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Research article

Compactness for iterated commutators of general bilinear fractional integral operators on Morrey spaces with non-doubling measures

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Abstract: In the present article, we obtain the compactness of iterated commutators generated by general bilinear fractional operator with RVMO functions on Morrey spaces with non-doubling measures.

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1. Introduction

For $0 < \alpha < n$, the fractional integral operator,

$$I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} d\mu(y),$$

has many applications in harmonic analysis, PDEs and the theory of Sobolev embeddings (see [2, 15]). For the boundedness of the fractional integral operator on function spaces, Adams [1] first obtained the boundedness of I_α on Morrey space. Recently it has been shown that many results of the classical singular integral theory also hold without assuming the doubling property, considerable attention has been paid to the study of when the underlying measures only satisfy the polynomial growth condition, namely, there exists a positive constant $C_0 > 0$ and $n \in (0, d]$ such that

$$\mu(B(x,l)) \le C_0 l^n, \tag{1.1}$$

for every $x \in \mathbb{R}^d$ and $l > 0$, $B(x,l)$ being the Euclidean ball with center x and radius l. In 2005, Sawano and Tanaka [19] introduced the definition of Morrey spaces under non-doubling measure conditions, and considered the boundedness of some classic operators including fractional integral

I_α on Morrey spaces. Let us recall the definition of Morrey spaces $\mathcal{M}^p_q(\mu)$, where μ satisfies the growth condition (1.1). Let $\kappa > 1$ and $1 \le q \le p < \infty$. We define the Morrey spaces $\mathcal{M}^p_q(\kappa,\mu)$ as

$$\mathcal{M}^p_q(\kappa,\mu) := \left\{ f \in L^q_{\text{loc}}(\mu) : \|f\|_{\mathcal{M}^p_q(\kappa,\mu)} < \infty \right\},$$

where the norm $\|f\|_{\mathcal{M}^p_q(\kappa,\mu)}$ is given by

$$\|f\|_{\mathcal{M}^p_q(\kappa,\mu)} := \sup_{Q \in \mathcal{Q}(\mu)} \mu(\kappa Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f|^q d\mu \right)^{\frac{1}{q}}. \tag{1.2}$$

Applying Hölder's inequality, it is easy to see that

$$L^p(\mu) = \mathcal{M}^p_p(\kappa,\mu) \subset \mathcal{M}^p_t(\kappa,\mu) \subset \mathcal{M}^p_s(\kappa,\mu),$$

for $1 \le s \le t \le p < \infty$. Moreover, the definition of Morrey spaces is independent of the constant $\kappa > 1$, and the norms for different choice of $\kappa > 1$ are equivalent, see [17, 19, 20] for details. So we will denote $\mathcal{M}^p_q(2,\mu)$ by $\mathcal{M}^p_q(\mu)$. One can see from the definition of Morrey spaces that, if $f_i \in \mathcal{M}^{p_i}_{s_i}(\mu)$, $1 \le s_i \le p_i < \infty, i = 1, 2$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$, we have $\|f_1 f_2\|_{\mathcal{M}^p_s(\mu)} \le \|f_1\|_{\mathcal{M}^{p_1}_{s_1}(\mu)} \|f_2\|_{\mathcal{M}^{p_2}_{s_2}(\mu)}$.

In recent decades, there have been many works [3, 12, 14, 24] about the properties of commutators. Chanillo [5] obtained the boundedness of the commutator $[b, I_\alpha]$ on the classical Lebesgue space, the operator $[b, I_\alpha]$ given by

$$[b, I_\alpha]f = bI_\alpha f - I_\alpha(bf), \quad b \in \text{BMO}.$$

Subsequently, Di Fazio and Ragusa [9] studied the boundedness of the commutator $[b, I_\alpha]$ on the classical Morrey space. These conclusions can be extended to the case of non-doubling measures. Tolsa [23] introduced the concept of BMO space with non-double measures and called it as RBMO. In order to describe RBMO, we use the follow notations. Here and below we mean by a "cube" a compact cube with the sides parallel to the coordinate axis. $Q(x, l(Q))$ will be the cube centered at x with side length $l(Q)$. Next, we define the set of doubling cubes:

$$\mathcal{Q}(\mu) := \{Q : Q \text{ is a cube with } \mu(Q) > 0\},$$

$$\mathcal{Q}(\mu, 2) := \{Q \in \mathcal{Q}(\mu) : 0 < \mu(2Q) \le c_1 \mu(Q)\},$$

where c_1 is a constant independent of $Q \in \mathcal{Q}(\mu)$. $\mathcal{Q}(\mu, 2)$ denotes the set of doubling cubes. For $Q \in \mathcal{Q}(\mu)$ be a cube which is not always doubling, one can find N large enough such that NQ is a doubling cube. Let $j \ge 0$ be the smallest integer such that $2^j Q$ is doubling, we denote this cube by Q^*.

Let $0 \le \gamma < n$. Denote by c_Q the center of $Q \in \mathcal{Q}(\mu)$. Then we define

$$K^{(\gamma)}_{Q,R} := 1 + \int_{l(Q)}^{l(R)} \left(\frac{\mu\left(B\left(c_Q, l\right)\right)}{l^n} \right)^{1-\frac{\gamma}{n}} \frac{dl}{l},$$

and $K_{Q,R} := K^{(0)}_{Q,R}$ for $Q, R \in \mathcal{Q}(\mu, 2)$ with $Q \subset R$. The coefficient $K_{Q,R}$ was introduced by Tolsa in [23] and the modified coefficient $K^{(\gamma)}_{Q,R}$ was defined later by Chen and Sawyer [7]. For $Q \in \mathcal{Q}(\mu, 2)$ we denote by $m_Q b$ the average of the $L^1_{loc}(\mu)$-function b, namely, $m_Q b = \frac{1}{\mu(Q)} \int_Q b(x) d\mu$. Let $\eta > 1$ be a fixed constant, we say that $b \in L^1_{loc}(\mu)$ is in RBMO(μ) if there exists a constant $\mathcal{A}$ such that

$$\frac{1}{\mu(\eta Q)} \int_Q \left| b(y) - m_{Q^*} b \right| d\mu(y) \le \mathcal{A},$$

for any cube Q, and

$$\left|m_Q b - m_R b\right| \leq \mathcal{A} K_{Q,R},$$

for any two doubling cubes $Q \subset R$. The minimal constant $\mathcal{A}$ is the RBMO(μ) norm of b, and it will be denoted by $\|b\|_*$.

By using an uniform boundedness property for I_α associated with non-doubling measures in $\mathbb{R}^d$, Betancor and Fariña [4] obtained the compactness of the commutator $[b, I_\alpha]$ when $b \in$ RVMO(μ), where RVMO(μ) denotes the closure of the space $C_c^\infty(\mathbb{R}^d)$ of the smooth and compact supported functions on $\mathbb{R}^d$, in RBMO(μ).

