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Research article

Inventory model with nonlinear price-dependent demand for non-instantaneous decaying items via advance payment and installment facility

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Abstract: Determining the joint pricing and ordering policy is a challenging task for policy-makers dealing with perishable items. This research deals with the inventory coordination for a decaying commodity under a non-linear price-sensitive demand structure where the policy-maker completes the payment partially in advance, exploiting the multiple installments facility to control supply disruptions. Moreover, an inventory-out situation is incorporated to make the model more representative; shortages are backlogged partially through a variable rate in exponential form, depending on the customer waiting times. Though the formulated inventory coordination creates a highly complex optimization problem, the existence of the joint optimal pricing and ordering policy is explored by developing several theoretical outcomes. Three numerical illustrations are adopted to ensure the effectiveness of the model in providing the joint optimal pricing and ordering policy for the decision manager. Furthermore, to visualize the concavity of the average profit of the policy manager, as well as to demonstrate the adequacy of the optimum condition, MATLAB software was utilized. Finally, sensitivity studies for known key parameters are done using graphic presentation and a few supportive guidelines for the manager are also shown. The inventory manager should motivate the supplier to allow a higher installment frequency to implement the prepayment regulation, thus reducing the capital cost against the prepayment amount.

Keywords: EOQ; non-linear price-dependent demand; non-instantaneous decay items; advance payment; partial backlogged shortages
Mathematics Subject Classification: 90B05

1. Introduction

Inventory control is a critical activity for a company organization in the domain of inventory management. Various realistic assumptions about inventory control characteristics like client demand, degradation rate, preservation, backordering (partial or complete) and so on are essential for successful inventory management. In addition, numerous business schemes (for example, advance payment, trade credit and price discount) are necessary to construct an accurate and realistic inventory model.

Proper control of inventory is an essential task for a business organization. For proper inventory management, different realistic assumptions regarding customer demand, deterioration rate, preservation, stock-out scenario, etc., need to be properly adopted during inventory analysis. Furthermore, to ensure the applicability of inventory modeling in the practical competitive businesses, many practical business strategies (viz., advance payment, permissible delay in payment and price discount) are needed to be taken into account when developing the inventory model. In inventory modeling, client demand is quite crucial, as it controls the main characteristics for any type of product management scheme. Customer demand can be influenced by various factors, such as time, stock, product quality and price. Product price has a considerable effect on demand because the demand of a product always changes as the selling price of the product increases or decreases. Consequently, to introduce a product to the market, a producer must be aware of the client's needs, as well as the product's price.

The success or failure of any practitioner in business largely depends on how they manage the stock of items. This activity becomes indispensable, especially when the items are perishable, because deteriorated items do not bring any revenue, as they increase the deterioration cost/opportunity cost for the practitioner. Moreover, not all items (for example, vegetables, fish and fruits) perish from the moment of storage, but after some time from the moment of storage, so they are known as non-immediate/non-instantaneous perishable/decaying items in stock analysis. As a result, the ordering decision for perishable products ultimately measures the success or failure of any practitioner to a great extent. On the other hand, suppliers/manufacturers of perishable items are very aware of the orders of their customers (retailers), as any order cancellation can cause a dramatic loss to their business. Apparently, they impose a prepayment strategy for their customers (retailers) to control the order cancellations. Under this prepayment strategy, the practitioner pays the supplier/manufacturer a percentage of the total acquisition price in advance, and the rest of the acquisition price is paid at the time of receiving the lot. However, the practitioner incurs an opportunity cost against the prepayment amount because they cannot earn any revenue before receiving the goods. Consequently, the following research question is raised: Is there any effect of prepayment strategy on the practitioner's order policy for non-instantaneous decay items? Under this prepayment strategy, the inventory decisions for the perishable items become much more complicated when the client demand varies with respect to a variable selling price. To the best of the authors' knowledge, there is no research work on inventory control to help practitioners by providing the optimal order policy for non-instantaneous decay items under the conditions of prepayment strategies and non-linear price-sensitive client demand. With the ambition of proper stock management, this study, for the first time, investigates the consequences of a prepayment scheme on the practitioner's ordering decision for non-instantaneous decay items while the customer demand changes non-linearly with respect to selling price.

2. Literature review

2.1. Price-dependent demand

Realizing the consequences of selling price, several researchers in the stock management area have developed inventory models that take into account price-dependent demand rates. In general, consumer demand shows an upward trend when the unit selling price falls. Nagaraju et al. [20] developed a two-tier inventory system for price-dependent demand that uses both centralized and decentralized tactics to optimize lot size and inventory selections. Ghoreshi et al. [9] explored the impact of client demand for linear pricing on a practitioner's profit within a trade credit scheme for deteriorated items. Banerjee and Agrawal [2] investigated price-dependent demand and proposed joint discounting and ordering decisions for degraded products with a short shelf life. For deteriorating items, Chen et al. [3] developed an optimal pricing and replenishment method for when client demand varies due to the resultant impacts of stock amount, price and product storage time. Adopting both perfect and imperfect products from a production process, Ruidas et al. [28] formulated a production model focusing on price- and stock-related customer demand. Considering the impact of carbon emissions on the environment, Ruidas et al. [29] developed another production model for client demand that is related to linear pricing uses the selling price as a decision variable. Recently, Ruidas et al. [30,31] studied two other production models, adopting a price-sensitive customer demand. The non-linear price-related demand structures reflect the behavior of the clients more appropriately than the linear form. Pando et al. [21] considered the consequences of price in an exponential structure on client demand; they found the best pricing strategy for a retailer to maximize returns on stock management. By considering a non-linear price-sensitive demand structure instead of the linear form, San-José et al. [33] described the optimal pricing policy for a practitioner to minimize profit per unit time. To provide optimal pricing and inventory decisions as related to perishable items for practitioners, this study also investigated client demand structure as related to non-linear pricing.

2.2. Non-instantaneous decay items

During stock-in, every perishable product/commodity deteriorates or loses usefulness due to many factors, such as temperature, dryness, perishability, spoilage, vaporization and time. As a result, the degradation effects play a significant role in inventory control system analysis. Ghare and Schreder [8], to our knowledge, were the first to use this approach to design an inventory modeling problem. Covert and Philip [4] improved this method by including two-parameter Weibull distributed degradation. Subsequently, several scholars conducted several studies based on static or variable degradation. Khan et al. [15] obtained closed-form optimal inventory policies for decay items, with the deterioration rate adopted as constant. Replacing the constant degradation rate by a time-varying degradation rate, Khan et al. [13] explored the best pricing and inventory decisions for a retailer, assuming expiration dates for perishable items. However, not all items (for example, vegetables, fish and fruits) perish from the moment of storage, but after some time from the moment of storage; they are are known as non-immediate/non-instantaneous perishable/decaying items in stock analysis. Ghoreshi et al. [9] explored the impact of client demand for a non-instantaneous decay rate on ordering decisions within a trade credit scheme. Yadav et al. [38] developed an inventory model for deteriorating electronic components within warehousing by using a genetic algorithm. Khan et al. [14] further investigated the effects of delayed decay initiation in a two-warehouse storage environment based on different decay initiation moments in different warehouses. Recently, Khan et al. [16] described a stock management scheme for non-instantaneous decay items under the conditions of an advance payment business scheme. However, these aforementioned authors assumed a constant or linear price-sensitive client demand structure while formulating the model. This study explores the non-instantaneous decay feature for perishable items under the conditions of non-linear price-sensitive demand for the first time.

2.3. Backordering

Due to volatility in the business markets, sometimes practitioners cannot fulfill the clients' desires on time. As a result, practitioners encounter a stock-out situation. During this stock-out period, a portion of customers either waits for the arrival of new products or moves elsewhere to fulfill their desires. When all clients wait for the new products to arrive, the situation is called "full backordering", which is not compatible for many perishable items. In addition, when only a portion of the customers wait to have their desires fulfilled, the situation is called "partial backordering", which is more compatible with many practical business situations. For degrading products with time-varying demand and holding costs, Dutta and Kumar [6] described a partial backordering environment in an inventory procedure. Jaggi et al. [10] investigated the impact of inflation and the time value of money on an inventory management system by considering perishable goods and partially backlog shortages. Under the conditions of price-sensitive demand and inventory age-associated holding costs, Rastogi and Singh [25] established an inventory model for fluctuating degrading and partial backlog pharmaceutical products. Rahman et al. [23] established the concept of a preservation facility and created an inventory model to account for price-sensitive demand and partial backordering. Udayakumar et al. [36] investigated a partial backlog shortage scenario in a practitioner's stock management scheme, focusing on deteriorating products under constant demand.

