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Research article

Analytic technique for solving temporal time-fractional gas dynamics equations with Caputo fractional derivative

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Abstract: Constructing mathematical models of fractional order for real-world problems and developing numeric-analytic solutions are extremely significant subjects in diverse fields of physics, applied mathematics and engineering problems. In this work, a novel analytical treatment technique called the Laplace residual power series (LRPS) technique is performed to produce approximate solutions for a non-linear time-fractional gas dynamics equation (FGDE) in a multiple fractional power series (MFPS) formula. The LRPS technique is a coupling of the RPS approach with the Laplace transform operator. The implementation of the proposed technique to handle time-FGDE models is introduced in detail. The MFPS solution for the target model is produced by solving it in the Laplace space by utilizing the limit concept with fewer computations and more accuracy. The applicability and performance of the technique have been validated via testing three attractive initial value problems for non-linear FGDEs. The impact of the fractional order β on the behavior of the MFPS approximate solutions is numerically and graphically described. The jth MFPS approximate solutions were found to be in full harmony with the exact solutions. The solutions obtained by the LRPS technique indicate and emphasize that the technique is easy to perform with computational efficiency for different kinds of time-fractional models in physical phenomena.

Keywords: Laplace transform; Caputo fractional derivative; fractional gas dynamics equation;

Laplace residual power series

Mathematics Subject Classification: 44A10, 35R11

1. Introduction

In the last decades, several scholars have made a lot of prominent contributions to the theory and applications of fractional differential equations (FDEs), due to their notable role in explaining several real-life phenomena that arise in the natural sciences, including mechanical systems, chaos synchronization, earthquake modeling, image processing, control theory and wave propagation phenomena (see, e.g., [1–4]). These phenomena and others can be described and reformulated as FDEs using fractional calculus. The most significant feature of using FDEs in the mentioned phenomena and others is their nonlocal property. This means that the differential operators provide an excellent tool for the description of memory and hereditary properties of various materials and processes. For more details, see [5–8].

Partial differential equations (PDEs) in the context of fractional derivatives are considered to be a powerful tool in mathematical modeling to understand and interpret some structures of physical phenomena that are complex and unpredictable due to external factors. For this, scholars have utilized them to simplify the controlling design without any loss of genetic information or memory effect, as well as to create a nature issue closely understandable. Besides that, many attempts have been successfully devoted to proposing reliable numerical techniques for handling the fractional PDEs of physical interest; we refer the reader to [9–15] and the references therein. The solutions of PDEs of fractional order provide outstanding insight into the behavior of some dynamic systems and many real-life problems like traffic flow, oscillation, earthquakes and gas dynamics [16,17], which can be reformulated as nonlinear PDEs in the context of fractional derivatives. So, it is necessary to form a convenient and applicable approach for finding the analytical solutions to these problems and others. Recently, numerous analytical and numerical approaches have been conducted by researchers to investigate and construct analytic-approximate solutions of FDEs and PDEs of fractional order, such as the residual power series (RPS) method [18–22], reproducing kernel (RK) method [23–25], unified method (UM) [26], Adomian decomposition method (ADM) [27,28] and homotopy perturbation method (HPM) [29].

In this work, a novel effective analytical technique [30], called the Laplace residual power series (LRPS) technique, has been used to study analytic-approximate solutions in the sense of the Caputo derivative of a nonlinear fractional gas dynamic equation (FGDE) in the form

$$\mathcal{D}_t^{\beta} u(x,t) - u(x,t) + u(x,t)\frac{\partial u(x,t)}{\partial x} + u^2(x,t) = g(x,t), \tag{1.1}$$

with the initial data $u(x,0) = u_0(x)$, for $t \geq 0, x \in \mathbb{R}, 0 < \beta \leq 1$, such that $u(x,t)$ is an unknown analytic function. For the integer case, $\beta = 1$, the FGDE (1.1) reduces to the standard GDE. It is a universal mathematical model that depends upon conservation laws that exist in engineering and physical practices, such as conservation of mass, conservation of momentum, conservation of energy, etc. The nonlinear FGDEs are applicable in the shock fronts, rarefactions and contact discontinuities. Due to the FGDEs having many applications in physics and engineering [31], different numeric-analytic techniques were exploited in recent years to investigate the solutions of FGDEs.

Kumar et al. [32] performed the homotopy perturbation transform technique for solving homogenous and non-homogenous FGDEs. Biazar and Eslami [33] presented the differential transform technique for solving FGDEs. Tamsir and Srivastava [34] considered FGDEs and utilized the fractional reduced differential transform technique for obtaining their solutions. Raja Balachandar et al. [35] proposed the shifted Legendre polynomial of fractional order technique to study the analytical solutions of FGDEs. Iyiola [36] obtained the series solutions of FGDEs using the q-homotopy analysis technique. Kumar and Rashidi [37] applied the fractional homotopy analysis transform technique to provide analytical solutions to homogenous and non-homogenous FGDEs.