In 2004, Garcia-Cuerva and Gatto [13] generalized the classic Hardy-Littlewood-Sobolev theorem, for $1 < p < \frac{n}{\alpha}$, $\|I_\alpha f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\mu)}$, where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. They also introduced a general class of fractional integrals operators K_α for the purpose of obtaining boundedness of the operators from $L^p(\mu)$ to $Lip(\alpha - \frac{n}{p})$, $\frac{n}{\alpha} < p \leq \infty$, which defined as follows

$$K_\alpha(f)(x) = \int_{\mathbb{R}^d} \mathfrak{K}_\alpha(x, y) f(y) d\mu(y), \tag{1.3}$$

the kernel function $\mathfrak{K}_\alpha : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is said to be a general fractional kernel of order α and regularity $\delta (0 < \delta \leq 1)$ if it satisfies the following two conditions:

$$|\mathfrak{K}_\alpha(x, y)| \leq \frac{C}{|x - y|^{n-\alpha}} \quad \text{for all } x \neq y, \tag{1.4}$$

and

$$\left|\mathfrak{K}_\alpha(x, y) - \mathfrak{K}_\alpha(x', y)\right| \leq C \frac{|x - x'|^\delta}{|x - y|^{n-\alpha+\delta}}, \tag{1.5}$$

for $|x-y| \geq 2|x-x'|$. We can see that I_α is a fractional integral operator with a kernel having regularity 1. In 2008, Sawano and Shirai [18] proved that if operator K_α with the kernel $\mathfrak{K}_\alpha$ which satisfying the conditions (1.4) and (1.5), then the commutator $[b, K_\alpha]$ with $b \in$ RVMO(μ) is a compact operator on the Morrey spaces with non-doubling measures. For the multilinear fractional integral $\mathcal{I}_\alpha$, we can also see [6, 8, 11, 16]. Tao and Zheng in [22] discussed the Morrey spaces boundedness of the iterated commutators $[b_1, b_2, \mathcal{I}_\alpha]$ under the setting with non-doubling measures for some indexes, which is defined by

$$[b_1, b_2, \mathcal{I}_\alpha](f_1, f_2)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(b_1(x) - b_1(y_1))(b_2(x) - b_2(y_2)) f_1(y_1) f_2(y_2)}{(|x - y_1| + |x - y_2|)^{2n-\alpha}} d\mu(y_1) d\mu(y_2), \tag{1.6}$$

where $b_1, b_2 \in$ RBMO(μ), and

$$\mathcal{I}_\alpha(f_1, f_2)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f_1(y_1) f_2(y_2)}{(|x - y_1| + |x - y_2|)^{2n-\alpha}} d\mu(y_1) d\mu(y_2).$$

For $b \in$ VMO, the BMO-closure of C_c^∞, Ding and Mei [10] established the compactness of the bilinear fractional integrals commutators $[\mathcal{I}_\alpha, b]_i (i = 1, 2)$ on Morrey spaces, the commutators of $\mathcal{I}_\alpha$ are defined by

$$[\mathcal{I}_\alpha, b]_1(f, g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{b(y) - b(x)}{(|x - y| + |x - z|)^{2n-\alpha}} f(y) g(z) dy dz,$$

$$[\mathcal{I}_{\alpha}, b]_2 (f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{b(z) - b(x)}{(|x-y| + |x-z|)^{2n-\alpha}} f(y)g(z)dydz.$$

In this paper, we consider a general bilinear factional integral operator $\mathcal{K}_\alpha$ based on (1.3), which is given by

$$\mathcal{K}_\alpha(f_1, f_2)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k_\alpha(x, y_1, y_2) f_1(y_1) f_2(y_2) d\mu(y_1) d\mu(y_2), \tag{1.7}$$

where $0 < \alpha < 2n$ and $0 < \delta \le 1$. A function $k_\alpha : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is said to be a general bilinear fractional kernel of order α and regularity δ if it satisfies the following two conditions:

$$|k_\alpha(x, y_1, y_2)| \le \frac{C}{(|x-y_1| + |x-y_2|)^{2n-\alpha}} \quad \text{for all } x \ne y_i, i = 1, 2, \tag{1.8}$$

$$|k_\alpha(x, y_1, y_2) - k_\alpha(x', y_1, y_2)| \le C \frac{|x-x'|^\delta}{(|x-y_1| + |x-y_2|)^{2n-\alpha+\delta}}, \tag{1.9}$$

for $\max\limits_{i=1,2} |x - y_i| \ge 2|x - x'| > 0$.

In this paper, under non-doubling measure conditions we will study the L^p-boundedness and Morrey-boundedness of the general bilinear factional integral operator $\mathcal{K}_\alpha$, and the Morrey-boundedness and compactness of $\mathcal{K}_\alpha$ associated commutators. Firstly, we give the boundedness of $\mathcal{K}_\alpha$ on Lebesgue spaces:

Lemma 1.1. *Let $0 < \alpha < 2n$, $f_i \in L^{p_i}(\mu)$ with $1 < p_i < \frac{\alpha}{n}$, $i = 1, 2$. If $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} > 0$, then there exists a constant $C > 0$ independent of f_1, f_2 such that*

$$\|\mathcal{K}_\alpha(f_1, f_2)\|_{L^p(\mu)} \le C \|f_1\|_{L^{p_1}(\mu)} \|f_2\|_{L^{p_2}(\mu)}.$$

Proof. By a simple inequality $(a + b)^c \ge a^{c/2} b^{c/2}$, $a, b, c > 0$, we have

$$\begin{aligned}
|\mathcal{K}_\alpha(f_1, f_2)(x)| &\le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |k_\alpha(x, y_1, y_2)||f_1(y_1)||f_2(y_2)| d\mu(y_1) d\mu(y_2) \\
&\le C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f_1(y_1)||f_2(y_2)|}{(|x-y_1| + |x-y_2|)^{2n-\alpha}} d\mu(y_1) d\mu(y_2) \\
&\le C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f_1(y_1)|}{|x-y_1|^{n-\frac{\alpha}{2}}} \frac{|f_2(y_2)|}{|x-y_2|^{n-\frac{\alpha}{2}}} d\mu(y_1) d\mu(y_2) \\
&\le C I_{\frac{\alpha}{2}}(|f_1|)(x) I_{\frac{\alpha}{2}}(|f_2|)(x).
\end{aligned}$$

Applying Hölder's inequality with $\frac{1}{t_1} = \frac{1}{p_1} - \frac{\alpha}{2n}$, $\frac{1}{t_2} = \frac{1}{p_2} - \frac{\alpha}{2n}$ and $\left\|I_{\frac{\alpha}{2}} f\right\|_{L^{t_i}(\mu)} \le C \|f\|_{L^{p_i}(\mu)}$, $i = 1, 2$ (see [13, Corollary3]), we get

$$\|\mathcal{K}_\alpha(f_1, f_2)\|_{L^p(\mu)} \le C \left\|I_{\frac{\alpha}{2}}(|f_1|)\right\|_{L^{t_1}(\mu)} \left\|I_{\frac{\alpha}{2}}(|f_2|)\right\|_{L^{t_2}(\mu)} \le C \|f_1\|_{L^{p_1}(\mu)} \|f_2\|_{L^{p_2}(\mu)}.$$

□

Next we will illustrate the boundedness properties of the general bilinear fractional integrals on Morrey spaces with non-doubling measures. For the convenience of the following, the modified fractional maximal operator is defined by

$$M_{\gamma, \frac{9}{8}} f(x) := \sup_{x \in Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{9}{8}Q\right)^{1-\frac{\gamma}{n}}} \int_Q |f(y)| d\mu(y), 0 \le \gamma < n,$$

and $Mf(x) := M_{0, \frac{9}{8}} f(x)$.

