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Research article

A new type of three dimensional metric spaces with applications to fractional differential equations

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Abstract: In this manuscript, we introduce a three dimension metric type spaces so called J-metric spaces. We prove the existence and uniqueness of a fixed point for self mappings in such spaces with different types of contractions. We use our result to prove the existence and uniqueness of a solution of the following fractional differential equations such as

$$(\mathcal{P}):\left\{\begin{array}{rcl} D^{\lambda}x(t) & = & f(t,x(t)) = Fx(t) \text{ if } t\in I_0=(0,T] \\ x(0) & = & x(T)=r \end{array}\right\}.$$

Moreover, we present other applications to systems of linear equations and Fredholm type integral equation.

Keywords: fixed point; J-metric space; fractional differential equation; system of linear equations
Mathematics Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

The importance of fixed point theory is due to its application in many fields, for example the existence and uniqueness of a solution of system of equations or fractional differential equations, integral equations. Also, it has many applications in economics, engineering and many more fields. The very first proof of existence and uniqueness of a fixed point was given by Banach [1] which was an inspiration to many researchers around the world to work in the field of fixed point theory See [2–10]. Generalizing Banach result is the focus of researchers now a days [11–22]. Lately, Beg et al. [23, 24] introduced the concept of S^{JS}-metric spaces, which is a three dimension metric space. However, given there defined triangle inequality, we do not see the point of the third component due to the fact that there is no use for it in their inequality, so basically it is a two dimension space. In this manuscript, we define J-metric spaces which are three dimension spaces where S^{JS}-metric spaces is a special case. Also, we present an application of our result to fractional differential equations along with an application to system of linear equations. First, we remind the reader of the definition of S^{JS}-metric spaces.

Definition 1.1. [23] Consider a nonempty set Ω and a function $J : \Omega^3 \to [0, \infty)$. Let us define the set

$$S(J, \Omega, \delta) = \{\{\delta_n\} \subset \Omega : \lim_{n\to\infty} J(\delta, \delta, \delta_n) = 0\}$$

for all $\delta \in \Omega$.

Definition 1.2. [23] Let Ω be a nonempty set and , $J : \Omega^3 \to [0, \infty)$ satisfy the following hypothesis:

(i) $J(\delta, \xi, \nu) = 0$ implies $\delta = \xi = \nu$ for any $\delta, \xi, \nu \in \Omega$;
(ii) There exists some $b > 0$ such that, for any $(\delta, \xi, \nu) \in \Omega^3$ and $\{\nu_n\} \in S(J, \Omega, \nu)$,

$$J(\delta, \xi, \nu) \leq b \limsup_{n\to\infty}(J(\delta, \delta, \nu_n) + J(\xi, \xi, \nu_n))$$

Then the pair (Ω, J) is called an S^{JS}-metric space.

Moreover, if J also satisfies $J(\delta, \delta, \xi) = J(\xi, \xi, \delta)$ for all $\delta, \xi \in \Omega$, then we call it a symmetric S^{JS}-metric space.

2. Main result

In this section, we introduce the notion of $J-$metric spaces, and prove fixed point theorems for self mappings in this new space.

Definition 2.1. Consider a nonempty set Ω and a function $J : \Omega^3 \to [0, \infty)$. Let us define the set

$$S(J, \Omega, \nu) = \{\{\nu_n\} \subset \Omega : \lim_{n\to\infty} J(\nu, \nu, \nu_n) = 0\}$$

for all $\nu \in \Omega$

Definition 2.2. Let Ω be a nonempty set and , $J : \Omega^3 \to [0, \infty)$ satisfy the following hypothesis:

(i) $J(\tau, \upsilon, \zeta) = 0$ implies $\tau = \upsilon = \zeta$ for any $\tau, \upsilon, \zeta \in \Omega$;

(ii) There exists some $b > 0$ such that, for any $(\tau, \upsilon, \zeta) \in \Omega^3$ and $\{\nu_n\} \in S(J, \Omega, \nu)$,

$$J(\tau, \upsilon, \zeta) \le b \limsup_{n\to\infty} \Big(J(\tau, \tau, \nu_n) + J(\upsilon, \upsilon, \nu_n) + J(\zeta, \zeta, \nu_n) \Big).$$

Then the pair (Ω, J) is called a J-metric space. Moreover, if $J(\tau, \tau, \upsilon) = J(\upsilon, \upsilon, \tau)$ for all $\tau, \upsilon \in \Omega$, then the pair (Ω, J) is called a symmetric J-metric space.

Remark 2.3. Note that, the following condition is not necessary true

$$J(x, y, z) = J(y, x, z) = J(z, y, x) = \cdots$$

Now, we present some of the topological properties of $J-$metric spaces.

Definition 2.4. Let (Ω, J) be an J-metric space. A sequence $\{\tau_n\} \subset \Omega$ is said to be convergent to an element $\tau \in \Omega$ if $\{\tau_n\} \in S(J, \Omega, \tau)$.

