Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(11): 20105–20125.
DOI: 10.3934/math.20221100
Received: 27 March 2022
Revised: 28 May 2022
Accepted: 18 July 2022
Published: 13 September 2022

Research article

Sobolev-type nonlinear Hilfer fractional stochastic differential equations with noninstantaneous impulsive

Mohamed Adel[1], M. Elsaid Ramadan[2], Hijaz Ahmad[3] and Thongchai Botmart[4,*]

[1] Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt
[2] Department of Mathematics, Faculty of Science, Islamic University of Madinah, Medina, Saudi Arabia
[3] Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy
[4] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen, 40002, Thailand

* **Correspondence:** Email: thongbo@kku.ac.th.

Abstract: The existence of a mild solution for nonlinear Hilfer fractional stochastic differential equations of the Sobolev type with non-instantaneous impulse in Hilbert space is investigated in this study. For nonlinear Hilfer fractional stochastic differential equations of Sobolev type with non-instantaneous impulsive conditions, sufficient criteria for controllability are established. Finally, an illustration of the acquired results is shown.

Keywords: Sobolev-type nonlinear Hilfer fractional stochastic differential equation; existence solutions; controllability; noninstantaneous impulsive condition
Mathematics Subject Classification: 26A33, 34A37, 34K37, 93B05

1. Introduction

Nonlinear stochastic differential equations play a very important role in formulation and analysis in mechanical, electrical, control engineering, physical sciences, economic and social sciences. Recently, stochastic fractional differential equations have been considered greatly by research community in various aspects due to its salient features for real world problems ([1–6]). Also, differential systems with impulses become an important area and many interesting works have been reported in ([7–9]). The Sobolev differential system is typically visible in the mathematical structure of numerous physical processes, such as fluid flow through cracked rocks and thermodynamics. Controllability problem for different kinds of fractional dynamical systems have been studied. The control hypothesis is an important area of mathematics that deals with the design and evaluation of control mechanisms. Controllability has had a significant impact on the development of modern mathematical control theory. Control system analysis and design frequently use the problem of dynamical system controllability. In recent years, fractional-order control systems defined by fractional-order differential equations have attracted a lot of attention ([10–13]) and the references therein.There are many interesting results on the existence and uniqueness of mild solutions for a class of Sobolev type fractional evolution equations [14].

According to the aforementioned literature review, the existence and exact controllability of the nonlinear Hilfer fractional stochastic differential equations of Sobolev-type have not been thoroughly investigated. Because of this, we think about the existence solution and controllability for the nonlinear Hilfer fractional stochastic differential equations of Sobolev-type with noninstantaneous impulsive condition of the form

$$\begin{cases} D_{0+}^{q,\jmath}[Zx(t) + G(t, x(\vartheta_1(t)))] = Ax(t) + f(t, x(\vartheta_2(t)), \int_0^t h(t, s)g_1(s, x(\vartheta_3(s)))ds) \\ +\sigma((t, x(\vartheta_4(t)), \int_0^t h(t, s)g_2(s, x(\vartheta_5(s)))ds)\frac{d\omega}{dt}, \quad t \in (s_i, t_{i+1}], \quad i \in [0, m], \\ x(t) = \xi_i(t, x(t)), \quad t \in (t_i, s_i], \quad i \in [1, m], \\ I_{0+}^{(1-q)(1-\jmath)}x(0) = x_0, \end{cases} \tag{1.1}$$

where $D_{0+}^{q,\jmath}$ is the Hilfer fractional derivative, $0 \leq q \leq 1$, $0 < \jmath < 1$, the state $x(\cdot)$ takes values in a separable Hilbert space X with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. The symbol A and Z are linear operators on X. Time interval $J = (0, b]$ where, t_i, s_i are fixed number satisfying $0 = s_0 < t_1 \leq s_1 \leq t_2 < \ldots < s_{m-1} < t_m \leq s_m \leq t_{m+1} = b$ and ξ_i is noninstataneous impulsive function for all $i = 1, 2, \ldots, m$, $\vartheta_i(t) : J \to J$, $i = 1, 2, 3, 4, 5$, are continuous functions. Let K be another separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_K$ and norm $\| \cdot \|_K$. Suppose $\{\omega(t)\}_{t \geq 0}$ is given K-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$. We are also employing the same notation $\| \cdot \|$ for the norm $L(K, X)$, where $L(K, X)$ denotes the space of all bounded linear operators from K into X. Also, $h : J \times J \to R$ is a continuous function and the mappings $G : J \times X \to X$, $f : J \times X \times X \to X$, $\sigma : J \times X \times X \to L_Q(K, X)$, $g_1 : J \times X \to X$ and $g_2 : J \times X \to X$ are nonlinear functions. Here $L_Q(K, X)$ denotes the space of all Q-Hilbert Schmidt operator from K into X.

To the best of our knowledge, there is no work reported on existence solution and controllability for nonlinear Hilfer fractional stochastic differential equations of Sobolev-type with noninstantaneous impulsive condition of the form (1.1). Thus, we will make the first attempt to study such problem in this paper. The presented work can be summarized as following:

Section 2 introduces some basic definitions and lemmas that will help you prove the important points. In Section 3, we show that mild solutions of nonlinear Hilfer fractional stochastic differential equations of the Sobolev type with non-instantaneousimpulsive conditionsexist and are unique. In Section 4, we prove that nonlinear Hilfer fractional stochastic differential equations with noninstantaneous impulsive circumstances are controllable. In the final Section 5, we consider an example to verify the theoretical results. The work is ended by Section 6, which is the conclusion.

2. Preliminaries

In this section, some definitions and results are given which will be used throughout this paper.

Definition 2.1. *[15]* The fractional integral operator of order $\jmath > 0$ for a function f can be defined as

$$I^{\jmath} f(t) = \frac{1}{\Gamma(\jmath)} \int_0^t \frac{f(s)}{(t-s)^{1-\jmath}} ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. *[16]* The Hilfer fractional derivative of order $0 \leq q \leq 1$ and $0 < \jmath < 1$ is defined as

$$D_{0+}^{q,\jmath} f(t) = I_{0+}^{q(1-\jmath)} \frac{d}{dt} I_{0+}^{(1-q)(1-\jmath)} f(t).$$

Let (Ω, Υ, P) be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\Upsilon_t \ : t \in J\}$ satisfying $\Upsilon_t \subset \Upsilon$. An X-valued random variable is an Υ-measurable function $x(t) : \Omega \to X$ and a collection of random variables $\Pi = \{x(t, \omega) : \Omega \to X | t \in J\}$ is called a stochastic process. Usually we suppress the dependence on $\omega \in \Omega$ and write $x(t)$ instead of $x(t, \omega)$ and $x(t) : J \to X$ in the place of Π. Let $\beta_n(t)$ $(n = 1, 2, ...)$ be a sequence of real valued one-dimensional standard Brownian motions mutually independent over (Ω, Υ, P). Set

$$\omega(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t) e_n, \ t \geq 0,$$

where λ_n, $(n = 1, 2, ...)$ are nonnegative real numbers and $\{e_n\}$ $(n = 1, 2, ...)$ is a complete orthonormal basis in K. Let $Q \in L(K, K)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite $Tr(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$, (Tr denotes the trace of the operator). Then the above K-valued stochastic process $\omega(t)$ is called Q-Wiener process.

We assume that $\Upsilon_t = \sigma\{\omega(s) : 0 \leq s \leq t\}$ is the σ-algebra generated by ω. For $\Psi \in L(K, X)$ we define $\| \Psi \|_Q^2 = \text{Tr}(\Psi Q \Psi^*) = \sum_{n=1}^{\infty} \| \sqrt{\lambda_n} \Psi e_n \|^2$. If $\| \Psi \|_Q^2 < \infty$, then Ψ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K, X)$ denote the space of all Q-Hilbert-Schmidt operators $\Psi : K \to X$. The completion $L_Q(K, X)$ of $L(K, X)$ with respect to the topology induced by the norm $\| \cdot \|_Q$ where $\|\Psi\|_Q^2 = \langle \Psi, \Psi \rangle$ is a Hilbert space with the above norm topology.

The collection of all strongly-measurable, square-integrable, X-valued random variables, denoted by $L_2(\Omega, X)$ is a Banach space equipped with norm $\|x(\cdot)\|_{L_2(\Omega,X)} = (E\|x(\cdot, \omega)\|^2)^{\frac{1}{2}}$, where the expectation, E is defined by $E(x) = \int_\Omega x(\omega) dP$.

Let $C(J, L_2(\Omega, X))$ be the Banach space of all continuous maps from J into $L_2(\Omega, X)$ satisfying the condition $\sup_{t \in J} E\|x(t)\|^2 < \infty$. Define $Y = C_{q,\jmath}((0, b], L_2(\Omega, X)) = \{x : x \in C((0, b], L_2(\Omega, X)) : \lim_{t \to 0^+} t^{(1-q)(1-\jmath)}\}$ endowed with the norm $\| \cdot \|_Y = (\sup_{t \in (0,b]} E\|t^{(1-q)(1-\jmath)} x(t)\|^2)^{\frac{1}{2}}$.

Obviously, Y is a Banach space.

Introduce the set $B_r = \{x \in Y : \|x\|_Y^2 \leq r\}$, where $r > 0$.

The operators $A : D(A) \subset X \to X$ and $Z : D(Z) \subset X \to X$ satisfy the following hypotheses:

$(H1)$ A and Z are closed linear operators.

$(H2)$ $D(Z) \subset D(A)$ and Z is bijective.

$(H3)$ $Z^{-1} : X \to D(Z)$ is continuous. Here, $(H1)$ and $(H2)$ together with the closed graph theorem imply the boundedness of the linear operator $AZ^{-1} : X \to X$.

$(H4)$ For each $t \in J$ and for $\lambda \in \rho(AZ^{-1})$, the resolvent of AZ^{-1}, the resolvent $R(\lambda, AZ^{-1})$ is compact operator.

Lemma 2.3. *[17]* Let $T(t)$ be a uniformly continuous semigroup generated by A. If the resolvent set $R(\lambda, A)$ of A is compact for every $\lambda \in \rho(A)$, then $T(t)$ is a compact semigroup.