Lemma 1.2. *[19, 20] Suppose that the parameters p, q, s, t and γ satisfy $1 < q \leq p < \infty$, $1 < t \leq s < \infty$, $0 \leq \gamma < n$, $\frac{1}{s} = \frac{1}{p} - \frac{\gamma}{n}$ and $\frac{q}{p} = \frac{t}{s}$. Then there exists a constant $C > 0$ such that, for all $f \in \mathcal{M}_q^p(\mu)$,*

$$\left\| M_{\gamma, \frac{9}{8}} f \right\|_{\mathcal{M}_t^s(\mu)} \leq C \|f\|_{\mathcal{M}_q^p(\mu)} .$$

Lemma 1.3. *[19] Let $1 < q \leq p < \infty$, $1 < t \leq s < \infty$, $0 < \alpha < n$ and $\frac{q}{p} = \frac{t}{s}$, $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$. Then there exists a constant $C > 0$ such that, for every $f \in \mathcal{M}_q^p(\mu)$,*

$$\|K_\alpha f\|_{\mathcal{M}_t^s(\mu)} \leq C \|f\|_{\mathcal{M}_q^p(\mu)} .$$

Lemma 1.4. *Support $0 < \alpha < 2n$, $1 < q_i \leq p_i < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} > 0$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{\alpha}{n} > 0$, $i = 1, 2$. Then, the operators $\mathcal{K}_\alpha$ are bounded from $\mathcal{M}_{q_1}^{p_1}(\mu) \times \mathcal{M}_{q_2}^{p_2}(\mu)$ to $\mathcal{M}_q^p(\mu)$, that is to say,*

$$\|\mathcal{K}_\alpha(f_1, f_2)\|_{\mathcal{M}_q^p(\mu)} \leq C \|f_1\|_{\mathcal{M}_{q_1}^{p_1}(\mu)} \|f_2\|_{\mathcal{M}_{q_2}^{p_2}(\mu)} ,$$

where $C > 0$ is the constant independent of f_1, f_2.

Proof. For all $y \in \mathbb{R}^d, y \neq x$, an elementary calculation yields

$$\frac{C}{|x-y|^{n-\alpha}} = \int_{|x-y|}^{\infty} l^{\alpha-n-1} dl = C \int_0^\infty \frac{\chi_{|x-y|<l}(y)}{l^n} l^{\alpha-1} dl. \tag{1.10}$$

Thus, by (1.10) and Fubini's theorem, we get

$$\begin{aligned}
&|\mathcal{K}_\alpha(f_1, f_2)(x)| \\
&\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |k_\alpha(x, y_1, y_2)||f_1(y_1)||f_2(y_2)| d\mu(y_1) d\mu(y_2) \\
&\leq C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f_1(y_1)||f_2(y_2)|}{(|x-y_1| + |x-y_2|)^{2n-\alpha}} d\mu(y_1) d\mu(y_2) \\
&\leq C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\int_{|x-y_1|+|x-y_2|}^{\infty} l^{\alpha-2n-1} dl \right) |f_1(y_1)||f_2(y_2)| d\mu(y_1) d\mu(y_2) \\
&= C \int_0^\infty \left(\frac{1}{l^n} \int_{|x-y_1|<l} |f_1(y_1)| d\mu(y_1) \right) \left(\frac{1}{l^n} \int_{|x-y_2|<l} |f_2(y_2)| d\mu(y_2) \right) l^{\alpha-1} dl \\
&= C \int_0^\epsilon \left(\frac{1}{l^n} \int_{|x-y_1|<l} |f_1(y_1)| d\mu(y_1) \right) \left(\frac{1}{l^n} \int_{|x-y_2|<l} |f_2(y_2)| d\mu(y_2) \right) l^{\alpha-1} dl \\
&\quad + C \int_\epsilon^\infty \left(\frac{1}{l^n} \int_{|x-y_1|<l} |f_1(y_1)| d\mu(y_1) \right) \left(\frac{1}{l^n} \int_{|x-y_2|<l} |f_2(y_2)| d\mu(y_2) \right) l^{\alpha-1} dl \\
&:= E_1 + E_2.
\end{aligned}$$

Applying the Hölder inequality and the growth condition (1.1), and letting cube $Q = Q(x, 2l)$, we get

$$\frac{1}{l^n} \int_{|x-y_i|<l} |f_i(y_i)| \, d\mu(y_i) \leq \frac{C}{\mu(\frac{9}{8}Q)} \int_Q |f_i(y_i)| \, d\mu(y_i) \leq M f_i(x), \quad i = 1, 2 \tag{1.11}$$

and

$$\begin{aligned}\frac{1}{l^n}\int_{|x-y_i|<l}|f_i(y_i)|\,d\mu(y_i) &\le \frac{C}{l^n}\int_Q |f_i(y_i)|\,d\mu(y_i)\\ &\le \frac{C}{l^n}\left(\int_Q |f_i(y_i)|^{q_i}\,d\mu(y_i)\right)^{\frac{1}{q_i}}\mu(Q)^{1-\frac{1}{q_i}} \\ &\le Cl^{-\frac{n}{p_i}}\|f_i\|_{\mathcal{M}^{p_i}_{q_i}(\mu)},\quad i=1,2.\end{aligned}\tag{1.12}$$

Then, by (1.11), we have

$$E_1 \le C\epsilon^{\alpha}Mf_1(x)Mf_2(x).$$

And by (1.12), we get

$$\begin{aligned}E_2 &\le C\int_\epsilon^\infty\left(l^{-\frac{n}{p_1}}\|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)}\right)\left(l^{-\frac{n}{p_1}}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}(\mu)}\right)l^{\alpha-1}dl\\ &\le C\epsilon^{-\frac{n}{p}}\|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}(\mu)}.\end{aligned}$$

By choosing $\epsilon=\left(\frac{\|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}(\mu)}}{Mf_1(x)Mf_2(x)}\right)^{\frac{n}{h}}$, where $\frac{1}{h}=\frac{1}{p_1}+\frac{1}{p_2}$, we can get

$$|\mathcal{K}_\alpha(f_1,f_2)(x)| \le C\left(\|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}(\mu)}\right)^{1-\frac{h}{p}}(Mf_1(x)Mf_2(x))^{\frac{h}{p}}.\tag{1.13}$$

For $\frac{1}{h}=\frac{1}{p_1}+\frac{1}{p_2}$, $\frac{1}{r}=\frac{1}{q_1}+\frac{1}{q_2}$, applying the Hölder inequality and Lemma 1.2, we obtain that

$$\begin{aligned}&|\mu(2Q)|^{\frac{1}{p}-\frac{1}{q}}\left\{\int_Q|Mf_1(y)Mf_2(y)|^{\frac{qh}{p}}\,d\mu(y)\right\}^{\frac{1}{q}}\\ &\le C|\mu(2Q)|^{\frac{1}{p}-\frac{h}{pr}}\left\{\int_Q|Mf_1(y)Mf_2(y)|^{r}\,d\mu(y)\right\}^{\frac{h}{pr}}\\ &\le C\left\{|\mu(2Q)|^{\frac{1}{h}-\frac{1}{r}}\left\{\int_Q|Mf_1(y)Mf_2(y)|^{r}\,d\mu(y)\right\}^{\frac{1}{r}}\right\}^{\frac{h}{p}}\\ &\le C\left\{|\mu(2Q)|^{\frac{1}{h}-\frac{1}{r}}\left(\int_Q|Mf_1(y)|^{q_1}d\mu(y)\right)^{\frac{1}{q_1}}\left(\int_Q|Mf_2(y)|^{q_2}d\mu(y)\right)^{\frac{1}{q_2}}\right\}^{\frac{h}{p}}\\ &\le C\left(\|Mf_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)}\|Mf_2\|_{\mathcal{M}^{p_2}_{q_2}(\mu)}\right)^{\frac{h}{p}} \le C\left(\|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}(\mu)}\right)^{\frac{h}{p}}.\end{aligned}\tag{1.14}$$

Thus, by (1.13) and (1.14), we get

$$\|\mathcal{K}_\alpha(f_1,f_2)\|_{\mathcal{M}^{p}_{q}(\mu)} \le C\|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}(\mu)}.$$

□

Now let $b_1, b_2 \in \mathrm{RBMO}(\mu)$, then the iterated commutators of general bilinear fractional integral operator $[b_1, b_2, \mathcal{K}_\alpha]$ is formally defined as

$$[b_1,b_2,\mathcal{K}_\alpha](f_1,f_2)(x)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(b_1(x)-b_1(y_1))(b_2(x)-b_2(y_2))k_\alpha(x,y_1,y_2)f_1(y_1)f_2(y_2)d\mu(y_1)d\mu(y_2).\tag{1.15}$$

Obviously, (1.15) is the form of (1.6) when $k_\alpha(x, y_1, y_2) = 1/(|x - y_1| + |x - y_2|)^{2n-\alpha}$. Inspired by [18] and [10, 22], we find that the boundedness and compactness of the iterated commutators of general bilinear fractional integral operators $[b_1, b_2, \mathcal{K}_\alpha]$ on the Morrey space with non-doubling measures have not been established. Our main results can be stated as follows.

