mathcal{N}_\delta}(r)) - \mathbb{T}_{\mathcal{N}_\delta} G(\widetilde{V}_{\mathcal{N}_\delta}(r))\Big]dr.\end{aligned} \tag{3.68}$$

Since (3.66) and (3.68), we receive

$$\Big\|\widetilde{V}^{\delta}_{\mathcal{N}_\delta}(\cdot,t)-u(\cdot,t)\Big\|_{L^2(\Omega)} \le \underbrace{\Big\|u(\cdot,t)-\widetilde{V}_{\mathcal{N}_\delta}(\cdot,t)\Big\|_{L^2(\Omega)}}_{\mathcal{B}_1}$$
$$+\underbrace{\Big\|\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}(f_\delta-f)\Big\|_{L^2(\Omega)}+\Big\|\mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}(g_\delta-g)\Big\|_{L^2(\Omega)}}_{\mathcal{B}_2}$$
$$+\underbrace{\Big\|\int_t^T \mathcal{P}_3(r-t)\Big[\mathbb{T}_{\mathcal{N}_\delta}G(\widetilde{V}^{\delta}_{\mathcal{N}_\delta}(r))-\mathbb{T}_{\mathcal{N}_\delta}G(\widetilde{V}_{\mathcal{N}_\delta}(r))\Big]dr\Big\|_{L^2(\Omega)}}_{\mathcal{B}_3}$$
$$+\underbrace{\Big\|\int_t^T \mathcal{P}_4(r-t)\Big[\mathbb{T}_{\mathcal{N}_\delta}G(\widetilde{V}^{\delta}_{\mathcal{N}_\delta}(r))-\mathbb{T}_{\mathcal{N}_\delta}G(\widetilde{V}_{\mathcal{N}_\delta}(r))\Big]dr\Big\|_{L^2(\Omega)}}_{\mathcal{B}_4}. \tag{3.69}$$

We rated $\Big\|\widetilde{V}^{\delta}_{\mathcal{N}_\delta}(\cdot,t)-u(\cdot,t)\Big\|_{L^2(\Omega)}$ through four steps as follows:
Step 1: Estimate of $\mathcal{B}_1$, we have

$$\begin{aligned}\mathcal{B}_1 \le \|u(\cdot,t)-\widetilde{V}_{\mathcal{N}_\delta}(\cdot,t)\|_{L^2(\Omega)} &= \Big(\sum_{k=\mathcal{N}_\delta+1}^{\infty}\exp\big(-2t\sqrt{\lambda_k}\big)\lambda_k^{-2\varsigma}\exp\big(2t\sqrt{\lambda_k}\big)\lambda_k^{2\varsigma}\Big|\langle u(t),e_k\rangle\Big|^2\Big)^{\frac{1}{2}}\\ &\le (\mathcal{N}_\delta)^{-\varsigma}\exp\big(-t\sqrt{\mathcal{N}_\delta}\big)M.\end{aligned} \tag{3.70}$$

Step 2: Estimate of $\mathcal{B}_2$, for $1<q<2$, we follow Lemma 2.1 in combination wiith the Sobolev embedding, one has

$$L^q(\Omega)\hookrightarrow \mathcal{H}^{\frac{N(2-q)}{4q}}(\Omega). \tag{3.71}$$