Proposition 2.5. *In a J-metric space (Ω, J), if $\{\tau_n\}$ converges to both τ_1 and τ_2, then $\tau_1 = \tau_2$.*

Proof. Assume that $\{\tau_n\}$ converges to both τ_1 and τ_2. Hence,

$$J(\tau_1, \tau_1, \tau_2) \le b \limsup_{n\to\infty} \Big(J(\tau_1, \tau_1, \tau_n) + J(\tau_1, \tau_1, \tau_n) + J(\tau_1, \tau_2, \tau_n) \Big),$$
$$b \limsup_{n\to\infty} \Big(2J(\tau_1, \tau_1, \tau_n) + J(\tau_2, \tau_2, \tau_n) \Big) = 0.$$

Thus,

$$J(\tau_1, \tau_1, \tau_2) = 0 \Rightarrow \tau_1 = \tau_2.$$

□

Definition 2.6. Let (Ω, J_1) and (Γ, J_2) be two J-metric spaces and $\sigma : \Omega \to \Gamma$ be a mapping. Then σ is called continuous at $a_0 \in \Omega$ if, for any $\varepsilon > 0$, there exists $\xi > 0$ such that, for any $\tau \in \Omega$, $J_2(\sigma a_0, \sigma a_0, \sigma\tau) < \varepsilon$ whenever $J_1(a_0, a_0, \tau) < \xi$.

Definition 2.7. (1) Let (Ω, J) be a J-metric space. A sequence $\tau_n \subset \Omega$ is said to be Cauchy if $\lim_{n,m\to\infty} J(\tau_n, \tau_n, \tau_m) = 0$.

(2) A J-metric space is said to be complete if every Cauchy sequence in Ω is convergent.

(3) In a J-metric space (Ω, J) if σ is continuous at $a_0 \in \Omega$ then for any sequence $\tau_n \in S(J, \Omega, a_0)$ implies $\{\sigma\tau_n\} \in S(J, \Omega, \sigma a_0)$.

Remark 2.8. Note that $S(J, \Omega, \delta)$ in some cases can be empty. The following example presents a nonempty set of $S(J, \Omega, \delta)$.

Example 2.9. Let $\Omega = \mathbb{R}$ and, $J : \Omega^3 \to [0, \infty)$ defined by $J(\delta, \xi, \nu) = |\delta - \xi| + |\xi - \nu|$ for all $\delta, \xi, \nu \in \mathbb{R}$. Let $\nu \in \mathbb{R}$ and the sequence (ν_n) such that $\nu_n = \nu + \frac{1}{n}$.

It is easy to see that $\lim_{n\to\infty} J(\nu, \nu + \frac{1}{n}, \nu + \frac{1}{n}) = 0$. Therefore, for every $\nu \in \mathbb{R}$ there exists a sequence $\nu_n = \nu + \frac{1}{n}$ such that $S(J, \Omega, \nu) \neq \emptyset$.

Next, we present two examples of J-metric spaces.

Example 2.10. Let $X = \mathbb{R}$ and $J(\tau, \upsilon, \zeta) = |\tau| + |\upsilon| + 2|\zeta|$ for all $\tau, \upsilon, \zeta \in X$.

We have $J(\tau, \upsilon, \zeta) = 0$ imply that $|\tau| + |\upsilon| + 2|\zeta| = 0$ which gives us $|\tau| = |\upsilon| = |\zeta| = 0$ then the first condition of the Definition 2.2 is satisfied. Also the symmetry of J is satisfied since we have $J(\tau, \tau, \upsilon) = 2|\tau| + 2|\upsilon| = J(\upsilon, \upsilon, \tau)$. Now, let's verfiy the triangle inequality. Let $\tau, \upsilon, \zeta \in X$ and ν_n a convergent sequence in X such that $\lim_{n\to\infty} J(\nu, \nu, \nu_n) = 0$, we have

$$\begin{aligned}
J(\tau, \upsilon, \zeta) &= |\tau| + |\upsilon| + 2|\zeta| \\
&\leq 4|\tau| + 4|\upsilon| + 4|\zeta| + 12|\nu_n| \\
&= 2(2|\tau| + 2|\nu_n| + 2|\upsilon| + 2|\nu_n| + 2|\zeta| + 2|\nu_n|) \\
&= 2(J(\tau, \tau, \nu_n) + J(\upsilon, \upsilon, \nu_n) + J(\zeta, \zeta, \nu_n)) \\
&\leq 2 \limsup_{n\to\infty} (J(\tau, \tau, \nu_n) + J(\upsilon, \upsilon, \nu_n) + J(\zeta, \zeta, \nu_n)).
\end{aligned}$$

Then, all the assumptions of Definition 2.2 are satisfied. Hence, J is a J-metric with $b = 2$.

Example 2.11. Let $X = [0, \infty)$ and $J(\tau, \upsilon, \zeta) = |\tau - \upsilon| + |\tau - \zeta|$ for all $\tau, \upsilon, \zeta \in X$. We have

- $J(\tau, \upsilon, \zeta) = 0$ imply that $|\tau| = |\upsilon| = |\zeta| = 0$.
- $J(\tau, \tau, \upsilon) = |\tau - \tau| + |\tau - \upsilon| = |\upsilon - \tau| = J(\upsilon, \upsilon, \tau)$.
- Let ν_n a convergent sequence in X such that $\lim_{n\to\infty} J(\nu, \nu, \nu_n) = 0$, we have

$$\begin{aligned}
J(\tau, \upsilon, \zeta) &= |\tau - \upsilon| + |\tau - \zeta| \\
&\leq |\tau - \nu_n + \nu_n - \upsilon| + |\tau - \nu_n + \nu_n - \zeta| \\
&\leq |\tau - \nu_n| + |\upsilon - \nu_n| + |\zeta - \nu_n| + |\tau - \nu_n| \\
&= 2|\tau - \nu_n| + |\upsilon - \nu_n| + |\zeta - \nu_n| \\
&\leq 2 \limsup_{n\to\infty} (|\tau - \nu_n| + |\upsilon - \nu_n| + |\zeta - \nu_n|) \\
&\leq 2 \limsup_{n\to\infty} (J(\tau, \tau, \nu_n) + J(\upsilon, \upsilon, \nu_n) + J(\zeta, \zeta, t_n)).
\end{aligned}$$

Then, all the assumptions of Definition 2.2 are satisfied. Hence, J is a J-metric with $b = 2$.