2.4. Advance payment

For security of payment, advance payment is an essential method in inventory management. It is a business plan wherein a merchant orders a product by paying a part of the entire purchase price in equal installments before receiving the product, and the remaining amount is paid after receiving the product. An advance payment strategy ensures payment, as well as timely delivery of products. However, the practitioner incurs an opportunity cost against the prepayment amount because they cannot earn any revenue before receiving the goods. Zhang [40] presented this notion in his model with a set per-payment cost. By combining advance payment with delayed payment, Zia and Taleizadeh [42] created an inventory model to achieve the best ordering strategy. Zhang et al. [41] investigated a two-stage supply chain system with an advance payment method. Li et al. [17] proposed a hybrid advance, cash and delay payment strategy, looking at cash flow analysis, for a degrading inventory system. Taleizadeh [35] further investigated the impact of advance payment conditions on the inventory systems. In addition, Khan et al. [11] proposed a two-warehouse inventory model under the conditions of an advance payment scheme, whereas Shaikh et al. [34] studied an advance payment inventory model with interval-valued inventory costs. Furthermore, Manna et al. [18] explored the prepayment method in an interval environment to handle uncertainty. Khan et al. [15] proposed a flexible prepayment scheme in accordance with the purchased quantity. Alshanbari et al. [1] investigated an advance payment system for degrading goods under the conditions of a capital cost reduction facility utilizing multiple installments. Duary et al. [5] proposed an inventory model with advance payment for a discount facility with a deteriorating item. To improve the financing efficiency through the supply chain, Pfohl and Gomm [22], Gelsomino et al. [7] and Xu et al. [37] conducted separate systematic literature review studies on supply chain financing. Recently, Marchi et al. [19] studied reverse factoring as a technique to improve cash flow in a supply chain. In order to clarify the research gaps of a prepayment mechanism under stock management, Table 1 is presented.

2.5. Our contributions

Table 1 reveals that only two studies [14,16] investigated the effects of a prepayment business strategy on ordering decisions for non-instantaneous decay items given a constant or time-sensitive client demand. However, they did not take into account the impacts of selling price on the clients' behavior. To the authors' best knowledge, there is no research work on inventory control to help practitioners by providing optimal ordering policies for non-instantaneous decay items under the conditions of prepayment strategies and price-sensitive client demand. The fundamental contribution of this study is to fill this research gap by designing an inventory model for a non-instantaneous decay item with non-linear price-dependent demand given an advance payment business policy. For the first time, this work investigates the consequences of a prepayment scheme on the practitioner's ordering decision for non-instantaneous decay items given that the customer demand changes non-linearly with respect to selling price. Moreover, the backlog rate varies based on how long consumers have been waiting, whereas a multiple installment facility is adopted to reduce the capital cost in the prepayment amount. To maximize the average profit, the existence of the optimal replenishing and pricing policies for the practitioner are explored theoretically. In addition, MATLAB software has been used to provide the graphic illustration of the average profit's concavity.

Table 1. Comparative assessment of prepayment regulation under non-linear price dependent demand.

Authors	Demand		Payment method for acquisition cost		Deterioration		Shortage	Backordering			Objective function	
	Constant / others	Price-dependent in	advance	cash	Instantaneous	Non-instantaneous		No	Fully	Partially with a	Cost Minimization	Profit maximization
Ghoreshi et al. [20]		Linear form		√		√	√			Variable rate		√
Zia and Taleizadeh [42]	Constant		√				√			Fixed rate		√
Banerjee and Aggarwal [2]		Non-linear form		√		√	√	√				√
Li et al. [17]		Exponential form	√		√							√
Taleizadeh [35]	Constant		√				√			Fixed rate	√	
San-José et al. [32]		Logit form		√			√		√			√
Khan et al. [11]	Constant		√		√		√			Fixed rate	√	
Shaikh et al. [34]	Constant		√		√		√			Fixed rate	√	
Khan et al. [14]	Constant		√			√	√			Fixed rate	√	
Manna et al. [18]		Linear form	√		√		√			Variable rate		√
Khan et al. [15]	Constant		√							Fixed rate		√
Alshanbari et al. [1]		Linear form	√	√	√		√			Variable rate		√
San-Jose et al. [32]		Non-linear form		√								√
Rahman et al. [24]	Stock sensitive		√		√		√			Variable rate		√
Khan et al. [16]	Time sensetive		√			√	√			Variable rate	√	
Duary et al. [5]	Time sensetive		√		√		√			Variable rate		√
Rezagholifam et al. [26]	Inventory level sensitive	Linear form		√		√						√
This study		Non-linear form	√	√		√	√			Variable rate		√

Finally, the results of varying various parameters in the proposed model are shown, leading to a useful conclusion and guidelines for the practitioner.

The remainder of the study is delineated as follows. The problem definition, along with all of the necessary hypotheses and notations, is delivered in Section 3. Section 4 exposes the problem formulation of the described inventory procedure. All analytical outcomes are derived for the optimal polices in Section 5, while Section 6 delivers numerical illustrations to verify the described inventory procedure. Following the sensitivity analysis, several managerial insights are derived in Section 7. Finally, Section 8 points out the conclusions of the study, along with several future research lines.

3. Problem definition

Consider a business situation in which a practitioner who stocks a non-instantaneous decay item from a supplier to satisfy a portion of the future client desires given that the clients' behavior varies non-linearly with respect to selling price. Due to the perishable nature of the item, the supplier does not want to face cancellation or postponement of any order from the practitioner, as any order cancellation or postponement would cause a dramatic loss to their business. To prevent order cancellation or postponement, the supplier imposes a business scheme known as a prepayment mechanism. With this arrangement, the practitioner has to confirm the order by paying a percentage of the acquisition cost in advance, and the remaining percentage is to be paid at the time of receipt of the order. In addition, the practitioner incurs an opportunity cost against the prepayment amount because they cannot earn any revenue before receiving the goods. The main purpose of this study is to provide a complete decision framework for the practitioner by establishing optimal pricing and ordering decisions for a non-instantaneous decay item under a prepayment business contract to maximize the average profit.

The following suppositions and notations are used to build the inventory model.

3.1. Hypotheses

(i) Nonlinear price-dependent demand is taken into account while developing the model. More precisely, the mathematical form of the demand structure is taken as $D(p) = ap^{-b}$, where $a > 0$ and $b > 1$ [36]. The parameter a reveals the highest possible customer demand without any effect of price, and the parameter b measures the reduced customer demand after taking into account the consequences of price.

(ii) The deterioration starts after the t_d unit of time from the moment of storage, while the rate of deterioration $\theta (0 < \theta << 1)$ is constant during the deterioration occurring period. The deterioration starting moment, t_d, can be assessed from past experiences and by using the maximum likelihood technique. In this study, for simplicity, the value of t_d is assumed as a known constant such that $0 \le t_d \le t_1$, where t_1 denotes the time point when stock becomes zero level [36].

(iii) Any deteriorated product is neither replaceable nor reparable [20].

(iv) The horizon of inventory planning is unlimited and the lead time is M unit of time, where the value of M is known and offered by the supplier [16].

(v) The practitioner has to confirm the order by paying a percentage K of the acquisition cost in advance with the help of n equal installments, and the remaining percentage is to be paid at the time of receipt of the order [16].

(vi) A stock-out situation is encountered by the practitioner and a variable backlogging rate (in exponential form) is taken into consideration based on the customer waiting times. Mathematically, this rate is represented as $e^{-\delta x}$, where x denotes the waiting time of the consumers and $\delta > 0$ is the backlogging parameter [39].

3.2. Notation

Notation	Unit	Description
A	\$/order	Replenishment cost
a	Constant	Demand parameter ($a > 0$)
b	Constant	Demand parameter ($b > 0$)
C_p	\$/unit	Purchase cost
h	\$/unit/unit of time	Holding cost
C_s	\$/unit/unit of time	Shortage cost
C_d	\$/unit/unit of time	Deterioration cost
C_1	\$/unit/unit of time	Lost sale cost
$I(t)$	Units	Level of inventory any time t
θ	Constant	Deterioration rate $(0 < \theta << 1)$
δ	Constant	Backlogging parameter $(\delta > 0)$
M	Unit of time	Lead time
n	Constant	Installment number to prepay
K	Constant	Fraction of the acquisition price to prepay $(K \in [0,1])$
t_d	Unit of time	Time at which deterioration starts
R	Units	Backlogged units
S	Units	Maximum inventory level
$\hat{X}$	\$/cycle	Total cyclic cost
X	\$/cycle	Total cyclic profit
$TP(p,t_1,t_2)$	\$/unit of time	Total profit per unit of time
	Decision variables	
p	\$/unit	Pre-unit selling price
t_1	Unit of time	Time point when stock becomes zero level
t_2	Unit of time	Shortage period

4. Problem derivation

Initially, an order is placed by the retailer with $(S + R)$ units of a sole item by reimbursing a certain part K of the purchase cost with n equal several part payment facility within the receiving time M, and the products are received after the payment of all of the remaining amount of the purchase price at time $t = 0$. Then, R units are utilized instantly to satisfy the backlogged demand and the level of inventory becomes S (see Figure 1).