This implies that

$$\begin{aligned}\mathcal{B}_2 \le&\Big\|\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_\delta}(f_\delta-f)\Big\|_{L^2(\Omega)}+\Big\|\mathcal{P}_2(T-t)\mathbb{T}_{\mathcal{N}_\delta}(g_\delta-g)\Big\|_{L^2(\Omega)}\\ \le&\sqrt{2}(\mathcal{N}_\delta)^{-\frac{N(2-q)}{4q}}\exp\big(\sqrt{\mathcal{N}_\delta}(T-t)\big)\Big\|f_\delta-f\Big\|_{\mathcal{H}^{\frac{N(2-q)}{4q}}(\Omega)}\\ &+\sqrt{2}(T-t)(\mathcal{N}_\delta)^{-\frac{N(2-q)}{4q}}\exp\big(\sqrt{\mathcal{N}_\delta}(T-t)\big)\Big\|g_\delta-g\Big\|_{\mathcal{H}^{\frac{N(2-q)}{4q}}(\Omega)}\\ \le&\sqrt{2}\,C_{10}(N,q)(\mathcal{N}_\delta)^{-\frac{N(2-q)}{4q}}\exp\big(\sqrt{\mathcal{N}_\delta}(T-t)\big)\Big\|f_\delta-f\Big\|_{L^q(\Omega)}\\ &+\sqrt{2}\,C_{10}(N,q)(T-t)(\mathcal{N}_\delta)^{-\frac{N(2-q)}{4q}}\exp\big(\sqrt{\mathcal{N}_\delta}(T-t)\big)\Big\|g_\delta-g\Big\|_{L^q(\Omega)}.\end{aligned} \tag{3.72}$$

Step 3: Estimate of $\mathcal{B}_3$, by using the similar argument as in (3.48), we can find that

$$\mathcal{B}_3 \le \mathcal{L}_f\int_t^T (r-t)\exp(\sqrt{\mathcal{N}_\delta}(r-t))\Big\|\mathcal{V}^{\delta}_{\mathcal{N}_\delta}(r)-\mathcal{V}_{\mathcal{N}_\delta}(r)\Big\|_{L^2(\Omega)}dr. \tag{3.73}$$

Step 4: Estimate of $\mathcal{B}_4$, by using the similar argument as in (3.49), one obtains

$$\mathcal{B}_4 \le \mathcal{L}_f \int_t^T \exp\left(\sqrt{\mathcal{N}_\delta}(r-t)\right)\left\|\mathcal{V}^\delta_{\mathcal{N}_\delta}(r) - \mathcal{V}_{\mathcal{N}_\delta}(r)\right\|_{L^2(\Omega)}. \tag{3.74}$$

Combining (3.66), estimation of Steps 1–4, we conclude that

$$\begin{aligned}\left\|\widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t) - u(\cdot,t)\right\|_{L^2(\Omega)} \le& (\mathcal{N}_\delta)^{-\varsigma}\exp(-t\mathcal{N}_\delta)M + 2\sqrt{2}C_{10}(N,q)(\mathcal{N}_\delta)^{-\frac{N(2-q)}{4q}}\exp\left(\sqrt{\mathcal{N}_\delta}(T-t)\right)\delta \\ &+ \mathcal{L}_f \int_t^T (r-t)\exp\left(\sqrt{\mathcal{N}_\delta}(r-t)\right)\left\|\mathcal{V}^\delta_{\mathcal{N}_\delta}(r) - \mathcal{V}_{\mathcal{N}_\delta}(r)\right\|_{L^2(\Omega)} dr \\ &+ \mathcal{L}_f \int_t^T \exp\left(\sqrt{\mathcal{N}_\delta}(r-t)\right)\left\|\mathcal{V}^\delta_{\mathcal{N}_\delta}(r) - \mathcal{V}_{\mathcal{N}_\delta}(r)\right\|_{L^2(\Omega)}. \end{aligned} \tag{3.75}$$

Multiplying both sides by $\exp\left(t\sqrt{\mathcal{N}_\delta}\right)$, we have

$$\begin{aligned}&\exp\left(t\sqrt{\mathcal{N}_\delta}\right)\left\|\widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t) - u(\cdot,t)\right\|_{L^2(\Omega)} \\ \le& (\mathcal{N}_\delta)^{-\varsigma}M + 2\sqrt{2}C_{10}(N,q)(\mathcal{N}_\delta)^{-\frac{N(2-q)}{4q}}\exp\left(\sqrt{\mathcal{N}_\delta}T\right)\delta \\ &+ 2\mathcal{L}_f\max\{T,1\}\int_t^T \exp\left(\sqrt{\mathcal{N}_\delta}r\right)\left\|\mathcal{V}^\delta_{\mathcal{N}_\delta}(r) - \mathcal{V}_{\mathcal{N}_\delta}(r)\right\|_{L^2(\Omega)} dr. \end{aligned} \tag{3.76}$$