Theorem 2.12. *Let (Ω, J) be a J-complete symmetric metric space and $\sigma : \Omega \to \Omega$ be a continuous mapping satisfying*

$$J(\sigma\tau, \sigma\upsilon, \sigma\zeta) \leq \psi(J(\tau, \upsilon, \zeta)) \quad \textit{for all } \tau, \upsilon, \zeta \in \Omega \tag{2.1}$$

where $\psi : [0, +\infty) \to [0, +\infty)$ is an increasing function such that, $\lim_{n\to\infty} \psi^n(t) = 0$ for each fixed $t > 0$. Then σ has a unique fixed point in Ω.

Proof. Let τ_0 be an arbitrary element in Ω. We define the sequence $\{\tau_n\}_{n\geq 0} \subset \Omega$ as follows

$$\tau_n = \sigma\tau_{n-1} = \sigma^n\tau_0, \quad n = 1, 2, ... \tag{2.2}$$

We will prove that $\{\tau_n\}$ is a Cauchy sequence in Ω. Let $n, m \in \mathbb{N}$, using (2.1) we obtain

$$\begin{aligned}
J(\tau_n, \tau_n, \tau_m) &\leq \psi(J(\tau_{n-1}, \tau_{n-1}, \tau_{m-1})) \\
&\leq \psi(J(\tau_{n-2}, \tau_{n-2}, \tau_{m-2}))
\end{aligned}$$

$$\leq \vdots$$
$$\leq \psi^n(J(\tau_0, \tau_0, \tau_{m-n})).$$

We assume w.l.o.g that $m = n + p$ for some constant $p \in \mathbb{N}$ we get

$$J(\tau_n, \tau_n, \tau_m) \leq \psi^n(J(\tau_0, \tau_0, \tau_p)). \tag{2.3}$$

By taking the limit in (2.3) as $n \longrightarrow \infty$ we obtain

$$\lim_{n\to\infty} J(\tau_n, \tau_n, \tau_m) = 0. \tag{2.4}$$

Therefore, $\{\tau_n\}$ is a Cauchy sequence in Ω and due to its completeness, there exists $\tau \in \Omega$ such that $\tau_k \to \tau$ as $k \to \infty$.

In addition, $\tau = \lim_{k\to\infty} \tau_k = \lim_{k\to\infty} \tau_{k+1} = \lim_{k\to\infty} \sigma\tau_k = \sigma\tau$. Thus, σ has τ as a fixed point.

Let τ_1 and τ_2 be two fixed points of σ.

$$J(\tau_1, \tau_1, \tau_2) = J(\sigma\tau_1, \sigma\tau_1, \sigma\tau_2) \leq \psi^n(J(\tau_1, \tau_1, \tau_2)). \tag{2.5}$$

Since, $\psi^n(t) < t$ for any $t > 0$, we obtain from (2.5), $J(\tau_1, \tau_1, \tau_2) < J(\tau_1, \tau_1, \tau_2)$, then $J(\tau_1, \tau_1, \tau_2) = 0$ and $\tau_1 = \tau_2$, and σ has a unique fixed point in Ω . □

Theorem 2.13. *Let (Ω, J) be a J- complete symmetric metric space and $\sigma : \Omega \to \Omega$ be a mapping satisfying*

$$J(\sigma\tau, \sigma\upsilon, \sigma\zeta) \leq \lambda(\tau, \upsilon, \zeta)J(\tau, \upsilon, \zeta) \quad \forall \tau, \upsilon, \zeta \in \Omega, \tag{2.6}$$

where $\lambda \in A = \{\lambda : \Omega^3 \to (0, 1), \lambda(f(\tau, \upsilon, \zeta)) \leq \lambda(\tau, \upsilon, \zeta)$ and $f : \Omega \to \Omega$ a given mapping$\}$. Then σ has a unique fixed point.

Proof. Let τ_0 be an arbitrary element in Ω. We construct the sequence $\{\tau_n$ as follows $\{\tau_n = \sigma^n\tau_0\}$. Let's prove that $\{\tau_n\}$ is a Cauchy sequence. For all natural numbers n, m, we suppose w.l.o.g that $n < m$ and assume that these exists a constant $p \in N$ such that $m = n + p$. By using (2.6) we have

$$\begin{aligned} J(\tau_n, \tau_n, \tau_m) &= J(\sigma\tau_{n-1}, \sigma\tau_{n-1}, \sigma\tau_{m-1}) \\ &\leq \lambda(\tau_{n-1}, \tau_{n-1}, \tau_{m-1})J(\tau_{n-1}, \tau_{n-1}, \tau_{m-1}) \\ &\leq \lambda^n(\tau_0, \tau_0, \tau_p)J(\tau_0, \tau_0, \tau_p). \end{aligned}$$