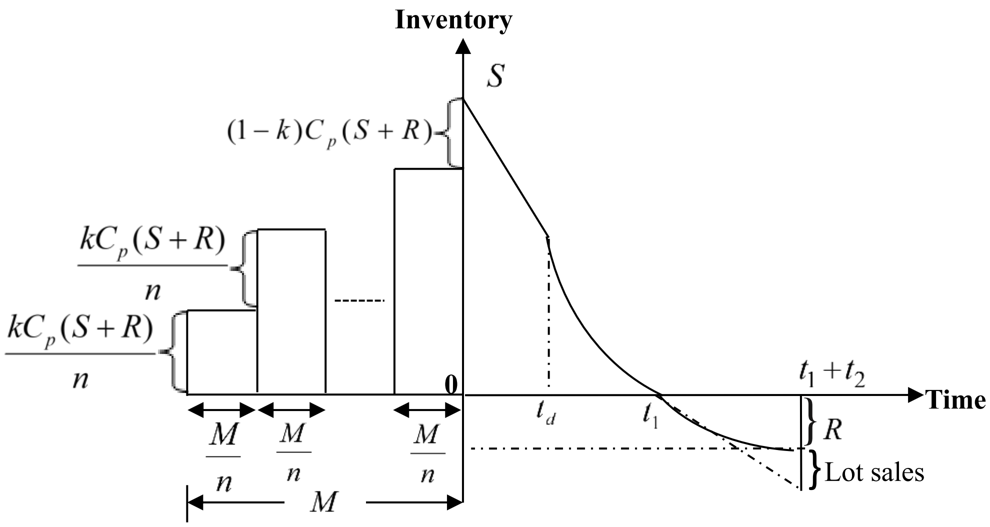

Figure 1. Graphic representation of inventory and paying mechanism.

The governing differential equations are as follows:

$$\frac{dI(t)}{dt} = -D(p), \qquad 0 < t \le t_d, \tag{1}$$

$$\frac{dI(t)}{dt} + \alpha I(t) = -D(p), \qquad t_d < t \le t_1, \tag{2}$$

$$\frac{dI(t)}{dt} = -D(p)e^{-\delta(t_1+t_2-t)}, \quad t_1 < t \le t_1 + t_2, \tag{3}$$

with the initial conditions $I(t) = S$ at $t = 0,$ $I(t) = 0$ at $t = t_1$ and $I(t) = -R$ at $t = t_1 + t_2$.

Using the initial condition of inventory level at $t = 0,$ from Eq (1), one has

$$I(t) = -D(p)t + S. \tag{4}$$

The inventory level for $t \in (t_d, t_1]$, given Eq (2) with $I(t) = 0$ at $t = t_1$, is

$$I(t) = \frac{D(p)}{\alpha}\left\{e^{\alpha(t_1-t)} - 1\right\}. \tag{5}$$

The shortage level for $t \in (t_1, t_1 + t_2]$, given Eq (3) with $I(t) = -R$ at $t = t_1 + t_2$, is

$$I(t)=\frac{D(p)}{\delta}\left\{1-e^{-\delta(t_1+t_2-t)}\right\}-R \quad (6)$$

The continuity property of the inventory level at $t=t_d$, as according to Eqs (4) and (5), provides

$$S=D(p)t_d+\frac{D(p)}{\alpha}\left\{e^{\alpha(t_1-t_d)}-1\right\} \quad (7)$$

Again, the continuity property of the inventory level at $t=t_1$, given Eqs (5) and (6), delivers

$$R=\frac{D(p)}{\delta}\left(1-e^{-\delta t_2}\right) \quad (8)$$

For a single business period, the practitioner bears the following costs:
Ordering cost $=A$:
Carrying cost of inventory

$$=h\int_0^{t_d}I(t)dt+h\int_{t_d}^{t_1}I(t)dt=h\left\{\frac{D(p)t_d^2}{2}+\frac{t_dD(p)}{\alpha}\left\{e^{\alpha(t_1-t_d)}-1\right\}\right\}+\frac{hD(p)}{\alpha^2}\left\{e^{\alpha(t_1-t_d)}-\alpha(t_1-t_d)-1\right\}.$$

Shortage cost

$$=-C_b\int_{t_1}^{t_1+t_2}I(t)dt=\frac{C_bD(p)}{\delta^2}\left(1-e^{-\delta t_2}-\delta t_2e^{-\delta t_2}\right).$$

Purchase cost

$$=C_P(S+R)=C_PD(p)\left[t_d+\frac{1}{\alpha}\left\{e^{\alpha(t_1-t_d)}-1\right\}+\frac{1}{\delta}\left(1-e^{-\delta t_2}\right)\right].$$

Capital cost

$$=I_C\left[(1+2+...+n)\frac{M}{n}\frac{KC_P(S+R)}{n}\right]$$

$$=I_CMKC_PD(p)\frac{(n+1)}{2n}\left[t_d+\frac{1}{\alpha}\left\{e^{\alpha(t_1-t_d)}-1\right\}+\frac{1}{\delta}\left(1-e^{-\delta t_2}\right)\right].$$

Lost sale cost

$$=C_1\int_{t_1}^{t_1+t_2}\left(1-e^{-\delta(t_1+t_2-t)}\right)Ddt=C_1D(p)\left\{t_2-\frac{1}{\delta}\left(1-e^{-\delta t_2}\right)\right\}.$$

As a result, the total cyclic cost for the practitioner is

$$\hat{X} = \begin{bmatrix} \langle \text{ordering cost} \rangle + \langle \text{carrying cost} \rangle + \langle \text{shortage cost} \rangle \\ + \langle \text{purchase cost} \rangle + \langle \text{capital cost} \rangle + \langle \text{lost sale cost} \rangle \end{bmatrix}$$

$$= A + h\left\{ \frac{D(p)t_d^2}{2} + \frac{t_d D(p)}{\alpha}\left\{ e^{\alpha(t_1 - t_d)} - 1 \right\} \right\} + \frac{hD(p)}{\alpha^2}\left\{ e^{\alpha(t_1 - t_d)} - \alpha(t_1 - t_d) - 1 \right\} + \frac{C_b D(p)}{\delta^2}\left(1 - e^{-\delta t_2} - \delta t_2 e^{-\delta t_2}\right)$$
$$+ \left\{ 1 + \frac{(n+1)}{2n} I_C MK \right\} C_P D(p)\left[t_d + \frac{1}{\alpha}\left\{ e^{\alpha(t_1 - t_d)} - 1 \right\} + \frac{1}{\delta}\left(1 - e^{-\delta t_2}\right) \right] + C_1 D(p)\left\{ t_2 - \frac{1}{\delta}\left(1 - e^{-\delta t_2}\right) \right\}.$$

The sales revenue for the practitioner within a single business period is

$$p\left[\int_0^{t_1} D(p)dt + \int_{t_1}^{t_1 + t_2} D(p)e^{-\delta(t_1 + t_2 - t)}dt \right] = pD(p)\left\{ t_1 + \frac{1}{\delta}\left(1 - e^{-\delta t_2}\right) \right\}.$$

Consequently, the profit for the practitioner within a single business period is

$$X = pD(p)\left\{ t_1 + \frac{1}{\delta}\left(1 - e^{-\delta t_2}\right) \right\} - A - h\left\{ \frac{D(p)t_d^2}{2} + \frac{t_d D(p)}{\alpha}\left\{ e^{\alpha(t_1 - t_d)} - 1 \right\} \right\}$$

$$- \frac{hD(p)}{\alpha^2}\left\{ e^{\alpha(t_1 - t_d)} - \alpha(t_1 - t_d) - 1 \right\} - \frac{C_b D(p)}{\delta^2}\left(1 - e^{-\delta t_2} - \delta t_2 e^{-\delta t_2}\right)$$

$$- \left\{ 1 + \frac{(n+1)}{2n} I_C MK \right\} C_P D(p)\left[t_d + \frac{1}{\alpha}\left\{ e^{\alpha(t_1 - t_d)} - 1 \right\} + \frac{1}{\delta}\left(1 - e^{-\delta t_2}\right) \right] - C_1 D(p)\left\{ t_2 - \frac{1}{\delta}\left(1 - e^{-\delta t_2}\right) \right\}.$$

Finally, the optimization problem for the entire system takes the form

$$\text{Maximize } TP(p, t_1, t_2) = \frac{X}{t_1 + t_2}, \tag{9}$$

which is subject to $0 \le t_d \le t_1,\, t_1 > 0$ and $t_2 > 0$.