Finding out the analytical-approximate solutions of non-linear time-PDEs of fractional order is a considerable matter for scholars to sense and study the physical and dynamic behaviors of nonlinear fractional models. Therefore, there is an imperious necessity for numeric-analytic techniques for creating precise solutions for both linear and nonlinear time-PDEs of fractional order. Motivated by this, the primary contribution of the present analysis is to generate an analytical-approximate solution in closed form compatible with the exact solution for standard-order $\beta = 1$ with no need for linearization, permutations or any physical assumptions in the meaning of the Caputo fractional derivative via extending the application of the LRPS technique. The novel solution technique has been suggested and proved by El-Ajou [30] for creating and analyzing the exact solitary solutions for a certain class of nonlinear time-FPDEs. Its hybrid technique associates the Laplace transform (LT) operator with the RPS scheme. The primary benefit of the present technique is to determine the unknown components of the proposed solutions by using limits in the Laplace space, which in turn reduces the required calculations and saves effort, in contrast to the RPS approach, which requires fractional differentiation in each phase [38–41]. The LRPS method had been successfully applied to create approximate series solutions in closed forms for different kinds of FDEs and time-fractional PDEs [42–45].

The rest of the current work is organized as follows: In Section 2, some basic definitions and theorems concerning fractional calculus, the Laplace transform and Laplace fractional expansion are revisited. In Section 3, the layout of the proposed technique for building the approximate solution of the considered fractional model (1.1) is presented. In Section 4, the LRPS technique is implemented for solutions of fractional gas problems to illustrate the applicability and performance in investigation of the solutions of time-PDEs of fractional order. Finally, some conclusions of our findings are drawn in Section 5.

2. Materials and methods

In this section, we review the primary definitions and theorems of fractional operators in the Riemann-Liouville and Caputo senses. Also, we review the primary definitions and theorems related to the Laplace transform, which will be used mainly in the next section.

Definition 2.1.[3] *For* $\beta \in \mathbb{R}^+$*, the Riemann-Liouville fractional integral operator for a real-valued function* $u(x,t)$ *is denoted by* $\mathcal{J}_t^\beta$ *and defined as*

$$\mathcal{J}_t^\beta u(x,t) = \begin{cases} \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t \frac{u(x,\eta)}{(t-\eta)^{1-\beta}} d\eta, & 0 \le \eta < t, \beta > 0, \\ u(x,t), & \beta = 0. \end{cases} \tag{2.1}$$

Definition 2.2. [3] *The time fractional derivative of order* $\beta > 0$, *for the function* $u(x,t)$ *in the Caputo case, is denoted by* $\mathcal{D}_t^{\beta}$ *and defined as*

$$\mathcal{D}_t^{\beta} u(x,t) = \begin{cases} \mathcal{J}_t^{n-\beta}\left(D_t^n u(x,t)\right) & 0 < n-1 < \beta \leq n, \\ D_t^n u(x,t), & \beta = n, \end{cases} \tag{2.2}$$

where $D_t^n = \frac{\partial^n}{\partial t^n}$, *for* $n \in \mathbb{N}$.

Definition 2.3. [43] *Assume that* $u(x,t): I \times [0,1] \to \mathbb{R}$. *The LT of* $u(x,t)$ *is defined as*

$$\mathcal{U}(x,s) = \mathcal{L}\{u(x,t)\} = \int_0^{\infty} u(x,t)\, e^{-st} dt, \qquad s > \rho, \tag{2.3}$$

where ρ *is the exponential order of* $u(x,t)$.
The inverse LT of the new function $\mathcal{U}(x,s)$ *is defined as*

$$u(x,t) = \mathcal{L}^{-1}\{\mathcal{U}(x,s)\} = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \mathcal{U}(x,s)\, e^{st} ds, \qquad \epsilon = \Re\mathrm{e}(s) > \epsilon_0, \tag{2.4}$$

with the following characteristics:

1) $\mathcal{L}\{t^{m\alpha}\} = \frac{\Gamma(m\alpha+1)}{s^{m\alpha+1}}$, $\alpha > -1$.
2) $\lim_{s \to \infty} s\mathcal{U}(x,s) = u(x,0)$.
3) $\mathcal{L}\{av(x,t) + bu(x,t)\} = aV(x,s) + b\mathcal{U}(x,s)$, *for any* $a, b \in \mathbb{R}$.
4) $\mathcal{L}^{-1}\{aV(x,s) + b\mathcal{U}(x,s)\} = av(x,t) + bu(x,t)$,

where $V(x,s) = \mathcal{L}\{v(x,t)\}$, *and* $\mathcal{U}(x,s) = \mathcal{L}\{u(x,t)\}$.