From the above fact, AZ^{-1} generates a compact semigroup $\{S(t), t > 0\}$ in X, which means that there exists $M > 1$ such that $\sup_{t\in J} \|S(t)\| \leq M$. We suppose that $0 \in \rho(AZ^{-1})$, the resolvent set of AZ^{-1} and $\|S(t)\| \leq M$ for some constant $M \geq 1$ and every $t > 0$. We define the fractional power $(AZ^{-1})^{-\gamma}$ by

$$(AZ^{-1})^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} S(t)dt, \ \gamma > 0.$$

For $\gamma \in (0, 1]$, $(AZ^{-1})^{\gamma}$ is a closed linear operator on its domain $D((AZ^{-1})^{\gamma})$. Furthermore, the subspace $D((AZ^{-1})^{\gamma})$ is dense in X. We will introduce the following basic properties of $(AZ^{-1})^{\gamma}$.

Theorem 2.4. (see *[18]*) The following results hold.

(i) Let $0 < \gamma \leq 1$, then $X_\gamma := D((AZ^{-1})^{\gamma})$ is a Banach space with the norm $\|x\|_\gamma = \|(AZ^{-1})^{\gamma} x\|$, $x \in X_\gamma$.

(ii) If $0 < \beta < \gamma \leq 1$, then $D((AZ^{-1})^{\gamma}) \hookrightarrow D((AZ^{-1})^{\beta})$ and the embedding is compact whenever the resolvent operator of (AZ^{-1}) is compact.

(iii) For every $0 < \gamma \leq 1$, there exists a positive constant C_γ such that

$$\|(AZ^{-1})^{\gamma} S(t)\| \leq \frac{C_\gamma}{t^\gamma}, \ 0 < t \leq b.$$

For $x \in X$, we define two families of operators $\{S_{q,J}(t) : t > 0\}$ and $\{P_J(t) : t > 0\}$ by

$$S_{q,J}(t) = I_{0+}^{q(1-J)} P_J(t), \ P_J(t) = t^{J-1} T_J(t), \ T_J(t) = \int_0^\infty J\theta \Psi_J(\theta) S(t^J \theta) d\theta, \tag{2.1}$$

where

$$\Psi_J(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-nJ)}, \ 0 < J < 1, \ \theta \in (0, \infty), \tag{2.2}$$

is a function of Wright-type which satisfies

$$\int_0^\infty \theta^\tau \Psi_J(\theta) d\theta = \frac{\Gamma(1+\tau)}{\Gamma(1+J\tau)}, \ \text{for } \theta \geq 0.$$

Lemma 2.5. (*[19]*, Propositions 2.15–2.17) The operators $S_{q,J}$ and P_J have the following properties.

(i) $\{P_J(t) : t > 0\}$ is continuous in the uniform operator topology.

(ii) For any fixed $t > 0$, $S_{q,J}(t)$ and $P_J(t)$ are linear and bounded operators, and

$$\|P_J(t)x\| \leq \frac{Mt^{J-1}}{\Gamma(J)} \|x\|, \ \|S_{q,J}(t)x\| \leq \frac{Mt^{(q-1)(1-J)}}{\Gamma(q(1-J)+J)} \|x\|. \tag{2.3}$$

(iii) $\{P_J(t) : t > 0\}$ and $\{S_{q,J}(t) : t > 0\}$ are strongly continuous.

By Theorem 2.4 and Lemma 2.5, we have

Lemma 2.6. For any $x \in X, \beta \in (0, 1)$ and $\delta \in (0, 1]$, we have

$$(AZ^{-1})T_J(t)x = (AZ^{-1})^{1-\beta} T_J(t)(AZ^{-1})^{\beta} x, \ 0 < t \leq b,$$

and

$$\|(AZ^{-1})^{\delta}T_{\jmath}(t)x\| \le \frac{\jmath C_{\delta}\Gamma(2-\delta)}{t^{\delta\jmath}\Gamma(1+\jmath(1-\delta))}\|x\|,\ 0 < t \le b.$$

Lemma 2.7. *[20]* (Burkholder-Davis-Gundy inequalities) Let $T > 0$ and $(M_t)_{0\le t\le T}$ be a continuous local martingale such that $M_0 = 0$. For every $0 < p < \infty$, there exist universal constants c_p and C_p, independent of T and $(M_t)_{0\le t\le T}$ such that $c_p\mathbb{E}\left(\langle M\rangle_T^{\frac{p}{2}}\right) \le \mathbb{E}\left(\left(\sup_{0\le t\le T}|M_t|\right)^p\right) \le C_p\mathbb{E}\left(\langle M\rangle_T^{\frac{p}{2}}\right)$.

3. Existence solution

In this section, we study the existence and uniqueness of mild solution for the nonlinear Hilfer fractional stochastic differential equations of Sobolev-type with noninstantaneous impulsive condition of the form (1.1).

Definition 3.1. (see *[19]*) An Υ_t-adapted stochastic process $x(t) : J \to X$ is a mild solution of the system (1.1) if the function $AZ^{-1}P_{\jmath}(t-s)G(s, x(\vartheta_1(s))),\ s \in (0, b)$ is integrable on $(0, b)$ and the following integral equation is verified:

$$\begin{aligned}
x(t) &= Z^{-1}S_{q,\jmath}(t)[Zx_0 + G(0, x(0))] - Z^{-1}G(t, x(\vartheta_1(t))) \\
&\quad + \int_0^t Z^{-1}AZ^{-1}P_{\jmath}(t-s)G(s, x(\vartheta_1(s))ds \\
&\quad + \int_0^t Z^{-1}P_{\jmath}(t-s)f(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau)ds \\
&\quad + \int_0^t Z^{-1}P_{\jmath}(t-s)\sigma(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau)d\omega(s), \quad t \in (0, t_1], \\
x(t) &= \xi_i(t, x(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \ldots, m, \\
x(t) &= Z^{-1}S_{q,\jmath}(t - s_i)\xi_i(s_i, x(s_i)) - Z^{-1}G(t, x(\vartheta_1(t)) \\
&\quad + \int_{s_i}^t Z^{-1}AZ^{-1}P_{\jmath}(t-s)G(s, x(\vartheta_1(s))ds \\
&\quad + \int_{s_i}^t Z^{-1}P_{\jmath}(t-s)f(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau)ds \\
&\quad + \int_{s_i}^t Z^{-1}P_{\jmath}(t-s)\sigma(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau)d\omega(s), t \in (s_i, t_{i+1}], \ i = 1, 2, \ldots, m.
\end{aligned} \tag{3.1}$$

In this paper we need the following assumptions.

$(H5)$ (i) The function $G : J \times X \to X$ is continuous and there exists constants $K_1 > 0$, $K_2 > 0$ such that for $t \in J$, $\vartheta_1(t) \in X$ we have

$$\begin{aligned}
E\|(AZ^{-1})^{\beta}G(t, x_1(\vartheta_1(t))) - (AZ^{-1})^{\beta}G(t, x_2(\vartheta_1(t)))\|^2 &\le K_1E\|x_1(\vartheta_1(t)) - x_2(\vartheta_1(t))\|^2, \\
K_2 &= E\|(AZ^{-1})^{\beta}G(t, 0)\|^2.
\end{aligned}$$

(ii) The function $f : J \times X \times X \to X$ is continuous and there exists constants $N_1 > 0$, $N_2 > 0$ such that for $t \in J$, $\vartheta_2(t), v_1(t), v_2(t) \in X$, we have

$$E\|f(t, x_1(\vartheta_2(t)), v_1(t)) - f(t, x_2(\vartheta_2(t)), v_2(t))\|^2 \le N_1[E\|x_1(\vartheta_2(t)) - x_2(\vartheta_2(t))\|^2 + E\|v_1(t) - v_2(t)\|^2],$$

$$N_2 = E\|f(t,0,0)\|^2.$$

(iii) The function $\sigma : J\times X\times X \to L_Q(K,X)$ is continuous and there exists constants $C_1 > 0, C_2 > 0$ such that for $t \in J$, $\vartheta_4(t), y_1(t), y_2(t) \in X$, we have

$$\begin{aligned} E\|\sigma(t,x_1(\vartheta_4(t)),y_1(t)) - \sigma(t,x_2(\vartheta_4(t)),y_2(t))\|_Q^2 &\le C_1[E\|x_1(\vartheta_4(t)) - x_2(\vartheta_4(t))\|^2 + E\|y_1(t) - y_2(t)\|^2], \\ C_2 &= E\|\sigma(t,0,0)\|_Q^2. \end{aligned}$$

(iv) The functions $\xi_i : (t_i, s_i]\times X \to X$ are continuous and there exist constants $C_7, C_8 > 0$, such that for all $t \in (t_i, s_i],\ i = 1,2,\dots,m$, $x, y \in X$, we have

$$\begin{aligned} E\|\xi_i(t,x) - \xi_i(t,y)\|^2 &\le C_7E\|x-y\|^2, \\ C_8 &= \|\xi_i(t,0)\|^2. \end{aligned}$$

$(H6)$ (i) $g_1 : J\times X \to X$ is continuous and there exist constants $C_3 > 0$, $C_4 > 0$ such that for $t \in J$ and $\vartheta_3(t) \in X$, we have

$$\begin{aligned} E\|g_1(t,x_1(\vartheta_3(t))) - g_1(t,x_2(\vartheta_3(t)))\|^2 &\le C_3E\|x_1(\vartheta_3(t)) - x_2(\vartheta_3(t))\|^2, \\ C_4 &= E\|g(t,0)\|^2. \end{aligned}$$

(ii) $g_2 : J\times X \to X$ is continuous and there exist constants $C_5 > 0$, $C_6 > 0$ such that for $t \in J$ and $\vartheta_5(t) \in X$, we have

$$\begin{aligned} E\|g_2(t,x_1(\vartheta_5(t))) - g_2(t,x_2(\vartheta_3(t)))\|^2 &\le C_5E\|x_1(\vartheta_3(t)) - x_2(\vartheta_3(t))\|^2, \\ C_6 &= E\|g_2(t,0)\|^2. \end{aligned}$$

$(H7)$ There exists a constant C such that $E|h(t,s)|^2 \le C$ for $(t,s) \in J\times J$.