Taking the limit as $n \to \infty$ and considering the property of λ into view, we obtain that $\lim_{n,m\to\infty} J(\tau_n, \tau_n, \tau_m) = 0$, that is $\{\tau_n\}$ is a Cauchy sequence. Then, by completeness of Ω, there exists $u \in \Omega$ such that

$$u = \lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \tau_{n-1}. \tag{2.7}$$

We claim that u is a fixed point of σ. From (2.7), we deduce that $\tau_n \in S(J, \Omega, u)$ and

$$\lim_{n\to\infty} J(u, u, \tau_n) = 0 \tag{2.8}$$

and

$$\lim_{n\to\infty} J(u, u, \tau_{n-1}) = 0. \tag{2.9}$$

By using the triangle inequality and taking into account (2.8) we get:

$$\begin{aligned} J(\sigma u, \sigma u, u) \leq & b \limsup_{n\to\infty} [2J(\sigma u, \sigma u, \tau_n) + J(u, u, \tau_n)] \\ =& 2b \limsup_{n\to\infty} J(\sigma u, \sigma u, \sigma\tau_{n-1}) \\ \leq & 2b \limsup_{n\to\infty} \lambda(u, u, \tau_{n-1}) J(u, u, \tau_{n-1}) \\ \leq & 2b\lambda(u, u, \tau_0) \limsup_{n\to\infty} J(u, u, \tau_{n-1}). \end{aligned} \tag{2.10}$$

Using (2.9) and (2.10) we obtain that $J(\sigma u, \sigma u, u) = 0$, that is $\sigma u = u$. Therefore u is a fixed point of σ.

Let, $\xi_1, \xi_2 \in \Omega$ be two fixed points of σ such that $\xi_1 \neq \xi_2$ that is $\sigma\xi_1 = \xi_1$ and $\sigma\xi_2 = \xi_2$

$$J(\xi_1, \xi_1, \xi_2) = J(\sigma\xi_1, \sigma\xi_1, \sigma\xi_2) \leq \lambda(\xi_1, \xi_1, \xi_2) J(\xi_1, \xi_1, \xi_2) < J(\xi_1, \xi_1, \xi_2).$$

Then, $J(\xi_1, \xi_1, \xi_2) = 0$ which implies that $\xi_1 = \xi_2$. □

Theorem 2.14. *Let (Ω, J) be a complete symmetric J-metric space and $\sigma : \Omega \to \Omega$ be a continuous mapping such that:*

$$J(\sigma\tau, \sigma\upsilon, \sigma\zeta) \leq \alpha J(\tau, \upsilon, \zeta) + \beta(\tau, \sigma\tau, \sigma\tau) + \gamma J(\upsilon, \sigma\upsilon, \sigma\upsilon) + \mu J(\zeta, \sigma\zeta, \sigma\zeta) \tag{2.11}$$

for all $\tau, \upsilon, \nu \in \Omega$ where

$$0 < \alpha + \beta < 1 - \gamma - \mu, \tag{2.12}$$

$$0 < \alpha < 1. \tag{2.13}$$

Then, there exists a unique fixed point of σ.

Proof. Let $\tau_0 \in \Omega$ be an arbitrary point of Ω and $\{\tau_n = \sigma^n \tau_0\}$ be a sequence in Ω. From (2.11) we have

$$\begin{aligned} J(\tau_n, \tau_{n+1}, \tau_{n+1}) =& J(\sigma\tau_{n-1}, \sigma\tau_n, \sigma\tau_n) \\ \leq & \alpha J(\tau_{n-1}, \tau_n, \tau_n) + \beta J(\tau_{n-1}, \tau_n, \tau_n) + \gamma J(\tau_n, \tau_{n+1}, \tau_{n+1}) + \mu J(\tau_n, \tau_{n+1}, \tau_{n+1}) \\ \leq & (\alpha + \beta) J(\tau_{n-1}, \tau_n, \tau_n) + (\gamma + \mu) J(\tau_n, \tau_{n+1}, \tau_{n+1}). \end{aligned}$$

Then

$$J(\tau_n, \tau_{n+1}, \tau_{n+1}) \leq \frac{\alpha + \beta}{1 - \gamma - \mu} J(\tau_{n-1}, \tau_n, \tau_n).$$

Taking $\lambda = \dfrac{\alpha + \beta}{1 - \gamma - \tau}$, then from (2.12) we have $0 < \lambda < 1$. By induction we get $J(\tau_n, \tau_{n+1}, \tau_{n+1}) \leq \lambda^n J(\tau_0, \tau_1, \tau_1)$ which gives that

$$\lim_{n\to\infty} J(\tau_n, \tau_{n+1}, \tau_{n+1}) = 0. \tag{2.14}$$

We denote $J_n = J(\tau_n, \tau_{n+1}, \tau_{n+1})$. For all $n, m \in N, n < m$ we assume w.l.o.g that there exists a fixed $p \in N$ such that $m = n + p$. we have