Lemma 2.1.[44] *Suppose that* $u(x,t): I \times [0,1] \to \mathbb{R}$ *is a real-valued function. Then,*

i. $\mathcal{L}\left\{\mathcal{D}_t^{\beta} u(x,t)\right\} = s^{\beta}\mathcal{U}(x,s) - \sum_{k=0}^{m-1} s^{\beta-k-1} u_t^{(k)}(x,0), \beta \in (n-1, n], n \in \mathbb{N}$.

ii. $\mathcal{L}\left\{\mathcal{D}_t^{m\beta} u(x,t)\right\} = s^{m\beta}\mathcal{U}(x,s) - \sum_{k=0}^{m-1} s^{(m-k)\beta-1} \mathcal{D}_t^{\beta k} u(x,0)$, $\beta \in (0,1], m \in \mathbb{N}$.

Theorem 2.1. [19,27] *Let* $u(x,t)$ *be infinitely* β*-th Caputo fractional differentiable at any point* $t \in (0, r^{\frac{1}{\beta}}]$, *where* $u(x,t)$ *has the following multiple fractional power series (MFPS):*

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x)\, t^{n\alpha}, \ \ t \geq 0, \beta \in (0,1]. \tag{2.5}$$

Then, the coefficients $u_n(x)$, *for* $n = 0,1,2,\dots$, *will be written in the form* $u_n(x) = \frac{\mathcal{D}_t^{n\alpha} u(x,0)}{\Gamma(n\alpha+1)}$ *such that* $\mathcal{D}_t^{n\beta} u(x,t)\,|_{t=0} = \mathcal{D}_t^{\beta}.\mathcal{D}_t^{\beta}.\mathcal{D}_t^{\beta} \dots \mathcal{D}_t^{\beta}\left(u(x,t)\right)|_{t=0}$ *(n-times).*

Theorem 2.2. [44] *Suppose that* $\mathcal{D}_t^{n\beta}u(x,t) \in C(0, r^{\frac{1}{\alpha}}]$, *in which* $u(x,t)$ *has the MFPS as in (2.5). Then, the new transform function* $\mathcal{U}(x,s)$, *in the Laplace space, could be expressed as in the following Laplace fractional series expansion (LFSE):*

$$\mathcal{U}(x,s) = \sum_{n=0}^{\infty} \frac{u_n(x)}{s^{n\alpha+1}}, s > 0\,, \beta \in (0,1], \tag{2.6}$$

where the coefficients $u_n(x) = \mathcal{D}_t^{n\beta}u(x,0)$.

Theorem 2.3. [44] *Suppose that* $\mathcal{U}(x,s)$ *is expanded in the LFSE form (2.6). If there exists* $\zeta(x) > 0$, *such that* $\left| s\,\mathcal{L}\{\mathcal{D}_t^{(n+1)\beta}u(x,t)\}\right| \leq \zeta(x)$, *for* $\beta \in (0,1]$, *then the remaining term* $R_n(x,s)$ *of LFSE (2.6) satisfies the following:*

$$|R_n(x,s)| \leq \frac{\zeta(x)}{s^{1+(n+1)\beta}}, \qquad x \in I, \qquad \eta_1 < s \leq \eta_2. \tag{2.7}$$

Theorem 2.4. [46] *If there is a constant* $\eta \in (0,1)$, *where* $\|u_{k+1}(x,t)\| \leq \eta\|u_k(x,t)\|$, $\forall k \in \mathbb{N}$ *and* $0 < t < \Re < 1$, *then the obtained approximate solution converges to an exact solution.*

Proof. For all $0 < t < \Re < 1$, we have

$$\|u(x,t) - u_k(x,t)\| = \left\| \sum_{i=k+1}^{\infty} u_i(x,t) \right\| \leq \sum_{i=k+1}^{\infty} \|u_i(x,t\| \leq \|\mathcal{F}(\zeta)\| \sum_{i=k+1}^{\infty} \eta^i = \frac{\eta^{k+1}}{1-\eta}\|\mathcal{F}(\zeta)\| \\ \to 0\,, k \to \infty.$$

3. Methodology of LRPS technique

In the current section, the main procedure of the LRPS technique for solving the non-linear time-fractional gas model (1.1) is introduced. Our novel technique depends basically on the running the LT to the both sides of the considered problem and converting it into the Laplace space, then providing the Laplace fractional series solution for the new problem via the residual error function (REF) with using the limit concept, and as a final step, we run the inverse LT to the resultant LFSE to find out the MFPS approximate solution to the main problem. The present technique gives the accurate approximate-analytic solutions in a rapidly convergent series with no need for linearization or any physical restriction. To reach our purpose, the subsequent algorithm summarizes the main steps to create the MFPS approximate solution of the non-linear time-fractional gas model (1.1).