$(H8)$ There exists a constant q such that for all $x_1, x_2 \in X$,

$$E\|x_1(\vartheta_i(t)) - x_2(\vartheta_i(t))\|^2 \le qE\|x_1(t) - x_2(t)\|^2,\ \text{for } i = 1,2,3,4,5.$$

$(H9)$ There exists a constant $r > 0$ such that

$$\begin{aligned} &\frac{25M^2\|Z^{-1}\|^2}{\Gamma^2(q(1-\jmath)+\jmath)}[\|Z\|^2E\|x_0\|^2 + M_0^2(K_1E\|x_0\|^2 + K_2) + (rC_7 + C_8)] + b^{2(1-q)(1-\jmath)}(rC_7 + C_8) \\ &+25\|Z^{-1}\|^2[rK_1 + K_2][M_0^2 + \frac{M^2C_{1-\beta}^2 b^{2\beta\jmath+2(1-q)(1-\jmath)}\Gamma^2(\beta)}{\Gamma^2(1+\jmath\beta)\Gamma^2(\jmath)}] + \frac{25\|Z^{-1}\|^2M^2b^{2-2q(1-\jmath)}}{\jmath^2\Gamma^2(\jmath)}\delta_1 \le r, \end{aligned}$$

where

$$\begin{aligned} \delta_1 &= (N_1(r + bC(rC_3 + C_4)) + N_2) + Tr(Q)\left(C_1(r + bC(rC_5 + C_6)) + C_2\right), \\ M_0 &= \|(AZ^{-1})^{-\beta}\|. \end{aligned}$$

Theorem 3.2. If the hypotheses $(H1)$–$(H9)$ are satisfied, then the system (1.1) has a mild solution on J provided that

$$\begin{aligned} \zeta &:= 25\|Z^{-1}\|^2\Bigg\{\left[M_0^2 + \frac{M^2C_{1-\beta}^2b^{2\beta\jmath}\Gamma^2(\beta)}{\Gamma^2(1+\jmath\beta)\Gamma^2(\jmath)}\right]qK_1 + \frac{M^2b^{2\jmath}q}{\jmath^2\Gamma^2(\jmath)}[N_1(1 + bCC_3) + C_1Tr(Q)(1 + bCC_5)] \\ &+\frac{M^2b^{2(q-1)(1-\jmath)}C_7}{\Gamma^2(q(1-\jmath)+\jmath)}\Bigg\} + C_7 < 1. \end{aligned}$$

Proof. Consider the operator Φ on Y defined as follows:

$$\begin{aligned}\Phi x(t) &= Z^{-1}S_{q,\jmath}(t)[Zx_0+G(0,x(0))]-Z^{-1}G(t,x(\vartheta_1(t))\\&+\int_0^t Z^{-1}AZ^{-1}P_\jmath(t-s)G(s,x(\vartheta_1(s))ds\\&+\int_0^t Z^{-1}P_\jmath(t-s)f(s,x(\vartheta_2(s)),\int_0^s h(s,\tau)g_1(\tau,x(\vartheta_3(\tau)))d\tau)ds\\&+\int_0^t Z^{-1}P_\jmath(t-s)\sigma(s,x(\vartheta_4(s)),\int_0^s h(s,\tau)g_2(\tau,x(\vartheta_5(\tau)))d\tau)d\omega(s),\quad t\in(0,t_1],\\ \Phi x(t) &= \xi_i(t,x(t)),\quad t\in(t_i,s_i],\quad i=1,2,\ldots,m,\\ \Phi x(t) &= Z^{-1}S_{q,\jmath}(t-s_i)\xi_i(s_i,x(s_i))-Z^{-1}G(t,x(\vartheta_1(t))\\&+\int_{s_i}^t Z^{-1}AZ^{-1}P_\jmath(t-s)G(s,x(\vartheta_1(s))ds\\&+\int_{s_i}^t Z^{-1}P_\jmath(t-s)f(s,x(\vartheta_2(s)),\int_0^s h(s,\tau)g_1(\tau,x(\vartheta_3(\tau)))d\tau)ds\\&+\int_{s_i}^t Z^{-1}P_\jmath(t-s)\sigma(s,x(\vartheta_4(s)),\int_0^s h(s,\tau)g_2(\tau,x(\vartheta_5(\tau)))d\tau)d\omega(s),t\in(s_i,t_{i+1}],\ i=1,2,\ldots,m.\end{aligned}$$

It will be shown that the operator Φ has a fixed point. This fixed point is then a mild solution of a system (1.1). For $x\in B_r$, we show that Φ maps B_r into itself. From Lemmas 2.5–2.7 together with Hölder inequality, we have for $t\in(0,t_1]$

$$\begin{aligned}\|\Phi x\|_Y^2 &\le 25\sup_{t\in J} t^{2(1-q)(1-\jmath)}\{E\|Z^{-1}S_{q,\jmath}(t)[Zx_0+G(0,x(0))]\|^2+E\|Z^{-1}G(t,x(\vartheta_1(t)))\|^2\\&+E\|\int_0^t Z^{-1}AZ^{-1}P_\jmath(t-s)G(s,x(\vartheta_1(s)))ds\|^2\\&+E\|\int_0^t Z^{-1}P_\jmath(t-s)f\left(s,x(\vartheta_2(s)),\int_0^s h(s,\tau)g_1(\tau,x(\vartheta_3(\tau)))d\tau\right)ds\|^2\\&+E\|\int_0^t Z^{-1}P_\jmath(t-s)\sigma\left(s,x(\vartheta_4(s)),\int_0^s h(s,\tau)g_2(\tau,x(\vartheta_5(\tau)))d\tau\right)d\omega(s)\|_Q^2\}\\&\le \frac{25M^2\|Z^{-1}\|^2}{\Gamma^2(q(1-\jmath)+\jmath)}[\|Z\|^2E\|x_0\|^2+M_0^2(K_1E\|x_0\|^2+K_2)]\\&+25\|Z^{-1}\|^2[rK_1+K_2][M_0^2+\frac{M^2C_{1-\beta}^2b^{2\beta\jmath+2(1-q)(1-\jmath)}\Gamma^2(\beta)}{\Gamma^2(1+\jmath\beta)\Gamma^2(\jmath)}]+\frac{25\|Z^{-1}\|^2M^2b^{2-2q(1-\jmath)}\delta_1}{\jmath^2\Gamma^2(\jmath)}\\&\le r,\end{aligned}$$

for $t\in(t_i,s_i]$

$$\begin{aligned}\|\Phi x\|_Y^2 &\le \sup_{t\in J} t^{2(1-q)(1-\jmath)}E\|\xi_i(t,x(t))\|^2\\&\le b^{2(1-q)(1-\jmath)}(rC_7+C_8)\\&\le r,\end{aligned}$$

and for $t \in (s_i, t_{i+1}]$

$$\begin{aligned}
\|\Phi x\|_Y^2 &\leq 25 \sup_{t\in J} t^{2(1-q)(1-\jmath)}\{E\|Z^{-1}S_{q,\jmath}(t-s_i)\xi_i(s_i, x(s_i))\|^2 + E\|Z^{-1}G(t, x(\vartheta_1(t)))\|^2 \\
&\quad + E\|\int_{s_i}^t Z^{-1}AZ^{-1}P_\jmath(t-s)G(s, x(\vartheta_1(s)))ds\|^2 \\
&\quad + E\|\int_{s_i}^t Z^{-1}P_\jmath(t-s)f\left(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau\right)ds\|^2 \\
&\quad + E\|\int_{s_i}^t Z^{-1}P_\jmath(t-s)\sigma\left(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau\right)d\omega(s)\|_Q^2\} \\
&\leq \frac{25M^2\|Z^{-1}\|^2}{\Gamma^2(q(1-\jmath)+\jmath)}(rC_7 + C_8) \\
&\quad + 25\|Z^{-1}\|^2[rK_1 + K_2][M_0^2 + \frac{M^2C_{1-\beta}^2 b^{2\beta\jmath+2(1-q)(1-\jmath)}\Gamma^2(\beta)}{\Gamma^2(1+\jmath\beta)\Gamma^2(\jmath)}] + \frac{25\|Z^{-1}\|^2M^2b^{2-2q(1-\jmath)}\delta_1}{\jmath^2\Gamma^2(\jmath)} \\
&\leq r.
\end{aligned}$$

Thus Φ maps B_r into itself.

We show that $(\Phi x)(t)$ is continuous on $[0, b]$ for any $x \in B_r$. Let $0 < t \leq b$ and $\epsilon > 0$ be sufficiently small, then for $t \in (0, t_1]$

$$\begin{aligned}
&\|(\Phi x)(\cdot+\epsilon) - (\Phi x)(\cdot)\|_Y^2 = \sup_{t\in J} t^{(1-q)(1-\jmath)}E\|(\Phi x)(t+\epsilon) - (\Phi x)(t)\|^2 \\
\leq\ & 5 \sup_{t\in J} t^{(1-q)(1-\jmath)}E\|Z^{-1}(S_{q,\jmath}(t+\epsilon) - S_{q,\jmath}(t))[Zx_0 + G(0, x(0))]\|^2 \\
&+5 \sup_{t\in J} t^{(1-q)(1-\jmath)}E\|Z^{-1}G(t+\epsilon, x(\vartheta_1(t+\epsilon))) - Z^{-1}G(t, x(\vartheta_1(t)))\|^2 \\
&+5 \sup_{t\in J} t^{(1-q)(1-\jmath)}E\|\int_0^{t+\epsilon} Z^{-1}AZ^{-1}P_\jmath(t+\epsilon-s)G(s, x(\vartheta_1(s)))ds \\
&-\int_0^t Z^{-1}AZ^{-1}P_\jmath(t-s)G(s, x(\vartheta_1(s)))ds\|^2 \\
&+5 \sup_{t\in J} t^{(1-q)(1-\jmath)}E\|\int_0^{t+\epsilon} Z^{-1}P_\jmath(t+\epsilon-s)f\left(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau\right)ds \\
&-\int_0^t Z^{-1}P_\jmath(t-s)f\left(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau\right)ds\|^2 \\
&+5 \sup_{t\in J} t^{(1-q)(1-\jmath)}\|\int_0^{t+\epsilon} Z^{-1}P_\jmath(t+\epsilon-s)\sigma\left(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau\right)d\omega(s) \\
&-\int_0^t Z^{-1}P_\jmath(t-s)\sigma\left(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau\right)d\omega(s)\|_Q^2,
\end{aligned} \tag{3.2}$$

for $t \in (t_i, s_i]$

$$\|(\Phi x)(\cdot+\epsilon) - (\Phi x)(\cdot)\|_Y^2 = \sup_{t\in J} t^{(1-q)(1-\jmath)}E\|(\Phi x)(t+\epsilon) - (\Phi x)(t)\|^2$$

$$\leq \sup_{t\in J} t^{2(1-q)(1-\jmath)} E\|\xi_i(t+\epsilon, x(t+\epsilon)) - \xi_i(t, x(t))\|^2, \tag{3.3}$$

