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Research article

Multi-super-stability of antiderivations in Banach algebras

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Abstract: In this study, we investigate the concept of antiderivations in Banach algebras and study multi-super-stability of antiderivations in Banach algebras, associated with functional inequalities.

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1. Introduction

Hyers [1] made a response to the question of Ulam in the context of Banach spaces in relation to additive mappings and was a considerable step towards further solutions in this area. Note the concept of stability is a major property in the qualitative theory of differential equations. Over the last few years, results have been presented on numerous types of differential equations.The approach proposed by Hyers [1] which provides the additive function is named a direct technique. This technique is a significant and helpful tool used to investigate the stability of different functional equations. In recent years, a number of research monographs and articles have been studied on diverse applications and generalizations of the HUS, like k-additive mappings, differential equations, Navier–Stokes equations, ODEs, and PDEs (see [2–4]). Also in recent years, the stability of different (integral and differential, others functional) equations and other subjects (such as C^*-ternary algebras, groups, flows and Banach algebras) have been investigated.Fixed–point methods are useful when examining stability and fixed point theory proposes vital tools for solving problems arising in different fields of functional analysis, like equilibrium problems, differential equations, and dynamical systems.

Assume Banach algebras $\mathscr{Q}$ and $\mathscr{Q}''$. Suppose $(\mathscr{Q}', \Delta)$ is a probability measure space and suppose

$(\mathscr{Q}, \mathfrak{B}_{\mathscr{Q}})$ and $(\mathscr{Q}'', \mathfrak{B}_{\mathscr{Q}''})$ are Borel measurable spaces. Then a map $f : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}''$ is a operator if $\{\wp : f(\wp, \alpha) \in \nu\} \in \Delta$ for each α in $\mathscr{Q}$ and $\nu \in \mathfrak{B}_{\mathscr{Q}''}$. Assume $\mho = (\mho_1, \ldots, \mho_m)$ and $\Omega = (\Omega_1, \ldots, \Omega_m), m \in \mathbb{N}$. Then we have

$$\mho \leq \Omega \Longleftrightarrow \mho_\iota \leq \Omega_\iota, \quad \iota = 1, \cdots, m;$$

and also

$$\mho \to 0 \Longleftrightarrow \mho_\iota \to 0, \quad \iota = 1, \cdots, m.$$

Definition 1.1 ([5]). Let $\nabla \neq \emptyset$ is a set and $d : \nabla^2 \to [0, +\infty]^m, m \in \mathbb{N}$, is a given mapping. If the following conditions are satisfied, then we say d is a generalized metric on ∇:

(1) For each $(g, g') \in \nabla \times \nabla$, we get

$$d(g, g') = \underbrace{(0, \cdots, 0)}_{m} \Longleftrightarrow g = g';$$

(2) For each $(g, g') \in \nabla \times \nabla$, we get

$$d(g', g) = d(g, g') \Longleftrightarrow g = g';$$

(3) For each $g, g', g'' \in \nabla$, we get

$$d(g, g'') + d(g'', g') \geq d(g', g).$$

Theorem 1.2 ([5]). *Assume the following assumptions:*

(1) $d : \nabla^2 \to [0, +\infty]^m, m \in \mathbb{N}$, and (∇, d) is a complete generalized metric space.

(2) $\mathscr{L} : \nabla \to \nabla$ is a strictly contractive mapping with Lipschitz constant $\mathcal{Z} < 1$.

Then for each $g \in \nabla$, either

$$d(\mathscr{L}^n g, \mathscr{L}^{n+1} g) = \overbrace{(+\infty, \cdots, +\infty)}^{m}$$

for each $n \in \mathbb{N} \cup \{0\}$ or there is a $n_0 \in \mathbb{N}$ such that

(1) $d(\mathscr{L}^n g, \mathscr{L}^{n+1} g) \leq \overbrace{(+\infty, \cdots, +\infty)}^{m}, \qquad \forall n \geq n_0$;

(2) The sequence $\{\mathscr{L}^n g\}$ converges to a fixed point $(g')^*$ of $\mathscr{L}$;

(3) $(g')^*$ is the unique fixed point of $\mathscr{L}$ in the set $\complement = \{g' \in \nabla \mid d(\mathscr{L}^{n_0} g, g') \leq \overbrace{(+\infty, \cdots, +\infty)}^{m}\}$;

(4) $d(g', (g')^*) \leq \frac{1}{1-\mathcal{Z}} d(g', \mathscr{L} g')$ for each $g' \in \complement$.

We use fixed-point way to study the multi-stability of antiderivations associated with the following inequality:

$$\begin{aligned}
&\operatorname{diag}\Big[\Big\| f_1(\Lambda, \varepsilon + \zeta + \varsigma) - f_1(\Lambda, \varepsilon + \varsigma) - f_1(\Lambda, \varepsilon + \zeta - \varsigma) + f_1(\Lambda, \varepsilon - \varsigma)) \Big\|, \ldots, \\
&\qquad \Big\| f_n(\Lambda, \varepsilon + \zeta + \varsigma) - f_n(\Lambda, \varepsilon + \varsigma) - f_n(\Lambda, \varepsilon + \zeta - \varsigma) + f_n(\Lambda, \varepsilon - \varsigma)) \Big\| \ \Big]_{n\times n} \\
&\leq \operatorname{diag}\Big[\Big\| \theta_1 \Big(f_1(\Lambda, \varepsilon - \zeta + \varsigma) - f_1(\Lambda, \varepsilon + \varsigma) - f_1(\Lambda, \varepsilon - \zeta - \varsigma) + f_1(\Lambda, \varepsilon - \varsigma) \Big) \Big\|, \ldots, \\
&\qquad \Big\| \theta_n \Big(f_n(\Lambda, \varepsilon - \zeta + \varsigma) - f_n(\Lambda, \varepsilon + \varsigma) - f_n(\Lambda, \varepsilon - \zeta - \varsigma) + f_n(\Lambda, \varepsilon - \varsigma) \Big) \Big\| \ \Big]_{n\times n}
\end{aligned} \tag{1.1}$$

for each $\varepsilon, \zeta, \varsigma \in \mathscr{Q}$, $\Lambda \in \mathscr{Q}'$ with $|\theta_1|, \ldots, |\theta_n| < 1$.

2. Proposing new control functions and the concept of multi stability

For this section we refer the reader [6, 7]. Assume $\Re(\nu)$ denotes the real part of ν if $\nu \in \mathbb{C}$. Also, let
(1) $\mathbb{Z}^+$ be the set of the positive integers;
(2) $\mathbb{Z}^-$ be the negative integer numbers;
(3) $\mathbb{R}_-$ be the negative real numbers;
(4) $\mathbb{R}_+$ be the positive real numbers.
We begin by defining various functions which will be needed later. The gamma function is given by

$$\Gamma(X) = \int_0^\infty e^{-Y} Y^{X-1} dY, \quad \Re(X) > 0, \quad X \in \mathbb{C}.$$

Euler's functional equation is given by

$$\Gamma(X+1) = X\Gamma(X), \quad \Re(X) > 0, \; X \in \mathbb{C}.$$

Theorem 2.1 ([6]). *If* $X \in \mathbb{N} \cup \{0\}$, *then*

$$\Gamma(X+1) = X!.$$

Theorem 2.2 ([6]). $\Gamma(0.5) = \pi^{0.5}$.

The Pochhammer symbol is

$$(\mathsf{C})_\jmath = \prod_{\imath=1}^{\jmath} (\mathsf{C} + \imath - 1) = \frac{\Gamma(\mathsf{C} + \jmath)}{\Gamma(\mathsf{C})} = \begin{cases} 1 & \jmath = 0 \\ \mathsf{C}(\mathsf{C}+1)\cdots(\mathsf{C}+\jmath-1) & \jmath \in \mathbb{N} \cup \{0\} \end{cases}$$

where $\mathsf{C} \in \mathbb{C}$ and $\jmath, \imath \in \mathbb{N}$.
Note that

$$\Gamma(\mathsf{C} + \jmath) = \mathsf{C}(\mathsf{C}+1)\cdots(\mathsf{C}+\jmath-1)\Gamma(\mathsf{C})$$

where $\jmath \in \mathbb{N} \cup \{0\}$.
The Gauss hypergeometric series [7] is given by

$$\begin{aligned} \varphi_1^{\circledS}(X) &:= {}_2\mathbb{F}_1(\alpha, \mathsf{B}; \mathsf{T}; X) \\ &= 1 + \frac{\alpha\mathsf{B}}{\mathsf{T}} X + \frac{\alpha(\alpha+1)\mathsf{B}(\mathsf{B}+1)}{\mathsf{T}(\mathsf{T}+1)} \frac{X^2}{2} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)_n (\mathsf{B})_n}{(\mathsf{T})_n} \frac{X^n}{n!}, \end{aligned} \tag{2.1}$$

where $\alpha, \mathsf{B}, \mathsf{T}, X \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, and $|X| < 1$.
Consider the Gauss differential equation

$$(X - X^2)\frac{d^2\omega}{dX^2} + (\mathsf{T} - (\alpha + \mathsf{B} + 1)X)\frac{d\omega}{dX} - \alpha\mathsf{B}\omega = 0, \tag{2.2}$$

where $\alpha, \mathsf{B}, X \in \mathbb{C}$, $\mathsf{T} \in \mathbb{C} \backslash (\mathbb{Z}^- \cup \{0\})$, and $|X| < 1$. The hypergeometric series is a solution of (2.2).

Theorem 2.3 ([6]). *Let* $\alpha, \mathsf{B}, \mathsf{T}, X \in \mathbb{C}$ *and* $|X| < 1$. *Then*

$$ {}_2\mathbb{F}_1(\alpha, \mathsf{B}; \mathsf{T}; X) = \frac{\Gamma(\mathsf{T})}{\Gamma(\mathsf{B})\Gamma(\mathsf{T}-\mathsf{B})} \int_0^1 Y^{\mathsf{B}-1}(1-Y)^{\mathsf{T}-\mathsf{B}-1}(1-XY)^{-\alpha} dY, $$

where $\Re(\mathsf{T}) > \Re(\mathsf{B}) > 0$.

Theorem 2.4. *If* $\Re(\mathsf{T}) > 0, |X| < 1$, *and* $|\arg(-X)| < \pi$, *then*

$$ {}_2\mathbb{F}_1(\alpha, \mathsf{B}; \mathsf{T}; X) = \frac{\Gamma(\mathsf{T})}{\Gamma(\alpha)\Gamma(\mathsf{B})} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\Gamma(\alpha+Y)\Gamma(\mathsf{B}+Y)\Gamma(-Y)}{\Gamma(\mathsf{T}+\mathsf{Y})} (-X)^Y dY, $$

We now present the Clausen hypergeometric series [7] and its properties:

$$ \begin{aligned} \varphi_2^{\circledS}(X) &:= {}_p\mathbb{F}_q((\alpha); (\mathsf{T}); X) \\ &= {}_p\mathbb{F}_q(\alpha, \cdots, \alpha_p; \mathsf{T}_1, \cdots, \mathsf{T}_q; X) \\ &= {}_p\mathbb{F}_q\left(\begin{smallmatrix}\alpha_1, \cdots, \alpha_p \\ \mathsf{T}_1, \cdots, \mathsf{T}_q\end{smallmatrix}; X\right) \\ &= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\mathsf{T}_1)_k \cdots (\mathsf{T}_q)_k} \frac{X^n}{n!}, \end{aligned} \tag{2.3} $$

where $p, n, q \in \mathbb{N} \cup \{0\}$ and $\alpha_n, X, \mathsf{T}_n \in \mathbb{C}$.