Step A: Apply the LT operator on both sides of the non-linear FGDE (1.1), that is,

$$\mathcal{L}\left\{\mathcal{D}_t^{\beta}u(x,t)\right\} - \mathcal{L}\{u(x,t)\} + \mathcal{L}\left\{u(x,t)\frac{\partial u(x,t)}{\partial x}\right\} + \mathcal{L}\{u^2(x,t)\} = \mathcal{L}\{g(x,t)\}. \tag{3.1}$$

Utilizing Lemma 2.1 and the initial condition of (1.1) yields $\mathcal{L}\left\{\mathcal{D}_t^{\beta}u(x,t)\right\} = s^{\beta}\mathcal{U}(x,s) - s^{1-\beta}u(x,0)$. So, Equation (3.2) becomes

$$\mathcal{U}(x,s)=\frac{u_0(x)}{s}+\frac{\mathcal{U}(x,s)}{s^\beta}-\frac{1}{s^\beta}\mathcal{L}\left\{\mathcal{L}^{-1}\left[\mathcal{U}(x,s)\frac{\partial}{\partial x}\mathcal{U}(x,s)\right]\right\} \\ -\frac{1}{s^\beta}\mathcal{L}\{[\mathcal{L}^{-1}\mathcal{U}(x,s)]^2\}-\frac{1}{s^\beta}G(x,s), \tag{3.2}$$

where $\mathcal{L}\{u(x,t)\}=\mathcal{U}(x,s)$, $\mathcal{L}\{g(x,t)\}=G(x,s)$.
Step B: According to the LRPS technique [30,45], the proposed solution of (3.2) has the following LFSE:

$$\mathcal{U}(x,s)=\sum_{m=0}^{\infty}\frac{u_m(x)}{s^{m\beta+1}}\ x\in I, s>0, 0<\ \beta\le 1. \tag{3.3}$$

Obviously, $u_0(x)=\lim_{s\to\infty}s\mathcal{U}(x,s)=u(x,0)$. So, the LFSE (3.3) can be expressed as

$$\mathcal{U}(x,s)=\frac{u_0(x)}{s}+\sum_{m=1}^{\infty}\frac{u_m(x)}{s^{m\beta+1}}\ x\in I, s>0, 0<\ \beta\le 1, \tag{3.4}$$

and the j-th Laplace series solution $\mathcal{U}_j(x,s)$, of (3.4), can be rewritten as

$$\mathcal{U}_j(x,s)=\frac{u_0(x)}{s}+\sum_{m=1}^{j}\frac{u_m(x)}{s^{m\beta+1}}\ x\in I, s>0, 0<\ \beta\le 1. \tag{3.5}$$

Step C: As in [30,45], the j-th Laplace-REF of (3.2) is defined as

$$\mathcal{L}\left\{Res^j\left(\mathcal{U}_j(x,s)\right)\right\} \\ =\mathcal{U}_j(x,s)-\frac{u_0(x)}{s}+\frac{\mathcal{U}_j(x,s)}{s^\beta} \\ -\frac{1}{s^\beta}\mathcal{L}\left\{\mathcal{L}^{-1}[\mathcal{U}_j(x,s)]\mathcal{L}^{-1}\left[\frac{\partial}{\partial x}\mathcal{U}_j(x,s)\right]\right\}-\frac{1}{s^\beta}\mathcal{L}\left\{\left[\mathcal{L}^{-1}\mathcal{U}_j(x,s)\right]^2\right\} \\ -\frac{1}{s^\beta}G(x,s). \tag{3.6}$$

The Laplace-REF is given as

$$\mathcal{L}\{Res(\mathcal{U}(x,s))\} \\ =\mathcal{U}(x,s)-\frac{u_0(x)}{s}+\frac{\mathcal{U}(x,s)}{s^\beta} \\ -\frac{1}{s^\beta}\mathcal{L}\left\{\mathcal{L}^{-1}[\mathcal{U}(x,s)]\mathcal{L}^{-1}\left[\frac{\partial}{\partial x}\mathcal{U}(x,s)\right]\right\}-\frac{1}{s^\beta}\mathcal{L}\{[\mathcal{L}^{-1}\mathcal{U}(x,s)]^2\} \\ -\frac{1}{s^\beta}G(x,s). \tag{3.7}$$

Consequently, we note that $\mathcal{L}\{Res(\mathcal{U}(x,s))\}=0$, and $\lim_{j\to\infty}\mathcal{L}\left\{Res^j\left(\mathcal{U}_j(x,s)\right)\right\}=\mathcal{L}\{Res(\mathcal{U}(x,s))\}$, for each $x\in I, s>0$. Further, one can use the following facts of the Laplace-REF, which are essential in finding the approximate solutions:

- $\lim_{s\to\infty} s\mathcal{L}\{Res(\mathcal{U}(x,s))\} = 0$, and $\lim_{s\to\infty} s\mathcal{L}\left\{Res^j\left(\mathcal{U}_j(x,s)\right)\right\} = 0$, for each $x \in I, s > 0$.
- $\lim_{s\to\infty} s^{j\beta+1}\mathcal{L}\left\{Res^j\left(\mathcal{U}_j(x,s)\right)\right\} = 0$, for each $x \in I, s > 0$, $j = 1,2,3,...$, and $\beta \in (0,1]$.

Step D: Collect the resulting unknown coefficients $u_m(x)$, for $m = 1,2,3,\dots,j$, in terms of the series expansion (3.5).