$$\begin{aligned}
J(\tau_n,\tau_n,\tau_m) &= J(\tau_n,\tau_n,\tau_{n+p}) = J(\sigma\tau_{n-1},\sigma\tau_{n-1},\sigma\tau_{n+p-1})\\
&\leq \alpha J(\tau_{n-1},\tau_{n-1},\tau_{n+p-1}) + \beta J(\tau_{n-1},\tau_n,\tau_n) + \gamma J(\tau_{n-1},\tau_n,\tau_n)\\
&+ \mu J(\tau_{n+p-1},\tau_{n+p},\tau_{n+p})\\
&= \alpha J(\tau_{n-1},\tau_{n-1},\tau_{n+p-1}) + (\beta+\gamma)J_{n-1} + \mu J_{n+p-1}\\
&\leq \alpha[\alpha J(\tau_{n-2},\tau_{n-2},\tau_{n+p-2}) + (\beta+\gamma)J_{n-2} + \tau J_{n+p-2}] + (\beta+\gamma)J_{n-1}\\
&+ \mu J_{n+P-1}\\
&= \alpha^2 J(\tau_{n-2},\tau_{n-2},\tau_{n+p-2}) + \alpha(\beta+\gamma)J_{n-2} + \alpha\tau J_{n+p-2} + (\beta+\gamma)J_{n-1}\\
&+ \mu J_{n+p-1}\\
&\vdots\\
&\leq \alpha^n J(\tau_0,\tau_0,\tau_p) + (\beta+\gamma)\sum_{k=1}^{n}\alpha^{k-1}J_{n-k} + \mu\sum_{k=1}^{n}\alpha^{k-1}J_{n+p-k}.
\end{aligned} \tag{2.15}$$

By taking the limit in (2.15) as $n \to \infty$ and using (2.13) and (2.14), we obtain

$$\lim_{n,m\to\infty} J(\tau_n,\tau_n,\tau_m) = 0.$$

Then, $\{\tau_n\}$ is a Cauchy sequence in Ω. By completeness, there exists $u \in \Omega$ such that $\tau_n \to u$ as $n \to \infty$ and

$$\lim_{n\to\infty} J(\tau_n,\tau_n,u) = \lim_{n,m\to\infty} J(\tau_n,\tau_m,u) = 0. \tag{2.16}$$

In addition, $u = \lim_{k\to\infty}\tau_k = \lim_{k\to\infty}\tau_{k+1} = \lim_{k\to\infty}\sigma\tau_k = \sigma u$. Therefore, σ has u as a fixed point.

Let $\xi_1,\xi_2 \in \Omega$ be two fixed point of σ, $\xi_1 \neq \xi_2$ that is $\sigma\xi_1 = \xi_1$, $\sigma\xi_2 = \xi_2$.

$$\begin{aligned}
J(\xi_1,\xi_1,\xi_2) =& J(\sigma\xi_1,\sigma\xi_1,\sigma\xi_2)\\
\leq& \alpha J(\xi_1,\xi_1,\xi_2) + (\beta+\gamma)J(\xi_1,\sigma\xi_1,\sigma\xi_1) + \mu J(\xi_2,\sigma\xi_2,\sigma\xi_2)\\
=& \alpha J(\xi_1,\xi_1,\xi_2) + (\beta+\gamma)J(\xi_1,\xi_1,\xi_1) + \mu J(\xi_2,\xi_2,\xi_2).
\end{aligned}$$

Then, $(1-\alpha)J(\xi_1,\xi_1,\xi_2) \leq 0$. Using (2.13) we conclude that $J(\xi_1,\xi_1,\xi_2) = 0$ that is $\xi_1 = \xi_2$. □

3. Applications

3.1. Fractional differential equation

There has been many applications of fixed point to fractional differential equations see [25]. In this section, we discuss the existence of a solution to the following problem:

$$(\mathcal{P}) : \left\{ \begin{array}{ccc} D^\lambda x(t) &=& f(t,x(t)) = Fx(t) \text{ if } t \in I_0 = (0,T] \\ x(0) &=& x(T) = r \end{array} \right\}$$

where $T > 0$ and $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, $I = [0,T]$ and $D^\lambda x$ denotes a Riemann-Liouville fractional derivative of x with $\lambda \in (0,1)$.

Let $C_{1-\lambda}(I,\mathbb{R})=\{f\in C((0,T],\mathbb{R}):t^{1-\lambda}\in C(I,\mathbb{R})\}$. We define the following weighted norm

$$\|f\|^{*}=\max_{[0,T]}t^{1-\lambda}|x(t)|.$$

Theorem 3.1. *Let* $\lambda\in(0,1)$*,* $f\in C(I\times\mathbb{R},\mathbb{R})$ *increasing and* $0<\alpha<1$*. In addition, we assume the following hypothesis:*

$$|f(u_1(t),v_1(t))-f(u_2(t),v_2(t))|\le\frac{\Gamma(2\lambda)}{T^{2\lambda-1}}\alpha|v_1-v_2|.$$

Then the problem $(\mathcal{P})$ *has a unique solution.*

Proof. Problem $(\mathcal{P})$ is equivalent to the problem $\mathcal{M}x(t)=x(t)$ where

$$\mathcal{M}x(t)=rt^{\lambda-1}+\frac{1}{\Gamma(\lambda)}\int_0^t(t-s)^{\lambda-1}Fx(s)ds.$$

In fact, proving that the operator $\mathcal{M}$ has a fixed point is sufficient to say that problem $\mathcal{P}$ has a unique solution. Indeed, $(A=C_{1-\lambda}(I,\mathbb{R}),J)$ is a complete $J-$metric space if we consider

$$J(x,y,z)=\max_{[0,T]}t^{1-\lambda}\Big(|x(t)-y(t)|+|x(t)-z(t)|\Big),\ x,y\in C_{1-\lambda}(J,\mathbb{R}).$$

The mapping $\mathcal{M}$ is increasing since f is increasing.