Theorem 1.5. *Let $0 < \alpha < 2n$, $1 < q_i \le p_i < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} > 0$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{\alpha}{n} > 0$, $i = 1, 2$. Suppose $\|\mu\| = \infty$ and $b_1, b_2 \in \mathrm{RBMO}(\mu)$. Then, the commutators $[b_1, b_2, \mathcal{K}_\alpha]$ are bounded from $\mathcal{M}^{p_1}_{q_1}(\mu) \times \mathcal{M}^{p_2}_{q_2}(\mu)$ to $\mathcal{M}^{p}_{q}(\mu)$, moreover,*

$$\|[b_1, b_2, \mathcal{K}_\alpha](f_1, f_2)\|_{\mathcal{M}^p_q(\mu)} \le C\|b_1\|_* \|b_2\|_* \|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)} \|f_2\|_{\mathcal{M}^{p_2}_{q_2}(\mu)},$$

where $C > 0$ is the constant independent of f_1, f_2, b_1, b_2.

By using the similar methods in [22, Theorem 1.4], we can obtain the proof of Theorem 1.5. Hence, we omit the details.

On the basis of Theorem 1.5, we will pay our attention to the compactness of $[b_1, b_2, \mathcal{K}_\alpha]$ on Morrey space with non-doubling measures. We have the following theorem.

Theorem 1.6. *Let $0 < \alpha < 2n$, $1 < q_i \le p_i < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} > 0$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{\alpha}{n} > 0$, $i = 1, 2$. Suppose $\|\mu\| = \infty$ and $b_1, b_2 \in \mathrm{RVMO}(\mu)$. Then, the commutator $[b_1, b_2, \mathcal{K}_\alpha]$ is a compact operator from $\mathcal{M}^{p_1}_{q_1}(\mu) \times \mathcal{M}^{p_2}_{q_2}(\mu)$ to $\mathcal{M}^{p}_{q}(\mu)$.*

2. Proof of Theorem 1.6

In order to prove Theorem 1.6, we also need a compactness criterion on spaces of $\mathcal{M}^{p_1}_{q_1}(\mu) \times \mathcal{M}^{p_2}_{q_2}(\mu)$.

Lemma 2.1. *[18, 21] Let $1 < q \le p < \infty$, $1 < q_i \le p_i < \infty$, $i = 1, 2$. Suppose $\mathcal{T}$ is a bilinear integral operator defined as follows*

$$\mathcal{T}(f_1, f_2)(x) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x, y_1, y_2) f_1(y_1) f_2(y_2) d\mu(y_1) d\mu(y_2), \tag{2.1}$$

where the kernel function $k \in L^\infty_c(\mu \otimes \mu \otimes \mu)$. Then $\mathcal{T}$ is a compact operator from $\mathcal{M}^{p_1}_{q_1}(\mu) \times \mathcal{M}^{p_2}_{q_2}(\mu)$ to $\mathcal{M}^{p}_{q}(\mu)$.

Lemma 2.2. *Let $x \in supp(\mu)$, $0 \le \alpha < n$, and let $\varrho > 0$ and $r > 0$. Then there exist constants $C_1, C_2 > 0$ such that*

$$\int_{|x-y|\le r} \frac{|f(y)|}{|x-y|^{n-\alpha-\varrho}} d\mu(y) \le C_1 r^\varrho M_\alpha f(x), \tag{2.2}$$

$$\int_{|x-y|> r} \frac{|f(y)|}{|x-y|^{n-\alpha+\varrho}} d\mu(y) \le C_2 r^{-\varrho} M_\alpha f(x). \tag{2.3}$$

Proof. We will use the same procedure as in the proof of ([13], Lemmas 2.1 and 2.2). If $n-\alpha \le \varrho$, (2.2) follows immediately from (1.1). If $\varrho < n - \alpha$, we write

$$\int_{|x-y|\le r} \frac{|f(y)|}{|x-y|^{n-\alpha-\varrho}} d\mu(y) = \sum_{j=0}^{\infty} \int_{2^{-j-1}r\le|x-y|<2^{-j}r} \frac{|f(y)|}{|x-y|^{n-\alpha-\varrho}} d\mu(y)$$

$$\leq \sum_{j=0}^{\infty} \frac{1}{(2^{-j-1}r)^{n-\alpha-\varrho}} \int_{|x-y|<2^{-j}r} |f(y)|d\mu(y)$$
$$= C_1 r^{\varrho} M_\alpha f(x).$$

On the other hand, with the similar ways as above we have

$$\begin{aligned}\int_{|x-y|>r} \frac{|f(y)|}{|x-y|^{n-\alpha+\varrho}} d\mu(y) &= \sum_{j=0}^{\infty} \int_{2^j r\leq|x-y|<2^{j+1}r} \frac{|f(y)|}{|x-y|^{n-\alpha+\varrho}} d\mu(y)\\ &\leq \sum_{j=0}^{\infty} \frac{1}{(2^j r)^{n-\alpha+\varrho}} \int_{|x-y|\leq 2^{j+1}r} |f(y)|d\mu(y)\\ &= C_2 r^{-\varrho} M_\alpha f(x).\end{aligned}$$

□

Recall the function space RVMO(μ), $b \in$ RVMO(μ) if and only if there exists a sequence of compactly supported smooth functions $\{b_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} \|b - b_j\|_* = 0$. In order to obtain the compactness in Theorem 1.6, we will establish some function approximations for the iterated commutator. Assume a function $\psi(t) \in C_0^\infty(\mathbb{R}^d)$ satisfying supp $\psi \subset [1,\infty)$ and $\psi(t) = 1$, $t \geq 2$. For $\varepsilon \leq 1$, we consider the truncation of $\mathcal{K}_\alpha$,

$$\mathcal{K}_\alpha^\varepsilon (f_1, f_2)(x) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi\left(\frac{|x-y_1|}{\varepsilon}\right)\psi\left(\frac{|x-y_2|}{\varepsilon}\right) k_\alpha (x, y_1, y_2) f_1 (y_1) f_2 (y_2) d\mu (y_1) d\mu (y_2). \quad (2.4)$$

Lemma 2.3. *Let $b_1, b_2 \in C_c^\infty(\mathbb{R}^d)$, the kernel function k_α satisfies* (1.8) *and* (1.9). *For $f_1 \in \mathcal{M}_{q_1}^{p_1}(\mu)$, $f_2 \in \mathcal{M}_{q_2}^{p_2}(\mu)$ with $1 < q_i \leq p_i < \infty$, $i = 1, 2$, and let $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} > 0$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{\alpha}{n} > 0$, then the iterated commutators $[b_1, b_2, \mathcal{K}_\alpha^\varepsilon]$ tends to $[b_1, b_2, \mathcal{K}_\alpha]$ uniformly in $\mathcal{M}_p^q(\mu)$ as $\varepsilon \to 0$.*