Step E: Lastly, take the inverse LT of the obtained LFSE in Step D to get the j-th MFPS solution $u_j(x,t)$.

Subsequently, we demonstrate the procedure of the proposed technique for solving the homogeneous type of the main problem (1.1). To find out the unknown coefficient $u_1(x)$, substitute $\mathcal{U}_1(x,s) = \frac{u_0(x)}{s} + \frac{u_1(x)}{s^{\beta+1}}$ into $\mathcal{L}\{Res^1(\mathcal{U}_1(x,s))\}$ of (3.6), i.e.,

$$\begin{aligned}
\mathcal{L}\{Res^1(\mathcal{U}_1(x,s))\}
&= \frac{u_1(x)}{s^{\beta+1}} - \frac{1}{s^\beta}\left(\frac{u_0(x)}{s} + \frac{u_1(x)}{s^{\beta+1}}\right) \\
&+ \frac{1}{s^\beta}\mathcal{L}\left\{\mathcal{L}^{-1}\left[\frac{u_0(x)}{s} + \frac{u_1(x)}{s^{\beta+1}}\right]\mathcal{L}^{-1}\left[\frac{u_0'(x)}{s} + \frac{u_1'(x)}{s^{\beta+1}}\right]\right\} \\
&+ \frac{1}{s^\beta}\mathcal{L}\left\{\left[\mathcal{L}^{-1}\left(\frac{u_0(x)}{s} + \frac{u_1(x)}{s^{\beta+1}}\right)\right]^2\right\} \\
&= \frac{u_1(x)}{s^{\beta+1}} - \frac{u_0(x)}{s^{\beta+1}} - \frac{u_1(x)}{s^{2\beta+1}} \\
&\quad + s^{-1-3\beta}\left(\frac{\Gamma(1+2\beta)u_1(x)\left(u_1(x)+u_1'(x)\right)}{\Gamma^2(1+\beta)}\right. \\
&\quad + s^\beta\left(s^\beta(u_0(x)^2 + u_1(x)u_0'(x)\right. \\
&\quad \left.\left. + u_0(x)\left(2u_1(x) + s^\beta u_0'(x) + u_1'(x)\right)\right)\right).
\end{aligned} \tag{3.8}$$

Then, multiplying both sides of (3.8) by $s^{\beta+1}$ yields

$$\begin{aligned}
s^{\beta+1}\mathcal{L}\{Res^1(\mathcal{U}_1(x,s))\}
&= u_1(x) + u_0(x)\left(-1 + u_0(x) + u_0'(x)\right) + \frac{\Gamma(1+2\beta)u_1(x)\left(u_1(x)+u_1'(x)\right)}{\Gamma^2(1+\beta)s^{2\beta}} \\
&+ \frac{1}{s^\beta}\left(u_1(x)\left(-1 + 2u_0(x) + u_0'(x)\right) + u_0(x)u_1'(x)\right).
\end{aligned} \tag{3.9}$$

Based on the fact that $\lim_{s\to\infty} s^{\beta+1}\mathcal{L}\{Res^1(\mathcal{U}_1(x,s))\} = 0$, we have

$$u_1(x) = u_0(x) - (u_0(x)^2 - u_0(x)u_0'(x). \tag{3.10}$$

Again, to obtain $u_2(x)$, consider the second Laplace-REF $\mathcal{L}\{Res^2(\mathcal{U}_2(x,s))\}$ as

$$\begin{aligned}\mathcal{L}&\{Res^2(\mathcal{U}_2(x,s))\}\\ &= \mathcal{U}_2(x,s) - \frac{u_0}{s} - \frac{\mathcal{U}_2(x,s)}{s^\beta} + \frac{1}{s^\beta}\mathcal{L}\left\{\mathcal{L}^{-1}[\mathcal{U}_2(x,s)]\mathcal{L}^{-1}\left[\frac{\partial}{\partial x}\mathcal{U}_2(x,s)\right]\right\}\\ &+ \frac{1}{s^\beta}\mathcal{L}\{[\mathcal{L}^{-1}\mathcal{U}_2(x,s)]^2\},\end{aligned} \tag{3.11}$$

where $\mathcal{U}_2(x,s) = \frac{u_0(x)}{s} + \frac{u_1(x)}{s^{\beta+1}} + \frac{u_2(x)}{s^{2\beta+1}}$. Thus, (3.11) can be reformulated as

$$\begin{aligned}\mathcal{L}&\{Res^2(\mathcal{U}_2(x,s))\}\\ &= \frac{u_1(x)}{s^{\beta+1}} + \frac{u_2(x)}{s^{2\beta+1}} - \frac{1}{s^\beta}\left(\frac{u_0(x)}{s} + \frac{u_1(x)}{s^{\beta+1}} + \frac{u_2(x)}{s^{2\beta+1}}\right)\\ &+ \frac{1}{s^\beta}\mathcal{L}\left\{\mathcal{L}^{-1}\left[\frac{u_0(x)}{s} + \frac{u_1(x)}{s^{\beta+1}} + \frac{u_2(x)}{s^{2\beta+1}}\right]\mathcal{L}^{-1}\left[\frac{u_0'(x)}{s} + \frac{u_1'(x)}{s^{\beta+1}} + \frac{u_2'(x)}{s^{2\beta+1}}\right]\right\}\\ &+ \frac{1}{s^\beta}\mathcal{L}\left\{\left[\mathcal{L}^{-1}\left(\frac{u_0(x)}{s} + \frac{u_1(x)}{s^{\beta+1}} + \frac{u_2(x)}{s^{2\beta+1}}\right)\right]^2\right\}.\end{aligned} \tag{3.12}$$