Applying the Grönwall's inequality, we get

$$\begin{aligned}&\exp\left(t\sqrt{\mathcal{N}_\delta}\right)\left\|\widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t) - u(\cdot,t)\right\|_{L^2(\Omega)} \\ \le& \left((\mathcal{N}_\delta)^{-\varsigma}M + 2\sqrt{2}C_{10}(N,q)(\mathcal{N}_\delta)^{-\frac{N(2-q)}{4q}}\exp\left(\sqrt{\mathcal{N}_\delta}T\right)\delta\right)\exp\left(2\mathcal{L}_f\max\{T,1\}(T-t)\right). \end{aligned} \tag{3.77}$$

This implies that

$$\begin{aligned}\left\|\widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t) - u(\cdot,t)\right\|_{L^2(\Omega)} \le& \exp\left(-t\sqrt{\mathcal{N}_\delta}\right)\left((\mathcal{N}_\delta)^{-\varsigma}M + 2\sqrt{2}C_{10}(N,q)(\mathcal{N}_\delta)^{-\frac{N(2-q)}{4q}}\exp\left(\sqrt{\mathcal{N}_\delta}T\right)\delta\right) \\ &\times\exp\left(2\mathcal{L}_f\max\{T,1\}(T-t)\right). \end{aligned} \tag{3.78}$$

Next, we have the following inequality:

$$\left\|\widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t) - u(\cdot,t)\right\|_{L^{\frac{2N}{N-4\rho}}(\Omega)} \le C_{11}(N,\rho)\left\|\widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t) - u(\cdot,t)\right\|_{\mathcal{H}^\rho(\Omega)}. \tag{3.79}$$

It is easy to see that

$$\left\|\widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t) - u(\cdot,t)\right\|_{\mathcal{H}^\rho(\Omega)} \le \underbrace{\left\|\widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t) - \widetilde{V}_{\mathcal{N}_\delta}(\cdot,t)\right\|_{\mathcal{H}^\rho(\Omega)}}_{\mathcal{E}_1} + \underbrace{\left\|\widetilde{V}_{\mathcal{N}_\delta}(\cdot,t) - u(\cdot,t)\right\|_{\mathcal{H}^\rho(\Omega)}}_{\mathcal{E}_2}. \tag{3.80}$$

Now, for $0 < \varsigma < \rho$, we immediately have a rating the first term of (3.80) as follows:

$$\begin{aligned}\mathcal{E}_1 \leq& \Big(\sum_{k=1}^{\mathcal{N}_\delta} \lambda_k^{2\rho}\Big|\langle \widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t) - \widetilde{V}_{\mathcal{N}_\delta}(\cdot,t), e_k\rangle\Big|^2\Big)^{\frac{1}{2}} \leq (\mathcal{N}_\delta)^\rho \Big\|\widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t) - \widetilde{V}_{\mathcal{N}_\delta}(\cdot,t)\Big\|_{L^2(\Omega)} \\ \leq& \exp\Big(-t\sqrt{\mathcal{N}_\delta}\Big)\Big((\mathcal{N}_\delta)^{\rho-\varsigma}M + 2\sqrt{2}C_{10}(N,q)(\mathcal{N}_\delta)^{\rho-\frac{N(2-q)}{4q}}\exp\big(\sqrt{\mathcal{N}_\delta}T\big)\delta\Big) \\ &\times \exp\Big(2\mathcal{L}_f \max\{T,1\}(T-t)\Big). \end{aligned} \tag{3.81}$$