Now, (2.3) is a solution of the following differential equation

$$ \Big(M(q, \eta, \mathsf{T}_n)\omega\Big)(X) - \Big(N(p, \eta, \alpha_n)\omega\Big)(X) = 0, $$

where

$$ \begin{aligned} &\Big(M(q, \eta, \mathsf{T}_n)\omega\Big)(X) \\ = &\Big(X\frac{d}{dX}\Big) \prod_{n=1}^{q} \Big(\Big(X\frac{d}{dX}\Big)\omega(X) + (\mathsf{T}_n - 1)\omega(X)\Big) = X\frac{d}{dX}\Big(\prod_{n=1}^{q}\Big(\Big(X\frac{d}{dX} + (\mathsf{T}_n - 1)\Big)\omega\Big)(X)\Big), \end{aligned} $$

and

$$ \Big(N(p, \eta, \alpha_n)\omega\Big)(X) = X \prod_{n=1}^{p} \Big(X\frac{d\omega(X)}{dX} + \alpha_n\omega(X)\Big) = X \prod_{n=1}^{p} \Big(\Big(X\frac{d}{dX} + \alpha_n\Big)\omega\Big)(X) $$

and $\alpha_n, X, \mathsf{T}_n \in \mathbb{C}$, $p, n, q \in \mathbb{N} \cup \{0\}$, and $|X| < 1$,

Theorem 2.5 ([6]). *Suppose* $\alpha_n \in \mathbb{C} \backslash (\mathbb{Z}^- \cup \{0\})$*:*

(1) The series converges only for $X = 0$, *if* $p > q + 1$.
(2) The series converges absolutely for $X \in \mathbb{C}$, *if* $p < q$.
(3) The series converges absolutely for $|X| < 1$ *and diverges for* $|X| = 1$ *and for* $|X| > 1$ *it converges absolutely for* $\Re\Big(\sum_{k=1}^{q} \mathsf{T}_k - \sum_{k=1}^{p} \alpha_k\Big) > 0$, *if* $p = q + 1$.

Assume the following notation [7]:

$$\Xi := -\sum_{k=1}^{q} b_k + \sum_{j=1}^{p} a_j, \tag{2.4}$$

$$\sigma := -\prod_{k=1}^{q} |b_k|^{-b_k} + \prod_{j=1}^{p} |a_j|^{-a_j}, \tag{2.5}$$

and

$$\chi := -\sum_{j=1}^{p} \kappa_j + \sum_{k=1}^{q} \vartheta_k + \frac{p-q}{2}, \tag{2.6}$$

where $\kappa_j, \vartheta_k \in \mathbb{C},\ k, j \in \mathbb{N},\ \ p, q \in \mathbb{N} \cup \{0\}$, and $b_k, a_j \in \mathbb{R}_+$.

The Wright generalized hypergeometric series is given by

$$\begin{aligned} \varphi_3^{\circledS}(X) &:= {}_p\mathbb{W}_q(X) \\ &= {}_p\mathbb{W}_q\left(^{(\kappa_1,a_1),\cdots,(\kappa_p,a_p)}_{(\vartheta_1,b_1),\cdots,(\vartheta_q,b_q)}; X\right) \\ &= {}_p\mathbb{W}_q\left(^{(\kappa_p,a_p)_{1,p}}_{(\vartheta_q,b_q)_{1,q}}; X\right) \\ &= \sum_{s=0}^{\infty} \frac{\left\{\prod_{j=1}^{p} \Gamma(\kappa_j + a_j s)\right\}}{\left\{\prod_{k=1}^{q} \Gamma(\vartheta_k + b_k s)\right\}} \frac{X^s}{s!}, \end{aligned} \tag{2.7}$$

where $j, s, k \in \mathbb{N}, X \in \mathbb{C},\ \Xi > -1, \kappa_j, \vartheta_k \in \mathbb{C},\ \ p, q \in \mathbb{N} \cup \{0\}$, and $b_k, a_j \in \mathbb{R}_+$.

Theorem 2.6 ([6]). *Suppose $X \in \mathbb{C},\ \vartheta_k, \kappa_j \in \mathbb{C},\ j, s, k \in \mathbb{N},\ b_k, a_j \in \mathbb{R}_+$, then*

(1) (2.7) *is absolutely convergent for each value of $|X| = \sigma$ and of $|X| < \sigma$, and $\Re(\chi) > 0.5$, if $\Xi + 1 = 0$.*
(2) (2.7) *is absolutely convergent for $X \in \mathbb{C}$, if $\Xi + 1 > 0$.*

Now, the Wright function is given by

$$\varphi_4^{\circledS}(X) := \mathbb{K}(\vartheta, b, X) = {}_0\mathbb{W}_1\left(^{-}_{(b,\vartheta)}; X\right) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\vartheta + bk)} \frac{X^k}{k!}, \tag{2.8}$$

where $X, \vartheta \in \mathbb{C}$, and $b \in \mathbb{R}$.

Theorem 2.7 ([6]). *Now* (2.8) *for $b \in \mathbb{C}$ ($b \in \mathbb{Z}^- \cup \{0\}$ if $\vartheta = 0$) and $\vartheta > -1$ is an entire function of type $\delta = (1 + \vartheta)|\vartheta|^{-\frac{\vartheta}{1+\vartheta}}$, and finite order $p = \frac{1}{1+\vartheta}$.*

Theorem 2.8 ([6]). *Now* (2.8) *is an entire function of X for each $b \in \mathbb{C}$ and $\vartheta > -1$.*

The Wright generalized Bessel function (Bessel-Maitland function) is given by

$$\varphi_5^{\circledS}(X) := \mathbb{J}(\kappa, a, X) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\kappa + 1 + ak)} \frac{(-X)^k}{k!} = {}_0\mathbb{W}_1\left({}^{-}_{(\kappa+1,b)}; -X\right),$$

where $\kappa, X \in \mathbb{C}$, and $a \in \mathbb{R}$.

Theorem 2.9 ([6]). *Suppose $X \in \mathbb{C}$, $j, s, k \in \mathbb{N}$, $a_j, b_k \in \mathbb{R}_+$, and $\kappa_j, \vartheta_k \in \mathbb{C}$. Then* (2.7) *is an entire function of X.*

Theorem 2.10 ([6]). *Suppose $b \in \mathbb{R}$ and $\vartheta \in \mathbb{C}$.*

(1) (2.8) *is absolutely convergent for all $|X| < 1$ and of $|X| = 1$, and $\Re(\chi) > 0.5$, if $b + 1 = 0$.*
(2) (2.8) *is absolutely convergent for $X \in \mathbb{C}$, if $b + 1 > 0$.*

Theorem 2.11 ([6]). *Suppose $b > -1$, $\vartheta \in \mathbb{C}$. Then* (2.8) *is an entire function of X.*

Theorem 2.12 ([6]). *Suppose $X \in \mathbb{C}$, $j, k, s \in \mathbb{N}$, $\kappa_j, \vartheta_k \in \mathbb{C}$, and $a_j, b_k \in \mathbb{R}_+$. Then*

$${}_p\mathbb{W}_q\left({}^{(\kappa_1,1),\cdots,(\kappa_p,1)}_{(\vartheta_1,1),\cdots,(\vartheta_q,1)}; X\right) = \frac{\prod_{j=1}^{p} \Gamma(\kappa_j)}{\prod_{k=1}^{p} \Gamma(\vartheta_k)} {}_p\mathbb{F}_q\left({}^{\kappa_1,\cdots,\kappa_p}_{\vartheta_1,\cdots,\vartheta_q}; X\right),$$

where $\Xi + 1 \geq 0$.

The shifted Wright generalized hypergeometric series [6] is given by

$$\begin{aligned}
&\varphi_6^{\circledS}(X) \\
&:= {}_p\mathbb{B}_q(X) \\
&= {}_p\mathbb{B}_q\left({}^{(\kappa_1,a_1;\vartheta_1,b_1),\cdots,(\kappa_p,a_p;\vartheta_p,b_p)}_{(\widehat{\kappa}_1,c_1;\widehat{\vartheta}_1,d_1),\cdots,(\widehat{\kappa}_p,c_p;\widehat{\vartheta}_p,d_p)}; X\right) \\
&= {}_p\mathbb{B}_q\left({}^{(\kappa_p,a_p;\vartheta_p,b_p)_{1,p}}_{(\widehat{\kappa}_p,c_p;\widehat{\vartheta}_p,d_p)_{1,q}}; X\right) \\
&= \sum_{k=0}^{\infty} \frac{\left\{\prod_{m=1}^{p} b(\kappa_m + a_m k; \vartheta_m + b_m k)\right\}}{\left\{\prod_{n=1}^{q} b(\widehat{\kappa}_n + c_n k; \widehat{\vartheta}_n + d_n k)\right\}} \frac{X^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{\prod_{m=1}^{p}\Big(\Gamma(\kappa_m + a_m k)\Gamma(\vartheta_m + b_m k)\Big) \prod_{n=1}^{q} \Gamma\Big((\widehat{\kappa}_n + \widehat{\vartheta}_n) + (c_n + d_n)k\Big)}{\prod_{m=1}^{p} \Gamma\Big((\vartheta_m + \kappa_m) + (b_m + a_m)k\Big) \prod_{n=1}^{q}\Big(\Gamma(\widehat{\kappa}_n + c_n k)\Gamma(\widehat{\vartheta}_n + d_n k)\Big)} \frac{X^k}{k!},
\end{aligned}$$

where $m, n \in \mathbb{N}, k \in \mathbb{N} \cup \{0\}, \kappa_m, \vartheta_m, \widehat{\kappa}_n, \widehat{\vartheta}_n, X \in \mathbb{C},\ p, q \in \mathbb{N} \cup \{0\}$, and $a_m, b_m, c_n, d_n \in \mathbb{R}_+$.

We have the following special cases:

$$\begin{aligned}
{}_0\mathbb{B}_0 &= e^X, \\
{}_1\mathbb{B}_0(X) &= {}_1\mathbb{B}_0\left({}^{(\kappa,a;\vartheta,b)}_{-}; X\right) \\
&= \sum_{k=0}^{\infty} b(\kappa + ak; \vartheta + bk) \frac{X^k}{k!}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{\Gamma(\kappa+ak)\Gamma(\vartheta+bk)}{\Gamma[(b+b)k+(\vartheta+\kappa)]}\frac{X^k}{k!} \\
&= {}_2\mathbb{W}_1\left({}^{(\kappa,b),(\vartheta,b)}_{(\vartheta+\kappa,b+b)}; X\right), \\
{}_0\mathbb{B}_1(X) &= {}_0\mathbb{B}_1\left({}^{-}_{(\kappa,a;\vartheta,b)}; X\right) \\
&= \sum_{k=0}^{\infty} \frac{1}{b(\kappa+ak;\vartheta+bk)}\frac{X^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{\Gamma[(b+b)k+(\vartheta+\kappa)]}{\Gamma(\vartheta+bk)\Gamma(\kappa+ak)}\frac{X^k}{k!} \\
&= {}_1\mathbb{W}_2\left({}^{(\vartheta+\kappa,b+b)}_{(\kappa,b),(\vartheta,b)}; X\right), \\
{}_1\mathbb{B}_1(X) &= {}_1\mathbb{B}_1\left({}^{(\widehat{\kappa},c;\widehat{\vartheta},d)}_{(\kappa,a;\vartheta,b)}; X\right) \\
&= \sum_{k=0}^{\infty} \frac{b(\kappa+ak;\vartheta+bk)}{b(\widehat{\kappa}+ck;\widehat{\vartheta}+dk)}\frac{X^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(\kappa+ak)\Gamma(\vartheta+bk)\Gamma[(\widehat{\kappa}+\widehat{\vartheta})+(c+d)k]}{\Gamma[(b+b)k+(\kappa+\vartheta)]\Gamma(\widehat{\vartheta}+dk)\Gamma(\widehat{\kappa}+ck)}\frac{X^k}{k!} \\
&= {}_3\mathbb{W}_3\left({}^{(\kappa,b),(\vartheta,b),(\widehat{\kappa}+\widehat{\vartheta},c+d)}_{(\widehat{\kappa},c),(\widehat{\vartheta},d),(\vartheta+\kappa,b+b)}; X\right),
\end{aligned}
$$

where $k \in \mathbb{N} \cup \{0\}, \kappa_m, \vartheta_m, \widehat{\kappa_n}, \widehat{\vartheta}_n, X \in \mathbb{C}$, and $a_m, b_m, c_n, d_n \in \mathbb{R}_+$.