Thereafter, multiply (3.12) by $s^{2\beta+1}$ and solve $\lim_{s\to\infty} s^{2\beta+1}\mathcal{L}\{Res^2(\mathcal{U}_2(x,s))\} = 0$ to get

$$u_2(x) = u_1(x) - 2u_0(x)u_1(x) - u_1(x)u_0'(x) - u_0(x)u_1'(x). \tag{3.13}$$

Similarly, for the third unknown coefficient, $u_3(x)$, substitute $\mathcal{U}_3(x,s) = \frac{u_0(x)}{s} + \sum_{m=1}^{3}\frac{u_m(x)}{s^{m\beta+1}}$ into the third Laplace-REF $\mathcal{L}\{Res^3(\mathcal{U}_3(x,s))\}$ of (3.6), and multiply the obtained fractional algebraic equation by $s^{3\beta+1}$. Next, by solving $\lim_{s\to\infty} s^{3\beta+1}\mathcal{L}\{Res^3(\mathcal{U}_3(x,s))\} = 0$, the third unknown function $u_3(x)$ will be given as

$$\begin{aligned}u_3(x) &= u_2(x) - 2u_0(x)u_2(x) - u_2(x)u_0'(x) - u_0(x)u_2'(x)\\ &\quad - \frac{\Gamma(1+2\beta)u_1(x)(u_1(x)+u_1'(x))}{\Gamma^2(1+\beta)}.\end{aligned} \tag{3.14}$$

Following the same manner in finding the fourth, fifth and sixth unknown coefficients, that is, $u_j(x)$ for $j = 4,5,6$, one can obtain, based on the fact $\lim_{s\to\infty} s^{j\beta+1}\mathcal{L}\left\{Res^j\left(\mathcal{U}_j(x,s)\right)\right\} = 0$, the following.

$$\begin{aligned}u_4(x) &= u_3(x) - 2u_0(x)u_3(x) - u_3(x)u_0'(x) - u_0(x)u_3'(x)\\ &\quad - \frac{\Gamma(1+3\beta)\left(u_2(x)u_1'(x) + u_1(x)(2u_2(x)+u_2'(x))\right)}{\Gamma(1+\beta)\Gamma(1+2\beta)}.\end{aligned} \tag{3.15}$$

$$u_5(x) = u_4(x) - 2u_0(x)u_4(x) - u_4(x)u_0'(x) - u_0(x)u_4'(x) - \frac{16^\beta \Gamma\left(\frac{1}{2} + 2\beta\right) u_2(x)\left(u_2(x) + u_2'(x)\right)}{\sqrt{\pi}\Gamma(1 + 2\beta)} - \frac{\Gamma(1 + 4\beta)\left(u_3(x)u_1'(x) + u_1(x)\left(2u_3(x) + u_3'(x)\right)\right)}{\Gamma(1 + \beta)\Gamma(1 + 3\beta)}.$$

$$u_6(x) = u_5(x) - 2u_0(x)u_5(x) - u_5(x)u_0'(x) - u_0(x)u_5'(x) - \frac{\Gamma(1 + 5\beta)\left(u_3(x)u_2'(x) + u_2(x)\left(2u_3(x) + u_3'(x)\right)\right)}{\Gamma(1 + 2\beta)\Gamma(1 + 3\beta)} - \frac{\Gamma(1 + 5\beta)\left(u_4(x)u_1'(x) + u_1(x)\left(2u_4(x) + u_4'(x)\right)\right)}{\Gamma(1 + \beta)\Gamma(1 + 4\beta)}.$$

More coefficients $u_j(x)$, for arbitrary j, can be computed in the same previous manner by employing the MATHEMATICA software package. Based on the obtained results of $u_j(x)$, we get the LFSE $\mathcal{U}(x,s)$ of the homogenous Laplace algebraic equation (3.2) in the form

$$\mathcal{U}(x,s) = \frac{u_0(x)}{s} + \frac{\left(u_0(x) - (u_0(x)^2 - u_0(x)u_0'(x)\right)}{s^{\beta+1}} + \frac{\left(u_1(x) - 2u_0(x)u_1(x) - u_1(x)u_0'(x) - u_0(x)u_1'(x)\right)}{s^{2\beta+1}} + \cdots. \tag{3.16}$$