$\mathcal{E}_2$ can be bounded as follows:

$$\mathcal{E}_2 \leq \left(\sum_{k=\mathcal{N}_\delta}^{\infty} \lambda_k^{-2n}\lambda_k^{2n+2\rho}\Big|\langle u(\cdot,t), e_k\rangle\Big|^2\right)^{\frac{1}{2}} \leq (\mathcal{N}_\delta)^{-n}\|u\|_{L^\infty(0,T;\mathcal{H}^{n+\rho}(\Omega))}. \tag{3.82}$$

From the observation above, we have

$$\begin{aligned} &\Big\|\widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t) - u(\cdot,t)\Big\|_{L^{\frac{2N}{N-4\rho}}(\Omega)} \\ \leq& C_{11}(N,\rho)\Big\|\widetilde{V}^\delta_{\mathcal{N}_\delta}(\cdot,t) - u(\cdot,t)\Big\|_{\mathcal{H}^\rho(\Omega)} \\ \leq& C_{11}(N,\rho)\Big[(\mathcal{N}_\delta)^{-n}\|u\|_{L^\infty(0,T;\mathcal{H}^{n+\rho}(\Omega))} + \exp\big(-t\sqrt{\mathcal{N}_\delta}\big) \\ &\Big((\mathcal{N}_\delta)^{\rho-\varsigma}M + 2\sqrt{2}C_{10}(N,q)(\mathcal{N}_\delta)^{\rho-\frac{N(2-q)}{4q}}\exp\big(\sqrt{\mathcal{N}_\delta}T\big)\delta\Big)\exp\Big(2\mathcal{L}_f\max\{T,1\}(T-t)\Big)\Big]. \end{aligned} \tag{3.83}$$

□

4. Numerical example

In this section, we carry out a numerical example in order to verify our proposed theory. In other words, we consider the stable property of the regularized solution based on the Fourier truncation method. First of all, we introduce some definitions to support the numerical implementation as follows. By choosing $\Omega \times (0,T) := (0,\pi)\times(0,1)$, we have the eigenvalues λ_k and the corresponding eigenvector e_k which are the complete orthonormal system of eigenfunctions forming an orthogonal basis such that $-\Delta e_k = \lambda_k e_k$ and $e_k|_{\partial\Omega} = 0$ for $k \in \mathbb{N}$. Here, we choose $\lambda_k = k^2\pi^2,\ e_k(x) = \sqrt{2}\sin(k\pi x)$. We are going to find a function $\mathcal{X}$ satisfied

$$\Delta^2 u \equiv u_{tttt} + 2u_{ttxx} + u_{xxxx} = G(x,t,u(x,t)), \quad (x,t) \in (0,1)\times(0,1], \tag{4.1}$$

under the boundary conditions

$$u(x,t) = \Delta u(x,t) = 0,\ (x,t) \in \{0,1\}\times(0,1), \tag{4.2}$$

and the boundary conditions at $T = 1$ as

$$\begin{cases} u(x,1) = f(x), & \dfrac{\partial u}{\partial t}(x,1) = 0, \quad x \in (0,1), \\ \Delta u(x,1) = g(x), & \dfrac{\partial \Delta u}{\partial t}(x,1) = 0, \quad x \in (0,1). \end{cases} \tag{4.3}$$

Finally, we use the finite difference method with the following partitions of temporal and spatial variable. For $x \in [0,1]$ and $t \in [0,1]$, let us consider the partitions $\mathcal{D}_\Omega \times \mathcal{D}_T$ as follows:

$$\mathcal{D}_\Omega := \left\{ x_1 = 0,\ x_2 = \frac{1}{N_\Omega},\ x_3 = \frac{2}{N_\Omega},\ \cdots,\ x_i = \frac{i-1}{N_\Omega}\ \cdots,\ x_{N_\Omega} = 1,\ \text{for } i = 1,2,...,N_\Omega, N_\Omega+1 \right\},$$

$$\mathcal{D}_T := \left\{ t_1 = 0,\ t_2 = \frac{1}{N_T},\ t_3 = \frac{2}{N_t},\ \cdots,\ t_j = \frac{j-1}{N_T}\ \cdots,\ t_{N_T} = 1,\ \text{for } j = 1,2,...,N_T, N_T+1 \right\}.$$