5. Theoretical derivation

To compute the optimum solution of Problem (9), the first-order partial derivatives of $TP(p, t_1, t_2)$ with respect to p, t_1 and t_2 should be calculated as

$$\frac{\partial TP(p, t_1, t_2)}{\partial t_1} = -\frac{X}{(t_1 + t_2)^2} + \frac{D(p)}{(t_1 + t_2)}\begin{bmatrix} p - ht_d e^{\alpha(t_1 - t_d)} - \dfrac{h}{\alpha}\left\{ e^{\alpha(t_1 - t_d)} - 1 \right\} \\ -\left\{ 1 + \dfrac{(n+1)}{2n} I_C MK \right\} C_P e^{\alpha(t_1 - t_d)} \end{bmatrix}, \tag{10}$$

$$\frac{\partial TP(p,t_1,t_2)}{\partial t_2} = -\frac{X}{(t_1+t_2)^2} + \frac{D(p)}{(t_1+t_2)}\left[\begin{array}{l} pe^{-\delta t_2} - C_b t_2 e^{-\delta t_2} - C_1\left(1-e^{-\delta t_2}\right) \\ -\left\{1+\frac{(n+1)}{2n} I_C MK\right\} C_P e^{-\delta t_2} \end{array}\right] \tag{11}$$

and

$$\frac{\partial TP(p,t_1,t_2)}{\partial p}$$

$$= \frac{abp^{-(b+1)}}{(t_1+t_2)}\left[\begin{array}{l} p\left(\frac{1}{b}-1\right)\left\{t_1+\frac{1}{\delta}\left(1-e^{-\delta t_2}\right)\right\} + h\left\{\frac{t_d^2}{2}+\frac{t_d}{\alpha}\left\{e^{\alpha(t_1-t_d)}-1\right\}\right\} + \frac{h}{\alpha^2}\left\{e^{\alpha(t_1-t_d)}-\alpha(t_1-t_d)-1\right\} \\ +\frac{C_b}{\delta^2}\left(1-e^{-\delta t_2}-\delta t_2 e^{-\delta t_2}\right) + \left\{1+\frac{(n+1)}{2n} I_C MK\right\} C_P\left[t_d+\frac{1}{\alpha}\left\{e^{\alpha(t_1-t_d)}-1\right\}+\frac{1}{\delta}\left(1-e^{-\delta t_2}\right)\right] \\ +C_1\left\{t_2-\frac{1}{\delta}\left(1-e^{-\delta t_2}\right)\right\} \end{array}\right]. \tag{12}$$

The optimality is explored in two ways. First, the optimum replenishing policy is explored for a given $p > 0$. For this purpose, the necessary conditions from Eqs (10) and (11) are

$$D(p)\left(t_1+t_2\right)\left[p - ht_d e^{\alpha(t_1-t_d)} - \frac{h}{\alpha}\left\{e^{\alpha(t_1-t_d)}-1\right\} - \left\{1+\frac{(n+1)}{2n} I_C MK\right\} C_P e^{\alpha(t_1-t_d)}\right] = X \tag{13}$$

and

$$D(p)\left(t_1+t_2\right)\left[pe^{-\delta t_2} - C_b t_2 e^{-\delta t_2} - C_1\left(1-e^{-\delta t_2}\right) - \left\{1+\frac{(n+1)}{2n} I_C MK\right\} C_P e^{-\delta t_2}\right] = X \tag{14}$$

Since the right-hand sides of both Eqs (13) and (14) are the same, one can find

$$\begin{aligned} &C_b t_2 e^{-\delta t_2} + \left(p+C_1\right)\left(1-e^{-\delta t_2}\right) + \left\{1+\frac{(n+1)}{2n} I_C MK\right\} C_P e^{-\delta t_2} \\ &\quad = ht_d e^{\alpha(t_1-t_d)} + \frac{h}{\alpha}\left\{e^{\alpha(t_1-t_d)}-1\right\} + \left\{1+\frac{(n+1)}{2n} I_C MK\right\} C_P e^{\alpha(t_1-t_d)}. \end{aligned} \tag{15}$$

From Eq (14), one can write

$$D(p)(t_1+t_2)\left[pe^{-\delta t_2} - C_b t_2 e^{-\delta t_2} - C_1\left(1-e^{-\delta t_2}\right) - \left\{1+\frac{(n+1)}{2n} I_C MK\right\} C_P e^{-\delta t_2}\right] pD(p)\left\{t_1 + \frac{1}{\delta}\left(1-e^{-\delta t_2}\right)\right\} - A$$

$$-h\left\{\frac{D(p)t_d^2}{2} + \frac{t_d D(p)}{\alpha}\left\{e^{\alpha(t_1-t_d)}-1\right\}\right\}$$

$$-\frac{hD(p)}{\alpha^2}\left\{e^{\alpha(t_1-t_d)}-\alpha(t_1-t_d)-1\right\}\frac{-C_b D(p)}{\delta^2}\left(1-e^{-\delta t_2}-\delta t_2 e^{-\delta t_2}\right)$$

$$-\left\{1+\frac{(n+1)}{2n} I_C MK\right\} C_P D(p)\left[t_d + \frac{1}{\alpha}\left\{e^{\alpha(t_1-t_d)}-1\right\} + \frac{1}{\delta}\left(1-e^{-\delta t_2}\right)\right]$$

$$-C_1 D(p)\left\{t_2 - \tfrac{1}{\delta}\left(1 - e^{-\delta t_2}\right)\right\}. \tag{16}$$

The first-order derivative of the left-hand side of Eq (15) with respect to t_2 is

$$\left[C_b(1-\delta t_2) + (p + C_1)\delta - \left\{1 + \tfrac{(n+1)}{2n} I_C MK\right\} C_P \delta\right] e^{-\delta t_2}.$$

Consequently, the left-hand side of Eq (15) is continuous; it increases for all $t_2 \in [0, \bar{t}_2]$ and decreases for all $t_2 \in$, where

$$\bar{t}_2 = \frac{C_b + (p+C_1)\delta - \left\{1 + \frac{(n+1)}{2n} I_C MK\right\} C_P \delta}{C_b \delta}.$$

Consequently, the left-hand side of Eq (15) has a maximum at $\bar{t}_2$, and the maximum is

$$\left(p + C_1\right)\delta\left\{C_P + \frac{(n+1)}{2n} I_C MKC_P - p - C_1\right\} + \left(\frac{C_b}{\delta} + p + C_1\right) e^{-\frac{C_b + (p+C_1)\delta - \left\{1 + \frac{(n+1)}{2n} I_C MK\right\} C_P \delta}{C_b}}.$$

Regarding the right-hand side, the right-hand side of Eq (15) is increasing for all $t_1 \in [t_d, \infty)$ and tends to infinity when $t_1 \to \infty$. As a result, there exists a unique $\bar{t}_1 \in$ such that

$$ht_d e^{\alpha(\bar{t}_1 - t_d)} + \frac{h}{\alpha}\left\{e^{\alpha(\bar{t}_1 - t_d)} - 1\right\} + \left\{1 + \frac{(n+1)}{2n} I_C MK\right\} C_P e^{\alpha(\bar{t}_1 - t_d)} (p + C_1)\delta\left\{C_P + \frac{(n+1)}{2n} I_C MKC_P - p - C_1\right\}$$

$$+ \left(\frac{C_b}{\delta} + p + C_1\right) e^{\frac{-C_b + (p+C_1)\delta - \left\{1 + \frac{(n+1)}{2n} I_C MK\right\} C_P \delta}{C_b}}.$$

Moreover, there exists a unique $t_1' \in [t_d, \infty)$ for any $t_2' \in (0, \bar{t}_2)$ such that Eq (15) is maintained. Similarly, for any given $t_2' \in (\bar{t}_2, \infty)$, there exists a unique $t_1' \in [t_d, \infty)$ such that Eq (15) is maintained. As a result, t_1 can be computed in terms of t_2 uniquely, and the expression of t_1 in terms of t_2, given Eq (15), is

$$t_1 = t_d + \frac{1}{\alpha}\ln\left[\frac{C_b t_2 e^{-\delta t_2} + \left(p + C_1\right)\left(1 - e^{-\delta t_2}\right) + \left\{1 + \frac{(n+1)}{2n} I_C MK\right\} C_P e^{-\delta t_2} + \frac{h}{\alpha}}{ht_d + \frac{h}{\alpha} + \left\{1 + \frac{(n+1)}{2n} I_C MK\right\} C_P}\right]. \tag{17}$$

Let us define the auxiliary function from Eq (16) as

$$\Psi(t_2) = D(p)(t_1 + t_2)\left[pe^{-\delta t_2} - C_b t_2 e^{-\delta t_2} - C_1\left(1 - e^{-\delta t_2}\right) - \left\{1 + \frac{(n+1)}{2n} I_C MK\right\} C_P e^{-\delta t_2}\right]$$

$$- pD(p)\left\{t_1 + \frac{1}{\delta}\left(1 - e^{-\delta t_2}\right)\right\}$$

$$+ A + h\left\{\frac{D(p)t_d^2}{2} + \frac{t_d D(p)}{\alpha}\left\{e^{\alpha(t_1 - t_d)} - 1\right\}\right\} + \frac{hD(p)}{\alpha^2}\left\{e^{\alpha(t_1 - t_d)} - \alpha(t_1 - t_d) - 1\right\} + C_1 D(p)\left\{t_2 - \frac{1}{\delta}\left(1 - e^{-\delta t_2}\right)\right\}$$

$$+\frac{C_b D(p)}{\delta^2}\left(1-e^{-\delta t_2}-\delta t_2 e^{-\delta t_2}\right)+\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P D(p)\left[t_d+\frac{1}{\alpha}\left\{e^{\alpha(t_1-t_d)}-1\right\}+\frac{1}{\delta}\left(1-e^{-\delta t_2}\right)\right]. \qquad (18)$$