Proof. One can see that

$$\begin{aligned}&[b_1, b_2, \mathcal{K}_\alpha](f_1, f_2)(x) - [b_1, b_2, \mathcal{K}_\alpha^\varepsilon](f_1, f_2)(x)\\ &= \lim_{\rho\to 0} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[\psi\left(\frac{|x-y_1|}{\varepsilon}\right)\psi\left(\frac{|x-y_2|}{\varepsilon}\right) - \psi\left(\frac{|x-y_1|}{\rho}\right)\psi\left(\frac{|x-y_2|}{\rho}\right)\right] \quad (2.5)\\ &\quad\times (b_1(x) - b_1(y_1))(b_2(x) - b_2(y_2))k_\alpha(x, y_1, y_2)f(y_1)f(y_2)d\mu(y_1)d\mu(y_2).\end{aligned}$$

In view of the size condition of the kernel k_α, the property of ψ, using Lebesgue's convergence theorem and Lemma 2.2, we obtain

$$\begin{aligned}&\left|[b_1, b_2, \mathcal{K}_\alpha](f_1, f_2)(x) - [b_1, b_2, \mathcal{K}_\alpha^\varepsilon](f_1, f_2)(x)\right|\\ &\leq C \lim_{\rho\to 0} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\psi\left(\frac{|x-y_1|}{\varepsilon}\right)\psi\left(\frac{|x-y_2|}{\varepsilon}\right) - \psi\left(\frac{|x-y_1|}{\rho}\right)\psi\left(\frac{|x-y_2|}{\rho}\right)\right|\\ &\quad\times \frac{|x-y_1||x-y_2||f_1(y_1)||f_2(y_2)|}{(|x-y_1| + |x-y_2|)^{2n-\alpha}} d\mu(y_1)d\mu(y_2)\\ &\leq C \int_{|x-y_1|\leq 2\varepsilon} \frac{|f_1 (y_1)|}{|x-y_1|^{n-1-\alpha_1}} d\mu (y_1) \int_{|x-y_2|\leq 2\varepsilon} \frac{|f_2 (y_2)|}{|x-y_2|^{n-1-\alpha_2}} d\mu (y_2)\end{aligned}$$

$$+C\int_{|x-y_1|\le 2\varepsilon}\frac{|f_1(y_1)|}{|x-y_1|^{n-2-\delta-\alpha_1}}d\mu(y_1)\int_{|x-y_2|>2\varepsilon}\frac{|f_2(y_2)|}{|x-y_2|^{n+\delta-\alpha_2}}d\mu(y_2)$$
$$+C\int_{|x-y_1|>2\varepsilon}\frac{|f_1(y_1)|}{|x-y_1|^{n+\delta-\alpha_1}}d\mu(y_1)\int_{|x-y_2|\le 2\varepsilon}\frac{|f_2(y_2)|}{|x-y_2|^{n-2-\delta-\alpha_2}}d\mu(y_2)$$
$$\le C\varepsilon^2 M_{\alpha_1}f_1(x)M_{\alpha_2}f_2(x),$$

where we have chosen $\delta > 0$ and $\alpha_1+\alpha_2=\alpha$ such that $\frac{1}{s_i}=\frac{1}{p_i}-\frac{\alpha_i}{n}>0$, $i=1,2$. We note that $\left\|M_{\alpha_i}f_i\right\|_{\mathcal{M}^{s_i}_{t_i}(\mu)}\le C\|f_i\|_{\mathcal{M}^{p_i}_{q_i}(\mu)}$ with $\frac{s_i}{t_i}=\frac{p_i}{q_i}$. Thus, we have that

$$\left\|([b_1,b_2,\mathcal{K}_\alpha]-[b_1,b_2,\mathcal{K}^\varepsilon_\alpha])(f_1,f_2)\right\|_{\mathcal{M}^p_q(\mu)}\le C\varepsilon^2\|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}(\mu)}.$$

Therefore, Lemma 2.3 holds. □

Lemma 2.4. *Under the same conditions and notations, let $\mathcal{K}^\varepsilon_\alpha$ be the truncation operator defined by (2.4). $\mathcal{K}^{\varepsilon,R}_\alpha$ be the truncation of $\mathcal{K}^\varepsilon_\alpha$,*

$$\mathcal{K}^{\varepsilon,R}_\alpha(f_1,f_2)(x):=\int_{|x-y_1|\le R}\int_{|x-y_2|\le R}\psi\left(\frac{|x-y_1|}{\varepsilon}\right)\psi\left(\frac{|x-y_2|}{\varepsilon}\right)k_\alpha(x,y_1,y_2)f(y_1)f(y_2)d\mu(y_1)d\mu(y_2).$$

Then, for $f_1\in\mathcal{M}^{p_1}_{p_1}(\mu)$, $f_2\in\mathcal{M}^{p_2}_{p_2}(\mu)$ with $1<q_i\le p_i<\infty$, $i=1,2$, the iterated commutator $[b_1,b_2,\mathcal{K}^{\varepsilon,R}_\alpha]$ tends to $[b_1,b_2,\mathcal{K}^\varepsilon_\alpha]$ uniformly in $\mathcal{M}^p_q(\mu)$ as $R\to\infty$. Moreover, there exist $\delta>0$ such that

$$\left\|([b_1,b_2,\mathcal{K}^{\varepsilon,R}_\alpha]-[b_1,b_2,\mathcal{K}^\varepsilon_\alpha])(f_1,f_2)\right\|_{\mathcal{M}^p_q(\mu)}\le C_\varepsilon R^{-\delta}\|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}(\mu)}.$$

Proof. Note $0<\alpha<2n$, $1<q\le p<\infty$, $1<q_i\le p_i<\infty$, $i=1,2$, $\frac{1}{h}=\frac{1}{p_1}+\frac{1}{p_2}$, $\frac{1}{r}=\frac{1}{q_1}+\frac{1}{q_2}$, $\frac{1}{p}=\frac{1}{h}-\frac{\alpha}{n}>0$, $\frac{1}{q}=\frac{1}{r}-\frac{\alpha}{n}>0$ and $b_1,b_2\in C^\infty_c(\mathbb{R}^d)$. Taking $f_1\in\mathcal{M}^{p_1}_{q_1}(\mu)$, $f_2\in\mathcal{M}^{p_2}_{q_2}(\mu)$, we obtain

$$[b_1,b_2,\mathcal{K}^\varepsilon_\alpha](f_1,f_2)(x)-[b_1,b_2,\mathcal{K}^{\varepsilon,R}_\alpha](f_1,f_2)(x)$$
$$=\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}-\int_{|x-y_1|\le R}\int_{|x-y_2|\le R)}\right)\psi\left(\frac{|x-y_1|}{\varepsilon}\right)\psi\left(\frac{|x-y_2|}{\varepsilon}\right)$$
$$\times(b_1(x)-b_1(y_1))(b_2(x)-b_2(y_2))k_\alpha(x,y_1,y_2)f(y_1)f(y_2)d\mu(y_1)d\mu(y_2).$$

Next we introduce the following two indices:

$$\alpha_1=\frac{1}{p_1}\left(\frac{1}{p_1}+\frac{1}{p_2}\right)^{-1}\alpha=\frac{h}{p_1}\alpha\quad\text{and}\quad\alpha_2=\frac{1}{p_2}\left(\frac{1}{p_1}+\frac{1}{p_2}\right)^{-1}\alpha=\frac{h}{p_2}\alpha.$$