Now, we define the Wright generalized hypergeometric series (see [6]) as follows

$$
\varphi_7^{\circledS}(X) := [{}_p\mathbb{W}_q]^n(X) = \sum_{s=0}^{n} \frac{\left\{\prod_{j=1}^{p}\Gamma(\kappa_j+a_js)\right\}}{\left\{\prod_{k=1}^{q}\Gamma(\vartheta_k+b_ks)\right\}}\frac{X^s}{s!},
$$

where $X, \kappa_j, \vartheta_k \in \mathbb{C}$, $s, j, k, q, p \in \mathbb{N}$, and $a_j, b_k \in \mathbb{R}_+$.

Let

$$
\mathrm{diag}[\rho_1, \cdots, \rho_n]_{n\times n} = \begin{bmatrix} \rho_1 & 0 & \dots & 0 \\ 0 & \rho_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \rho_n \end{bmatrix}_{n\times n}.
$$

Note that $\rho := \mathrm{diag}[\rho_1, \cdots, \rho_n] \leq \varrho := \mathrm{diag}[\varrho_1, \cdots, \varrho_n]$ if $\rho_i \leq \varrho_i$ for each $1 \leq i \leq n$.

We denote $\mathfrak{W}[X]$ as

$$
\mathrm{diag}\Big[\varphi_1^{\circledS}(X), \cdots, \varphi_n^{\circledS}(X)\Big]_{n\times n}.
$$

A HUR-stability with control functions $\mathfrak{W}[X]$, is called multi-stability.

3. Investigating multi-stability and super-multi-stability problem associated to (1.1)

We now propose the notion of antiderivations in Banach algebras and introduce the super-multi-stability of antiderivations in algebras Banach, associated to (1.1).

Throughout this section, let $\mathscr{Q}$ be a complex Banach algebra and that $\theta_1, \cdots, \theta_n \in \mathbb{C}\backslash\{0\}$ with $|\theta_1|, \ldots, |\theta_n| < 1$.

3.1. *Multi stability of the* $(\theta_1, \cdots, \theta_n)$*-functional inequality* (1.1)

In this subsection, we study the multi stability of the additive $(\theta_1, \cdots, \theta_n)$-functional inequality (1.1).

Lemma 3.1. *Suppose* $f_i : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}$ $(i = 1, \ldots, n \in \mathbb{N})$ *are mappings satisfying* $f_i(\Lambda, 0) = 0$ *and* (1.1) *for each* $\varepsilon, \zeta, \varsigma \in \mathscr{Q}$, *and* $\Lambda \in \mathscr{Q}'$. *Then the mappings* $f_i : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}$, $(i = 1, \ldots, n \in \mathbb{N})$ *are additive (the usual definition is at the end of the proof).*

Proof. Assume that $f_i : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}$ $(i = 1, \ldots, n \in \mathbb{N})$ satisfies (1.1).

Replacing ζ by $-\zeta$ in (1.1), we get

$$\begin{aligned} &\operatorname{diag}\Big[\Big\| f_1(\Lambda, \varepsilon - \zeta + \varsigma) - f_1(\Lambda, \varepsilon + \varsigma) - f_1(\Lambda, \varepsilon - \zeta - \varsigma) + f_1(\Lambda, \varepsilon - \varsigma) \Big\|, \ldots, \\ &\qquad \Big\| f_n(\Lambda, \varepsilon - \zeta + \varsigma) - f_n(\Lambda, \varepsilon + \varsigma) - f_n(\Lambda, \varepsilon - \zeta - \varsigma) + f_n(\Lambda, \varepsilon - \varsigma) \Big\| \Big] \\ &\leq \operatorname{diag}\Big[\Big\| \theta_1 \Big(f_1(\Lambda, \varepsilon + \zeta + \varsigma) - f_1(\Lambda, \varepsilon + \varsigma) - f_1(\Lambda, \varepsilon + \zeta - \varsigma) + f_1(\Lambda, \varepsilon - \varsigma) \Big) \Big\|, \ldots, \\ &\qquad \Big\| \theta_n \Big(f_n(\Lambda, \varepsilon + \zeta + \varsigma) - f_n(\Lambda, \varepsilon + \varsigma) - f_n(\Lambda, \varepsilon + \zeta - \varsigma) + f_n(\Lambda, \varepsilon - \varsigma) \Big) \Big\| \Big] \end{aligned} \tag{3.1}$$

for each $\varepsilon, \zeta, \varsigma \in \mathscr{Q}$, and $\Lambda \in \mathscr{Q}'$. According to (1.1) and (3.1) we have

$$\begin{aligned} &\operatorname{diag}\Big[\Big\| f_1(\Lambda, \varepsilon + \zeta + \varsigma) - f_1(\Lambda, \varepsilon + \varsigma) - f_1(\Lambda, \varepsilon + \zeta - \varsigma) + f_1(\Lambda, \varepsilon - \varsigma) \Big\|, \ldots, \\ &\qquad \Big\| f_n(\Lambda, \varepsilon + \zeta + \varsigma) - f_n(\Lambda, \varepsilon + \varsigma) - f_n(\Lambda, \varepsilon + \zeta - \varsigma) + f_n(\Lambda, \varepsilon - \varsigma) \Big\| \Big] \\ &\leq \operatorname{diag}\Big[\Big\| \theta_1^2 \Big(f_1(\Lambda, \varepsilon + \zeta + \varsigma) - f_1(\Lambda, \varepsilon + \varsigma) - f_1(\Lambda, \varepsilon + \zeta - \varsigma) + f_1(\Lambda, \varepsilon - \varsigma) \Big) \Big\|, \ldots, \\ &\qquad \Big\| \theta_n^2 \Big(f_n(\Lambda, \varepsilon + \zeta + \varsigma) - f_n(\Lambda, \varepsilon + \varsigma) - f_n(\Lambda, \varepsilon + \zeta - \varsigma) + f_n(\Lambda, \varepsilon - \varsigma) \Big) \Big\| \Big] \end{aligned}$$

and so

$$f_i(\Lambda, \varepsilon + \zeta + \varsigma) - f_i(\Lambda, \varepsilon + \varsigma) - f_i(\Lambda, \varepsilon + \zeta - \varsigma) + f_i(\Lambda, \varepsilon - \varsigma) = 0, \quad i = 1, \ldots, n \tag{3.2}$$

for each $\varepsilon, \zeta, \varsigma \in \mathscr{Q}$, $\Lambda \in \mathscr{Q}'$, since $|\theta_i| < 1$ $(i = 1, \ldots, n)$.

Letting $\varsigma = \varepsilon$ in (3.2),

$$f_i(\Lambda, 2\varepsilon + \zeta) - f_i(\Lambda, 2\varepsilon) - f_i(\Lambda, \zeta) = 0, \quad i = 1, \ldots, n$$

for each $\varepsilon, \zeta \in \mathscr{Q}$, $\Lambda \in \mathscr{Q}'$. Thus f_i $(i = 1, \ldots, n)$ are additive. □

Throughout the paper, let $\varphi_{j_i} : (\mathscr{Q})^{3i} \to [0,\infty)^i$, $1 \le i \le n$, $1 \le j_i \le n$, and $n \in \mathbb{N}$. Notice that $\mathfrak{M} := \operatorname{diag}[\varphi_{j_1}, \ldots, \varphi_{j_n}]$ is a matrix valued control function such that $\varphi_{j_1}(\varphi_{j_n})$ represents the element at the $1^{th}(n^{th})$ row and $1^{th}(n^{th})$ column of the matrix $\mathfrak{M}$ and φ_{j_i} demonstrates the j_ith given control function.

Theorem 3.2. *Let $(\varphi_{j_1}, \ldots, \varphi_{j_n}) : (\mathscr{Q} \times \mathscr{Q} \times \mathscr{Q})^n \to [0,\infty)^n$ $(1 \le j_1, \ldots, j_n \le n)$, be functions such that there exists an $(\mathcal{T}_1, \ldots, \mathcal{T}_n) < \underbrace{(1, \ldots, 1)}_{n}$ with*

$$\begin{aligned}
&\operatorname{diag}\Big[\ \varphi_{j_1}(\frac{\varepsilon}{2}, \frac{\zeta}{2}, \frac{\varsigma}{2}), \ldots, \varphi_{j_n}(\frac{\varepsilon}{2}, \frac{\zeta}{2}, \frac{\varsigma}{2})\ \Big] \\
&\le \operatorname{diag}\Big[\ \frac{\mathcal{T}_1}{2}\varphi_{j_1}(\varepsilon, \zeta, \varsigma), \ldots, \frac{\mathcal{T}_n}{2}\varphi_{j_n}(\varepsilon, \zeta, \varsigma)\ \Big],
\end{aligned} \tag{3.3}$$

for all $\varepsilon, \zeta, \varsigma \in \mathscr{Q}$. Suppose $f_i : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}$ $(i = 1, \ldots, n)$ are mappings satisfying $f_i(\Lambda, 0) = 0$ and

$$\begin{aligned}
\operatorname{diag}\Big[\ &\Big\|f_1(\Lambda, \varepsilon + \zeta + \varsigma) - f_1(\Lambda, \varepsilon + \varsigma) - f_1(\Lambda, \varepsilon + \zeta - \varsigma) + f_1(\Lambda, \varepsilon - \varsigma)\Big\| \\
&\quad + |\theta_1| \underbrace{\varphi_{j_1}}_{1 \le j_1 \le n}(\varepsilon, -\zeta, \varsigma), \ldots, \\
&\Big\|f_n(\Lambda, \varepsilon + \zeta + \varsigma) - f_n(\Lambda, \varepsilon + \varsigma) - f_n(\Lambda, \varepsilon + \zeta - \varsigma) + f_n(\Lambda, \varepsilon - \varsigma)\Big\| \\
&\quad + |\theta_n| \underbrace{\varphi_{j_n}}_{1 \le j_n \le n}(\varepsilon, -\zeta, \varsigma)\ \Big] \\
\le \operatorname{diag}\Big[\ &\Big\|\theta_1\Big(f_1(\Lambda, \varepsilon - \zeta + \varsigma) - f_1(\Lambda, \varepsilon + \varsigma) - f_1(\Lambda, \varepsilon - \zeta - \varsigma) + f_1(\Lambda, \varepsilon - \varsigma)\Big)\Big\| \\
&\quad + \underbrace{\varphi_{j_1}}_{1 \le j_1 \le n}(\varepsilon, \zeta, \varsigma), \ldots, \\
&\Big\|\theta_n\Big(f_n(\Lambda, \varepsilon - \zeta + \varsigma) - f_n(\Lambda, \varepsilon + \varsigma) - f_n(\Lambda, \varepsilon - \zeta - \varsigma) + f_n(\Lambda, \varepsilon - \varsigma)\Big)\Big\| \\
&\quad + \underbrace{\varphi_{j_n}}_{1 \le j_n \le n}(\varepsilon, \zeta, \varsigma)\ \Big],
\end{aligned} \tag{3.4}$$

for each $\varepsilon, \zeta, \varsigma \in \mathscr{Q}$ and $\Lambda \in \mathscr{Q}'$. Then there exist unique additive mappings $f_i' : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}$ such that

$$\begin{aligned}
&\operatorname{diag}\Big[\ \Big\|f_1(\Lambda, \varepsilon) - f_1'(\Lambda, \varepsilon)\Big\|, \ldots, \Big\|f_n(\Lambda, \varepsilon) - f_n'(\Lambda, \varepsilon)\Big\|\ \Big]_{n \times n} \\
&\le \operatorname{diag}\Big[\ \frac{\mathcal{T}_1}{2(1 - \mathcal{T}_1)} \underbrace{\varphi_{j_1}}_{1 \le j_1 \le n}\Big(\frac{\varepsilon}{2}, \varepsilon, \frac{\varepsilon}{2}\Big), \ldots, \frac{\mathcal{T}_n}{2(1 - \mathcal{T}_n)} \underbrace{\varphi_{j_n}}_{1 \le j_n \le n}\Big(\frac{\varepsilon}{2}, \varepsilon, \frac{\varepsilon}{2}\Big)\ \Big]_{n \times n},
\end{aligned} \tag{3.5}$$

for each $\varepsilon \in \mathscr{Q}$, and $\Lambda \in \mathscr{Q}'$.