The first-order derivative of $\Psi(t_2)$ with respect to t_2 is

$$\begin{aligned}\frac{d\Psi(t_2)}{dt_2}&=D(p)\left(\frac{dt_1}{dt_2}+1\right)\left[pe^{-\delta t_2}-C_b t_2 e^{-\delta t_2}-C_1\left(1-e^{-\delta t_2}\right)-\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P e^{-\delta t_2}\right]\\&+D(p)\left(t_1+t_2\right)\left[-\delta pe^{-\delta t_2}-C_b\left(e^{-\delta t_2}-\delta t_2 e^{-\delta t_2}\right)-C_1\delta e^{-\delta t_2}+\delta\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P e^{-\delta t_2}\right]\\&-pD(p)\left\{\frac{dt_1}{dt_2}+e^{-\delta t_2}\right\}+ht_d D(p)e^{\alpha(t_1-t_d)}\frac{dt_1}{dt_2}+\frac{hD(p)}{\alpha}\left\{e^{\alpha(t_1-t_d)}-1\right\}\frac{dt_1}{dt_2}\\&+C_b D(p)t_2 e^{-\delta t_2}+\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P D(p)\left[e^{\alpha(t_1-t_d)}\frac{dt_1}{dt_2}+e^{-\delta t_2}\right]+C_1 D(p)\left\{1-e^{-\delta t_2}\right\},\end{aligned}$$

i.e.,

$$\frac{d\Psi(t_2)}{dt_2}=-D(p)\left(t_1+t_2\right)\left[C_b\left(1-\delta t_2\right)+\delta\left(p+C_1\right)-\delta\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P\right]e^{-\delta t_2}. \qquad (19)$$

Now, the existence of the optimal replenishing policy can be observed from the following theorem:

Theorem. For any $p>0$,

(a) if $\Psi(\mathring{t}_2)<0$, then the optimal point (t_1^*,t_2^*) is unique and maximizes $TP(t_1,t_2|p)$ given $t_2\in(0,\mathring{t}_2)$.

(b) if $\Psi(\mathring{t}_2)\geq 0$, the optimal value of t_2 is $t_2^*\rightarrow\infty$.

Proof. See Appendix.

For any $p>0$, Theorem (a) shows that, if $\Psi(\mathring{t}_2)<0$, then $TP(t_1,t_2|p)$ holds its global maximum value at the single point (t_1^*,t_2^*), where $t_2\in(0,\mathring{t}_2)$. Now, at the point (t_1^*,t_2^*), from Eq (14), one can find

$$TP(t_1^*,t_2^*|p)=D(p)\left[pe^{-\delta t_2^*}-C_b t_2 e^{-\delta t_2^*}-C_1\left(1-e^{-\delta t_2^*}\right)-\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P e^{-\delta t_2^*}\right].$$

Again, Theorem (b) shows that, if $\Psi(\mathring{t}_2)\geq 0$, then $t_2^*\rightarrow\infty$. Then, from Eq (17), one can compute

$$t_1^*=t_d+\frac{1}{\alpha}\ln\left[\frac{p+C_1+\frac{h}{\alpha}}{ht_d+\frac{h}{\alpha}+\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P}\right]$$

and $\lim_{t_2^*\rightarrow\infty}TP(t_1^*,t_2^*|p)=-C_1 D(p)$. This yields that the given $p>0$ provides a negative profit; hence, this $p>0$ is unstable and it should be improved.

Now, the unique optimal selling price per unit is computed after analyzing its existence. For any

$t_1^* > t_d$ and $t_2^* > 0$, taking the first-order derivative with respect to p and performing some simplifications, the necessary condition to compute p^* is

$$\frac{dTP(p|t_1^*,t_2^*)}{dp}=\left[D(p)+D'(p)\left\{p-C_P-C_P\frac{(n+1)}{2n}I_CMK\right\}\right]\left\{1+\frac{1-\delta t_2^*-e^{-\delta t_2^*}}{\delta\left(t_1^*+t_2^*\right)}\right\}$$

$$-\frac{D'(p)}{\left(t_1^*+t_2^*\right)}\begin{bmatrix}\frac{C_b}{\delta^2}\left(1-e^{-\delta t_2^*}-\delta t_2 e^{-\delta t_2^*}\right)-\frac{C_1}{\delta}\left(1-\delta t_2^*-e^{-\delta t_2^*}\right)\\ h\left\{\frac{t_d^2}{2}+\frac{t_d}{\alpha}\left\{e^{\alpha(t_1^*-t_d)}-1\right\}\right\}+\frac{h}{\alpha^2}\left\{e^{\alpha(t_1^*-t_d)}-\alpha(t_1^*-t_d)-1\right\}\\ +\left\{1+\frac{(n+1)}{2n}I_CMK\right\}C_P\left[t_d+\frac{1}{\alpha}\left\{e^{\alpha(t_1^*-t_d)}-1\right\}\right]-\left\{1+\frac{(n+1)}{2n}I_CMK\right\}C_Pt_1^*\end{bmatrix}=0 . \tag{20}$$

Since

$$\left\{1+\frac{(n+1)}{2n}I_CMK\right\}C_P\left[t_d+\frac{1}{\alpha}\left\{e^{\alpha(t_1^*-t_d)}-1\right\}\right]-\left\{1+\frac{(n+1)}{2n}I_CMK\right\}C_Pt_1^*$$

$$=\left\{1+\frac{(n+1)}{2n}I_CMK\right\}C_P\left[t_d+\frac{1}{\alpha}\left\{1+\alpha(t_1^*-t_d)+\frac{1}{2}\alpha^2(t_1^*-t_d)^2+...-1\right\}-t_1^*\right]>0 ,$$

$$1-e^{-\delta t_2^*}-\delta t_2 e^{-\delta t_2^*}>0,\quad 1-\delta t_2^*-e^{-\delta t_2^*}<0,\quad e^{\alpha(t_1^*-t_d)}-1$$

and

$$e^{\alpha(t_1^*-t_d)}-\alpha(t_1^*-t_d)-1>0 ,$$

the coefficient of $\frac{D'(p)}{(t_1^*+t_2^*)}$ is always positive. In addition, $D'(p)=-abp^{-(b+1)}$ is negative always. Consequently, the necessary condition (i.e., Eq (20)) is solvable for p if

$$D(p)+D'(p)\left\{p-C_P-C_P\frac{(n+1)}{2n}I_CMK\right\}<0 ,$$

that is,

$$p(1-b)+bC_P\left\{1+\frac{(n+1)}{2n}I_CMK\right\}<0$$

as

$$1+\frac{1-\delta t_2^*-e^{-\delta t_2^*}}{\delta\left(t_1^*+t_2^*\right)}=\frac{t_2^*}{\left(t_1^*+t_2^*\right)}+\frac{1-e^{-\delta t_2^*}}{\delta\left(t_1^*+t_2^*\right)}>0 .$$

Furthermore, if the per unit gross profit including the capital cost $D(p)\left\{p-C_P-C_P\frac{(n+1)}{2n}I_C MK\right\}$ is strictly concave in p, then its derivative $D(p)+D'(p)\left\{p-C_P-C_P\frac{(n+1)}{2n}I_C MK\right\}$ is strictly decreasing; hence,

$$2D'(p)+D''(p)\left\{p-C_P-C_P\frac{(n+1)}{2n}I_C MK\right\}<0.$$

As a result, the second-order derivative with respect to p is

$$\frac{d^2TP(p|t_1^*,t_2^*)}{dp^2}=\left\{2D'(p)+D''(p)\left[p-C_P\left\{1+\frac{(n+1)}{2n}I_C MK\right\}\right]\right\}\left\{1+\frac{1-\delta t_2^*-e^{-\delta t_2^*}}{\delta\left(t_1^*+t_2^*\right)}\right\}$$
$$-\frac{D''(p)}{\left(t_1^*+t_2^*\right)}\begin{bmatrix}\frac{C_b}{\delta^2}\left(1-e^{-\delta t_2^*}-\delta t_2 e^{-\delta t_2^*}\right)-\frac{C_1}{\delta}\left(1-\delta t_2^*-e^{-\delta t_2^*}\right)\\ h\left\{\frac{t_d^2}{2}+\frac{t_d}{\alpha}\left\{e^{\alpha(t_1^*-t_d)}-1\right\}\right\}+\frac{h}{\alpha^2}\left\{e^{\alpha(t_1^*-t_d)}-\alpha(t_1^*-t_d)-1\right\}\\ +\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P\left[t_d+\frac{1}{\alpha}\left\{e^{\alpha(t_1^*-t_d)}-1\right\}\right]-\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P t_1^*\end{bmatrix}<0. \quad (21)$$

Therefore, a single selling price p^* can be found which maximizes the total profit of the practitioner. In addition, the selling price p which satisfies the equation

$$p(1-b)+bC_P\left\{1+\frac{(n+1)}{2n}I_C MK\right\}=0$$

is the lower bound of the optimal selling price (say, p_l). As a result,

$$p_l=\frac{bC_P\left\{1+\frac{(n+1)}{2n}I_C MK\right\}}{b-1}.$$

Summarizing the theoretical outcomes, the following algorithm is established to achieve the optimal solution for the retailer.