and for $t \in (s_i, t_{i+1}]$

$$
\begin{aligned}
&\|(\Phi x)(\cdot+\epsilon) - (\Phi x)(\cdot)\|_Y^2 = \sup_{t\in J} t^{(1-q)(1-\jmath)} E\|(\Phi x)(t+\epsilon) - (\Phi x)(t)\|^2 \\
\leq\ & 5\sup_{t\in J} t^{(1-q)(1-\jmath)} E\|Z^{-1}(S_{q,\jmath}(t+\epsilon-s_i) - S_{q,\jmath}(t-s_i))\xi_i(s_i, x(s_i))\|^2 \\
&+5\sup_{t\in J} t^{(1-q)(1-\jmath)} E\|Z^{-1}G(t+\epsilon, x(\vartheta_1(t+\epsilon))) - Z^{-1}G(t, x(\vartheta_1(t)))\|^2 \\
&+5\sup_{t\in J} t^{(1-q)(1-\jmath)} E\| \int_{s_i}^{t+\epsilon} Z^{-1}AZ^{-1}P_\jmath(t+\epsilon-s)G(s, x(\vartheta_1(s)))ds \\
&- \int_{s_i}^{t} Z^{-1}AZ^{-1}P_\jmath(t-s)G(s, x(\vartheta_1(s)))ds\|^2 \\
&+5\sup_{t\in J} t^{(1-q)(1-\jmath)} E\| \int_{s_i}^{t+\epsilon} Z^{-1}P_\jmath(t+\epsilon-s) f\left(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau\right)ds \\
&- \int_{s_i}^{t} Z^{-1}P_\jmath(t-s) f\left(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau\right)ds\|^2 \\
&+5\sup_{t\in J} t^{(1-q)(1-\jmath)} \| \int_{s_i}^{t+\epsilon} Z^{-1}P_\jmath(t+\epsilon-s) \sigma\left(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau\right)d\omega(s) \\
&- \int_{s_i}^{t} Z^{-1}P_\jmath(t-s) \sigma\left(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau\right)d\omega(s)\|_Q^2.
\end{aligned}
\tag{3.4}
$$

Clearly, the right hand sides of (3.2)–(3.4) are tends to zero as $\epsilon \to 0$. Hence, $(\Phi x)(t)$ is continuous on $[0, b]$.

Next for $x_1, x_2 \in B_r$, we show that Φ is a contraction mapping. From Lemmas 2.5–2.7 together with Hölder inequality, we obtain for $t \in (0, t_1]$

$$
\begin{aligned}
&E\|(\Phi x_1)(t) - (\Phi x_2)(t)\|^2 \\
\leq\ & 25\|Z^{-1}\|^2\{\|(AZ^{-1})^{-\beta}\|^2 E\|(AZ^{-1})^\beta G(t, x_1(\vartheta_1(t))) - (AZ^{-1})^\beta G(t, x_2(\vartheta_1(t)))\|^2 \\
&+E\| \int_0^t (t-s)^{\jmath-1}(AZ^{-1})^{1-\beta} T_\jmath(t-s)[(AZ^{-1})^\beta G(s, x_1(\vartheta_1(s))) - (AZ^{-1})^\beta G(s, x_2(\vartheta_1(s)))]ds\|^2 \\
&+E\| \int_0^t P_\jmath(t-s)[f(s, x_1(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x_1(\vartheta_3(\tau)))d\tau) \\
&-f(s, x_2(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x_2(\vartheta_3(\tau)))d\tau)]ds\|^2 \\
&+E\| \int_0^t P_\jmath(t-s)[\sigma(s, x_1(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x_1(\vartheta_5(\tau)))d\tau) \\
&-\sigma(s, x_2(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x_2(\vartheta_5(\tau)))d\tau)]d\omega(s)\|^2\} \\
\leq\ & 25\|Z^{-1}\|^2\Bigg\{\left[M_0^2 + \frac{M^2 C_{1-\beta}^2 b^{2\beta\jmath}\Gamma^2(\beta)}{\Gamma^2(1+\jmath\beta)\Gamma^2(\jmath)}\right]qK_1 + \frac{M^2 b^{2\jmath} q}{\jmath^2\Gamma^2(\jmath)}[N_1(1+bCC_3)
\end{aligned}
$$

$$+C_1Tr(Q)(1+bCC_5)]\Big\}E\|x_1(t)-x_2(t)\|^2, \tag{3.5}$$

for $t \in (t_i, s_i]$

$$\begin{aligned} & E\|(\Phi x_1)(t)-(\Phi x_2)(t)\|^2 \\ \leq\ & E\|\xi_i(t,x_1(t))-\xi_i(t,x_2(t))\|^2 \leq C_7\ E\|x_1(t)-x_2(t)\|^2, \end{aligned} \tag{3.6}$$

and for $t \in (s_i, t_{i+1}]$

$$\begin{aligned} & E\|(\Phi x_1)(t)-(\Phi x_2)(t)\|^2 \\ \leq\ & 25\|Z^{-1}\|^2\{E\|S_{q,J}(t-s_i)(\xi_i(s_i,x_1(s_i))-\xi_i(s_i,x_2(s_i)))\|^2 \\ & +\|(AZ^{-1})^{-\beta}\|^2E\|(AZ^{-1})^{\beta}G(t,x_1(\vartheta_1(t)))-(AZ^{-1})^{\beta}G(t,x_2(\vartheta_1(t)))\|^2 \\ & +E\|\int_{s_i}^t(t-s)^{J-1}(AZ^{-1})^{1-\beta}T_J(t-s)[(AZ^{-1})^{\beta}G(s,x_1(\vartheta_1(s)))-(AZ^{-1})^{\beta}G(s,x_2(\vartheta_1(s)))]ds\|^2 \\ & +E\|\int_{s_i}^t P_J(t-s)[f(s,x_1(\vartheta_2(s)),\int_0^s h(s,\tau)g_1(\tau,x_1(\vartheta_3(\tau)))d\tau) \\ & -f(s,x_2(\vartheta_2(s)),\int_0^s h(s,\tau)g_1(\tau,x_2(\vartheta_3(\tau)))d\tau)]ds\|^2 \\ & +E\|\int_{s_i}^t P_J(t-s)[\sigma(s,x_1(\vartheta_4(s)),\int_0^s h(s,\tau)g_2(\tau,x_1(\vartheta_5(\tau)))d\tau) \\ & -\sigma(s,x_2(\vartheta_4(s)),\int_0^s h(s,\tau)g_2(\tau,x_2(\vartheta_5(\tau)))d\tau)]d\omega(s)\|^2\} \\ \leq\ & 25\|Z^{-1}\|^2\Big\{\left[M_0^2+\frac{M^2C_{1-\beta}^2b^{2\beta J}\Gamma^2(\beta)}{\Gamma^2(1+J\beta)\Gamma^2(J)}\right]qK_1+\frac{M^2b^{2J}q}{J^2\Gamma^2(J)}[N_1(1+bCC_3)+C_1Tr(Q)(1+bCC_5)] \\ & +\frac{M^2b^{2(q-1)(1-J)}C_7}{\Gamma^2(q(1-J)+J)}\Big\}E\|x_1(t)-x_2(t)\|^2. \end{aligned} \tag{3.7}$$

Therefore, by combining (3.5)–(3.7) , we get

$$\sup_{t\in J} t^{2(1-q)(1-J)}E\|(\Phi x_1)(t)-(\Phi x_2)(t)\|^2 \leq \zeta \sup_{t\in J} t^{2(1-q)(1-J)}E\|x_1(t)-x_2(t)\|^2.$$

This implies that

$$\|\Phi x_1-\Phi x_2\|_Y^2 \leq \zeta\|x_1-x_2\|_Y^2.$$

Then, Φ is a contraction mapping and hence there exists unique fixed point $x \in B_r$ such that $\Phi x = x$. Hence, any fixed point of Φ is a mild solution of (1.1) on J. The proof is completed.

4. Controllability results

In this section, we study the controllability of the following Sobolev-type nonlinear Hilfer fractional stochastic differential equations with noninstantaneous impulsive condition:

$$\begin{cases} D_{0+}^{q,J}[Zx(t)+G(t,x(\vartheta_1(t)))] = Ax(t)+Bu(t)+f(t,x(\vartheta_2(t)),\int_0^t h(t,s)g_1(s,x(\vartheta_3(s)))ds) \\ +\sigma(t,x(\vartheta_4(t)),\int_0^t h(t,s)g_2(s,x(\vartheta_5(s)))ds)\frac{d\omega}{dt}, \quad t\in(s_i,t_{i+1}], \quad i\in[0,m] \\ x(t)=\xi_i(t,x(t)), \quad t\in(t_i,s_i], \quad i\in[1,m] \\ I_{0+}^{(1-q)(1-J)}x(0)=x_0, \end{cases} \tag{4.1}$$

where $B : U \to X$ is a bounded linear operator and the control function $u \in L_2(J, U)$, the Hilbert space of admissible control functions with U a Hilbert space.

Definition 4.1. An Υ_t-adapted stochastic process $x(t) : J \to X$ is a mild solution of the system (4.1) if the function $AZ^{-1}P_\jmath(t-s)G(s, x(\vartheta_1(s))),\ s \in (0, b)$ is integrable on $(0, b)$ and the following integral equation is verified:

$$
\begin{aligned}
x(t) &= Z^{-1}S_{q,\jmath}(t)[Zx_0 + G(0, x(0))] - Z^{-1}G(t, x(\vartheta_1(t))) \\
&\quad + \int_0^t Z^{-1}AZ^{-1}P_\jmath(t-s)G(s, x(\vartheta_1(s))ds + \int_0^t Z^{-1}P_\jmath(t-s)Bu(s)ds \\
&\quad + \int_0^t Z^{-1}P_\jmath(t-s)f(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau)ds \\
&\quad + \int_0^t Z^{-1}P_\jmath(t-s)\sigma(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau)d\omega(s), \quad t \in (0, t_1] \\
x(t) &= \xi_i(t, x(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \ldots, m \\
x(t) &= Z^{-1}S_{q,\jmath}(t-s_i)\xi_i(s_i, x(s_i)) - Z^{-1}G(t, x(\vartheta_1(t))) \\
&\quad + \int_{s_i}^t Z^{-1}AZ^{-1}P_\jmath(t-s)G(s, x(\vartheta_1(s))ds + \int_{s_i}^t Z^{-1}P_\jmath(t-s)Bu(s)ds \\
&\quad + \int_{s_i}^t Z^{-1}P_\jmath(t-s)f(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau)ds \\
&\quad + \int_{s_i}^t Z^{-1}P_\jmath(t-s)\sigma(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau)d\omega(s), t \in (s_i, t_{i+1}], \ i = 1, 2, \ldots, m.
\end{aligned}
\tag{4.2}
$$

Definition 4.2. The system (4.1) is said to be controllable on J, if for every $x_0, x_1 \in X$, there exists a control $u \in L_2(J, U)$ such that the mild solution $x(t)$ of the system (4.1) satisfies $x(b) = x_1$, where x_1 and b are the preassigned terminal state and time respectively.