Proof. Replacing ζ by $-\zeta$ in (3.4), we get

$$
\begin{aligned}
&\operatorname{diag}\Big[\ \Big\|f_1(\Lambda,\varepsilon-\zeta+\varsigma)-f_1(\Lambda,\varepsilon+\varsigma)-f_1(\Lambda,\varepsilon-\zeta-\varsigma)+f_1(\Lambda,\varepsilon-\varsigma)\Big\| \\
&\qquad +|\theta_1|\underbrace{\varphi_{j_1}}_{1\le j_1\le n}(\varepsilon,\zeta,\varsigma),\dots, \\
&\qquad \Big\|f_n(\Lambda,\varepsilon-\zeta+\varsigma)-f_n(\Lambda,\varepsilon+\varsigma)-f_n(\Lambda,\varepsilon-\zeta-\varsigma)+f_n(\Lambda,\varepsilon-\varsigma)\Big\| \\
&\qquad +|\theta_n|\underbrace{\varphi_{j_n}}_{1\le j_n\le n}(\varepsilon,\zeta,\varsigma)\ \Big] \\
&\le \operatorname{diag}\Big[\ \Big\|\theta_1(f_1(\Lambda,\varepsilon+\zeta+\varsigma)-f_1(\Lambda,\varepsilon+\varsigma)-f_1(\Lambda,\varepsilon+\zeta-\varsigma)+f_1(\Lambda,\varepsilon-\varsigma))\Big\| \\
&\qquad +\underbrace{\varphi_{j_1}}_{1\le j_1\le n}(\varepsilon,-\zeta,\varsigma),\dots, \\
&\qquad \Big\|\theta_n(f_n(\Lambda,\varepsilon+\zeta+\varsigma)-f_n(\Lambda,\varepsilon+\varsigma)-f_n(\Lambda,\varepsilon+\zeta-\varsigma)+f_n(\Lambda,\varepsilon-\varsigma))\Big\| \\
&\qquad +\underbrace{\varphi_{j_n}}_{1\le j_n\le n}(\varepsilon,-\zeta,\varsigma)\ \Big],
\end{aligned}
\tag{3.6}
$$

for each $\varepsilon,\zeta,\varsigma\in\mathscr{Q}$, and $\Lambda\in\mathscr{Q}'$. According to (3.4) and (3.6) we have

$$
\begin{aligned}
&\operatorname{diag}\Big[\ \Big\|f_1(\Lambda,\varepsilon+\zeta+\varsigma)-f_1(\Lambda,\varepsilon+\varsigma)-f_1(\Lambda,\varepsilon+\zeta-\varsigma)+f_1(\Lambda,\varepsilon-\varsigma)\Big\|,\dots, \\
&\qquad \Big\|f_n(\Lambda,\varepsilon+\zeta+\varsigma)-f_n(\Lambda,\varepsilon+\varsigma)-f_n(\Lambda,\varepsilon+\zeta-\varsigma)+f_n(\Lambda,\varepsilon-\varsigma)\Big\|\ \Big]_{n\times n} \\
&\le \operatorname{diag}\Big[\ \underbrace{\varphi_{j_1}}_{1\le j_1\le n}(\varepsilon,\zeta,\varsigma),\dots,\underbrace{\varphi_{j_n}}_{1\le j_n\le n}(\varepsilon,\zeta,\varsigma)\ \Big]_{n\times n},
\end{aligned}
\tag{3.7}
$$

for each $\varepsilon,\zeta,\varsigma\in\mathscr{Q}$, and $\Lambda\in\mathscr{Q}'$.

Letting $\varepsilon=\varsigma=\frac{\sigma}{2}$ and $\zeta=\sigma$ in (3.7), we get

$$
\begin{aligned}
&\operatorname{diag}\Big[\ \Big\|f_1(\Lambda,2\sigma)-2f_1(\Lambda,\sigma)\Big\|,\dots,\Big\|f_n(\Lambda,2\sigma)-2f_n(\Lambda,\sigma)\Big\|\ \Big]_{n\times n} \\
&\le \operatorname{diag}\Big[\ \underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\sigma}{2},\sigma,\frac{\sigma}{2}\right),\dots,\underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\sigma}{2},\sigma,\frac{\sigma}{2}\right)\ \Big]_{n\times n},
\end{aligned}
\tag{3.8}
$$

for each $\sigma\in\mathscr{Q}$, and $\Lambda\in\mathscr{Q}'$.

Let $\hbar=(\hbar_1,\dots,\hbar_n)$ and $\hbar'=(\hbar'_1,\dots,\hbar'_n)$.

Now, consider the set

$$
\nabla := \{\hbar:(\mathscr{Q}'\times\mathscr{Q})^n\to\mathscr{Q}^n:\hbar(\Lambda,0)=\overbrace{(0,\dots,0)}^{n}\}
$$

and define the generalized metric on ∇ by

$$d(\hbar,\hbar') = \inf\Big\{(\mu_1,\ldots,\mu_n)\in\mathbb{R}^n_+ :$$
$$\operatorname{diag}\Big[\ \|\hbar_1(\Lambda,\varepsilon)-\hbar'_1(\Lambda,\varepsilon)\|,\ldots\|\hbar_n(\Lambda,\varepsilon)-\hbar'_n(\Lambda,\varepsilon)\|\ \Big]$$
$$\leq \operatorname{diag}\Big[\ \mu_1\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}\left(\frac{\varepsilon}{2},\varepsilon,\frac{\varepsilon}{2}\right),\ldots,\mu_n\underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}\left(\frac{\varepsilon}{2},\varepsilon,\frac{\varepsilon}{2}\right)\ \Big],\forall\varepsilon\in\mathscr{Q},\Lambda\in\mathscr{Q}'\Big\},$$

where $\inf\emptyset = \underbrace{(+\infty,\ldots,+\infty)}_{n}$.

Now (∇,d) is complete (also, see [8]).

Let $\mathscr{L} := (\mathscr{L}_1,\ldots,\mathscr{L}_n)$. Now, we consider the linear mapping $\mathscr{L}:\nabla\to\nabla$ s.t.

$$\mathscr{L}_i\hbar_i(\Lambda,\varepsilon) := 2\hbar_i\left(\Lambda,\frac{\varepsilon}{2}\right),\quad i=1,\ldots,n$$

for each $\varepsilon\in\mathscr{Q}$, and $\Lambda\in\mathscr{Q}'$.

Let $\hbar,\hbar'\in\nabla$ be given s.t. $d(\hbar,\hbar')=(\varepsilon_1,\ldots,\varepsilon_n)$. Then

$$\operatorname{diag}\Big[\ \|\hbar_1(\Lambda,\varepsilon)-\hbar'_1(\Lambda,\varepsilon)\|,\ldots,\|\hbar_n(\Lambda,\varepsilon)-\hbar'_n(\Lambda,\varepsilon)\|\ \Big]$$
$$\leq \operatorname{diag}\Big[\ \varepsilon_1\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}\left(\tfrac{\varepsilon}{2},\varepsilon,\tfrac{\varepsilon}{2}\right),\ldots,\varepsilon_n\underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}\left(\tfrac{\varepsilon}{2},\varepsilon,\tfrac{\varepsilon}{2}\right)\ \Big],$$

for each $\varepsilon\in\mathscr{Q}$, and $\Lambda\in\mathscr{Q}'$. Hence

$$\operatorname{diag}\Big[\ \|\mathscr{L}_1\hbar_1(\Lambda,\varepsilon)-\mathscr{L}_1\hbar'_1(\Lambda,\varepsilon)\|,\ldots,\|\mathscr{L}_n\hbar_n(\Lambda,\varepsilon)-\mathscr{L}_n\hbar'_n(\Lambda,\varepsilon)\|\ \Big]$$
$$=\operatorname{diag}\Big[\ \left\|2\hbar_1\left(\Lambda,\frac{\varepsilon}{2}\right)-2\hbar'_1\left(\Lambda,\frac{\varepsilon}{2}\right)\right\|,\ldots,\left\|2\hbar_n\left(\Lambda,\frac{\varepsilon}{2}\right)-2\hbar'_n\left(\Lambda,\frac{\varepsilon}{2}\right)\right\|\ \Big]$$
$$\leq\operatorname{diag}\Big[\ 2\varepsilon_1\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}\left(\frac{\varepsilon}{4},\frac{\varepsilon}{2},\frac{\varepsilon}{4}\right),\ldots,2\varepsilon_n\underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}\left(\frac{\varepsilon}{4},\frac{\varepsilon}{2},\frac{\varepsilon}{4}\right)\ \Big]$$
$$\leq\operatorname{diag}\Big[\ \mathcal{T}_1\varepsilon_1\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}\left(\frac{\varepsilon}{2},\varepsilon,\frac{\varepsilon}{2}\right),\ldots,\mathcal{T}_n\varepsilon_n\underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}\left(\frac{\varepsilon}{2},\varepsilon,\frac{\varepsilon}{2}\right)\ \Big],$$

for each $\varepsilon\in\mathscr{Q}$, and $\Lambda\in\mathscr{Q}'$. Thus $d(\hbar,\hbar')=\underbrace{(\varepsilon_1,\ldots,\varepsilon_n)}_{n}$ implies that

$$d(\mathscr{L}\hbar(\Lambda,\varepsilon),\mathscr{L}\hbar'(\Lambda,\varepsilon))\leq(\mathcal{T}_1\varepsilon_1,\ldots,\mathcal{T}_n\varepsilon_n).$$

Hence

$$d(\mathscr{L}\hbar(\varepsilon),\mathscr{L}\hbar'(\varepsilon))\leq(\mathcal{T}_1,\ldots,\mathcal{T}_n)d(\hbar,\hbar'),$$

for each $\hbar,\hbar'\in\nabla$. According to (3.8), we get

$$\begin{aligned}
&\operatorname{diag}\Big[\ \Big\|f_1(\Lambda,\varepsilon)-2f_1(\Lambda,\frac{\varepsilon}{2})\Big\|,\ldots,\Big\|f_n(\Lambda,\varepsilon)-2f_n(\Lambda,\frac{\varepsilon}{2})\Big\|\ \Big]_{n\times n}\\
&\leq \operatorname{diag}\Big[\ \underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}\Big(\frac{\varepsilon}{4},\frac{\varepsilon}{2},\frac{\varepsilon}{4}\Big),\ldots,\underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}\Big(\frac{\varepsilon}{4},\frac{\varepsilon}{2},\frac{\varepsilon}{4}\Big)\ \Big]_{n\times n}\\
&\leq \operatorname{diag}\Big[\ \frac{\mathcal{T}_1}{2}\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}\Big(\frac{\varepsilon}{2},\varepsilon,\frac{\varepsilon}{2}\Big),\ldots,\frac{\mathcal{T}_n}{2}\underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}\Big(\frac{\varepsilon}{2},\varepsilon,\frac{\varepsilon}{2}\Big)\ \Big]_{n\times n},
\end{aligned}$$

for each $\varepsilon\in\mathscr{Q},\Lambda\in\mathscr{Q}'$, so $d(f,\mathscr{L}f)\leq(\frac{\mathcal{T}_1}{2},\ldots,\frac{\mathcal{T}_n}{2})$.