Algorithm

Step 1:	Input the values of known parameters.
Step 2:	Start with setting $j=1$ and $p=p^{(j)}=\dfrac{bC_P\left\{1+\dfrac{(n+1)}{2n}I_CMK\right\}}{b-1}$, which is the solution of $p(1-b)+bC_P\left\{1+\dfrac{(n+1)}{2n}I_CMK\right\}=0$.
Step 3:	Put $p^{(j)}$ and $t_2=\frac{C_b+(p^{(j)}+C_1)\delta-\left\{1+\frac{(n+1)}{2n}I_CMK\right\}C_P\delta}{C_b\delta}$ into Eq (15) to compute the corresponding value of t_1, that is, t_1. Utilizing these values, given Eq (18), find the value of $\Psi(t_2)$.
Step 4:	If $\Psi(t_2)<0$, then go to Step 5; otherwise, set $p^{(j+1)}=p^{(j)}+\kappa$, where κ is a sufficiently small positive number, and go to Step 3 after updating j by $j+1$.
Step 5:	From Eq (18), find the value of t_2^* by solving $\Psi(t_2)=0$. Determine the corresponding t_1^* from Eq (15) and then go to Step 5.
Step 6:	Compute the value of p from Eq (20). Set $p^{(j+1)}=p$.
Step 7:	If $\left\|p^{(j+1)}-p^{(j+1)}\right\|$ is sufficiently small, go to Step 8; otherwise, set $j=j+1$ and go to Step 3.
Step 8:	Report the optimal solution: $p^*=p^{(j+1)}$, t_1^* and t_2^* with the maximum profit $TP\left(p^*,t_1^*,t_2^*\right)$.
Step 9:	End

In the next section, we illustrate the described model by solving three numerical examples.

6. Numerical illustration

Utilizing the solution algorithm in the previous section, three numerical examples were solved while LINGO 18 software was used to solve all the necessary equations during the implementation of the algorithm. In addition, MATLAB software was used to illustrate the concavity of the objective function by using the same numerical case to evaluate the applicability of the suggested model.

Example 1. Let $A=\$200$/order, $a=3500$, $b=1.5$, $C_p=\$30$/unit, $C_1=\$10$/unit, $C_h=\$1$/unit, $\alpha=0.05$, $\delta=0.4$, $M=0.25$ months, $n=20$, $I_c=0.01$/\$/month, $K=0.4$, $t_d=0.2$ month and $C_s=\$15$/unit. Therefore, adopting the algorithm, the optimal solutions for this example are $p^*=115.8991$, $t_1^*=6.5999$ months, $t_2^*=0.3964$ months, $S^*=21.7184$ units, $R^*=1.0284$ units, $Q^*=22.7468$ units and $TP(p^*,t_1^*,t_2^*)=\$187.2284$.

The concavity of the objective function for Example 1 is shown pairwise with respect to the independent decision variables (p , t_1 and t_2) in Figures 2–4.

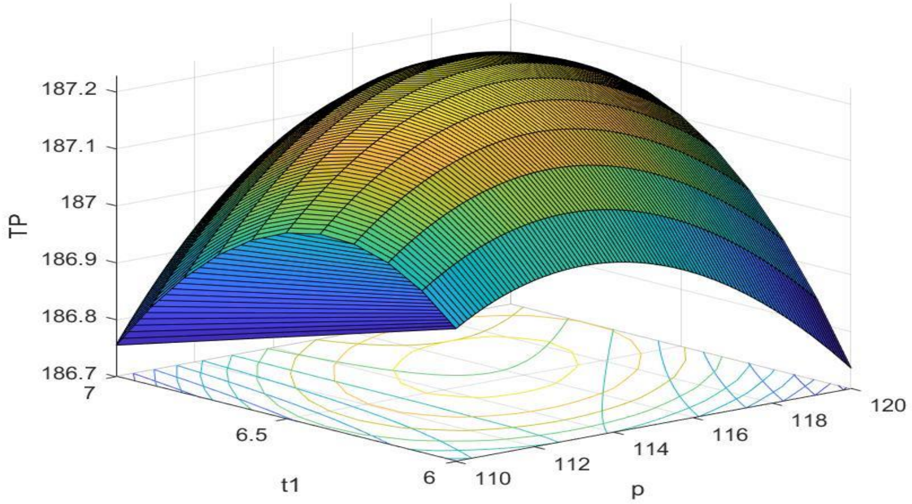

Figure 2. Concavity of the average profit function with respect to p and t_1.

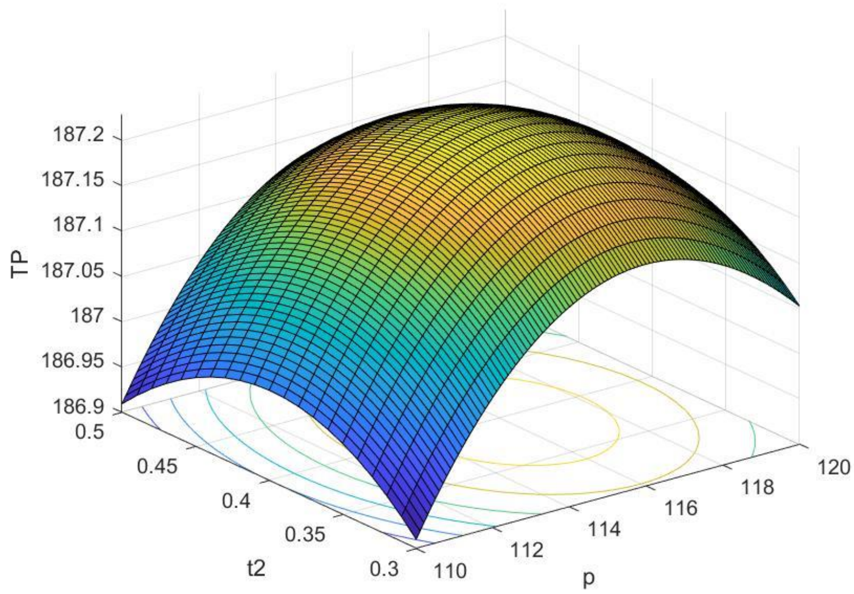

Figure 3. Concavity of the average profit function with respect to p and t_2.

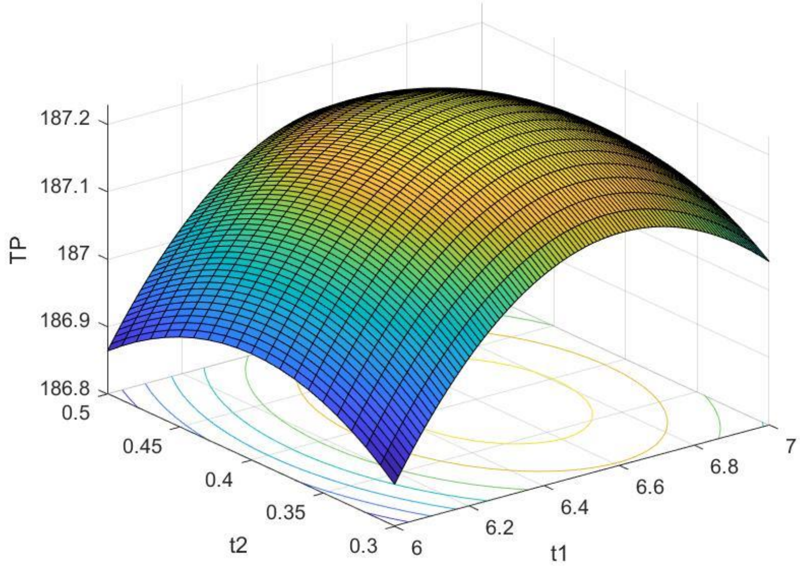

Figure 4. Concavity of the average profit function with respect to t_1 and t_2.

Example 2. Let $A=\$250$/order, $a=2500$, $b=1.2$, $C_p=\$35$/unit, $C_1=\$10$/unit, $C_h=\$1$/unit, $\alpha=0.05$, $\delta=0.4$, $M=0.5$ months, $n=20$, $I_c=0.01$/\$/month, $K=0.5$, $t_d=0.5$ month and $C_s=\$15$ /unit. Hence, adopting the algorithm, the optimal solutions for this example are $p^*=266.3658$, $t_1^*=6.8282$ months, $t_2^*=0.1985$ months, $S^*=24.4005$ units, $R^*=0.5862$ units, $Q^*=24.9867$ units and $TP(p^*,t_1^*,t_2^*)=\$645.4862$.