Finally, by applying the inverse LT of the resultant equation (3.16), the MFPS approximate solution of the fractional gas model (1.1) will be represented as

$$u(x,t) = u_0(x) + \frac{\left(u_0(x) - (u_0(x)^2 - u_0(x)u_0'(x)\right)}{\Gamma(1 + \beta)} t^\beta + \frac{\left(u_1(x) - 2u_0(x)u_1(x) - u_1(x)u_0'(x) - u_0(x)u_1'(x)\right)}{\Gamma(1 + 2\beta)} t^{2\beta} + \cdots. \tag{3.17}$$

4. Simulation and test problems

In this section, the LRPS technique is profitably applied in view of the Caputo derivative for investigating the analytical-approximate solution of three time-nonlinear FGDEs subject to suitable initial data. Some graphical and numerical simulations are achieved for the solved problems, which confirmed the efficiency and applicability of the proposed technique. It is worth mentioning that all computations and numerical and graphical simulations of the obtained results were accomplished utilizing Mathematica 12.

Problem 4.1. Consider the following homogeneous non-linear fractional gas model [32,34]:

$$\mathcal{D}_t^{\beta}u(x,t)-u(x,t)+u(x,t)\frac{\partial}{\partial x}u(x,t)+u^2(x,t)=0, t\geq 0, x\in\mathbb{R}, 0<\beta\leq 1, \tag{4.1}$$

with the initial data

$$u(x,0)=e^{-x}. \tag{4.2}$$

For the standard case $\beta=1$, the exact solution of (4.1) and (4.2) is $u(x,t)=e^{t-x}$. In light of the previous discussion of the LRPS technique, we apply the LT to (4.1). Then, by using the initial data (4.2), we get

$$\mathcal{U}(x,s)=\frac{e^{-x}}{s}+\frac{\mathcal{U}(x,s)}{s^{\beta}}-\frac{1}{s^{\beta}}\mathcal{L}\left\{\mathcal{L}^{-1}\left[\mathcal{U}(x,s)\frac{\partial}{\partial x}\mathcal{U}(x,s)\right]\right\}-\frac{1}{s^{\beta}}\mathcal{L}\{[\mathcal{L}^{-1}\mathcal{U}(x,s)]^2\}. \tag{4.3}$$

To create the approximate solution of (4.3), let the j-th LFSE $\mathcal{U}_j(x,s)$ be given as

$$\mathcal{U}_j(x,s)=\frac{e^{-x}}{s}+\sum_{m=1}^{j}\frac{u_m(x)}{s^{m\beta+1}}, \tag{4.4}$$

where the unknown coefficients u_m, $m=1,2,\dots,j$, are obtained through the construction of the following j-th Laplace REF of the time-FGDE (4.3) and (4.4) as

$$\begin{aligned}\mathcal{L}&\left\{Res^j\left(\mathcal{U}_j(x,s)\right)\right\}\\ &=\sum_{m=1}^{j}\frac{u_m(x)}{s^{m\beta+1}}-\frac{1}{s^{\beta}}\left(\frac{e^{-x}}{s}+\sum_{m=1}^{j}\frac{u_m(x)}{s^{m\beta+1}}\right)\\ &+\frac{1}{s^{\beta}}\mathcal{L}\left\{\mathcal{L}^{-1}\left[\frac{e^{-x}}{s}+\sum_{m=1}^{j}\frac{u_m(x)}{s^{m\beta+1}}\right]\mathcal{L}^{-1}\left[-\frac{e^{-x}}{s}+\sum_{m=1}^{j}\frac{u'_m(x)}{s^{m\beta+1}}\right]\right\}\\ &+\frac{1}{s^{\beta}}\mathcal{L}\left\{\left[\mathcal{L}^{-1}\left(\frac{e^{-x}}{s}+\sum_{m=1}^{j}\frac{u_m(x)}{s^{m\beta+1}}\right)\right]^2\right\}.\end{aligned} \tag{4.5}$$

Next, by using the formulas in (3.10) and (3.13)–(3.15), the unknown coefficients for $m=1,2,\dots,6$ will be $u_m(x)=e^{-x}$. Therefore, the sixth L-FSE $\mathcal{U}_6(x,s)$ is written as

$$\mathcal{U}_6(x,s)=\frac{e^{-x}}{s}+\frac{e^{-x}}{s^{\beta+1}}+\frac{e^{-x}}{s^{2\beta+1}}+\frac{e^{-x}}{s^{3\beta+1}}+\frac{e^{-x}}{s^{4\beta+1}}+\frac{e^{-x}}{s^{5\beta+1}}+\frac{e^{-x}}{s^{6\beta+1}}. \tag{4.6}$$

So, the infinite series solution $\mathcal{U}(x,s)$ of the Laplace algebraic equation (4.3) can be written as

$$\mathcal{U}(x,s)=e^{-x}\left(\frac{1}{s}+\frac{1}{s^{\beta+1}}+\frac{1}{s^{2\beta+1}}+\frac{1}{s^{3\beta+1}}+\cdots+\frac{1}{s^{j\beta+1}}+\cdots\right). \tag{4.7}$$

Consequently, the MFPS approximate solution $u(x,t)$ of the fractional gas model (4.1) and (4.2) takes the following infinite series form:

$$u(x,t) = e^{-x}\left(1 + \frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{3\beta}}{\Gamma(3\beta+1)} + \cdots + \frac{t^{j\beta}}{\Gamma(j\beta+1)} + \cdots\right). \tag{4.8}$$

If we replace $\beta = 1$ in the MFPS formula (4.8), we obtain $u(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^j}{j!} + \cdots\right)$, which is the same as the obtained results in [32,34].