Now, we must prove that $\mathcal{M}$ is a contraction map. Let $x,y,z\in C_{1-\lambda}(J,\mathbb{R})$, $0<\lambda<1$.

$$\begin{aligned}J(\mathcal{M}x,\mathcal{M}y,\mathcal{M}z) &= \max_{[0,T]}t^{1-\lambda}\Big(|\mathcal{M}x(t)-\mathcal{M}y(t)|+|\mathcal{M}x(t)-\mathcal{M}z(t)|\Big)\\ &\le \frac{1}{\Gamma(\lambda)}\max_{t\in[0,T]}t^{1-\lambda}\int_0^t(t-s)^{\lambda-1}\Big(|f(t,x(s))-f(t,y(s))|\\ &+ |f(t,x(s))-f(t,z(s))|\Big)ds.\end{aligned}$$

Subsequently, by the hypothesis of the theorem we have

$$\begin{aligned}J(\mathcal{M}x,\mathcal{M}y,\mathcal{M}z) &\le \frac{1}{\Gamma(\lambda)}\max_{t\in[0,T]}t^{1-\lambda}\int_0^t(t-s)^{\lambda-1}\frac{\Gamma(2\lambda)}{T^{2\lambda-1}}\Big[\alpha|x(s)-y(s)|\\ &+\alpha|x(s)-z(s)|\Big]ds\\ &\le \frac{1}{\Gamma(\lambda)}\max_{t\in[0,T]}t^{1-\lambda}\int_0^t(t-s)^{\lambda-1}\frac{\Gamma(2\lambda)}{T^{2\lambda-1}}\Big[\alpha\|x-y\|^{*}s^{\lambda-1}\\ &+\alpha\|x-z\|^{*}s^{\lambda-1}\Big]ds\\ &\le \frac{1}{\Gamma(\lambda)}\max_{t\in[0,T]}t^{1-\lambda}\alpha(\|x-y\|^{*}+\|x-z\|^{*})\frac{\Gamma(2\lambda)}{T^{2\lambda-1}}\int_0^t(t-s)^{\lambda-1}s^{\lambda-1}ds.\end{aligned}$$

From the Riemann-Liouville fractional integral we have

$$\int_0^t(t-s)^{\lambda-1}s^{\lambda-1}ds=\frac{\Gamma(\lambda)}{\Gamma(2\lambda)}t^{2\lambda-1}.$$

Therefore, we have

$$J(\mathcal{M}x,\mathcal{M}y,\mathcal{M}z)\le\alpha J(x,y,z).$$

Therefore, σ satisfies the conditions of Theorem 2.12 with $\psi(t)=\alpha t$.

Thus, by Theorem 2.12, we deduce that $\mathcal{M}$ has a unique fixed point which leads us to conclude that equation $((\mathcal{P}))$ has a unique solution as required. □

3.2. Fredholm type integral equation

Let $\Omega = C([0,1],\mathbb{R})$ and consider the following Fredholm type integral equation

$$f(\nu) = \int_0^1 \chi(\delta,\xi,f(\nu))ds \text{ for } \delta,\xi \in [0,1] \tag{3.1}$$

where $\chi : [0,1]^3 \longrightarrow \mathbb{R}$ is a continuous function.

Let $J : \Omega^3 \longrightarrow [0,\infty)$ defined by $J(f,g,h) = |\frac{\sup_{t\in[0,1]}(f(t),g(t)) - h(t)}{2}|$. It is not diffuclt to see that (Ω, J) is a complete $J-$metric space.

Theorem 3.2. *If $f,g,h \in \Omega$ satisfy the following condition*

$$|\chi(\delta,\xi,f(\nu)) - \chi(\delta,\xi,h(\nu))| \leq \lambda|f(\nu) - h(\nu)| \textit{ for some } \lambda \in (0,1),$$

then Eq (3.1) has a unique solution.

Proof. Let $\sigma : \Omega \longrightarrow \Omega$ defined by $\sigma f(\nu) = \int_0^1 \chi(\delta,\xi,f(\nu))ds$ then

$$\begin{aligned} J(\sigma f,\sigma g,\sigma h) &= \Big|\frac{\sup_{t\in[0,1]}\big(\sigma f(t),\sigma g(t)\big) - \sigma h(t)}{2}\Big| \\ &= \frac{1}{2}\Big|\sup_{t\in[0,1]}\Big(\int_0^1 \chi(\delta,\xi,f(\nu))ds, \int_0^1 \chi(\delta,\xi,g(\nu))ds\Big) - \int_0^1 \chi(\delta,\xi,h(\nu))ds\Big|. \end{aligned}$$