Example 3. Let $A=\$250$/order, $a=2200$, $b=1.4$, $C_p=\$55$/unit, $C_1=\$15$/unit, $C_h=\$1.5$ /unit, $\alpha=0.07$, $\delta=0.5$, $M=0.8$ months, $n=25$, $I_c=0.05$/\$/month, $K=0.45$, $t_d=0.4$ months and $C_s=\$17$/unit. Then, adopting the algorithm, the optimal solutions for this example are $p^*=283.5804$, $t_1^*=8.5979$ months, $t_2^*=0.5030$ months, $S^*=9.2970$ units, $R^*=0.3604$ units, $Q^*=9.6574$ units and $TP(p^*,t_1^*,t_2^*)=\$135.6230$.

7. Sensitivity analysis

This section deals with the variation patterns of optimal policies p^*,t_1^*,t_2^*,S^*,R^*, along with the profit per unit of time due to the changes of the original value of the key parameters in Example 1. This process is conducted by altering only a single parameter at a time from -20% to +20% and keeping the values of the rest of the parameters as the same. The corresponding computational results are illustrated graphically in Figures 5–8.

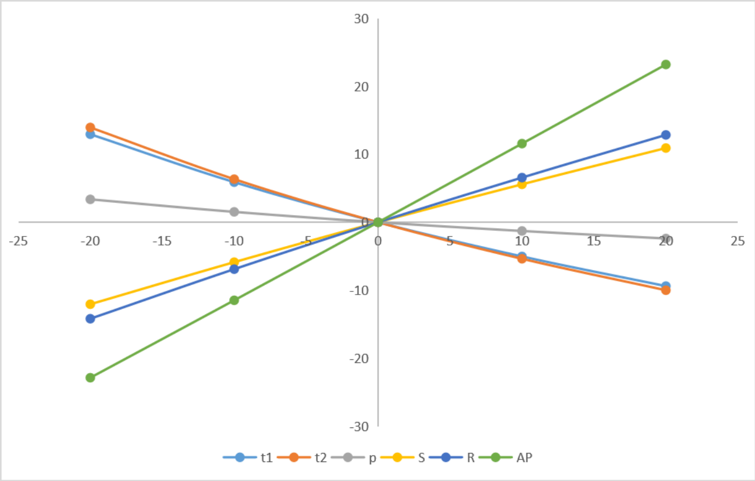

Figure 5. Percent change with respect to a.

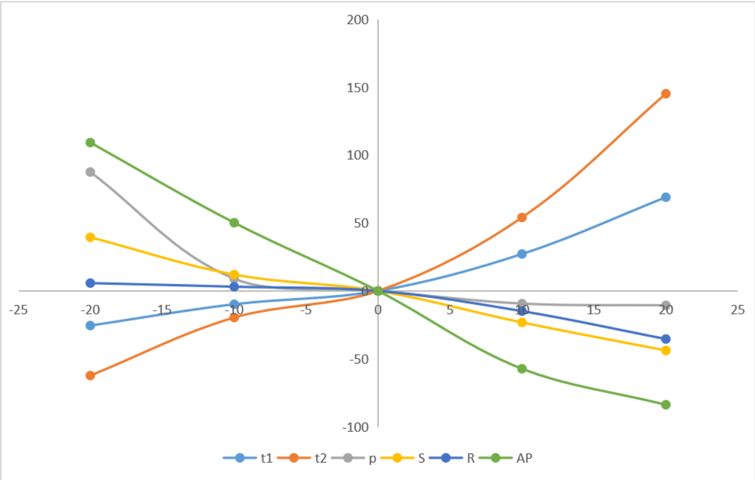

Figure 6. Percent change with respect to b.

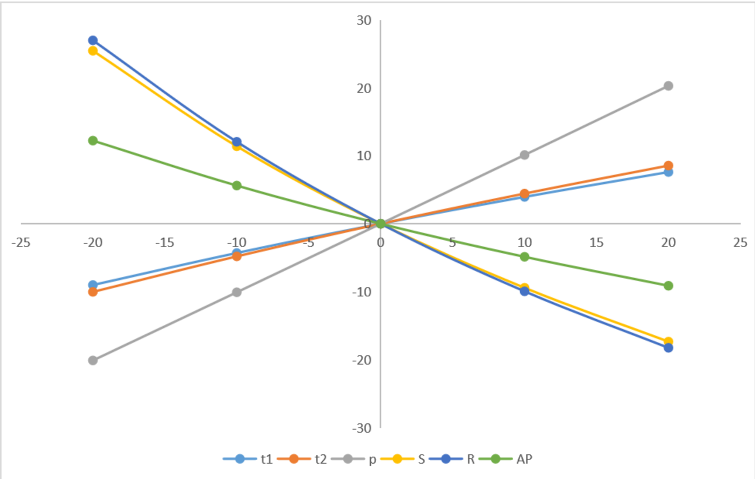

Figure 7. Percent change with respect to C_p.

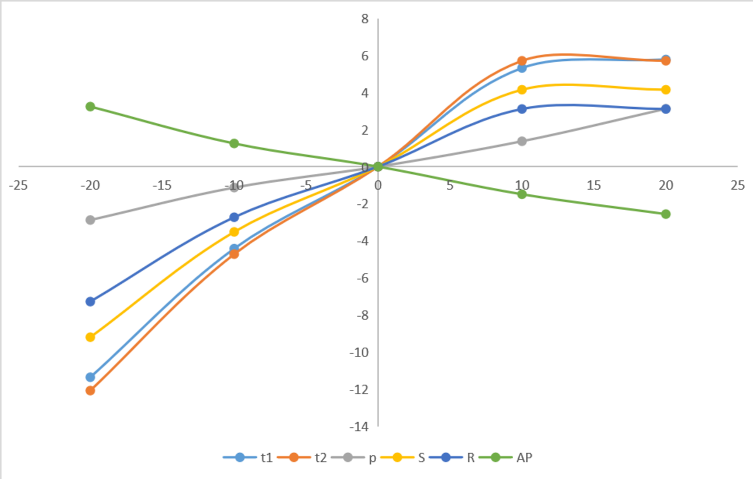

Figure 8. Percent change with respect to A.

From the sensitivity analysis in Figure 5, it is concluded that the profit per unit time (TP) is equally sensitive, whereas the positive stock period (t_1) and stock-out period (t_2) are relatively responsive and the initial stock (S) and highest shortage (R) are less responsive for changing the value of a. The optimal selling price for the retailer falls gradually when the demand parameter a increases. Though the optimal selling price decreases, the maximum profit per unit of time increases. This observation suggests that the decision manager should implement some effective marketing strategies (for example, attractive advertising campaigns) to increase potential market demand and then reduce the selling price slightly.

According to Figure 6, the profit per unit time (TP) and stock-out period t_2 are highly responsive, whereas the positive stock period (t_1), initial stock (S) and highest shortage level (R) are equally responsive in terms of altering the value of the price-sensitive parameter b. The variation of the price-sensitive parameter (b) reveals the dramatic consequences on the stock-out period. When the parameter b increases, the client demand decreases; hence, the manager stocks a relatively small amount. As a result, the stock-out period increases significantly. On the other hand, the profit of the retailer falls as client demand falls considerably when the parameter b increases.

Figure 7 exposes that profit per unit of time (TP) is equally sensitive, while both the initial stock (S) and maximum shortage level (R) are highly sensitive with respect to the variation of the per unit acquisition price (C_p). On the another hand, the positive stock period (t_1) and stock-out period (t_2) are relatively sensitive when C_p increases or decreases. The optimal selling price for the retailer increases when the per unit acquisition price increases. However, the profit per unit time decreases. In this case, increasing the per unit selling price does not ensure an increase of the profit. This outcome suggests that the decision manager should negotiate the per unit acquisition price with the supplier. Moreover, when the per unit acquisition price (C_p) rises, the selling price (p) also increases; hence, the client demand $D(p)$ falls considerably. As a result, both the initial stock (S) and maximum shortage level (R) falls when the per unit acquisition price increases.

Figure 8 indicates that the profit per unit of time (TP) has only negative consequences due to the variation of the ordering cost per order (A), while all of the rest (independent and dependent) of the decision variables have positive impacts with respect to the changes of A. When the ordering cost increases, the decision manager purchases more quantities so that the ordering cost against each unit falls and, therefore, the stock-in period (t_1) and entire cycle length increase. To face a higher ordering cost per order, the decision manager sets up a higher selling price for each unit. In this case, though the unit selling price increases, the profit per unit time decreases.

Taking into account the behavior with respect to the variations of the inventory parameters, the following management insights have been drawn for the inventory manager.

(a) As both the demand parameters (a and b) have the greatest consequences for the manager's profit per unit of time, the decision manager should implement some effective marketing strategies (for example, attractive advertising campaigns) to increase potential market demand and then reduce the selling price slightly.

(b) Though the optimal selling price for the retailer increases, the profit per unit time decreases when the per unit acquisition price increases. In this case, increasing the per unit selling price does not ensure an increase of the profit. This outcome suggests that the decision manager should negotiate the per unit acquisition price with the supplier.

(c) When the ordering cost for placing every order increases, the decision manager should purchase more quantities so that the ordering cost against each unit decreases.