To establish the result, we need the following additional hypotheses

$(H10)$ The linear operator W from U into X defined by

$$Wu = \int_0^b Z^{-1}P_\jmath(b-s)Bu(s)ds$$

has an inverse bounded operator W^{-1} which takes values in $L_2(J, U)\setminus \ker W$, where the kernel space of W is defined by $\ker W = \{x \in L_2(J, U) : Wx = 0\}$ and B is bounded operator.

$(H11)$ There exists a constant $r > 0$ such that

$$
\begin{aligned}
&\left[\frac{\delta_3+\delta_4}{\Gamma^2(q(1-\jmath)+\jmath)} + \left(M_0^2 + \frac{M^2C_{1-\beta}^2 b^{2\beta\jmath+2(1-q)(1-\jmath)}\Gamma^2(\beta)}{\Gamma^2(1+\jmath\beta)\Gamma^2(\jmath)}\right)\delta_2 + \frac{36\|Z^{-1}\|^2M^2b^{2-2q(1-\jmath)}\delta_1}{\jmath^2\Gamma^2(\jmath)}\right] \\
&\times\left[1 + \frac{M^2b^{2\jmath}\|Z^{-1}\|^2\|B\|^2\|W^{-1}\|^2}{\jmath^2\Gamma^2(\jmath)}\right] + \frac{36M^2b^{2-2q(1-\jmath)}\|Z^{-1}\|^2\|B\|^2\|W^{-1}\|^2E\|x_1\|^2}{\jmath^2\Gamma^2(\jmath)} \\
&+b^{2(1-q)(1-\jmath)}(rC_7 + C_8) \le r,
\end{aligned}
$$

where

$$\delta_2 = 36\|Z^{-1}\|^2[rK_1 + K_2],$$

$$\begin{aligned}\delta_3 &= 36M^2\|Z^{-1}\|^2[\|Z\|^2E\|x_0\|^2 + M_0^2(K_1E\|x_0\|^2 + K_2)],\\ \delta_4 &= 36M^2\|Z^{-1}\|^2(rC_7 + C_8).\end{aligned}$$

Theorem 4.3. If the hypotheses $(H1)$–$(H8)$, $(H10)$ and $(H11)$ are satisfied then, the system (4.1) is controllable on J. provided that

$$\varrho := 36\|Z^{-1}\|^2\Bigg\{\Bigg[M_0^2 + \frac{M^2C_{1-\beta}^2 b^{2\beta\jmath}\Gamma^2(\beta)}{\Gamma^2(1+\jmath\beta)\Gamma^2(\jmath)}\Bigg]qK_1 + \frac{M^2b^{2\jmath}q}{\jmath^2\Gamma^2(\jmath)}[N_1(1+bCC_3) + C_1Tr(Q)(1+bCC_5)]$$
$$+\frac{M^2b^{2(q-1)(1-\jmath)}C_7}{\Gamma^2(q(1-\jmath)+\jmath)}\Bigg\}\Bigg\{1 + \frac{36M^2b^{2\jmath}\|Z^{-1}\|^4\|B\|^2\|W^{-1}\|^2}{\jmath^2\Gamma^2(\jmath)}\Bigg\} + c_7 < 1.$$

Proof. Using the assumption $(H10)$, define the control

$$\begin{aligned}u(t) &= W^{-1}\{x_1 - Z^{-1}S_{q,\jmath}(t)[Zx_0 + G(0,x(0))] + Z^{-1}G(b, x(\vartheta_1(b)))\\
&\quad - \int_0^b Z^{-1}AZ^{-1}P_\jmath(b-s)G(s, x(\vartheta_1(s)))ds\\
&\quad - \int_0^b Z^{-1}P_\jmath(b-s)f(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau)ds\\
&\quad - \int_0^b Z^{-1}P_\jmath(b-s)\sigma(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau)d\omega(s)\}\\
&\qquad , \quad t\in(0,t_1],\\
u(t) &= W^{-1}\{x_1 - Z^{-1}S_{q,\jmath}(t-s_i)\xi_i(s_i, x(s_i)) + Z^{-1}G(b, x(\vartheta_1(b)))\\
&\quad - \int_{s_i}^b Z^{-1}AZ^{-1}P_\jmath(b-s)G(s, x(\vartheta_1(s)))ds\\
&\quad - \int_{s_i}^b Z^{-1}P_\jmath(b-s)f(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau)ds\\
&\quad - \int_{s_i}^b Z^{-1}P_\jmath(b-s)\sigma(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau)d\omega(s)\}\\
&\qquad , \quad t\in(s_i,t_{i+1}].\end{aligned}$$

Consider the operator Φ^* on Y defined as follows:

$$\begin{aligned}\Phi^*x(t) &= Z^{-1}S_{q,\jmath}(t)[Zx_0 + G(0,x(0))] - Z^{-1}G(t, x(\vartheta_1(t))\\
&\quad + \int_0^t Z^{-1}AZ^{-1}P_\jmath(t-s)G(s, x(\vartheta_1(s))ds + \int_0^t Z^{-1}P_\jmath(t-s)Bu(s)ds\\
&\quad + \int_0^t Z^{-1}P_\jmath(t-s)f\left(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau\right)ds\\
&\quad + \int_0^t Z^{-1}P_\jmath(t-s)\sigma(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau)d\omega(s), \qquad t\in(0,t_1],\\
\Phi^*x(t) &= \xi_i(t, x(t)), \quad t\in(t_i,s_i], \quad i = 1,2,\ldots,m,\\
\Phi^*x(t) &= Z^{-1}S_{q,\jmath}(t-s_i)\xi_i(s_i, x(s_i)) - Z^{-1}G(t, x(\vartheta_1(t))\end{aligned}$$

$$+\int_{s_i}^{t} Z^{-1}AZ^{-1}P_{\jmath}(t-s)G(s,x(\vartheta_1(s))ds + \int_{s_i}^{t} Z^{-1}P_{\jmath}(t-s)Bu(s)ds$$
$$+\int_{s_i}^{t} Z^{-1}P_{\jmath}(t-s)f\left(s,x(\vartheta_2(s)),\int_0^s h(s,\tau)g_1(\tau,x(\vartheta_3(\tau)))d\tau\right)ds$$
$$+\int_{s_i}^{t} Z^{-1}P_{\jmath}(t-s)\sigma(s,x(\vartheta_4(s)),\int_0^s h(s,\tau)g_2(\tau,x(\vartheta_5(\tau)))d\tau)d\omega(s), t\in(s_i,t_{i+1}],\ i=1,2,\ldots,m.$$

It will be shown that the operator Φ^* has a fixed point. This fixed point is then a mild solution of a system (4.1). For $x \in B_r$, we show that Φ^* maps B_r into itself. From Lemmas 2.5–2.7 together with Hölder inequality, yields for $t \in (0, t_1]$

$$\begin{aligned}
\|\Phi^* x\|_Y^2 \leq\ & 36\sup_{t\in J} t^{2(1-q)(1-\jmath)}\{E\|Z^{-1}S_{q,\jmath}(t)[Zx_0+G(0,x(0))]\|^2 + E\|Z^{-1}G(t,x(\vartheta_1(t))\|^2 \\
&+E\|\int_0^t Z^{-1}AZ^{-1}P_{\jmath}(t-s)G(s,x(\vartheta_1(s))ds\|^2 + E\|\int_0^t Z^{-1}P_{\jmath}(t-s)Bu(s)ds\|^2 \\
&+E\|\int_0^t Z^{-1}P_{\jmath}(t-s)f\left(s,x(\vartheta_2(s)),\int_0^s h(s,\tau)g_1(\tau,x(\vartheta_3(\tau)))d\tau\right)ds\|^2 \\
&+E\|\int_0^t Z^{-1}P_{\jmath}(t-s)\sigma\left(s,x(\vartheta_4(s)),\int_0^s h(s,\tau)g_2(\tau,x(\vartheta_5(\tau)))d\tau\right)d\omega(s)\|_Q^2
\end{aligned}$$

$$\begin{aligned}
\leq\ & \Bigg\{\frac{36M^2\|Z^{-1}\|^2}{\Gamma^2(q(1-\jmath)+\jmath)}[\|Z\|^2E\|x_0\|^2 + M_0^2(K_1E\|x_0\|^2+K_2)] \\
&+36\|Z^{-1}\|^2[rK_1+K_2][M_0^2 + \frac{M^2C_{1-\beta}^2 b^{2\beta\jmath+2(1-q)(1-\jmath)}\Gamma^2(\beta)}{\Gamma^2(1+\jmath\beta)\Gamma^2(\jmath)}] \\
&+\frac{36\|Z^{-1}\|^2M^2b^{2-2q(1-\jmath)}}{\jmath^2\Gamma^2(\jmath)}[(N_1(r+bC(rC_3+C_4))+N_2) \\
&+Tr(Q)\,(C_1(r+bC(rC_5+C_6))+C_2)]\Bigg\}\left[1+\frac{M^2b^{2\jmath}\|Z^{-1}\|^2\|B\|^2\|W^{-1}\|^2}{\jmath^2\Gamma^2(\jmath)}\right] \\
&+\frac{36M^2b^{2-2q(1-\jmath)}\|Z^{-1}\|^2\|B\|^2\|W^{-1}\|^2}{\jmath^2\Gamma^2(\jmath)}E\|x_1\|^2 \\
=\ & \left[\frac{\delta_3}{\Gamma^2(q(1-\jmath)+\jmath)} + \left(M_0^2 + \frac{M^2C_{1-\beta}^2 b^{2\beta\jmath+2(1-q)(1-\jmath)}\Gamma^2(\beta)}{\Gamma^2(1+\jmath\beta)\Gamma^2(\jmath)}\right)\delta_2 + \frac{49\|Z^{-1}\|^2M^2b^{2-2q(1-\jmath)}\delta_1}{\jmath^2\Gamma^2(\jmath)}\right] \\
&\times\left[1+\frac{M^2b^{2\jmath}\|Z^{-1}\|^2\|B\|^2\|W^{-1}\|^2}{\jmath^2\Gamma^2(\jmath)}\right] + \frac{49M^2b^{2-2q(1-\jmath)}\|Z^{-1}\|^2\|B\|^2\|W^{-1}\|^2E\|x_1\|^2}{\jmath^2\Gamma^2(\jmath)} \leq r.
\end{aligned}$$