According to Theorem 1.2 there exist mappings $f_i':\mathscr{Q}\to\mathscr{Q}(i=1,\ldots,n)$ satisfying the following:

(1) f' is a fixed point of $\mathscr{L}$, i.e.

$$f'(\Lambda,\varepsilon)=2f'\Big(\Lambda,\frac{\varepsilon}{2}\Big),\tag{3.9}$$

for each $\varepsilon\in\mathscr{Q}$, and $\Lambda\in\mathscr{Q}'$. The mapping f' is a unique fixed point of $\mathscr{L}$ in the set

$$\Bbbk=\{\hbar\in\nabla: d(f,\hbar)<\infty\}.$$

This implies that f' is a unique mapping satisfying (3.9) s.t. there exist $\mu_1,\ldots,\mu_n\in(0,\infty)$ satisfying

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|f_1(\Lambda,\varepsilon)-f_1'(\Lambda,\varepsilon)\|,\ldots,\|f_n(\Lambda,\varepsilon)-f_n'(\Lambda,\varepsilon)\|\ \Big]\\
&\leq \operatorname{diag}\Big[\ \mu_1\underbrace{\varphi_{j_1}}_{0\leq j_1\leq n}\Big(\frac{\varepsilon}{2},\varepsilon,\frac{\varepsilon}{2}\Big),\ldots,\mu_n\underbrace{\varphi_{j_n}}_{0\leq j_n\leq n}\Big(\frac{\varepsilon}{2},\varepsilon,\frac{\varepsilon}{2}\Big)\Big],
\end{aligned}$$

for each $\varepsilon\in\mathscr{Q}$, and $\Lambda\in\mathscr{Q}'$.

(2) Since $\lim_{n\to\infty}d(\mathscr{L}^n f,f')=0$,

$$\lim_{n\to\infty}2^n f_i\Big(\Lambda,\frac{\varepsilon}{2^n}\Big)=f_i'(\Lambda,\varepsilon),\ \ \forall i=1,\ldots,n\tag{3.10}$$

for each $\varepsilon\in\mathscr{Q}$, and $\Lambda\in\mathscr{Q}'$.

(3) $d(f,f')\leq(\frac{1}{1-\mathcal{T}_1},\ldots,\frac{1}{1-\mathcal{T}_n})d(f,\mathscr{L}f)$, which implies

$$\begin{aligned}
&\operatorname{diag}\Big[\ \Big\|f_1(\Lambda,\varepsilon)-f_1'(\Lambda,\varepsilon)\Big\|,\ldots,\Big\|f_n(\Lambda,\varepsilon)-f_n'(\Lambda,\varepsilon)\Big\|\ \Big]_{n\times n}\\
&\leq \operatorname{diag}\Big[\ \frac{\mathcal{T}_1}{2(1-\mathcal{T}_1)}\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}\Big(\frac{\varepsilon}{2},\varepsilon,\frac{\varepsilon}{2}\Big),\ldots,\frac{\mathcal{T}_n}{2(1-\mathcal{T}_n)}\underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}\Big(\frac{\varepsilon}{2},\varepsilon,\frac{\varepsilon}{2}\Big)\ \Big]_{n\times n},
\end{aligned}$$

for each $\varepsilon\in\mathscr{Q}$, and $\Lambda\in\mathscr{Q}'$. According to (3.3) and (3.4) we have

$$\begin{aligned}
\operatorname{diag}\Big[\ &\Big\|f_1'(\Lambda,\varepsilon+\zeta+\varsigma)-f_1'(\Lambda,\varepsilon+\varsigma)-f_1'(\Lambda,\varepsilon+\zeta-\varsigma)+f_1'(\Lambda,\varepsilon-\varsigma)\Big\|\ ,\ldots,\\
&\Big\|f_n'(\Lambda,\varepsilon+\zeta+\varsigma)-f_n'(\Lambda,\varepsilon+\varsigma)-f_n'(\Lambda,\varepsilon+\zeta-\varsigma)+f_n'(\Lambda,\varepsilon-\varsigma)\Big\|\ \Big]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{diag}\Big[\ \lim_{n\to\infty} 2^n \Big\| f_1\left(\Lambda, \frac{\varepsilon+\zeta+\varsigma}{2^n}\right) - f_1\left(\Lambda, \frac{\varepsilon+\varsigma}{2^n}\right) \\
&\qquad - f_1\left(\Lambda, \frac{\varepsilon+\zeta-\varsigma}{2^n}\right) + f_1\left(\Lambda, \frac{\varepsilon-\varsigma}{2^n}\right)\Big\|\ ,\ldots, \\
&\qquad \lim_{n\to\infty} 2^n \Big\| f_n\left(\Lambda, \frac{\varepsilon+\zeta+\varsigma}{2^n}\right) - f_n\left(\Lambda, \frac{\varepsilon+\varsigma}{2^n}\right) \\
&\qquad - f_n\left(\Lambda, \frac{\varepsilon+\zeta-\varsigma}{2^n}\right) + f_n\left(\Lambda, \frac{\varepsilon-\varsigma}{2^n}\right)\Big\|\ \Big] \\
&\le \mathrm{diag}\Big[\ \lim_{n\to\infty} 2^n |\theta_1| \Big\| f_1\left(\Lambda, \frac{\varepsilon-\zeta+\varsigma}{2^n}\right) - f_1\left(\Lambda, \frac{\varepsilon+\varsigma}{2^n}\right) \\
&\qquad - f_1\left(\Lambda, \frac{\varepsilon-\zeta-\varsigma}{2^n}\right) + f_1\left(\Lambda, \frac{\varepsilon-\varsigma}{2^n}\right)\Big\| \\
&\qquad + \lim_{n\to\infty} 2^n \left(\underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\varepsilon}{2^n}, \frac{\zeta}{2^n}, \frac{\varsigma}{2^n}\right) - \theta_1 \underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\varepsilon}{2^n}, \frac{-\zeta}{2^n}, \frac{\varsigma}{2^n}\right)\right)\ ,\ldots, \\
&\qquad \lim_{n\to\infty} 2^n |\theta_n| \Big\| f_n\left(\Lambda, \frac{\varepsilon-\zeta+\varsigma}{2^n}\right) - f_n\left(\Lambda, \frac{\varepsilon+\varsigma}{2^n}\right) \\
&\qquad - f_n\left(\Lambda, \frac{\varepsilon-\zeta-\varsigma}{2^n}\right) + f_n\left(\Lambda, \frac{\varepsilon-\varsigma}{2^n}\right)\Big\| \\
&\qquad + \lim_{n\to\infty} 2^n \left(\underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\varepsilon}{2^n}, \frac{\zeta}{2^n}, \frac{\varsigma}{2^n}\right) - \theta_n \underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\varepsilon}{2^n}, \frac{-\zeta}{2^n}, \frac{\varsigma}{2^n}\right)\right)\ \Big] \\
&\le \mathrm{diag}\Big[\ \Big\|\theta_n(f_1'(\Lambda, \varepsilon-\zeta+\varsigma) - f_1'(\Lambda, \varepsilon+\varsigma) - f_1'(\Lambda, \varepsilon-\zeta-\varsigma) + f_1'(\Lambda, \varepsilon-\varsigma))\Big\|, \ldots, \\
&\qquad \Big\|\theta_n(f_n'(\Lambda, \varepsilon-\zeta+\varsigma) - f_n'(\Lambda, \varepsilon+\varsigma) - f_n'(\Lambda, \varepsilon-\zeta-\varsigma) + f_n'(\Lambda, \varepsilon-\varsigma))\Big\|\ \Big]
\end{aligned}
$$

for each $\varepsilon, \zeta, \varsigma \in \mathscr{Q}$, and $\Lambda \in \mathscr{Q}'$. According to Lemma 3.1, the mapping f_i' $(i = 1, \ldots, n)$ is additive. □

3.2. *Super-multi-stability of antiderivations in Banach algebras*

Definition 3.3. Assume $\mathscr{Q}$ is a complex Banach algebra. A $\mathbb{C}$-linear mapping $\mathscr{G} : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}$ is called an antiderivation if it satisfies

$$\mathscr{G}(\Lambda, \varepsilon)\mathscr{G}(\Lambda, \zeta) = \mathscr{G}(\Lambda, \mathscr{G}(\Lambda, \varepsilon)\zeta) + \mathscr{G}(\Lambda, \varepsilon\mathscr{G}(\Lambda, \zeta))$$

for each $\varepsilon, \zeta \in \mathscr{Q}$ and $\Lambda \in \mathscr{Q}'$.

Example 3.4. Suppose Q_m is the collection of all polynomials of degree m with complex coefficients and

$$Q = \{q_m \in Q_m | q(\Lambda, 0) = 0, m \in \mathbb{N}\}.$$

Define $\mathscr{G} : \mathscr{Q}' \times Q \to Q$ by

$$\mathscr{G}(\Lambda, \sum_{k=1}^{n} b_k \chi^k) = \sum_{k=1}^{n} \frac{b_k}{k} \chi^k$$

and $\mathscr{G}(\Lambda, 0) = 0$. Then $\mathscr{G}$ is an antiderivation.

Example 3.5. Consider the collection of all continuous functions on $\mathbb{R}$, represented by $C(\mathbb{R})$.
Define $\mathscr{G} : \mathscr{Q}' \times C(\mathbb{R}) \to C(\mathbb{R})$ by

$$\mathscr{G}(\Lambda, g(\varepsilon)) = \int_0^{\varepsilon} g(\tau)d\tau$$

for each $\tau \in \mathbb{R}$. Then $\mathscr{G}$ is an antiderivation.