(d) As

$$\frac{dTP}{dM}=-\left\{\frac{(n+1)}{2n}I_C K\right\}C_P D(p)\left[t_d+\frac{1}{\alpha}\left\{e^{\alpha(t_1-t_d)}-1\right\}+\frac{1}{\delta}\left(1-e^{-\delta t_2}\right)\right]<0,$$

the inventory manager's profit is strictly decreasing with respect to the allowed lead time from the supplier. When the lead time (M) increases, the inventory manger's capital cost or opportunity cost from the prepayment amount increases; hence, the profit per unit of time falls. This observation suggests that the decision manager should negotiate the lead time with the supplier.

(e) Since

$$\frac{dTP}{dK}=-\left\{\frac{(n+1)}{2n}I_C M\right\}C_P D(p)\left[t_d+\frac{1}{\alpha}\left\{e^{\alpha(t_1-t_d)}-1\right\}+\frac{1}{\delta}\left(1-e^{-\delta t_2}\right)\right]<0,$$

the profit per unit of time is strictly decreasing when the fraction of the acquisition price to prepay increases. When the fraction of the acquisition price to prepay (K) increases, the inventory manger's capital cost or opportunity cost from the prepayment amount increases; hence, the profit per unit of time decreases. Therefore, the decision manager should negotiate with the supplier to reduce the fraction of the acquisition price for prepayment.

(f) As

$$\frac{dTP}{dn}=\frac{I_C K C_P D(p)}{2n^2}\left[t_d+\frac{1}{\alpha}\left\{e^{\alpha(t_1-t_d)}-1\right\}+\frac{1}{\delta}\left(1-e^{-\delta t_2}\right)\right]>0,$$

the profit per unit of time shows an upward trend when the installment frequency increases. A higher installment number for prepayment helps the manager to reduce the capital cost or opportunity cost against the prepayment amount. Consequently, the inventory manger should motivate the supplier to allow a higher installment frequency to implement the prepayment regulation.

8. Conclusions

In this study, a stock control system was investigated for degrading products with non-linear price-dependent client demand and advanced payment regulations with a multiple installment plan given exponential partial backordering. To achieve the optimal joint pricing and replenishment schedule for the retailer, some important characteristics were explored. The nature of the decision variables in the formulated inventory problem is complex; hence, the optimal values of these variables cannot be obtained via a simple algebraic approach. As a result, the optimal pricing and replenishment scheduling are achieved by employing a numerical technique in an algorithmic manner. Three numerical examples were used to ensure the efficacy of the algorithm; the concavity of the average profit function was then visually shown pairwise with respect to the decision variables by using MATLAB software. To obtain management insights into optimal policies, the impacts of key parameters were explored by altering the values of these parameters in an inventory problem. Precisely, the decision manager should implement some effective marketing strategies (for example, attractive advertising campaigns) to increase potential market demand and then reduce the selling price slightly in order to increase the profit per unit of time. When the ordering cost for placing every order increases, the decision manager should purchase more quantities so that the ordering cost against each unit decreases. The inventory manager should motivate the supplier to allow a higher installment frequency to implement the prepayment regulation so that the capital cost or opportunity cost against the prepayment amount is reduced and, hence, the profit per unit of time is increased. Moreover, to increase the profit, the decision manager should negotiate with the supplier to reduce both the lead time and fraction of the acquisition price for accomplishing the prepayment.

This work considers the installment frequency as an input parameter to complete the prepayment without incurring any cost against each installment, which is a limitation of this study. Therefore, an interesting extension of this work would be to investigate the optimal installment frequency for the retailer.

No effect of product storage time on client demand was considered while developing the model. However, the behavior of clients depends not only on the unit selling price, but also on the storage period of the deteriorating products. Hence, another immediate future line of this study is to investigate the combined effects of price and time on the demand structure, either in additive or multiplicative form. It might also be an interesting expansion of a model without shortage to loosen the zero-ending situation by using a non-ending inventory model.

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Conflicts of interest

We have no conflicts with competing interests to disclose.

Appendix

Proof of the theorem.

(a) Since $C_b(1-\delta t_2)+\delta(p+C_1)-\delta\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P$ is positive for all $t_2 \in$ and negative for all $t_2 \in (\bar{t}_2,\infty)$, Eq (19) reveals that $\Psi(t_2)$ is strictly decreasing for all $t_2 \in$ and increasing for all $t_2 \in (\bar{t}_2,\infty)$. As a result, $\Psi(t_2)$ has a minimum at $t_2=\bar{t}_2$, and the minimum is $\Psi(\bar{t}_2)$. First, the interval $[0,\bar{t}_2]$ is considered. When $t_2=0$, Eq (17) shows that $t_1<t_d$, which contradicts the restriction $t_1 \geq t_d$. As a result, when $t_2=0$, $t_1=t_d$. Thus, one has

$$\Psi(0)=A+h\frac{D(p)t_d^2}{2}+\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P D(p)t_d>0.$$

Because $\Psi(t_2)$ is strictly decreasing for all $t_2 \in$, under the condition of $\Psi(\bar{t}_2)<0$, the intermediate value theorem ensures a unique solution $t_2 \in (0,\bar{t}_2)$ such that $\Psi(t_2^*)=0$. Now, exploiting

$$C_b\left(1-\delta t_2\right)+\left(p+C_1\right)\delta-\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P\delta>0$$

for all $t_2<\bar{t}_2$, one has

$$\left.\frac{\partial^2 TP(t_1,t_2|p)}{\partial t_1^2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)}=-\frac{D(p)}{\left(t_1^*+t_2^*\right)}\left[\alpha h t_d e^{\alpha(t_1^*-t_d)}+he^{\alpha(t_1^*-t_d)}\alpha\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P e^{\alpha(t_1^*-t_d)}\right]<0,$$

$$\left.\frac{\partial^2 TP(t_1,t_2|p)}{\partial t_2^2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)}=-\frac{D(p)}{\left(t_1^*+t_2^*\right)}\left[C_b\left(1-\delta t_2^*\right)+\left(p+C_1\right)\delta-\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P\delta\right]e^{-\delta t_2^*}<0.$$

And,

$$\left.\frac{\partial^2 TP(t_1,t_2|p)}{\partial t_1\partial t_2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)}=\left.\frac{\partial^2 TP(t_1,t_2|p)}{\partial t_2\partial t_1}\right|_{(t_1,t_2)=(t_1^*,t_2^*)}=0.$$

Therefore, the determinant of the Hessian matrix at (t_1^*,t_2^*) is

$$\frac{\{D(p)\}^2e^{-\delta t_2^*}}{(t_1^*+t_2^*)^2}\left[\alpha h t_d e^{\alpha(t_1^*-t_d)}+he^{\alpha(t_1^*-t_d)}\alpha\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P e^{\alpha(t_1^*-t_d)}\right]\begin{bmatrix}C_b(1-\delta t_2^*)+(p+C_1)\delta\\ -\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P\delta\end{bmatrix}>0.$$

Since the first and second principal minors of the Hessian matrix are negative and positive, respectively, (t_1^*,t_2^*) is the optimal solution.

Again, the interval is considered now. Because $\Psi(t_2)$ is increasing for all $t_2 \in (\bar{t}_2,\infty)$, there are two possible cases corresponding to the condition $\Psi(\bar{t}_2)<0$, namely, (i) there is no $t_2 \in (\bar{t}_2,\infty)$ such that $\Psi(t_2)=0$, and (ii) there is a single point $\hat{t}_2$ in the interval $(\bar{t}_2,\infty)$ such that $\Psi(\hat{t}_2)=0$. For the first possibility, the optimal solution for t_2 does not exist. On the another hand, for the second possibility, the determinant of the Hessian matrix at (t_1^*,t_2^*) becomes negative as

$$C_b(1-\delta t_2)+(p+C_1)\delta-\left\{1+\frac{(n+1)}{2n}I_C MK\right\}C_P\delta<0$$

for all $t_2 > \bar{t}_2$. As a result, the point (t_1^*, t_2^*) is not the optimal solution.

(b) Because $\Psi(t_2)$ has a global minimum at $t_2 = \bar{t}_2$, $\Psi(t_2)$ is always positive for all t_2 given the condition $\Psi(\bar{t}_2) > 0$. Now, exploiting Eqs (11) and (18), one has $\frac{\partial TP(t_1,t_2|p)}{\partial t_2} = \frac{\Psi(t_2)}{(t_1+t_2)^2} > 0$. This highlights that $TP(t_1, t_2|p)$ has a higher value for higher values of t_2. Therefore, $TP(t_1, t_2|p)$ achieves the maximum value at the point $t_2^* \to \infty$. Again, when $\Psi(\bar{t}_2) = 0$, $\left.\frac{\partial TP(t_1,t_2|p)}{\partial t_2}\right|_{t_2=\bar{t}_2} = 0$. Since $TP(t_1, t_2|p)$ is increasing in both the intervals $(0, \bar{t}_2)$ and $(\bar{t}_2, \infty)$ in this case, the point $t_2 = \bar{t}_2$ is an inflection point; hence, $TP(t_1, t_2|p)$ achieves the maximum value at the point $t_2^* \to \infty$.

This completes the proof.

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