for $t \in (t_i, s_i]$

$$\begin{aligned}
\|\Phi^* x\|_Y^2 \leq\ & \sup_{t\in J} t^{2(1-q)(1-\jmath)}E\|\xi_i(t,x(t))\|^2 \\
\leq\ & b^{2(1-q)(1-\jmath)}(rC_7+C_8) \\
\leq\ & r,
\end{aligned}$$

and for $t \in (s_i, t_{i+1}]$

$$\begin{aligned}
\|\Phi^* x\|_Y^2 \leq\ & 36 \sup_{t\in J} t^{2(1-q)(1-\jmath)} \{E\|Z^{-1}S_{q,\jmath}(t-s_i)\xi_i(s_i, x(s_i))\|^2 + E\|Z^{-1}G(t, x(\vartheta_1(t))\|^2 \\
& + E\|\int_{s_i}^t Z^{-1}AZ^{-1}P_\jmath(t-s)G(s, x(\vartheta_1(s))ds\|^2 + E\|\int_{s_i}^t Z^{-1}P_\jmath(t-s)Bu(s)ds\|^2 \\
& + E\|\int_{s_i}^t Z^{-1}P_\jmath(t-s) f\left(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau\right)ds\|^2 \\
& + E\|\int_{s_i}^t Z^{-1}P_\jmath(t-s)\sigma\left(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau\right)d\omega(s)\|_Q^2
\end{aligned}$$

$$\begin{aligned}
\leq\ & \left\{\frac{36M^2\|Z^{-1}\|^2}{\Gamma^2(q(1-\jmath)+\jmath)}(rC_7 + C_8) \right.\\
& + 36\|Z^{-1}\|^2[rK_1 + K_2][M_0^2 + \frac{M^2C_{1-\beta}^2 b^{2\beta\jmath+2(1-q)(1-\jmath)}\Gamma^2(\beta)}{\Gamma^2(1+\jmath\beta)\Gamma^2(\jmath)}] \\
& + \frac{36\|Z^{-1}\|^2M^2b^{2-2q(1-\jmath)}}{\jmath^2\Gamma^2(\jmath)}[(N_1(r + bC(rC_3 + C_4)) + N_2) \\
& \left. + Tr(Q)\,(C_1(r + bC(rC_5 + C_6)) + C_2)]\right\}\left[1 + \frac{M^2b^{2\jmath}\|Z^{-1}\|^2\|B\|^2\|W^{-1}\|^2}{\jmath^2\Gamma^2(\jmath)}\right] \\
& + \frac{36M^2b^{2-2q(1-\jmath)}\|Z^{-1}\|^2\|B\|^2\|W^{-1}\|^2}{\jmath^2\Gamma^2(\jmath)}E\|x_1\|^2 \\
=\ & \left[\frac{\delta_4}{\Gamma^2(q(1-\jmath)+\jmath)} + \left(M_0^2 + \frac{M^2C_{1-\beta}^2 b^{2\beta\jmath+2(1-q)(1-\jmath)}\Gamma^2(\beta)}{\Gamma^2(1+\jmath\beta)\Gamma^2(\jmath)}\right)\delta_2 + \frac{49\|Z^{-1}\|^2M^2b^{2-2q(1-\jmath)}\delta_1}{\jmath^2\Gamma^2(\jmath)}\right] \\
& \times\left[1 + \frac{M^2b^{2\jmath}\|Z^{-1}\|^2\|B\|^2\|W^{-1}\|^2}{\jmath^2\Gamma^2(\jmath)}\right] + \frac{49M^2b^{2-2q(1-\jmath)}\|Z^{-1}\|^2\|B\|^2\|W^{-1}\|^2E\|x_1\|^2}{\jmath^2\Gamma^2(\jmath)} \leq r.
\end{aligned}$$

Thus Φ^* maps B_r into itself.

We show that $(\Phi^* x)(t)$ is continuous on $[0, b]$ for any $x \in B_r$. Let $0 < t \leq b$ and $\epsilon > 0$ be sufficiently small, then,then for $t \in (0, t_1]$

$$\begin{aligned}
& \|(\Phi^* x)(\cdot + \epsilon) - (\Phi^* x)(\cdot)\|_Y^2 = \sup_{t\in J} t^{(1-q)(1-\jmath)}E\|(\Phi^* x)(t+\epsilon) - (\Phi^* x)(t)\|^2 \\
\leq\ & 6\sup_{t\in J} t^{(1-q)(1-\jmath)}E\|Z^{-1}(S_{q,\jmath}(t+\epsilon) - S_{q,\jmath}(t))[Zx_0 + G(0, x(0))]\|^2 \\
& + 6\sup_{t\in J} t^{(1-q)(1-\jmath)}E\|Z^{-1}G(t+\epsilon, x(\vartheta_1(t+\epsilon))) - Z^{-1}G(t, x(\vartheta_1(t)))\|^2 \\
& + 6\sup_{t\in J} t^{(1-q)(1-\jmath)}E\|\int_0^{t+\epsilon} Z^{-1}AZ^{-1}P_\jmath(t+\epsilon-s)G(s, x(\vartheta_1(s)))ds \\
& - \int_0^t Z^{-1}AZ^{-1}P_\jmath(t-s)G(s, x(\vartheta_1(s)))ds\|^2 \\
& + 6\sup_{t\in J} t^{(1-q)(1-\jmath)}E\|\int_0^{t+\epsilon} Z^{-1}P_\jmath(t+\epsilon-s)Bu(s)ds - \int_0^t Z^{-1}P_\jmath(t-s)Bu(s)ds\|^2
\end{aligned}$$

$$+6\sup_{t\in J} t^{(1-q)(1-j)}E\|\int_0^{t+\epsilon} Z^{-1}P_J(t+\epsilon-s)f\left(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau\right)ds$$
$$-\int_0^t Z^{-1}P_J(t-s)f\left(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau\right)ds\|^2$$
$$+6\sup_{t\in J} t^{(1-q)(1-j)}\|\int_0^{t+\epsilon} Z^{-1}P_J(t+\epsilon-s)\sigma\left(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau\right)d\omega(s)$$
$$-\int_0^t Z^{-1}P_J(t-s)\sigma\left(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau\right)d\omega(s)\|_Q^2, \tag{4.3}$$

for $t \in (t_i, s_i]$

$$\|(\Phi^* x)(\cdot+\epsilon) - (\Phi^* x)(\cdot)\|_Y^2 = \sup_{t\in J} t^{(1-q)(1-j)}E\|(\Phi^* x)(t+\epsilon) - (\Phi^* x)(t)\|^2$$
$$\leq \sup_{t\in J} t^{2(1-q)(1-j)}E\|\xi_i(t+\epsilon, x(t+\epsilon)) - \xi_i(t, x(t))\|^2 \tag{4.4}$$

and for $t \in (s_i, t_{i+1}]$

$$\|(\Phi^* x)(\cdot+\epsilon) - (\Phi^* x)(\cdot)\|_Y^2 = \sup_{t\in J} t^{(1-q)(1-j)}E\|(\Phi^* x)(t+\epsilon) - (\Phi^* x)(t)\|^2$$
$$\leq 6\sup_{t\in J} t^{(1-q)(1-j)}E\|Z^{-1}(S_{q,J}(t+\epsilon-s_i) - S_{q,J}(t-s_i))\xi_i(s_i, x(s_i))\|^2$$
$$+6\sup_{t\in J} t^{(1-q)(1-j)}E\|Z^{-1}G(t+\epsilon, x(\vartheta_1(t+\epsilon))) - Z^{-1}G(t, x(\vartheta_1(t)))\|^2$$
$$+6\sup_{t\in J} t^{(1-q)(1-j)}E\|\int_{s_i}^{t+\epsilon} Z^{-1}AZ^{-1}P_J(t+\epsilon-s)G(s, x(\vartheta_1(s)))ds$$
$$-\int_{s_i}^t Z^{-1}AZ^{-1}P_J(t-s)G(s, x(\vartheta_1(s)))ds\|^2$$
$$+6\sup_{t\in J} t^{(1-q)(1-j)}E\|\int_{s_i}^{t+\epsilon} Z^{-1}P_J(t+\epsilon-s)Bu(s)ds - \int_{s_i}^t Z^{-1}P_J(t-s)Bu(s)ds\|^2$$
$$+6\sup_{t\in J} t^{(1-q)(1-j)}E\|\int_{s_i}^{t+\epsilon} Z^{-1}P_J(t+\epsilon-s)f\left(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau\right)ds$$
$$-\int_{s_i}^t Z^{-1}P_J(t-s)f\left(s, x(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x(\vartheta_3(\tau)))d\tau\right)ds\|^2$$
$$+6\sup_{t\in J} t^{(1-q)(1-j)}\|\int_{s_i}^{t+\epsilon} Z^{-1}P_J(t+\epsilon-s)\sigma\left(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau\right)d\omega(s)$$
$$-\int_{s_i}^t Z^{-1}P_J(t-s)\sigma\left(s, x(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x(\vartheta_5(\tau)))d\tau\right)d\omega(s)\|_Q^2, \tag{4.5}$$

Clearly, the right hand sides of (4.3)–(4.5) are tends to zero as $\epsilon \to 0$. Hence, $(\Phi^* x)(t)$ is continuous on $[0, b]$.