Lemma 3.6. *[9] Suppose $\mathscr{Q}$ is complex Banach algebra and suppose $f : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}$ is an additive mapping s.t. $f(\Lambda, \Im\varepsilon) = \Im f(\Lambda, \varepsilon)$ for each $\Im \in \mathbb{T}^1 := \{\eta \in \mathbb{C} : |\eta| = 1\}$ and each $\varepsilon \in \mathscr{Q}$. Then f is $\mathbb{C}$-linear.*

Theorem 3.7. *Suppose $\varphi_{j_1}, \ldots, \varphi_{j_n} : \mathscr{Q}^3 \to [0, \infty)$, $(1 \leq j_1, \ldots, j_n \leq n)$, are functions.*

(i) If there exist $(\mathcal{T}_1, \ldots, \mathcal{T}_n) < (1, \ldots, 1)$ satisfying

$$\begin{aligned}
&\operatorname{diag}\Big[\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}\left(\frac{\varepsilon}{\Im}, \frac{\zeta}{\Im}, \frac{\varsigma}{\Im}\right), \ldots, \underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}\left(\frac{\varepsilon}{\Im}, \frac{\zeta}{\Im}, \frac{\varsigma}{\Im}\right) \Big] \\
&\leq \operatorname{diag}\Big[\frac{\mathcal{T}_1}{2} \underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}(2\varepsilon, 2\zeta, 2\varsigma), \ldots, \frac{\mathcal{T}_n}{2} \underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}(2\varepsilon, 2\zeta, 2\varsigma) \Big],
\end{aligned} \tag{3.11}$$

and if $f_i : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}$, $(i = 1, \ldots, n)$, are mappings satisfying $f_i(\Lambda, 0) = 0$ and

$$\begin{aligned}
&\operatorname{diag}\Big[\Big\|\Im f_1(\Lambda, \varepsilon+\zeta+\varsigma) - f_1(\Lambda, \Im(\varepsilon+\varsigma)) - f_1(\Lambda, \Im(\varepsilon+\zeta-\varsigma)) + \Im f_1(\Lambda, \varepsilon-\varsigma)\Big\| \\
&\qquad + |\theta_1| \underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}(\varepsilon, -\zeta, \varsigma), \ldots, \\
&\quad \Big\|\Im f_n(\Lambda, \varepsilon+\zeta+\varsigma) - f_n(\Lambda, \Im(\varepsilon+\varsigma)) - f_n(\Lambda, \Im(\varepsilon+\zeta-\varsigma)) + \Im f_n(\Lambda, \varepsilon-\varsigma)\Big\| \\
&\qquad + |\theta_n| \underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}(\varepsilon, -\zeta, \varsigma) \Big] \\
&\leq \operatorname{diag}\Big[\Big\|\theta_1(\Im f_1(\Lambda, \varepsilon-\zeta+\varsigma) - f_1(\Lambda, \Im(\varepsilon+\varsigma)) - f_1(\Lambda, \Im(\varepsilon-\zeta-\varsigma)) + \Im f_1(\Lambda, \varepsilon-\varsigma))\Big\| \\
&\qquad + \underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}(\varepsilon, \zeta, \varsigma), \ldots, \\
&\quad \Big\|\theta_n(\Im f_n(\Lambda, \varepsilon-\zeta+\varsigma) - f_n(\Lambda, \Im(\varepsilon+\varsigma)) - f_n(\Lambda, \Im(\varepsilon-\zeta-\varsigma)) + \Im f_n(\Lambda, \varepsilon-\varsigma))\Big\| \\
&\qquad + \underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}(\varepsilon, \zeta, \varsigma) \Big],
\end{aligned} \tag{3.12}$$

for each $\Im \in \mathbb{T}^1$ and all $\varepsilon, \zeta, \varsigma \in \mathscr{Q}, \Lambda \in \mathscr{Q}'$, then there exist unique $\mathbb{C}$-linear mappings $\mathscr{G}_i : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}$, $(i = 1, \ldots, n)$, s.t.

$$\operatorname{diag}\Big[\ \Big\|f_1(\Lambda,\varepsilon)-\mathscr{G}_1(\Lambda,\varepsilon)\Big\|,\ldots,\Big\|f_n(\Lambda,\varepsilon)-\mathscr{G}_n(\Lambda,\varepsilon)\Big\|\ \Big]$$
$$\leq \operatorname{diag}\Big[\ \frac{\mathcal{T}_1}{2(1-\mathcal{T}_1)}\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}\Big(\frac{\varepsilon}{2},\varepsilon,-\frac{\varepsilon}{2}\Big),\ldots,\frac{\mathcal{T}_n}{2(1-\mathcal{T}_n)}\underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}\Big(\frac{\varepsilon}{2},\varepsilon,-\frac{\varepsilon}{2}\Big)\ \Big], \tag{3.13}$$

for each $\varepsilon\in\mathscr{Q},\Lambda\in\mathscr{Q}'$.

(ii) In addition, if $(\mathcal{T}_1,\ldots,\mathcal{T}_n)<(\frac{1}{2},\ldots,\frac{1}{2})$ *and* $f_i,(i=1,\ldots,n)$, *are continuous and satisfy* $f_i(\Lambda,2\varepsilon)=2f_i(\Lambda,\varepsilon)$ *and*

$$\operatorname{diag}\Big[\ \Big\|f_1(\Lambda,\varepsilon)f_1(\Lambda,\zeta)-f_1(\Lambda,f_1(\Lambda,\varepsilon)\zeta)-f_1(\Lambda,\varepsilon f_1(\Lambda,\zeta))\Big\|,\ldots,$$
$$\Big\|f_n(\Lambda,\varepsilon)f_n(\Lambda,\zeta)-f_n(\Lambda,f_n(\Lambda,\varepsilon)\zeta)-f_n(\Lambda,\varepsilon f_n(\Lambda,\zeta))\Big\|\ \Big]$$
$$\leq \operatorname{diag}\Big[\ \underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}(\varepsilon,\zeta,\varepsilon),\ldots,\underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}(\varepsilon,\zeta,\varepsilon)\ \Big], \tag{3.14}$$

for each $\varepsilon,\zeta\in\mathscr{Q}$, *then* $f_i:\mathscr{Q}'\times\mathscr{Q}\to\mathscr{Q}$ *are antiderivations.*

Proof. By a similar method used in Theorem 3.2 the proof of (i) is straightforward. Now, we prove (ii).

(ii) Since $\mathscr{G}_i=f_i,(i=1,\ldots,n)$, are continuous and $\mathbb{C}$-linear, we conclude from (3.11) and (3.14) that

$$\operatorname{diag}\Big[\ \|\mathscr{G}_1(\Lambda,\varepsilon)\mathscr{G}_1(\Lambda,\zeta)-\mathscr{G}_1(\Lambda,\mathscr{G}_1(\Lambda,\varepsilon)\zeta)-\mathscr{G}_1(\Lambda,\varepsilon\mathscr{G}_1(\Lambda,\zeta))\|$$
$$,\ldots,\|\mathscr{G}_n(\Lambda,\varepsilon)\mathscr{G}_n(\Lambda,\zeta)-\mathscr{G}_n(\Lambda,\mathscr{G}_n(\Lambda,\varepsilon)\zeta)-\mathscr{G}_n(\Lambda,\varepsilon\mathscr{G}_n(\Lambda,\zeta))\|\ \Big]$$
$$=\operatorname{diag}\Big[\ \lim_{m\to\infty}4^m\Big\|\mathfrak{I}^m\Big(f_1\Big(\Lambda,\frac{\varepsilon}{2^m\mathfrak{I}^m}\Big)f_1\Big(\Lambda,\frac{\zeta}{2^m\mathfrak{I}^m}\Big)-\mathscr{G}_1\Big(\Lambda,f_1\Big(\Lambda,\frac{\varepsilon}{2^m\mathfrak{I}^m}\Big)\frac{\zeta}{2^m\mathfrak{I}^m}\Big)$$
$$-\mathscr{G}_1\Big(\Lambda,\frac{\varepsilon}{2^m\mathfrak{I}^m}f_1\Big(\Lambda,\frac{\zeta}{2^m\mathfrak{I}^m}\Big)\Big)\Big)\Big\|,\ldots,$$
$$\lim_{m\to\infty}4^m\Big\|\mathfrak{I}^m\Big(f_n\Big(\Lambda,\frac{\varepsilon}{2^m\mathfrak{I}^m}\Big)f_n\Big(\Lambda,\frac{\zeta}{2^m\mathfrak{I}^m}\Big)-\mathscr{G}_n\Big(\Lambda,f_n\Big(\Lambda,\frac{\varepsilon}{2^m\mathfrak{I}^m}\Big)\frac{\zeta}{2^m\mathfrak{I}^m}\Big)$$
$$-\mathscr{G}_n\Big(\Lambda,\frac{\varepsilon}{2^m\lambda^m}f_n\Big(\Lambda,\frac{\zeta}{2^m\mathfrak{I}^m}\Big)\Big)\Big)\Big\|\ \Big]$$
$$=\operatorname{diag}\Big[\ \lim_{m\to\infty}4^m\Big\|\mathfrak{I}^m\Big(f_1\Big(\Lambda,\frac{\varepsilon}{2^m\mathfrak{I}^m}\Big)f_1\Big(\Lambda,\frac{\zeta}{2^m\mathfrak{I}^m}\Big)-f_1\Big(\Lambda,f_1\Big(\Lambda,\frac{\varepsilon}{2^m\mathfrak{I}^m}\Big)\frac{\zeta}{2^m\mathfrak{I}^m}\Big)$$
$$-f_1\Big(\Lambda,\frac{\varepsilon}{2^m\mathfrak{I}^m}f_1\Big(\Lambda,\frac{\zeta}{2^m\mathfrak{I}^m}\Big)\Big)\Big)\Big\|,\ldots,$$
$$\lim_{m\to\infty}4^m\Big\|\mathfrak{I}^m\Big(f_n\Big(\Lambda,\frac{\varepsilon}{2^m\mathfrak{I}^m}\Big)f_n\Big(\Lambda,\frac{\zeta}{2^m\mathfrak{I}^m}\Big)-f_n\Big(\Lambda,f_1\Big(\Lambda,\frac{\varepsilon}{2^m\mathfrak{I}^m}\Big)\frac{\zeta}{2^m\mathfrak{I}^m}\Big)$$
$$-f_n\Big(\Lambda,\frac{\varepsilon}{2^m\mathfrak{I}^m}f_n\Big(\Lambda,\frac{\zeta}{2^m\mathfrak{I}^m}\Big)\Big)\Big)\Big\|\ \Big]$$
$$\leq\operatorname{diag}\Big[\ \lim_{m\to\infty}2^{2m}\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}\Big(\frac{\varepsilon}{2^m\mathfrak{I}^m},\frac{\zeta}{2^m\mathfrak{I}^m},\frac{\varepsilon}{2^m\mathfrak{I}^m}\Big),\ldots,$$

$$\lim_{m\to\infty} 2^{2m} \underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\varepsilon}{2^m\Im^m}, \frac{\zeta}{2^m\Im^m}, \frac{\varepsilon}{2^m\Im^m}\right) \Big]$$

$$\le \operatorname{diag}\Big[\lim_{m\to\infty}(2\mathcal{T}_1)^m \underbrace{\varphi_{j_1}}_{1\le j_1\le n}(\varepsilon,\zeta,\varepsilon), \ldots, \lim_{m\to\infty}(2\mathcal{T}_n)^m \underbrace{\varphi_{j_n}}_{1\le j_n\le n}(\varepsilon,\zeta,\varepsilon)\Big],$$

for each $\Im \in \mathbb{T}^1$ and each $\varepsilon, \zeta \in \mathscr{Q}, \Lambda \in \mathscr{Q}'$. Since $\underbrace{(2\mathcal{T}_1, \cdots, 2\mathcal{T}_n)}_{n} < \underbrace{(1, \cdots, 1)}_{n}$, the $\mathbb{C}$-linear mappings $\mathscr{G}_i, (i = 1, \cdots, n)$, are antiderivations. Thus the mappings $f_i : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}, (i = 1, \cdots, n)$, are antiderivations. □

3.3. Super-multi-stability of continuous antiderivations in Banach algebras

In this subsection, we investigate the super-multi-stability of continuous antiderivations in Banach algebras.

Theorem 3.8. *Consider* $\underbrace{\varphi_{j_1}}_{1\le j_1\le n}, \cdots, \underbrace{\varphi_{j_n}}_{1\le j_n\le n} : \mathscr{Q}^3 \to [0,\infty)$.