Then α_1 and α_2 satisfy

$$\alpha=\alpha_1+\alpha_2,\quad 0<\alpha_1<\frac{n}{p_1},\quad 0<\alpha_2<\frac{n}{p_2}.$$

In fact,

$$\alpha_i-\frac{n}{p_i}=n\left(\frac{h}{p_i}\frac{\alpha}{n}-\frac{1}{p_i}\right)<n\left(\frac{h}{p_i}\frac{1}{h}-\frac{1}{p_i}\right)=0,\quad i=1,2.$$

Now, since $p_1,p_2>1$, we see that $n>\max(\alpha_1,\alpha_2)$. In particular, this yields

$$(|x-y_1|+|x-y_2|)^{2n-\alpha}=(|x-y_1|+|x-y_2|)^{(n-\alpha_1)+(n-\alpha_2)}\ge|x-y_1|^{n-\alpha_1}|x-y_2|^{n-\alpha_2}.$$

For $b_1, b_2 \in C_c^\infty(\mathbb{R}^d)$, there exist $Q_1 \supset \operatorname{supp} b_1$ and $Q_2 \supset \operatorname{supp} b_2$. We can see that the kernel satisfies

$$\begin{aligned}&\left|\psi\left(\frac{|x-y_1|}{\varepsilon}\right)\right|\left|\psi\left(\frac{|x-y_2|}{\varepsilon}\right)\right| |(b_1(x)-b_1(y_1))|\,|(b_2(x)-b_2(y_2))|\,|k_\alpha(x,y_1,y_2)|\\ &\leq C\frac{\chi_{\mathbb{R}^d\setminus B(x,\varepsilon)}(y_1)\left(\chi_{Q_1}(x)+\chi_{Q_1}(y_1)\right)\chi_{\mathbb{R}^d\setminus B(x,\varepsilon)}(y_2)\left(\chi_{Q_2}(x)+\chi_{Q_2}(y_2)\right)}{|x-y_1|^{n-\alpha_1}\,|x-y_2|^{n-\alpha_2}},\end{aligned}$$

that is

$$\begin{aligned}&\left|[b_1,b_2,\mathcal{K}_\alpha^\varepsilon](f_1,f_2)(x)-[b_1,b_2,\mathcal{K}_\alpha^{\varepsilon,R}](f_1,f_2)(x)\right|\\ &\leq C\left(\int_{|x-y_1|\geq R}\int_{|x-y_2|\geq R}+\int_{|x-y_1|\geq R}\int_{\varepsilon\leq|x-y_2|<R}+\int_{\varepsilon\leq|x-y_1|<R}\int_{|x-y_2|\geq R}\right)\\ &\quad\times\frac{(\chi_{Q_1}(x)+\chi_{Q_1}(y_1))(\chi_{Q_2}(x)+\chi_{Q_2}(y_2))|f_1(y_1)||f_2(y_2)|}{|x-y_1|^{n-\alpha_1}\,|x-y_2|^{n-\alpha_2}}d\mu(y_1)d\mu(y_2)\\ &:= D_1+D_2+D_3.\end{aligned}$$

To estimate D_1, applying (1.10), the Fubini theorem and (1.12), we get

$$\begin{aligned}D_1&=C\int_{|x-y_1|\geq R}(\chi_{Q_1}(x)+\chi_{Q_1}(y_1))\frac{|f_1(y_1)|}{|x-y_1|^{n-\alpha_1}}d\mu(y_1)\\ &\quad\times\int_{|x-y_2|\geq R}(\chi_{Q_2}(x)+\chi_{Q_2}(y_2))\frac{|f_2(y_2)|}{|x-y_2|^{n-\alpha_2}}d\mu(y_2)\\ &=C\int_{|x-y_1|\geq R}(\chi_{Q_1}(x)+\chi_{Q_1}(y_1))\left(\int_0^\infty\frac{\chi_{|x-y_1|<l_1}(y_1)}{l_1^n}l_1^{\alpha_1-1}dl_1\right)|f_1(y_1)|d\mu(y_1)\\ &\quad\times\int_{|x-y_2|\geq R}(\chi_{Q_2}(x)+\chi_{Q_2}(y_2))\left(\int_0^\infty\frac{\chi_{|x-y_2|<l_2}(y_2)}{l_2^n}l_2^{\alpha_2-1}dl_2\right)|f_2(y_2)|d\mu(y_2)\\ &=C\int_R^\infty\left(\frac{1}{l_1^n}\int_{|x-y_1|<l_1}(\chi_{Q_1}(x)+\chi_{Q_1}(y_1))|f_1(y_1)|d\mu(y_1)\right)l_1^{\alpha_1-1}dl_1\\ &\quad\times\int_R^\infty\left(\frac{1}{l_2^n}\int_{|x-y_2|<l_2}(\chi_{Q_2}(x)+\chi_{Q_2}(y_2))|f_2(y_2)|d\mu(y_2)\right)l_2^{\alpha_2-1}dl_2\\ &\leq C\left[\chi_{Q_1}(x)\int_R^\infty l_1^{-\frac{n}{p_1}+\alpha_1-1}dl_1\,\|f_1\|_{\mathcal{M}_{q_1}^{p_1}(\mu)}+\int_R^\infty\left(\int_{|x-y_1|<l_1}\chi_{Q_1}(y_1)\,|f_1(y_1)|\,d\mu(y_1)\right)l_1^{-n+\alpha_1-1}dl_1\right]\\ &\quad\times\left[\chi_{Q_2}(x)\int_R^\infty l_2^{-\frac{n}{p_2}+\alpha_2-1}dl_2\,\|f_2\|_{\mathcal{M}_{q_2}^{p_2}(\mu)}+\int_R^\infty\left(\int_{|x-y_1|<l_2}\chi_{Q_2}(y_2)\,|f_2(y_2)|\,d\mu(y_2)\right)l_2^{-n+\alpha_2-1}dl_2\right]\\ &\leq C\left[\chi_{Q_1}(x)R^{-\frac{n}{p_1}+\alpha_1}\,\|f_1\|_{\mathcal{M}_{q_1}^{p_1}(\mu)}\int_R^\infty\left(\int_{Q_1}\chi_{|x-y_1|<l_1}(y_1)\,|f_1(y_1)|\,d\mu(y_1)\right)l_1^{-n+\alpha_1-1}dl_1\right]\\ &\quad\times\left[\chi_{Q_2}(x)R^{-\frac{n}{p_2}+\alpha_2}\,\|f_2\|_{\mathcal{M}_{q_2}^{p_2}(\mu)}\int_R^\infty\left(\int_{Q_2}\chi_{|x-y_1|<l_2}(y_2)\,|f_2(y_2)|\,d\mu(y_2)\right)l_2^{-n+\alpha_2-1}dl_2\right].\end{aligned}$$