In Python software, the solutions can be re-write by the following form matrix:

$$\begin{bmatrix} u(x_1,t_1) & u(x_1,t_2) & u(x_1,t_3) & \cdots & u(x_1,t_{N_T+1}) \\ u(x_2,t_1) & u(x_2,t_2) & u(x_2,t_3) & \cdots & u(x_2,t_{N_T+1}) \\ u(x_3,t_1) & u(x_3,t_2) & u(x_3,t_3) & \cdots & u(x_3,t_{N_T+1}) \\ \vdots & \vdots & \vdots & \vdots & \\ u(x_{N_\Omega+1},t_1) & u(x_{N_\Omega+1},t_1) & u(x_{N_\Omega+1},t_2) & \cdots & u(x_{N_\Omega+1},t_{N_T+1}) \end{bmatrix}_{(N_\Omega+1)\times(N_T+1)}$$

In this example, by choosing the solution $u(x,t) = \cosh(1-t)\sin(\pi x)$ to test the proposed results with the following input data:

$$\begin{cases} G(x,t) = (\pi^4 - 2\pi^2 + 1)\cosh(1-t)\sin(\pi x), & (x,t) \in (0,1)\times(0,1), \\ f(x) = \sin(\pi x), & x \in (0,1), \\ g(x) = -\pi^2 \sin(\pi x), & x \in (0,1). \end{cases} \tag{4.4}$$

During the measurement of electromagnetic fields in applications of environmental and geophysical imaging, the exact data is approximated by the function $f_\delta,\ g_\delta,\ G_\delta$ as follows:

$$\|f_\delta - f\|_{L^q(\Omega)} + \|g_\delta - g\|_{L^q(\Omega)} + \left\|G_\delta - G\right\|_{L^\infty(0,T;L^q(\Omega))} \le \delta, \tag{4.5}$$

where

$$\begin{cases} G_\delta(\cdot) = G(\cdot) + \delta\,\mathrm{rand}(\cdot)/3, \\ f_\delta(\cdot) = f(\cdot) + \delta\,\mathrm{rand}(\cdot)/3, \\ g_\delta(\cdot) = g(\cdot) + \delta\,\mathrm{rand}(\cdot)/3. \end{cases} \tag{4.6}$$

By applying the Fourier truncation method, we provide a regularization solution as follows:

$$\begin{aligned} u_\delta^{\mathcal{N}_{tr}}(x,t) = & \sum_{k=1}^{\mathcal{N}_{tr}} \cosh\left(\sqrt{\lambda_k}(T-t)\right)\langle f_\delta, e_k\rangle e_k(x) + \sum_{k=1}^{\mathcal{N}_{tr}} \frac{(T-t)\sinh\left(\sqrt{\lambda_k}(T-t)\right)}{2\sqrt{\lambda_k}}\langle g_\delta, e_k\rangle e_k(x) \\ & + \sum_{k=1}^{\mathcal{N}_{tr}} \left(\int_t^T \frac{(r-t)\cosh\left(\sqrt{\lambda_k}(r-t)\right)}{2\lambda_k}\langle G_\delta(\cdot,t), e_k\rangle dr \right) e_k(x) \\ & + \sum_{k=1}^{\mathcal{N}_{tr}} \left(\int_t^T \frac{\sinh\left(\sqrt{\lambda_k}(r-t)\right)}{2\lambda_k\sqrt{\lambda_k}}\langle G_\delta(\cdot,t), e_k\rangle dr \right) e_k(x), \end{aligned} \tag{4.7}$$

where $\mathcal{N}_{tr}$ is a parameter regularization.