Next for $x_1, x_2 \in B_r$, we show that Φ^* is a contraction mapping. From Lemmas 2.5–2.7 together with Hölder inequality, we obtain for $t \in (0, t_1]$

$$E\|(\Phi^* x_1)(t) - (\Phi^* x_2)(t)\|^2$$

$$\leq \ 36\|Z^{-1}\|^2\{\|(AZ^{-1})^{-\beta}\|^2 E\|(AZ^{-1})^{\beta}G(t,x_1(\vartheta_1(t))) - (AZ^{-1})^{\beta}G(t,x_2(\vartheta_1(t)))\|^2$$
$$+E\|\int_0^t (t-s)^{\jmath-1}(AZ^{-1})^{1-\beta}T_\jmath(t-s)[(AZ^{-1})^{\beta}G(s,x_1(\vartheta_1(s))) - (AZ^{-1})^{\beta}G(s,x_2(\vartheta_1(s)))]ds\|^2$$
$$+E\|\int_0^t P_\jmath(t-s)[f(s,x_1(\vartheta_2(s)),\int_0^s h(s,\tau)g_1(\tau,x_1(\vartheta_3(\tau)))d\tau)$$
$$-f(s,x_2(\vartheta_2(s)),\int_0^s h(s,\tau)g_1(\tau,x_2(\vartheta_3(\tau)))d\tau)]ds\|^2$$
$$+E\|\int_0^t P_\jmath(t-s)[\sigma(s,x_1(\vartheta_4(s)),\int_0^s h(s,\tau)g_2(\tau,x_1(\vartheta_5(\tau)))d\tau)$$
$$-\sigma(s,x_2(\vartheta_4(s)),\int_0^s h(s,\tau)g_2(\tau,x_2(\vartheta_5(\tau)))d\tau)]d\omega(s)\|^2\}$$
$$+36E\|\int_0^t Z^{-1}P_\jmath(t-s)BW^{-1}\{Z^{-1}(AZ^{-1})^{-\beta}[(AZ^{-1})^{\beta}G(b,x_1(\vartheta_1(b)))$$
$$-(AZ^{-1})^{\beta}G(b,x_2(\vartheta_1(b)))] - \int_0^b Z^{-1}(b-s)^{\jmath-1}(AZ^{-1})^{1-\beta}T_\jmath(b-s)[(AZ^{-1})^{\beta}G(s,x_1(\vartheta_1(s)))$$
$$-(AZ^{-1})^{\beta}G(s,x_2(\vartheta_1(s)))]ds - \int_0^b Z^{-1}P_\jmath(b-\tau)[f(\tau,x_1(\vartheta_2(\tau)),\int_0^\tau h(\tau,\eta)g_1(\eta,x_1(\vartheta_3(\eta)))d\eta)$$
$$-f(\tau,x_2(\vartheta_2(\tau)),\int_0^\tau h(\tau,\eta)g_1(\eta,x_2(\vartheta_3(\eta)))d\eta)]d\tau$$
$$-\int_0^b Z^{-1}P_\jmath(b-\tau)[\sigma(\tau,x_1(\vartheta_4(\tau)),\int_0^\tau h(\tau,\eta)g_2(\eta,x_1(\vartheta_5(\eta)))d\eta)$$
$$-\sigma(\tau,x_2(\vartheta_4(\tau)),\int_0^\tau h(\tau,\eta)g_2(\eta,x_2(\vartheta_5(\eta)))d\eta)d\omega(\tau)]\}ds\|^2$$
$$\leq \ 36\|Z^{-1}\|^2\Big\{\left[M_0^2 + \frac{M^2C_{1-\beta}^2 b^{2\beta\jmath}\Gamma^2(\beta)}{\Gamma^2(1+\jmath\beta)\Gamma^2(\jmath)}\right]qK_1 + \frac{M^2b^{2\jmath}q}{\jmath^2\Gamma^2(\jmath)}[N_1(1+bCC_3)+C_1Tr(Q)(1+bCC_5)]\Big\}$$
$$\times\Big\{1+\frac{49M^2b^{2\jmath}\|Z^{-1}\|^4\|B\|^2\|W^{-1}\|^2}{\jmath^2\Gamma^2(\jmath)}\Big\}E\|x_1(t)-x_2(t)\|^2, \tag{4.6}$$

for $t \in (t_i, s_i]$

$$E\|(\Phi^* x_1)(t) - (\Phi^* x_2)(t)\|^2$$
$$\leq \ E\|\xi_i(t,x_1(t)) - \xi_i(t,x_2(t))\|^2 \leq C_7\ E\|x_1(t)-x_2(t)\|^2, \tag{4.7}$$

and for $t \in (s_i, t_{i+1}]$

$$E\|(\Phi^* x_1)(t) - (\Phi^* x_2)(t)\|^2$$
$$\leq \ 36\|Z^{-1}\|^2\{E\|S_{q,J}(t-s_i)(\xi_i(s_i,x_1(s_i)) - \xi_i(s_i,x_2(s_i)))\|^2$$
$$+\|(AZ^{-1})^{-\beta}\|^2 E\|(AZ^{-1})^{\beta}G(t,x_1(\vartheta_1(t))) - (AZ^{-1})^{\beta}G(t,x_2(\vartheta_1(t)))\|^2$$
$$+E\|\int_{s_i}^t (t-s)^{\jmath-1}(AZ^{-1})^{1-\beta}T_\jmath(t-s)[(AZ^{-1})^{\beta}G(s,x_1(\vartheta_1(s))) - (AZ^{-1})^{\beta}G(s,x_2(\vartheta_1(s)))]ds\|^2$$
$$+E\|\int_{s_i}^t P_\jmath(t-s)[f(s,x_1(\vartheta_2(s)),\int_0^s h(s,\tau)g_1(\tau,x_1(\vartheta_3(\tau)))d\tau)$$

$$
\begin{aligned}
&-f(s, x_2(\vartheta_2(s)), \int_0^s h(s,\tau)g_1(\tau, x_2(\vartheta_3(\tau)))d\tau)]ds\|^2 \\
&+E\|\int_{s_i}^t P_J(t-s)[\sigma(s, x_1(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x_1(\vartheta_5(\tau)))d\tau) \\
&-\sigma(s, x_2(\vartheta_4(s)), \int_0^s h(s,\tau)g_2(\tau, x_2(\vartheta_5(\tau)))d\tau)]d\omega(s)\|^2\} \\
&+36E\|\int_{s_i}^t Z^{-1}P_J(t-s)BW^{-1}\{-Z^{-1}S_{q,J}(b-s_i)(\xi_i(s_i, x_1(s_i))-\xi_i(s_i, x_2(s_i))) \\
&+Z^{-1}(AZ^{-1})^{-\beta}[(AZ^{-1})^{\beta}G(b, x_1(\vartheta_1(b))) \\
&-(AZ^{-1})^{\beta}G(b, x_2(\vartheta_1(b)))] - \int_{s_i}^b Z^{-1}(b-s)^{J-1}(AZ^{-1})^{1-\beta}T_J(b-s)[(AZ^{-1})^{\beta}G(s, x_1(\vartheta_1(s))) \\
&-(AZ^{-1})^{\beta}G(s, x_2(\vartheta_1(s)))]ds - \int_{s_i}^b Z^{-1}P_J(b-\tau)[f(\tau, x_1(\vartheta_2(\tau)), \int_0^\tau h(\tau,\eta)g_1(\eta, x_1(\vartheta_3(\eta)))d\eta) \\
&-f(\tau, x_2(\vartheta_2(\tau)), \int_0^\tau h(\tau,\eta)g_1(\eta, x_2(\vartheta_3(\eta)))d\eta)]d\tau \\
&-\int_{s_i}^b Z^{-1}P_J(b-\tau)[\sigma(\tau, x_1(\vartheta_4(\tau)), \int_0^\tau h(\tau,\eta)g_2(\eta, x_1(\vartheta_5(\eta)))d\eta) \\
&-\sigma(\tau, x_2(\vartheta_4(\tau)), \int_0^\tau h(\tau,\eta)g_2(\eta, x_2(\vartheta_5(\eta)))d\eta)d\omega(\tau)]\}ds\|^2 \\
\leq\ & 36\|Z^{-1}\|^2\Big\{\Big[M_0^2 + \frac{M^2C_{1-\beta}^2 b^{2\beta J}\Gamma^2(\beta)}{\Gamma^2(1+J\beta)\Gamma^2(J)}\Big]qK_1 + \frac{M^2b^{2J}q}{J^2\Gamma^2(J)}[N_1(1+bCC_3)+C_1Tr(Q)(1+bCC_5)] \\
&+\frac{M^2b^{2(q-1)(1-J)}C_7}{\Gamma^2(q(1-J)+J)}\Big\}\Big\{1+\frac{36M^2b^{2J}\|Z^{-1}\|^4\|B\|^2\|W^{-1}\|^2}{J^2\Gamma^2(J)}\Big\}E\|x_1(t)-x_2(t)\|^2, \qquad (4.8)
\end{aligned}
$$

Therefore, by combining (4.6)–(4.8), we get

$$\sup_{t\in J} t^{2(1-q)(1-J)}E\|(\Phi^*x_1)(t)-(\Phi^*x_2)(t)\|^2 \leq \varrho \sup_{t\in J} t^{2(1-q)(1-J)}E\|x_1(t)-x_2(t)\|^2.$$

This implies that

$$\|\Phi^*x_1 - \Phi^*x_2\|_Y^2 \leq \varrho\|x_1-x_2\|_Y^2.$$

Then, Φ^* is a contraction mapping and hence there exists unique fixed point $x \in B_r$ such that $\Phi^*x(t) = x(t)$. Therefore the system (4.1) has a mild solution satisfying $x(b) = x_1$. Thus, system (4.1) is controllable on J.

5. Application

In this section, we present an example to illustrate our main result. Let us consider the following Sobolev-type Hilfer fractional stochastic partial differential equation with noninstantaneous impulsive condition

$$
\begin{cases}
D_{0+}^{\frac{1}{3},\frac{3}{5}}[x(t,y)-x_{yy}(t,y)+\tilde{G}(t,x(t-\rho_1,y))] = \frac{\partial^2 x(t,y)}{\partial y^2} + \upsilon(t,y) + x(t-\rho_2,y) \\
+\int_0^t \sin x(s-\rho_3,y)ds + x(t-\rho_4,y) + \int_0^t e^{x(s-\rho_5,y)} d\omega(s), \\
\ t \in (0,\frac{1}{5}]\bigcup(\frac{2}{5},\frac{3}{5}]\bigcup(\frac{4}{4},1],\ 0\le z\le \pi, \\
x(t,0)=x(t,\pi)=0,\ t\in(0,1], \\
x(t,y)=\frac{1}{5}e^{-(t-\frac{1}{5})}\frac{\|x(t,y)\|}{1+\|x(t,y)\|},\ \ t\in(\frac{1}{5},\frac{2}{5}],\ 0\le y\le\pi, \\
x(t,y)=\frac{3}{5}e^{-(t-\frac{3}{5})}\frac{\|x(t,y)\|}{1+\|x(t,y)\|},\ \ t\in(\frac{3}{5},\frac{4}{5}],\ 0\le y\le\pi, \\
I_{0+}^{\frac{4}{15}}(x(0,y)) = x_0(y),\ 0\le y\le\pi,
\end{cases}
\tag{5.1}
$$

where $D_{0+}^{\frac{1}{3},\frac{3}{5}}$ is the Hilfer fractional derivative of order $q=\frac{1}{3},\ \jmath=\frac{3}{5}$.