Furthermore, using the Minkowski inequality, we can deduce that

$$\begin{aligned}&\|D_1\|_{\mathcal{M}_q^p(\mu)}\\ &\leq CR^{-\frac{n}{p_1}+\alpha_1-\frac{n}{p_2}+\alpha_2}\left\|\chi_{Q_1}(x)\chi_{Q_2}(x)\right\|_{\mathcal{M}_q^p(\mu)}\|f_1\|_{\mathcal{M}_{q_1}^{p_1}(\mu)}\|f_2\|_{\mathcal{M}_{q_2}^{p_2}(\mu)}\end{aligned}$$

$$
\begin{aligned}
&+ CR^{-\frac{n}{p_1}+\alpha_1} \|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)} \left\| \chi_{Q_1}(x) \int_R^\infty \left(\int_{Q_2} \chi_{|x-y_1|<l_2}(y_2)\, |f_2(y_2)|\, d\mu(y_2) \right) l_2^{-n+\alpha_2-1} dl_2 \right\|_{\mathcal{M}^p_q(\mu)} \\
&+ CR^{-\frac{n}{p_2}+\alpha_2} \|f_2\|_{\mathcal{M}^{p_2}_{q_2}(\mu)} \left\| \chi_{Q_2}(x) \int_R^\infty \left(\int_{Q_1} \chi_{|x-y_1|<l_1}(y_1)\, |f_1(y_1)|\, d\mu(y_1) \right) l_1^{-n+\alpha_1-1} dl_1 \right\|_{\mathcal{M}^p_q(\mu)} \\
&+ C \left\| \int_R^\infty \left(\int_{Q_1} \chi_{|x-y_1|<l_1}(y_1)\, |f_1(y_1)|\, d\mu(y_1) \right) l_1^{-n+\alpha_1-1} dl_1 \right. \\
&\qquad \left. \times \int_R^\infty \left(\int_{Q_2} \chi_{|x-y_1|<l_2}(y_2)\, |f_2(y_2)|\, d\mu(y_2) \right) l_2^{-n+\alpha_2-1} dl_2 \right\|_{\mathcal{M}^p_q(\mu)} \\
&:= D_{11} + D_{12} + D_{13} + D_{14}.
\end{aligned}
$$

It is easy to get that

$$
D_{11} \le CR^{-\frac{n}{p}} \|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)} \|f_2\|_{\mathcal{M}^{p_2}_{q_2}(\mu)},
$$

since

$$
\begin{aligned}
\left\| \chi_{Q_1}(x)\chi_{Q_2}(x) \right\|_{\mathcal{M}^p_q(\mu)} &= \sup_{Q \in \mathcal{Q}(\mu)} \mu(2Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \left| \chi_{Q_1}\chi_{Q_2} \right|^q d\mu \right)^{\frac{1}{q}} \\
&= \sup_{Q \in \mathcal{Q}(\mu)} \frac{\mu\,(Q \cap Q_1 \cap Q_2)^{\frac{1}{q}}}{\mu(2Q)^{\frac{1}{q}-\frac{1}{p}}} \\
&\le C\mu\,(Q_1 \cap Q_2)^{\frac{1}{p}} < C.
\end{aligned}
$$

For D_{12}, using the generalized Minkowski inequality, $\mathcal{M}^q_p(\mu) \supset L^p(\mu)$ and the growth condition (1.1), we have the follow estimate

$$
\begin{aligned}
D_{12} &= CR^{-\frac{n}{p_1}+\alpha_1} \|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)} \left\| \int_R^\infty \left(\int_{Q_2} \chi_{Q_1}(x)\chi_{|x-y_1|<l_2}(y_2)\, |f_2(y_2)|\, d\mu(y_2) \right) l_2^{-n+\alpha_2-1} dl_2 \right\|_{\mathcal{M}^p_q(\mu)} \\
&\le CR^{-\frac{n}{p_1}+\alpha_1} \|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)} \int_R^\infty \left(\int_{Q_2} \left\| \chi_{Q_1}(\cdot)\chi_{|\cdot-y_1|<l_2}(y_2) \right\|_{\mathcal{M}^p_q(\mu)} |f_2(y_2)|\, d\mu(y_2) \right) l_2^{-n+\alpha_2-1} dl_2 \\
&\le CR^{-\frac{n}{p_1}+\alpha_1} \int_R^\infty \sup_{y_2 \in \mathbb{R}^d} \left\| \chi_{B(\cdot,l_2)}(y_2) \right\|_{L^p(\mu)} l_2^{-n+\alpha_2-1} dl_2\, \|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)} \|f_2\|_{\mathcal{M}^{p_2}_1(\mu)} \\
&\le CR^{-\frac{n}{p_1}+\alpha_1} \int_R^\infty l_2^{\frac{n}{p}-n+\alpha_2-1} dl_2\, \|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)} \|f_2\|_{\mathcal{M}^{p_2}_1(\mu)} \\
&\le CR^{\frac{n}{p_2}-n} \|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)} \|f_2\|_{\mathcal{M}^{p_2}_1(\mu)}.
\end{aligned}
$$

Similarly, we have

$$
D_{13} \le CR^{\frac{n}{p_1}-n} \|f_1\|_{\mathcal{M}^{p_1}_1(\mu)} \|f_2\|_{\mathcal{M}^{p_2}_{q_2}(\mu)}.
$$

Applying the Hölder inequality, we obtain

$$
\begin{aligned}
D_{14} &\le C \int_R^\infty \int_R^\infty \int_{Q_1} \int_{Q_2} \left\| \chi_{B(\cdot,l_1)}(y_1)\chi_{B(\cdot,l_2)}(y_2) \right\|_{\mathcal{M}^p_q(\mu)} |f_2(y_2)|\, d\mu(y_2)\, |f_1(y_1)|\, d\mu(y_1) l_2^{-n+\alpha_2-1} dl_2 l_1^{-n+\alpha_1-1} dl_1 \\
&\le C \int_R^\infty \int_R^\infty \left\| \chi_{B(\cdot,l_1)}(y_1)\chi_{B(\cdot,l_2)}(y_2) \right\|_{L^p} \int_{Q_1} \int_{Q_2} |f_2(y_2)|\, d\mu(y_2)\, |f_1(y_1)|\, d\mu(y_1) l_2^{-n+\alpha_2-1} dl_2 l_1^{-n+\alpha_1-1} dl_1
\end{aligned}
$$

$$\leq C\int_R^\infty \sup_{y_1\in\mathbb{R}^d}\left\|\chi_{B(\cdot,l_1)}(y_1)\right\|_{L^{w_1}} l_1^{-n+\alpha_1-1}dl_1 \int_R^\infty \sup_{y_2\in\mathbb{R}^d}\left\|\chi_{B(\cdot,l_2)}(y_2)\right\|_{L^{w_2}} l_2^{-n+\alpha_2-1}dl_2 \,\|f_1\|_{\mathcal{M}_1^{p_1}(\mu)}\,\|f_2\|_{\mathcal{M}_1^{p_2}(\mu)}$$
$$\leq CR^{\frac{n}{w_1}-n+\alpha_1+\frac{n}{w_2}-n+\alpha_2}\,\|f_1\|_{\mathcal{M}_1^{p_1}(\mu)}\,\|f_2\|_{\mathcal{M}_1^{p_2}(\mu)}$$
$$\leq CR^{\frac{n}{p}-2n+\alpha}\,\|f_1\|_{\mathcal{M}_1^{p_1}(\mu)}\,\|f_2\|_{\mathcal{M}_1^{p_2}(\mu)},$$

where $w_1, w_2,$ and p satisfy $\frac{1}{w_1}=\frac{1}{p_1}-\frac{\alpha_1}{n}$, $\frac{1}{w_2}=\frac{1}{p_2}-\frac{\alpha_2}{n}$ and $\frac{1}{p}=\frac{1}{w_1}+\frac{1}{w_2}$.
Therefore, recalling that $\mathcal{M}_1^{p_i}\supset\mathcal{M}_{q_i}^{p_i}$, we obtain the norm estimate of D_1,

$$\|D_1\|_{\mathcal{M}_q^p}\leq C\left(R^{-\frac{n}{p}}+R^{-\frac{n}{p_2}-n}+R^{-\frac{n}{p_1}-n}+R^{-\frac{n}{p}-2n+\alpha}\right)\|f_1\|_{\mathcal{M}_{q_1}^{p_1}(\mu)}\,\|f_2\|_{\mathcal{M}_{q_2}^{p_2}(\mu)}\,.$$