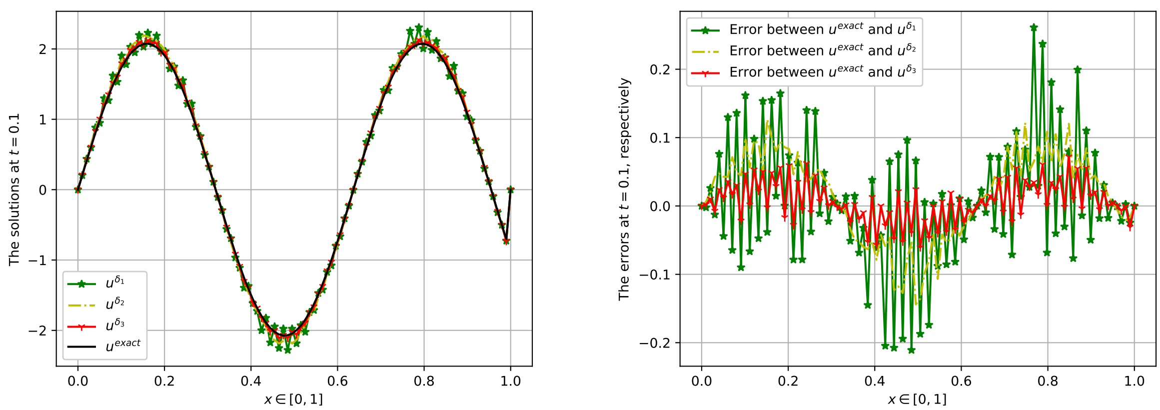
We use the following estimation to evaluate the error between the regularized and exact solution at a certain time t:

$$\mathcal{E}_{\delta}^{N_{tr}}(t) = \sqrt{\frac{\sum_{i=1}^{N_\Omega+1} \left| u_{\delta}^{N_{tr}}(x_i, t) - u(x_i, t) \right|^2}{N_\Omega + 1}}.$$

Table 1 and Figures 1–5 show the error estimate between the exact and regularized solutions at five observation times $t \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ with $\delta \in \{0.5, 0.05, 0.005\}$, respectively. Figure 6 presents the 3D graphs of the exact and regularized solutions, it shows they are quite similar. Overall, it shows that the error becomes smaller as the noise δ is decreased. It also shows the regularized solution is converged to the exact solution in this example.

Table 1. The error estimation between the regularized and exact solutions with $N_\Omega = 100$, $N_T = 100$.

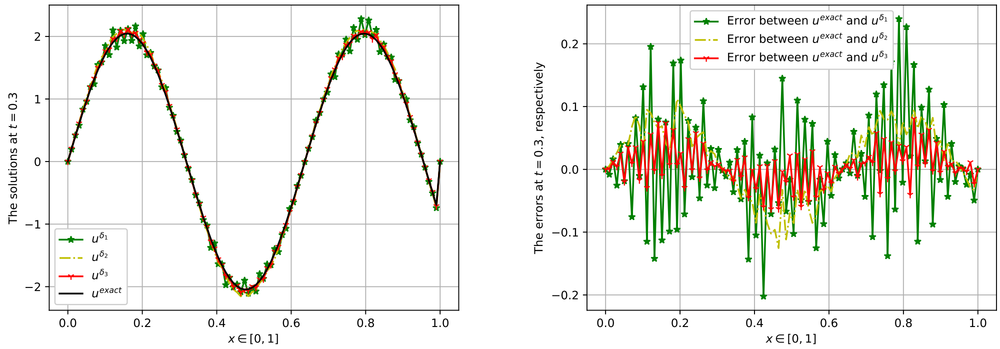
t	$\delta_1 = 0.5$	$\delta_2 = 0.05$	$\delta_3 = 0.005$
0.1	0.09600230092017747	0.06839424680602106	0.035043593672148474
0.3	0.09297773184568403	0.06548087722374361	0.030044924035854536
0.5	0.08631247486802868	0.05829101885421206	0.031096834115623428
0.7	0.08256109054607924	0.04950471042879639	0.028282438899535974
0.9	0.08226034798718103	0.03610621656184547	0.024246403928042365