Let $X = U = L_2([0,\pi])$, define the operator $Z : D(Z) \subset X \to X$ and $A : D(A) \subset X \to X$ by $Zx = x - x_{yy}$ and $Ax = x_{yy}$ where domains $D(Z)$ and $D(A)$ are given by $\{x \in X : x,\ x_y$ are absolutely continuous, $x_{yy} \in X,\ x(0) = x(\pi) = 0\}$. Then A and Z can be written as

$$
\begin{aligned}
Ax &= -\sum_{n=1}^{\infty} n^2\langle x, x_n\rangle x_n,\ x\in D(A),\\
Zx &= \sum_{n=1}^{\infty}(1+n^2)\langle x, x_n\rangle x_n,\ x\in D(Z).
\end{aligned}
$$

Furthermore, for $x \in X$ we have

$$
\begin{aligned}
Z^{-1}x &= \sum_{n=1}^{\infty}\frac{1}{1+n^2}\langle x, x_n\rangle x_n,\\
AZ^{-1}x &= \sum_{n=1}^{\infty}\frac{-n^2}{1+n^2}\langle x, x_n\rangle x_n.
\end{aligned}
$$

It is known that AZ^{-1} is self-adjoin and has the eigenvalues $\lambda_n = -n^2\pi^2,\ n \in N$, with the corresponding normalized eigenvectors $e_n(\wp) = \sqrt{2}\ \sin(n\pi\wp)$. Furthermore, AZ^{-1} generates a uniformly strongly continuous semigroup of bounded linear operators $S(t), t \ge 0$, on X which is given by

$$
S(t)x = \sum_{n=1}^{\infty} e^{\frac{-n^2}{1+n^2}t}\langle x, x_n\rangle x_n,\ x\in X,
$$

with $\|S(t)\| \le e^{-t} \le 1$.
Moreover, the two operators $S_{\frac{1}{3},\frac{3}{5}}(t)$ and $P_{\frac{3}{5}}(t)$ can be defined by

$$
S_{\frac{1}{3},\frac{3}{5}}(t)x = \frac{3}{5\Gamma(\frac{2}{15})}\int_0^t\int_0^{\infty}\theta(t-s)^{\frac{-13}{15}}s^{\frac{-2}{5}}\Psi_{\frac{3}{5}}(\theta)T(s^{\frac{3}{5}}\theta)xd\theta ds,
$$

$$
P_{\frac{3}{5}}(t)x = \frac{3}{5}\int_0^{\infty}\theta t^{\frac{-2}{5}}\Psi_{\frac{3}{5}}(\theta)T(s^{\frac{3}{5}}\theta)xd\theta.
$$

Clearly,

$$\|P_{\frac{3}{5}}(t)\| \leq \frac{1}{\Gamma(\frac{3}{5})}, \quad \|S_{\frac{1}{3},\frac{3}{5}}(t)\| \leq \frac{1}{\Gamma(\frac{11}{15})}.$$

We define the bounded operator $B : U \to X$ by $B = I$.
Also, We define the following functions:

$$\begin{aligned}
x(t)y &= x(t, y), \\
\xi_1(t, x(t, y)) &= \frac{1}{5} e^{-(t-\frac{1}{5})} \frac{\|x(t, y)\|}{1 + \|x(t, y)\|}, \\
\xi_2(t, x(t, y)) &= \frac{3}{5} e^{-(t-\frac{3}{5})} \frac{\|x(t, y)\|}{1 + \|x(t, y)\|}, \\
G(t, x(\vartheta_1(t)))(y) &= \tilde{G}(t, x(t - \rho_1, y)), \\
\int_0^t h(t, s) g_1(s, x(\vartheta_3(s)))(y) ds &= \int_0^t \sin x(s - \rho_3, y) ds, \\
\int_0^t h(t, s) g_2(s, x(\vartheta_5(s)))(y) ds &= \int_0^t e^{x(s-\rho_5, y)} ds \\
f\left(t, x(\vartheta_2(t)), \int_0^t h(t, s) g_1(s, x(\vartheta_3(s))) ds\right)(y) &= x(t - \rho_2, y) + \int_0^t \sin x(s - \rho_3, y) ds, \\
\sigma\left(t, x(\vartheta_4(t)), \int_0^t h(t, s) g_2(s, x(\vartheta_5(s))) ds\right)(y) &= x(t - \rho_4, y) + \int_0^t e^{x(s-\rho_5, y)} ds,
\end{aligned}$$

where $h(t, s) = 1$.

Hence, with the above choices, system (5.1) can be rewritten in the abstract form of (4.1). On the other hand, all the hypotheses of Theorem 4 are satisfied and

$$\begin{aligned}
\varrho := 36\|Z^{-1}\|^2 \Bigg\{ &\left[M_0^2 + \frac{M^2 C_{1-\beta}^2 b^{2\beta\jmath} \Gamma^2(\beta)}{\Gamma^2(1 + \jmath\beta)\Gamma^2(\jmath)} \right] q K_1 + \frac{M^2 b^{2\jmath} q}{\jmath^2 \Gamma^2(\jmath)} [N_1(1 + bCC_3) + C_1 Tr(Q)(1 + bCC_5)] \\
&+ \frac{M^2 b^{2(q-1)(1-\jmath)} C_7}{\Gamma^2(q(1 - \jmath) + \jmath)} \Bigg\} \left\{ 1 + \frac{36 M^2 b^{2\jmath} \|Z^{-1}\|^4 \|B\|^2 \|W^{-1}\|^2}{\jmath^2 \Gamma^2(\jmath)} \right\} + c_7 < 1.
\end{aligned}$$

Thus, we can conclude that the Sobolev-type Hilfer fractional stochastic partial differential inclusions with noninstantaneous impulsive condition (5.1) is controllable on $(0, 1]$.

6. Conclusions

In this paper, we show that there is a moderate solution for nonlinear Hilfer fractional stochastic differential equations of Sobolev type with non-instantaneous impulsive in Hilbert space. For nonlinear Hilfer fractional stochastic differential equations of Sobolev type with non-instantaneous impulsive circumstances, we established suitable controllability criteria. To demonstrate the acquired results, an example is given.

Conflict of interest

The author declares that they have no competing interests.

References

1. X. Mao, *Stochastic differential equations and their applications*, Horwood, Chichester, 1997.
2. A. Atangana, S. I. Araz, *Fractional stochastic differential equations: Applications to Covid-19 modeling*, Springer Singapore, 2022. https://doi.org/10.1007/978-981-19-0729-6
3. F. A. Rihan, C. Rajivganthi, P. Muthukumar, Fractional stochastic differential equations with Hilfer fractional derivative: Poisson iumps and optimal control, *Discrete Dyn. Nat. Soc.*, **2017** (2017), 5394528. https://doi.org/10.1155/2017/5394528
4. H. Ahmad, N. Alam, M. Omri, New computational results for a prototype of an excitable system, *Results Phys.*, **28** (2021), 104666. https://doi.org/10.1016/j.rinp.2021.104666
5. M. Adel, Numerical simulations for the variable order two-dimensional reaction sub diffusion equation: Linear and nonlinear, *Fractals*, **30** (2022), 2240019. https://doi.org/10.1142/S0218348X22400199
6. M. Adel, M. Elsaid, An efficient approach for solving fractional variable order reaction sub-diffusion equation base on Hermite formula, *Fractals*, **30** (2022), 2240020. https://doi.org/10.1142/S0218348X22400205
7. M. M. Khader, J. F. Gómez-Aguilar, M. Adel, Numerical study for the fractional RL, RC, and RLC electrical circuits using Legendre pseudo-spectral method, *Int. J. Circ. Theor. App.*, **49** (2021), 3266–3285. https://doi.org/10.1002/cta.3103
8. M. Adel, H. M. Srivastava, M. M. Khader, Implementation of an accurate method for the analysis and simulation of electrical R-L circuits, *Math. Method. Appl. Sci.*, 2022. https://doi.org/10.1002/mma.8062
9. H. M. Ahmed, M. M. El-Borai, A. S. O. El Bab, M. E. Ramadan, Approximate controllability of noninstantaneous impulsive Hilfer fractional integrodifferential equations with fractional Brownian motion, *Bound Value Probl.*, **2020** (2020), 1–25. https://doi.org/10.1186/s13661-020-01418-0
10. J. P. Dauer, N. I. Mahmudov, Exact null controllability of semilinear integrodifferential systems in Hilbert spaces, *J. Math. Anal. Appl.*, **299** (2004), 322–332. https://doi.org/10.1016/j.jmaa.2004.01.050
11. H. M. Ahmed, M. M. El-Borai, A. S. O. El Bab, M. E. Ramadan, Boundary controllability of nonlocal Hilfer fractional stochastic differential systems with fractional Brownian motion and Poisson jumps, *Adv. Differ. Equ.*, **2019** (2019), 82. https://doi.org/10.1186/s13662-019-2028-1
12. P. Muthukumar, K. Thiagu, Existence of solutions and approximate controllability of fractional nonlocal neutral impulsive stochastic differential equations of order $1 < q < 2$ with infinite delay and Poisson jumps, *J. Dyn. Control Syst.*, **23** (2017), 213–235. https://doi.org/10.1007/s10883-015-9309-0

13. A. Chadha, S. N. Bora, Approximate controllability of impulsive neutral stochastic differential equations driven by Poisson jumps, *J. Dyn. Control Syst.*, **24** (2018), 101–128. https://doi.org/10.1007/s10883-016-9348-1
14. H. M. Ahmed, M. M. El-Borai, A. S. O. El Bab, M. E. Ramadan, Controllability and constrained controllability for nonlocal Hilfer fractional differential systems with Clarke's subdifferential, *J. Inequal. Appl.*, **2019** (2019), 233. https://doi.org/10.1186/s13660-019-2184-6
15. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, New York: John Wiley, 1993.
16. R. Hilfer, *Applications of fractional calculus in physics*, Singapore: World Scientific, 2000. https://doi.org/10.1142/3779
17. R. F. Curtain, H. Zwart, *An introduction to infinite dimensional linear systems theory*, New York: Springer, 1995. https://doi.org/10.1007/978-1-4612-4224-6
18. A. Pazy, Semigroups of linear operators and applications to partial differential equations, In: *Applied mathematical sciences*, New York: Springer, 1983. https://doi.org/10.1007/978-1-4612-5561-1
19. H. Gu, J. J. Trujillo, Existence of mild solution for evolution equation with Hilfer fractional derivative, *Appl. Math. Comput.*, **257** (2015), 344–354. https://doi.org/10.1016/j.amc.2014.10.083
20. I. Yaroslavtsev, Burkholder-Davis-Gundy inequalities in UMD banach spaces, *Commun. Math. Phys.*, **379** (2020), 417–459. https://doi.org/10.1007/s00220-020-03845-7