(i) If there exist $(\mathcal{T}_1, \cdots, \mathcal{T}_n) \prec \overbrace{(1, \cdots, 1)}^{n}$ *satisfying*

$$\operatorname{diag}\Big[\underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\varepsilon}{\Im}, \frac{\zeta}{\Im}, \frac{\varsigma}{\Im}\right), \cdots, \underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\varepsilon}{\Im}, \frac{\zeta}{\Im}, \frac{\varsigma}{\Im}\right) \Big]$$
$$\le \operatorname{diag}\Big[\frac{\mathcal{T}_1}{2} \underbrace{\varphi_{j_1}}_{1\le j_1\le n}(2\varepsilon, 2\zeta, 2\varsigma), \cdots, \frac{\mathcal{T}_n}{2} \underbrace{\varphi_{j_n}}_{1\le j_n\le n}(2\varepsilon, 2\zeta, 2\varsigma) \Big] \tag{3.15}$$

and if $f_i : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}, (i = 1, \cdots, n)$, *are mappings satisfying* $f_i(\Lambda, 0) = 0$ *and*

$$\operatorname{diag}\Big[\Big\|\Im f_1(\Lambda, \varepsilon+\zeta+\varsigma) - f_1(\Lambda, \Im(\varepsilon+\varsigma)) - f_1(\Lambda, \Im(\varepsilon+\zeta-\varsigma)) + \Im f_1(\Lambda, \varepsilon-\varsigma)\Big\|$$
$$+ |\theta_1| \underbrace{\varphi_{j_1}}_{1\le j_1\le n}(\varepsilon, -\zeta, \varsigma) \ , \cdots,$$
$$\Big\|\Im f_n(\Lambda, \varepsilon+\zeta+\varsigma) - f_n(\Lambda, \Im(\varepsilon+\varsigma)) - f_n(\Lambda, \Im(\varepsilon+\zeta-\varsigma)) + \Im f_n(\Lambda, \varepsilon-\varsigma)\Big\|$$
$$+ |\theta_n| \underbrace{\varphi_{j_n}}_{1\le j_n\le n}(\varepsilon, -\zeta, \varsigma) \Big]$$
$$\le \operatorname{diag}\Big[\Big\|\theta_1(\Im f_1(\Lambda, \varepsilon-\zeta+\varsigma) - f_1(\Lambda, \Im(\varepsilon+\varsigma)) - f_1(\Lambda, \Im(\varepsilon-\zeta-\varsigma)) + \Im f_1(\Lambda, \varepsilon-\varsigma))\Big\|$$
$$+ \underbrace{\varphi_{j_1}}_{1\le j_1\le n}(\varepsilon, \zeta, \varsigma) \ , \cdots,$$
$$\Big\|\theta_n(\Im f_n(\Lambda, \varepsilon-\zeta+\varsigma) - f_n(\Lambda, \Im(\varepsilon+\varsigma)) - f_n(\Lambda, \Im(\varepsilon-\zeta-\varsigma)) + \Im f_n(\Lambda, \varepsilon-\varsigma))\Big\|$$
$$+ \underbrace{\varphi_{j_n}}_{1\le j_n\le n}(\varepsilon, \zeta, \varsigma) \Big], \tag{3.16}$$

for each $\mathfrak{J} \in \mathbb{C} - \overline{\mathbb{T}}^1$ *and each* $\varepsilon, \zeta, \varsigma \in \mathscr{Q}$, $\Lambda \in \mathscr{Q}'$, *then there are unique* $\mathbb{C}$*-linear mappings* $\mathscr{G}_i : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}, (i = 1, \cdots, n)$, *s.t.*

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|f_1(\Lambda, \varepsilon) - \mathscr{G}_1(\Lambda, \varepsilon)\|, \cdots, \|f_n(\Lambda, \varepsilon) - \mathscr{G}_n(\Lambda, \varepsilon)\|\Big] \\
&\leq \operatorname{diag}\Big[\ \frac{\mathcal{T}_1}{2(1-\mathcal{T}_1)} \underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}\Big(\frac{\varepsilon}{2}, \varepsilon, -\frac{\varepsilon}{2}\Big), \cdots, \frac{\mathcal{T}_n}{2(1-\mathcal{T}_n)} \underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}\Big(\frac{\varepsilon}{2}, \varepsilon, -\frac{\varepsilon}{2}\Big)\Big], \qquad (3.17)
\end{aligned}$$

for each $\varepsilon \in \mathscr{Q}$, $\Lambda \in \mathscr{Q}'$.

(ii) Furthermore, if $(\mathcal{T}_1, \cdots, \mathcal{T}_n) \prec (\overbrace{\frac{1}{2}, \cdots, \frac{1}{2}}^{n})$, $\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}, \cdots, \underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}$ *are continuous functions and also* $f_i, (i = 1, \cdots, n)$, *are continuous and satisfy* $f_i(\Lambda, 2\varepsilon) = 2f_i(\Lambda, \varepsilon)$ *and*

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|f_1(\Lambda, \varepsilon)f_1(\Lambda, \zeta) - f_1(\Lambda, f_1(\Lambda, \varepsilon)\zeta) - f_1(\Lambda, \varepsilon f_1(\Lambda, \zeta))\| \\
&\qquad , \cdots, \|f_n(\Lambda, \varepsilon)f_n(\Lambda, \zeta) - f_n(\Lambda, f_n(\Lambda, \varepsilon)\zeta) - f_n(\Lambda, \varepsilon f_n(\Lambda, \zeta))\|\ \Big] \\
&\leq \operatorname{diag}\Big[\ \underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}(\varepsilon, \zeta, \varepsilon), \cdots, \underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}(\varepsilon, \zeta, \varepsilon)\ \Big],
\end{aligned}$$

for each $\varepsilon, \zeta \in \mathscr{Q}$, $\Lambda \in \mathscr{Q}'$, *then* $f_i : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}$ *are continuous antiderivations.*

Proof. Using the same reasoning as in the proof of Theorem 3.7, we obtain the desired result. □

3.4. Application

Here, let $n = 7$.

Corollary 3.9. *Suppose* $f_i : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}$ $(i = 1, \ldots, n)$ *are mappings satisfying* $f_i(\Lambda, 0) = 0$ *and*

$$\begin{aligned}
&diag\Big[\ \Big\|f_1(\Lambda, \varepsilon+\zeta+\varsigma) - f_1(\Lambda, \varepsilon+\varsigma) - f_1(\Lambda, \varepsilon+\zeta-\varsigma) + f_1(\Lambda, \varepsilon-\varsigma)\Big\|, \ldots, \\
&\qquad \Big\|f_n(\Lambda, \varepsilon+\zeta+\varsigma) - f_n(\Lambda, \varepsilon+\varsigma) - f_n(\Lambda, \varepsilon+\zeta-\varsigma) + f_n(\Lambda, \varepsilon-\varsigma)\Big\|\ \Big] \\
&\leq diag\Big[\ \Big\|\theta_1(f_1(\Lambda, \varepsilon-\zeta+\varsigma) - f_1(\Lambda, \varepsilon+\varsigma) - f_1(\Lambda, \varepsilon-\zeta-\varsigma) + f_1(\Lambda, \varepsilon-\varsigma))\Big\| \\
&\qquad + (1-|\theta_1|)\underbrace{\varphi^{\circledS}_{j_1}}_{1\leq j_1\leq n}(\|\varepsilon^2\| + \|\zeta^2\| + \|\varsigma^2\|), \ldots, \\
&\qquad \Big\|\theta_n(f_n(\Lambda, \varepsilon-\zeta+\varsigma) - f_n(\Lambda, \varepsilon+\varsigma) - f_n(\Lambda, \varepsilon-\zeta-\varsigma) + f_n(\Lambda, \varepsilon-\varsigma))\Big\| \\
&\qquad + (1-|\theta_n|)\underbrace{\varphi^{\circledS}_{j_n}}_{1\leq j_n\leq n}(\|\varepsilon^2\| + \|\zeta^2\| + \|\varsigma^2\|)\ \Big]
\end{aligned}$$

for each $\varepsilon, \zeta, \varsigma \in \mathscr{Q}$, and $\Lambda \in \mathscr{Q}'$. Then there are unique additive mappings $f_i' : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}$ s.t.

$$diag\Big[\ \|f_1(\Lambda,\varepsilon) - f_1'(\Lambda,\varepsilon)\|, \ldots, \|f_n(\Lambda,\varepsilon) - f_n'(\Lambda,\varepsilon)\|\ \Big]_{n\times n} \\ \leq diag\Big[\ \underbrace{\varphi_{j_1}^{\circledS}}_{1\leq j_1\leq n}(\|\varepsilon\|^2), \ldots, \underbrace{\varphi_{j_n}^{\circledS}}_{1\leq j_n\leq n}(\|\varepsilon\|^2)\ \Big],$$

for each $\varepsilon \in \mathscr{Q}$ and $\Lambda \in \mathscr{Q}'$.

Proof. The proof follows from Theorem 3.2 by letting

$$\mathrm{diag}\Big[\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}(\varepsilon,\zeta,\varsigma), \ldots, \underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}(\varepsilon,\zeta,\varsigma)\Big] \\ := \mathrm{diag}\Big[\underbrace{\varphi_{j_1}^{\circledS}}_{1\leq j_1\leq n}\Big(\|\varepsilon^2\| + \|\zeta^2\| + \|\varsigma^2\|\Big), \ldots, \underbrace{\varphi_{j_n}^{\circledS}}_{1\leq j_n\leq n}\Big(\|\varepsilon^2\| + \|\zeta^2\| + \|\varsigma^2\|\Big)\Big],$$

for each $\varepsilon, \zeta, \varsigma \in \mathscr{Q}$. Choosing $(\mathscr{T}_1, \ldots, \mathscr{T}_n) = (\frac{4}{7}, \ldots, \frac{4}{7})$, we obtain the desired result. □

Corollary 3.10. *Suppose $f_i : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}, (i = 1, \ldots, n)$ are mappings satisfying $f_i(\Lambda, 0) = 0$ and*

$$diag\Big[\ \Big\|f_1(\Lambda,\varepsilon+\zeta+\varsigma) - f_1(\Lambda,\varepsilon+\varsigma) - f_1(\Lambda,\varepsilon+\zeta-\varsigma) + f_1(\Lambda,\varepsilon-\varsigma)\Big\|, \ldots, \\ \Big\|f_n(\Lambda,\varepsilon+\zeta+\varsigma) - f_n(\Lambda,\varepsilon+\varsigma) - f_n(\Lambda,\varepsilon+\zeta-\varsigma) + f_n(\Lambda,\varepsilon-\varsigma)\Big\|\ \Big] \\ \leq diag\Big[\ \Big\|\theta_1(f_1(\Lambda,\varepsilon-\zeta+\varsigma) - f_1(\Lambda,\varepsilon+\varsigma) - f_1(\Lambda,\varepsilon-\zeta-\varsigma) + f_1(\Lambda,\varepsilon-\varsigma))\Big\| \\ + (1-|\theta_1|)\underbrace{\varphi_{j_1}^{\circledS}}_{1\leq j_1\leq n}(\|\varepsilon\zeta\varsigma\|)\ , \ldots, \\ \Big\|\theta_n(f_n(\Lambda,\varepsilon-\zeta+\varsigma) - f_n(\Lambda,\varepsilon+\varsigma) - f_n(\Lambda,\varepsilon-\zeta-\varsigma) + f_n(\Lambda,\varepsilon-\varsigma))\Big\| \\ + (1-|\theta_n|)\underbrace{\varphi_{j_n}^{\circledS}}_{1\leq j_n\leq n}(\|\varepsilon\zeta\varsigma\|)\ \Big]$$

for each $\varepsilon, \zeta, \varsigma \in \mathscr{Q}$ and $\Lambda \in \mathscr{Q}'$. Then there are unique additive mappings $f_i' : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}, (i = 1, \ldots, n)$, s.t.

$$diag\Big[\ \|f_1(\Lambda,\varepsilon) - f_1'(\Lambda,\varepsilon)\|, \ldots, \|f_n(\Lambda,\varepsilon) - f_n'(\Lambda,\varepsilon)\|\ \Big] \\ \leq diag\Big[\ \underbrace{\varphi_{j_1}^{\circledS}}_{1\leq j_1\leq n}(\|\varepsilon\|^3), \ldots, \underbrace{\varphi_{j_n}^{\circledS}}_{1\leq j_n\leq n}(\|\varepsilon\|^3)\ \Big],$$

for each $\varepsilon \in \mathscr{Q}$ and $\Lambda \in \mathscr{Q}'$.