We now estimate D_2. Similarly, using (1.10), the Fubini theorem and (1.12), we get

$$D_2 = C\int_{|x-y_1|\geq R}(\chi_{Q_1}(x)+\chi_{Q_1}(y_1))\frac{|f_1(y_1)|}{|x-y_1|^{n-\alpha_1}}d\mu(y_1)\int_{\varepsilon\leq|x-y_2|<R}(\chi_{Q_2}(x)+\chi_{Q_2}(y_2))\frac{|f_2(y_2)|}{|x-y_2|^{n-\alpha_2}}d\mu(y_2)$$
$$= C\int_{|x-y_1|\geq R}(\chi_{Q_1}(x)+\chi_{Q_1}(y_1))\left(\int_0^\infty\frac{\chi_{|x-y_1|<l_1}(y_1)}{l_1^n}l_1^{\alpha_1-1}dl_1\right)|f_1(y_1)|d\mu(y_1)$$
$$\times\int_{\varepsilon\leq|x-y_2|<R}(\chi_{Q_2}(x)+\chi_{Q_2}(y_2))\left(\int_0^\infty\frac{\chi_{|x-y_2|<l_2}(y_2)}{l_2^n}l^{\alpha_2-1}dl_2\right)|f_2(y_2)|d\mu(y_2)$$
$$= C\int_R^\infty\left(\frac{1}{l_1^n}\int_{|x-y_1|<l_1}(\chi_{Q_1}(x)+\chi_{Q_1}(y_1))|f_1(y_1)|d\mu(y_1)\right)l_1^{\alpha_1-1}dl_1$$
$$\times\left(\int_R^\infty\left(\frac{1}{l_2^n}\int_{|x-y_2|<R}(\chi_{Q_2}(x)+\chi_{Q_2}(y_2))|f_2(y_2)|d\mu(y_2)\right)l_2^{\alpha_2-1}dl_2\right.$$
$$\left.+\int_\varepsilon^R\left(\frac{1}{l_2^n}\int_{|x-y_2|<l_2}(\chi_{Q_2}(x)+\chi_{Q_2}(y_2))|f_2(y_2)|d\mu(y_2)\right)l_2^{\alpha_2-1}dl_2\right)$$
$$= C\int_R^\infty\left(\frac{1}{l_1^n}\int_{|x-y_1|<l_1}(\chi_{Q_1}(x)+\chi_{Q_1}(y_1))|f_1(y_1)|d\mu(y_1)\right)l_1^{\alpha_1-1}dl_1$$
$$\times\int_\varepsilon^\infty\left(\frac{1}{l_2^n}\int_{|x-y_2|<l_2}(\chi_{Q_2}(x)+\chi_{Q_2}(y_2))|f_2(y_2)|d\mu(y_2)\right)l_2^{\alpha_2-1}dl_2$$
$$\leq C\left[\chi_{Q_1}(x)R^{-\frac{n}{p_1}+\alpha_1}\,\|f_1\|_{\mathcal{M}_{q_1}^{p_1}(\mu)}+\int_R^\infty\left(\int_{Q_1}\chi_{|x-y_1|<l_1}(y_1)\,|f_1(y_1)|\,d\mu(y_1)\right)l_1^{-n+\alpha_1-1}dl_1\right]$$
$$\times\left[\chi_{Q_2}(x)\varepsilon^{-\frac{n}{p_2}+\alpha_2}\,\|f_2\|_{\mathcal{M}_{q_2}^{p_2}(\mu)}+\int_\varepsilon^\infty\left(\int_{Q_2}\chi_{|x-y_1|<l_2}(y_2)\,|f_2(y_2)|\,d\mu(y_2)\right)l_2^{-n+\alpha_2-1}dl_2\right].$$

Thus, we get the norm estimate of D_2,

$$\|D_2\|_{\mathcal{M}_q^p(\mu)}\leq C\left(R^{-\frac{n}{p_1}+\alpha_1}\varepsilon^{-\frac{n}{p_2}+\alpha_2}+R^{-\frac{n}{p_1}+\alpha_1}\varepsilon^{\frac{n}{p}-n+\alpha_2}+\varepsilon^{-\frac{n}{p_2}+\alpha_2}R^{\frac{n}{p}-n+\alpha_1}\right.$$
$$\left.+R^{\frac{n}{w_1}-n+\alpha_1}\varepsilon^{\frac{n}{w_2}-n+\alpha_2}\right)\|f_1\|_{\mathcal{M}_{q_1}^{p_1}(\mu)}\,\|f_2\|_{\mathcal{M}_{q_2}^{p_2}(\mu)},$$

where w_1, w_2, p satisfy $\frac{1}{w_1}=\frac{1}{p_1}-\frac{\alpha_1}{n}$, $\frac{1}{w_2}=\frac{1}{p_2}-\frac{\alpha_2}{n}$ and $\frac{1}{p}=\frac{1}{w_1}+\frac{1}{w_2}$.

The same estimate holds for D_3 by above argument with the role of y_1 and y_2 inter changed,

$$\|D_3\|_{\mathcal{M}_q^p(\mu)}\leq C\left(\varepsilon^{-\frac{n}{p_1}+\alpha_1}R^{-\frac{n}{p_2}+\alpha_2}+\varepsilon^{-\frac{n}{p_1}+\alpha_1}R^{\frac{n}{p}-n+\alpha_2}+R^{-\frac{n}{p_2}+\alpha_2}\varepsilon^{\frac{n}{p}-n+\alpha_1}\right.$$

$$+\varepsilon^{\frac{n}{w_1}-n+\alpha_1}R^{\frac{n}{w_2}-n+\alpha_2}\Big)\|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}(\mu)},$$

where w_1, w_2, p satisfy $\frac{1}{w_1}=\frac{1}{p_1}-\frac{\alpha_1}{n}$, $\frac{1}{w_2}=\frac{1}{p_2}-\frac{\alpha_2}{n}$ and $\frac{1}{p}=\frac{1}{w_1}+\frac{1}{w_2}$.

Thus, we obtain that

$$\left\|([b_1,b_2,\mathcal{K}^{\varepsilon,R}_{\alpha}]-[b_1,b_2,\mathcal{K}^{\varepsilon}_{\alpha}])(f_1,f_2)\right\|_{\mathcal{M}^{p}_{q}(\mu)}\leq C_{\varepsilon}R^{-\delta}\|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mu)}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}(\mu)},$$

for some constants $\delta>0$, and the constant C_{ε} independent of R. □

Proof of Theorem 1.6. Lemmas 2.3 and 2.4 reduce the proof of Theorem 1.6 to proving the compactness for $[b_1,b_2,\mathcal{K}^{\varepsilon,R}_{\alpha}]$ if $b_1,b_2\in C^{\infty}_{c}(\mathbb{R}^d)$ on product Morrey spaces. We can also get the compactness for the operator $[b_1,b_2,\mathcal{K}^{\varepsilon,R}_{\alpha}]$ by using Lemma 2.1. □

3. Conclusions

In this paper, we mainly obtain the Morrey-boundedness and compactness of the iterated commutators of general bilinear fractional integral operators $\mathcal{K}_{\alpha}$ under the non-doubling measure conditions. Instead of establishing of the compactness of the iterated commutators directly we proved it by means of two truncation.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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