Assume w.l.o.g that $\chi(\delta,\xi,f(\nu)) > \chi(\delta,\xi,g(\nu))$, then we obtain

$$\begin{aligned} J(\sigma f,\sigma g,\sigma h) &= \frac{1}{2}\Big|\int_0^1 \chi(\delta,\xi,f(\nu))ds - \int_0^1 \chi(\delta,\xi,h(\nu))ds\Big| \\ &= \frac{1}{2}\Big|\int_0^1 \Big[\chi(\delta,\xi,f(\nu)) - \chi(\delta,\xi,h(\nu))\Big]ds\Big| \\ &\leq \frac{1}{2}\int_0^1 \Big|\chi(\delta,\xi,f(\nu)) - \chi(\delta,\xi,h(\nu))\Big|ds \\ &\leq \frac{1}{2}\int_0^1 \lambda|f(\nu) - h(\nu)|ds \\ &\leq \frac{\lambda}{2}\Big|f(\nu) - h(\nu)\Big| \\ &\leq \frac{\lambda}{2}\Big|\sup_{t\in[0,1]}(f(\nu),g(\nu)) - h(\nu)\Big| \\ &= \lambda\Big|\frac{\sup_{t\in[0,1]}\big(f(\nu),g(\nu)\big) - h(\nu)}{2}\Big| \\ &= \lambda J(f,g,h). \end{aligned}$$

Therefore, σ satisfies the conditions of Theorem 2.12 with $\psi(t) = \lambda t$.

Thus, by Theorem 2.12, we deduce that σ has a unique fixed point which leads us to conclude that Eq (3.1) has a unique solution as desired. □

3.3. Linear system of equations

Consider the set $\Omega = \mathbb{R}^n$ where $\mathbb{R}$ is the set of real numbers and n a positive integer. Now, consider the symmetric $J-$metric space (Ω, J) defined by

$$J(\delta, \xi, \nu) = \max_{1\leq i\leq n} |\delta_i - \xi_i| + |\delta_i - \nu_i|$$

for all $\delta = (\delta_1, ..., \delta_n), \xi = (\xi_1, ..., \xi_n), \nu = (\nu_1, ..., \nu_n) \in \Omega$.

Theorem 3.3. *Consider the following system*

$$\begin{cases} s_{11}\delta_1 + s_{12}\delta_2 + s_{13}\delta_3 + s_{1n}\delta_n = r_1 \\ s_{21}\delta_1 + s_{22}\delta_2 + s_{23}\delta_3 + s_{2n}\delta_n = r_2 \\ \vdots \\ s_{n1}\delta_1 + s_{n2}\delta_2 + s_{n3}\delta_3 + s_{nn}\delta_n = r_n \end{cases}$$

if $\theta = \max_{1\leq i\leq n}\Big(\sum_{j=1, j\neq i}^{n} |s_{ij}| + |1 + s_{ii}|\Big) < 1$, *then the above linear system has a unique solution.*

Proof. Consider the map $\sigma : \Omega \to \Omega$ defined by $\sigma\delta = (B + I_n)\delta - r$ where

$$B = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{pmatrix},$$

$\delta = (\delta_1, \delta_2, \cdots, \delta_n); \xi = (\xi_1, \xi_2, \cdots, \xi_n)$ and $\nu = (\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{R}^n$, I_n is the identity matrix for $n \times n$ matrices and $r = (r_1, r_2, \cdots, r_n) \in \mathbb{C}^n$. Let us prove that $J(\sigma\delta, \sigma\xi, \sigma\nu) \leq \theta J(\delta, \xi, \nu)$, $\forall \delta, \xi, \nu \in \mathbb{R}^n$.

We denote by

$$\tilde{B} = B + I_n = (\tilde{b}_{ij}), \quad i, j = 1, ..., n,$$

with

$$\tilde{b}_{ij} = \begin{cases} s_{ij}, j \neq i \\ 1 + s_{ii}, j = i \end{cases}.$$

Hence,

$$\max_{1\leq i\leq n} \sum_{j=1}^{n} |\tilde{b}_{ij}| = \max_{1\leq i\leq n}\Big(\sum_{j=1, j\neq i}^{n} |s_{ij}| + |1 + s_{ii}|\Big) = \theta < 1.$$

On the other hand, for all $i = 1, ..., n$, we have

$$(\sigma\delta)_i - (\sigma\xi)_i = \sum_{j=1}^{n} \tilde{b}_{ij}(\delta_j - \xi_j), \tag{3.2}$$

$$(\sigma\delta)_i - (\sigma\nu)_i = \sum_{j=1}^{n} \tilde{b}_{ij}(\delta_j - \nu_j). \tag{3.3}$$

Therefore, using (3.2) and (3.3) we get

$$
\begin{aligned}
J(\sigma\delta, \sigma\xi, \sigma\nu) &= \max_{1\le i\le n} \left(|(\sigma\delta)_i - (\sigma\xi)_i| + |(\sigma\delta)_i - (\sigma\nu)_i|\right) \\
&\le \max_{1\le i\le n} \Big(\sum_{j=1}^{n} |\tilde{b}_{ij}||\delta_j - \xi_j| + \sum_{j=1}^{n} |\tilde{b}_{ij}||\delta_j - \nu_j|\Big) \\
&\le \max_{1\le i\le n} \sum_{j=1}^{n} |\tilde{b}_{ij}| \max_{1\le k\le n} \left(|\delta_k - \xi_k| + |\delta_k - \nu_k|\right) \\
&= \theta J(\delta, \xi, \nu) = \Phi(J(\delta, \xi, \nu)),
\end{aligned}
$$

where, $\Phi(t) = \theta t$, $\forall t \ge 0$. Note that, all the hypotheses of Theorem 2.12 are satisfied. Thus, σ has a unique fixed point. Therefore, the above linear system has a unique solution as desired. □