Proof. The proof follows from Theorem 3.2 by letting

$$\mathrm{diag}\Big[\underbrace{\varphi_{j_1}}_{1\le j_1\le n}(\varepsilon,\zeta,\varsigma),\dots,\underbrace{\varphi_{j_n}}_{1\le j_n\le n}(\varepsilon,\zeta,\varsigma)\Big] := \mathrm{diag}\Big[\underbrace{\varphi^{\circledS}_{j_1}}_{1\le j_1\le n}(\|\varepsilon\zeta\varsigma\|),\dots,\underbrace{\varphi^{\circledS}_{j_n}}_{1\le j_n\le n}(\|\varepsilon\zeta\varsigma\|)\Big],$$

for each $\varepsilon,\zeta,\varsigma\in\mathcal{Q}$ and $\Lambda\in\mathcal{Q}'$. Choosing $(\mathcal{T}_1,\dots,\mathcal{T}_n)=(\frac{8}{9},\dots,\frac{8}{9})$, we obtain the desired result. □

Corollary 3.11. *Let $f_i:\mathcal{Q}'\times\mathcal{Q}\to\mathcal{Q},(i=1,\dots,n)$ be odd mappings satisfying*

$$\begin{aligned}
&diag\Big[\Big\|f_1(\Lambda,\varepsilon+\zeta+\varsigma)-f_1(\Lambda,\varepsilon+\varsigma)-f_1(\Lambda,\varepsilon+\zeta-\varsigma)+f_1(\Lambda,\varepsilon-\varsigma)\Big\|\\
&\quad,\dots,\Big\|f_n(\Lambda,\varepsilon+\zeta+\varsigma)-f_n(\Lambda,\varepsilon+\varsigma)-f_n(\Lambda,\varepsilon+\zeta-\varsigma)+f_n(\Lambda,\varepsilon-\varsigma)\Big\|\Big]\\
&\le diag\Big[\Big\|\theta_1[f_1(\Lambda,\varepsilon-\zeta+\varsigma)-f_1(\Lambda,\varepsilon+\varsigma)-f_1(\Lambda,\varepsilon-\zeta-\varsigma)+f_1(\Lambda,\varepsilon-\varsigma)]\Big\|\\
&\qquad+(1-|\theta_1|)\underbrace{\varphi^{\circledS}_{j_1}}_{1\le j_1\le n}(\|\varepsilon\zeta\varsigma\|)\\
&\quad,\dots,\Big\|\theta_n[f_n(\Lambda,\varepsilon-\zeta+\varsigma)-f_n(\Lambda,\varepsilon+\varsigma)-f_n(\Lambda,\varepsilon-\zeta-\varsigma)+f_n(\Lambda,\varepsilon-\varsigma)]\Big\|\\
&\qquad+(1-|\theta_n|)\underbrace{\varphi^{\circledS}_{j_n}}_{1\le j_n\le n}(\|\varepsilon\zeta\varsigma\|)\Big],
\end{aligned}\tag{3.18}$$

for each $\varepsilon,\zeta,\varsigma\in\mathcal{Q}$ and $\Lambda\in\mathcal{Q}'$. Then $f_i(i=1,\dots,n)$ are additive.

Proof. Putting $\varepsilon=0$ in (3.18), we get

$$\begin{aligned}
&\mathrm{diag}\Big[\Big\|f_1(\Lambda,\zeta+\varsigma)-f_1(\Lambda,\varsigma)-f_1(\Lambda,\zeta-\varsigma)+f_1(\Lambda,-\varsigma)\Big\|,\dots,\\
&\quad\Big\|f_n(\Lambda,\zeta+\varsigma)-f_n(\Lambda,\varsigma)-f_n(\Lambda,\zeta-\varsigma)+f_n(\Lambda,-\varsigma)\Big\|\Big]\\
&\le\mathrm{diag}\Big[\Big\|\theta_1(f_1(\Lambda,-\zeta+\varsigma)-f_1(\Lambda,\varsigma)-f_1(\Lambda,-\zeta-\varsigma)+f_1(\Lambda,-\varsigma))\Big\|,\dots,\\
&\quad\Big\|\theta_n(f_n(\Lambda,-\zeta+\varsigma)-f_n(\Lambda,\varsigma)-f_n(\Lambda,-\zeta-\varsigma)+f_n(\Lambda,-\varsigma))\Big\|\Big],
\end{aligned}\tag{3.19}$$

for each $\zeta,\varsigma\in\mathcal{Q}$ and $\Lambda\in\mathcal{Q}'$. Replacing ζ by $-\zeta$ in (3.19), we have

$$\begin{aligned}
&\mathrm{diag}\Big[\Big\|f_1(\Lambda,-\zeta+\varsigma)-f_1(\Lambda,\varsigma)-f_1(\Lambda,-\zeta-\varsigma)+f_1(\Lambda,-\varsigma)\Big\|,\dots,\\
&\quad\Big\|f_n(\Lambda,-\zeta+\varsigma)-f_n(\Lambda,\varsigma)-f_n(\Lambda,-\zeta-\varsigma)+f_n(\Lambda,-\varsigma)\Big\|\Big]\\
&\le\mathrm{diag}\Big[\Big\|\theta_1[f_1(\Lambda,\zeta+\varsigma)-f_1(\Lambda,\varsigma)-f_1(\Lambda,\zeta-\varsigma)+f_1(\Lambda,-\varsigma)]\Big\|,\dots,\\
&\quad\Big\|\theta_n[f_n(\Lambda,\zeta+\varsigma)-f_n(\Lambda,\varsigma)-f_n(\Lambda,\zeta-\varsigma)+f_n(\Lambda,-\varsigma)]\Big\|\Big],
\end{aligned}\tag{3.20}$$

for each $\zeta, \varsigma \in \mathscr{Q}$ and $\Lambda \in \mathscr{Q}'$. From (3.19) and (3.20), it follows that

$$f_i(\Lambda, \zeta + \varsigma) - f_i(\Lambda, \varsigma) - f_i(\Lambda, \zeta - \varsigma) + f_i(\Lambda, -\varsigma) = 0, \quad i = 1, \ldots, n$$

for each $\zeta, \varsigma \in \mathscr{Q}$ and $\Lambda \in \mathscr{Q}'$. Since $f_i, (i = 1, \ldots, n)$, are odd mappings,

$$f_i(\varsigma + \zeta) + f_i(\varsigma - \zeta) - 2f_i(\varsigma) = 0, \quad i = 1, \ldots, n$$

for each $\zeta, \varsigma \in \mathscr{Q}$ and $\Lambda \in \mathscr{Q}'$. Thus the mappings $f_i, (i = 1, \ldots, n)$, are additive. □

Corollary 3.12. *Suppose $f_i : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}, (i = 1, \cdots, n)$, are mappings satisfying $f_i(\Lambda, 0) = 0$ and*

$$\begin{aligned}
&diag\Big[\Big\|f_1(\Lambda, \varepsilon + \zeta + \varsigma) - f_1(\Lambda, \varepsilon + \varsigma) - f_1(\Lambda, \varepsilon + \zeta - \varsigma) + f_1(\Lambda, \varepsilon - \varsigma)\Big\|, \cdots, \\
&\qquad \Big\|f_n(\Lambda, \varepsilon + \zeta + \varsigma) - f_n(\Lambda, \varepsilon + \varsigma) - f_n(\Lambda, \varepsilon + \zeta - \varsigma) + f_n(\Lambda, \varepsilon - \varsigma)\Big\| \Big] \\
&\leq diag\Big[\Big\|\theta_1(f_1(\Lambda, \varepsilon - \zeta + \varsigma) - f_1(\Lambda, \varepsilon + \varsigma) - f_1(\Lambda, \varepsilon - \zeta - \varsigma) + f_1(\Lambda, \varepsilon - \varsigma))\Big\| \\
&\qquad + (1 - |\theta_1|) \underbrace{\varphi^{\circledS}_{j_1}}_{1 \leq j_1 \leq n} \Big(\|\varepsilon^4 + \zeta^4 + \varsigma^4\|\Big), \cdots, \\
&\qquad \Big\|\theta_n(f_n(\Lambda, \varepsilon - \zeta + \varsigma) - f_n(\Lambda, \varepsilon + \varsigma) - f_n(\Lambda, \varepsilon - \zeta - \varsigma) + f_n(\Lambda, \varepsilon - \varsigma))\Big\| \\
&\qquad + (1 - |\theta_n|) \underbrace{\varphi^{\circledS}_{j_n}}_{1 \leq j_n \leq n} \Big(\|\varepsilon^4 + \zeta^4 + \varsigma^4\|\Big) \Big]
\end{aligned}$$

and

$$\begin{aligned}
&diag\Big[\Big\|f_1(\Lambda, \varepsilon) f_1(\Lambda, \zeta) - f_1(\Lambda, f_1(\Lambda, \varepsilon)\zeta) - f_1(\Lambda, \varepsilon f_1(\Lambda, \zeta))\Big\|, \cdots, \\
&\qquad \Big\|f_n(\Lambda, \varepsilon) f_n(\Lambda, \zeta) - f_n(\Lambda, f_n(\Lambda, \varepsilon)\zeta) - f_n(\Lambda, \varepsilon f_n(\Lambda, \zeta))\Big\| \Big] \\
&\leq diag\Big[\underbrace{\varphi^{\circledS}_{j_1}}_{1 \leq j_1 \leq n} \Big(\|2\varepsilon^4 + \zeta^4\|\Big), \cdots, \underbrace{\varphi^{\circledS}_{j_n}}_{1 \leq j_n \leq n} \Big(\|2\varepsilon^4 + \zeta^4\|\Big)\Big]_{n \times n},
\end{aligned}$$

for each $\varepsilon, \zeta, \varsigma \in \mathscr{Q}, \Lambda \in \mathscr{Q}'$. If $f_i(\Lambda, 2\varepsilon) = 2f_i(\Lambda, \varepsilon)$ for each $\varepsilon, \zeta, \varsigma \in \mathscr{Q}, \Lambda \in \mathscr{Q}'$, and $f_i, (i = 1, \cdots, n)$, are continuous, then the mappings $f_i : \mathscr{Q}' \times \mathscr{Q} \to \mathscr{Q}, (i = 1, \cdots, n)$, are antiderivations.

Proof. The proof follows from Theorem 3.7 by letting

$$\begin{aligned}
&\text{diag}\Big[\underbrace{\varphi_{j_1}}_{1 \leq j_1 \leq n}(\varepsilon, \zeta, \varsigma), \cdots, \underbrace{\varphi_{j_n}}_{1 \leq j_n \leq n}(\varepsilon, \zeta, \varsigma)\Big] \\
&:= \text{diag}\Big[\underbrace{\varphi^{\circledS}_{j_1}}_{1 \leq j_1 \leq n} \Big(\|\varepsilon^4 + \zeta^4 + \varsigma^4\|\Big), \cdots, \underbrace{\varphi^{\circledS}_{j_n}}_{1 \leq j_n \leq n} \Big(\|\varepsilon^4 + \zeta^4 + \varsigma^4\|\Big) \Big]_{n \times n}
\end{aligned}$$

for each $\varepsilon, \zeta, \varsigma \in \mathscr{Q}$. Choosing $(\mathcal{T}_1, \cdots, \mathcal{T}_n) = \overbrace{(\frac{8}{17}, \cdots, \frac{8}{17})}^{n}$, we obtain the desired result. □

4. Conclusions

In this study, we investigated the concept of antiderivations in Banach algebras and study multi-super-stability of antiderivations in Banach algebras, associated with functional inequalities.

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Conflict of interest

The authors declare that they have no competing interests.

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