(a) The solution at $t = 0.1$

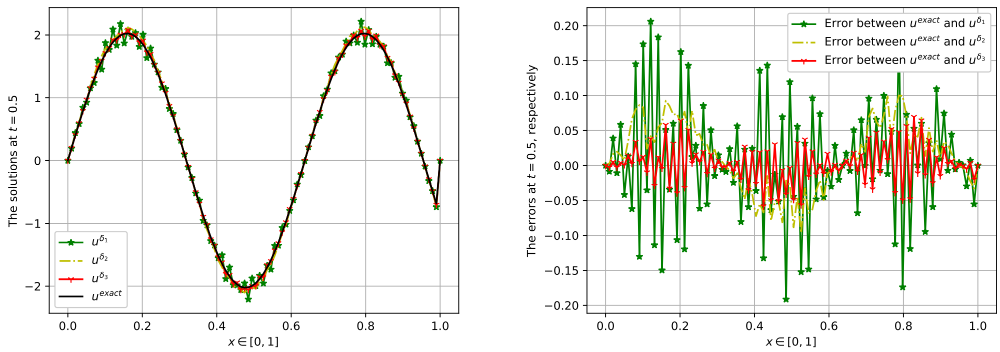
(b) The error at $t = 0.1$

Figure 1. The solutions at $t = 0.1$ and the error between exact and regularized solutions.



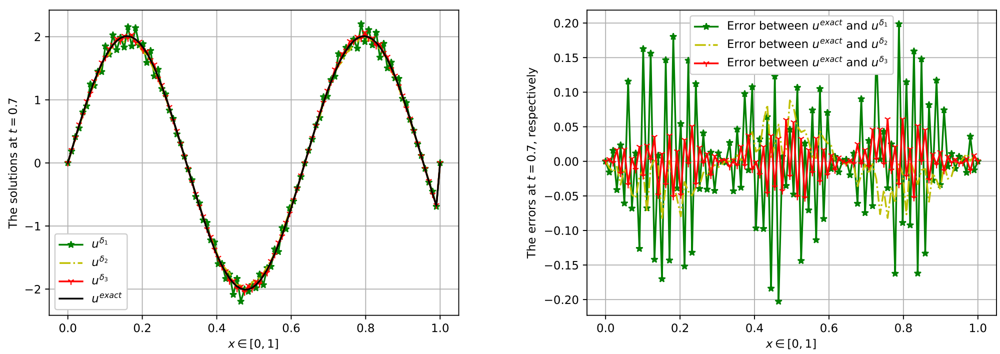
(a) The solution at $t = 0.3$ (b) The error at $t = 0.3$

Figure 2. The solutions at $t = 0.3$ and the error between exact and regularized solutions.



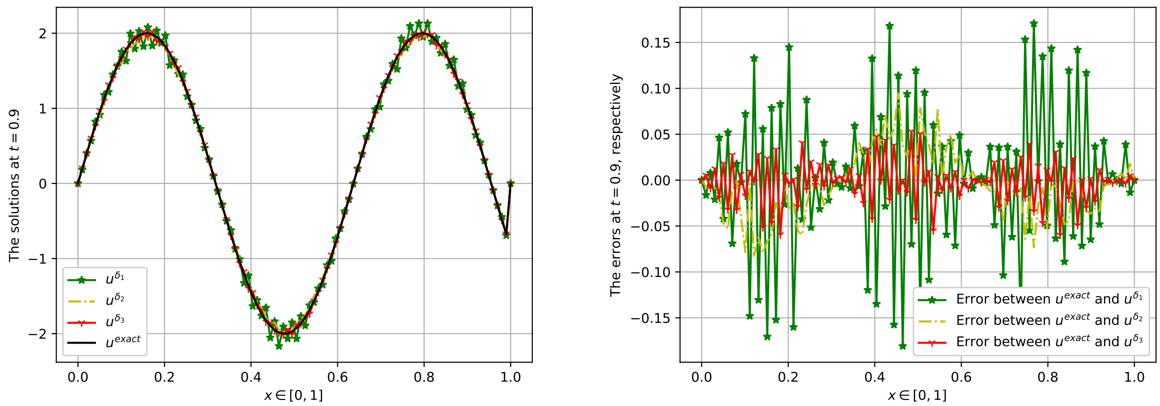
(a) The solution at $t = 0.5$ (b) The error at $t = 0.5$