4. Conclusions

In this manuscript, we introduced a new type of metric spaces called $J-$metric spaces, which is a three dimension metric space, we proved the existence and uniqueness of a fixed point for self mapping in such space under different types of metric spaces. Moreover, we presented an application of our results to solving system of linear equations and a fractional differential equation using fixed point theory approach.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.*, **3** (1922), 133–181.
2. T. Kamran, M. Samreen, Q. UL Ain, A generalization of b-metric space and some fixed point theorems, *Mathematics*, **5** (2017), 19. https://doi.org/10.3390/math5020019
3. P. Debnath, Z. D. Mitrović, H. M. Srivastava, Fixed points of some asymptotically regular multivalued mappings satisfying a Kannan-type condition, *Axioms*, **10** (2021), 24. https://doi.org/10.3390/axioms10010024
4. P. Debnath, H. M. Srivastava, Global optimization and common best proximity points for some multivalued contractive pairs of mappings, *Axioms*, **9** (2020), 102. https://doi.org/10.3390/axioms9030102

5. P. Debnath, B. S. Choudhury, M. Neog, Fixed set of set valued mappings with set valued domain in terms of start set on a metric space with a graph, *Fixed Point Theory Appl.*, **2017** (2016), 5. https://doi.org/10.1186/s13663-017-0598-8
6. P. Debnath, Banach, Kannan, Chatterjea, and Reich-type contractive inequalities for multivalued mappings and their common fixed points, *Math. Meth. Appl. Sci.*, **45** (2022), 1587–1596. https://doi.org/10.1002/mma.7875
7. P. Debnath, Optimization through best proximity points for multivalued F-contractions, *Miskolc Math. Notes*, **22** (2021), 143–151. https://doi.org/10.18514/MMN.2021.3355
8. P. Debnath, Set-valued Meir-Keeler, Geraghty and Edelstein type fixed point results in b-metric spaces, *Rend. Circ. Mat. Palermo, II. Ser.*, **70** (2021), 1389–1398. https://doi.org/10.1007/s12215-020-00561-y
9. P. Debnath, N. Konwar, S. Radenović, *Metric fixed point theory*, Applications in Science, Engineering and Behavioural Sciences, Singapore: Springer, 2021. https://doi.org/10.1007/978-981-16-4896-0
10. P. Debnath, H. M. Srivastava, P. Kumam, B. Hazarika, *Fixed point theory and fractional calculus*, Recent Advances and Applications, Singapore: Springer, 2022. https://doi.org/10.1007/978-981-19-0668-8
11. S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorems in S-metric spaces, *Mat. Vesnik*, **64** (2012), 258–266.
12. N. M. Mlaiki, α-ψ-contractive mapping on S-metric space, *Math. Sci. Lett.*, **4** (2015), 9–12. http://dx.doi.org/10.12785/msl/040103
13. N. M. Mlaiki, Common fixed points in complex S-metric space, *Adv. Fixed Point Theory*, **4** (2014), 509–524.
14. N. Souayah, N. Mlaiki, A fixed point theorem in S_b-metric spaces, *J. Math. Comput. Sci.*, **16** (2016), 131–139. http://dx.doi.org/10.22436/jmcs.016.02.01
15. N. Souayah, A fixed point in partial S_b-metric spaces, *An. Şt. Univ. Ovidius Constanţa*, **24** (2016), 351–362. http://dx.doi.org/10.1515/auom-2016-0062
16. N. Souayah, N. Mlaiki, A coincident point principle for two weakly compatible mappings in partial S-metric spaces, *J. Nonlinear Sci. Appl.*, **9** (2016), 2217–2223.
17. H. Aydi, W. Shatanawi, C. Vetro, On generalized weakly G-contraction mapping in G-metric spaces, *Comput. Math. Appl.*, **62** (11), 4222–4229. https://doi.org/10.1016/j.camwa.2011.10.007
18. F. Gu, W. Shatanawi, Some new results on common coupled fixed points of two hybrid pairs of mappings in partial metric spaces, *J. Nonlinear Funct. Anal.*, **2019** (2019), 13. https://doi.org/10.23952/jnfa.2019.13
19. S. Romaguera, P. Tirado, A characterization of quasi-metric completeness in terms of α-Ψ-contractive mappings having fixed points, *Mathematics*, **8** (2020), 16. https://doi.org/10.3390/math8010016
20. S. Romaguera, On the correlation between Banach contraction principle and Caristi's fixed point theorem in b-metric spaces, *Mathematics*, **10** (2022), 136. https://doi.org/10.3390/math10010136

21. N. Savanovi, I. D. Arandelovi, Z. D. Mitrovi, The results on coincidence and common fixed points for a new type multivalued mappings in b-metric spaces, *Mathematics*, **10** (2022), 856. https://doi.org/10.3390/math10060856

22. H. Huang, V. Todorčević, S. Radenović, Remarks on recent results for generalized F-contractions, *Mathematics*, **10** (2022), 768. https://doi.org/10.3390/math10050768

23. I. Beg, K. Roy, M. Saha, S^{JS}-metric and topological spaces, *J. Math. Extension*, **15** (2021), 1–16. https://doi.org/10.30495/JME.2021.1589

24. K. Roy, M. Saha, I. Beg, Fixed point of contractive mappings of integral type over an S^{JS}-metric space, *Tamkang J. Math.*, **52** (2021), 267–280.

25. Z. Jia, X. Liu, Stability in measure for uncertain fractional differential equations with jumps, *U.P.B. Sci. Bull., Ser. A*, **84** (2022), 145–154.