Figure 3. The solutions at $t = 0.5$ and the error between exact and regularized solutions.



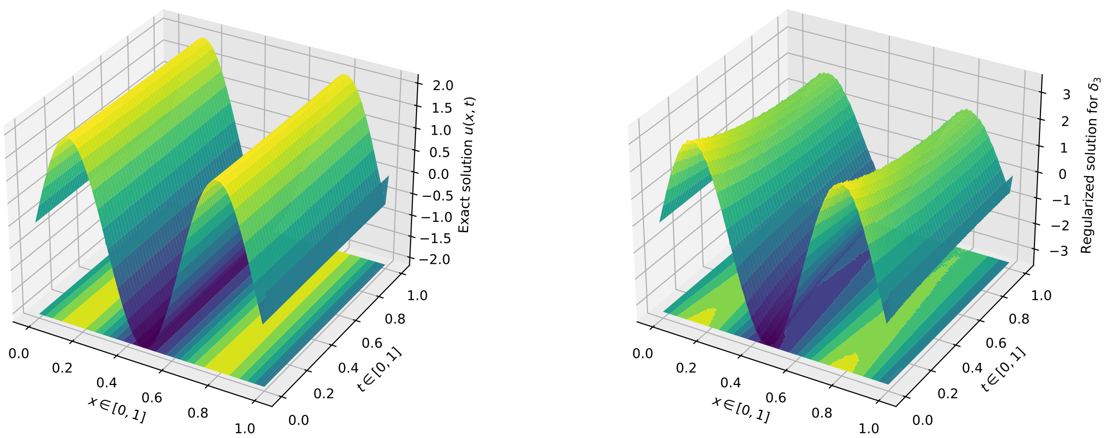
(a) The solution at $t = 0.7$ (b) The error at $t = 0.7$

Figure 4. The solutions at $t = 0.7$ and the error between exact and regularized solutions.



(a) The solution at $t = 0.9$ (b) The error at $t = 0.9$

Figure 5. The solutions at $t = 0.9$ and the error between exact and regularized solutions.



(a) The exact solution u (b) The regularized solution $\delta = 0.005$

Figure 6. The 3D solutions on $\mathcal{D}_\Omega \times \mathcal{D}_T \in (0, 1) \times (0, 1)$ for $\delta = 0.005$.

5. Conclusions

In this study, we focused on the final value problem of an inverse problem for both linear and nonlinear bi-harmonic equations. The regularized method for the biharmonic equation is proposed using the Fourier series truncation method and the terminal input data in $L^q(\Omega)$ for $q \neq 2$. The error between the exact and regularized solutions is estimated in $L^q(\Omega)$ using the embedding between $L^q(\Omega)$ and Hilbert scale spaces $\mathcal{H}^\rho(\Omega)$. The proposoed method has been verified by a numerical example; wherein, the regularized solution is well converged to the exact solution. It shows that our proposal method is capable of solving the final value problem of an inverse problem for both linear and nonlinear biharmonic equations.

Acknowledgements

The authors would like to thank for the support from the National Research Foundation of Korea under grant number NRF-2020K1A3A1A05101625, and from the Institute of Construction and Environmental Engineering at Seoul National University.

Conflict of interest

The authors declare no conflicts of interest.

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