Table 1 shows the absolute errors $|u(x,t) - u_{10}(x,t)|$, which are obtained for the fractional gas model (4.1) and (4.2) for some selected various values of t with step size 0.15 and fixed values of x at $n = 10$. Graphically, the solution behaviors of the tenth MFPS approximate solution at different values of fractional order β are plotted in Figure 1 against the exact solution when $\beta = 1$. One can see from this graph that, as the parameter β increases on its domain, the obtained approximate solutions converge continuously to the standard case for $\beta = 1$. Figure 2 demonstrates the comparison of the geometric behaviors between the exact solution and the obtained tenth MFPS approximate solution of (4.1) and (4.2) at various β values for $(x,t) \in [0,1]^2$. From these 3D surface plots, we see that the solution behaviors for different Caputo fractional derivatives on their domain are in close agreement with each other, particularly for classical derivatives.

Table 1. Numerical simulation for fractional gas model (4.1) and (4.2) at $\beta = 1$ and $n = 10$ with different values of t.

x_i	t_i	$u(x,t)$	$u_{10}(x,t)$	$\lvert u(x,t) - u_{10}(x,t)\rvert$
-1	0.15	3.158192909689767	3.158192909689767	0.0
	0.30	3.669296667619244	3.669296667619121	$1.23900889 \times 10^{-13}$
	0.45	4.263114515168817	4.263114515157978	$1.08393294 \times 10^{-11}$
	0.60	4.953032424395115	4.953032424135106	$2.60008903 \times 10^{-10}$
	0.75	5.754602676005731	5.754602672938858	$3.06687298 \times 10^{-9}$
	0.90	6.685894442279268	6.685894419187874	$2.30913946 \times 10^{-8}$
1	0.15	0.427414931948733	0.427414931948727	$5.55111512 \times 10^{-17}$
	0.30	0.496585303791409	0.496585303791393	$1.67088565 \times 10^{-14}$
	0.45	0.576949810380487	0.576949810379019	$1.46693768 \times 10^{-12}$
	0.60	0.670320046035639	0.670320046000451	$3.51882967 \times 10^{-11}$
	0.75	0.778800783071405	0.778800782656349	$4.15055989 \times 10^{-10}$
	0.90	0.904837418035959	0.904837414910879	$3.12508052 \times 10^{-9}$

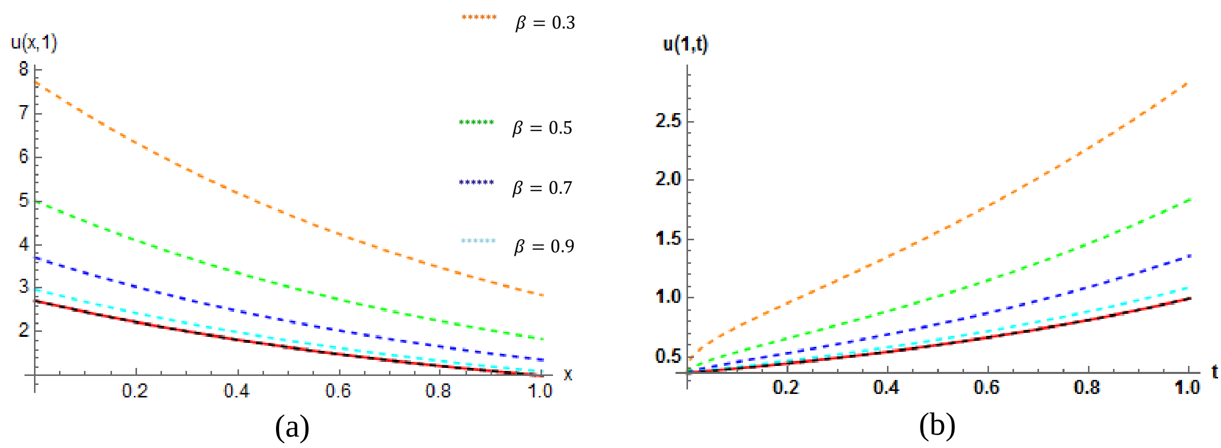

Figure 1. (a) Plots of exact and tenth MFPS solutions at various β values and $t = 1$. (b) Plots of exact and tenth MFPS solutions at various β values and $x = 1$ for Problem 4.1.

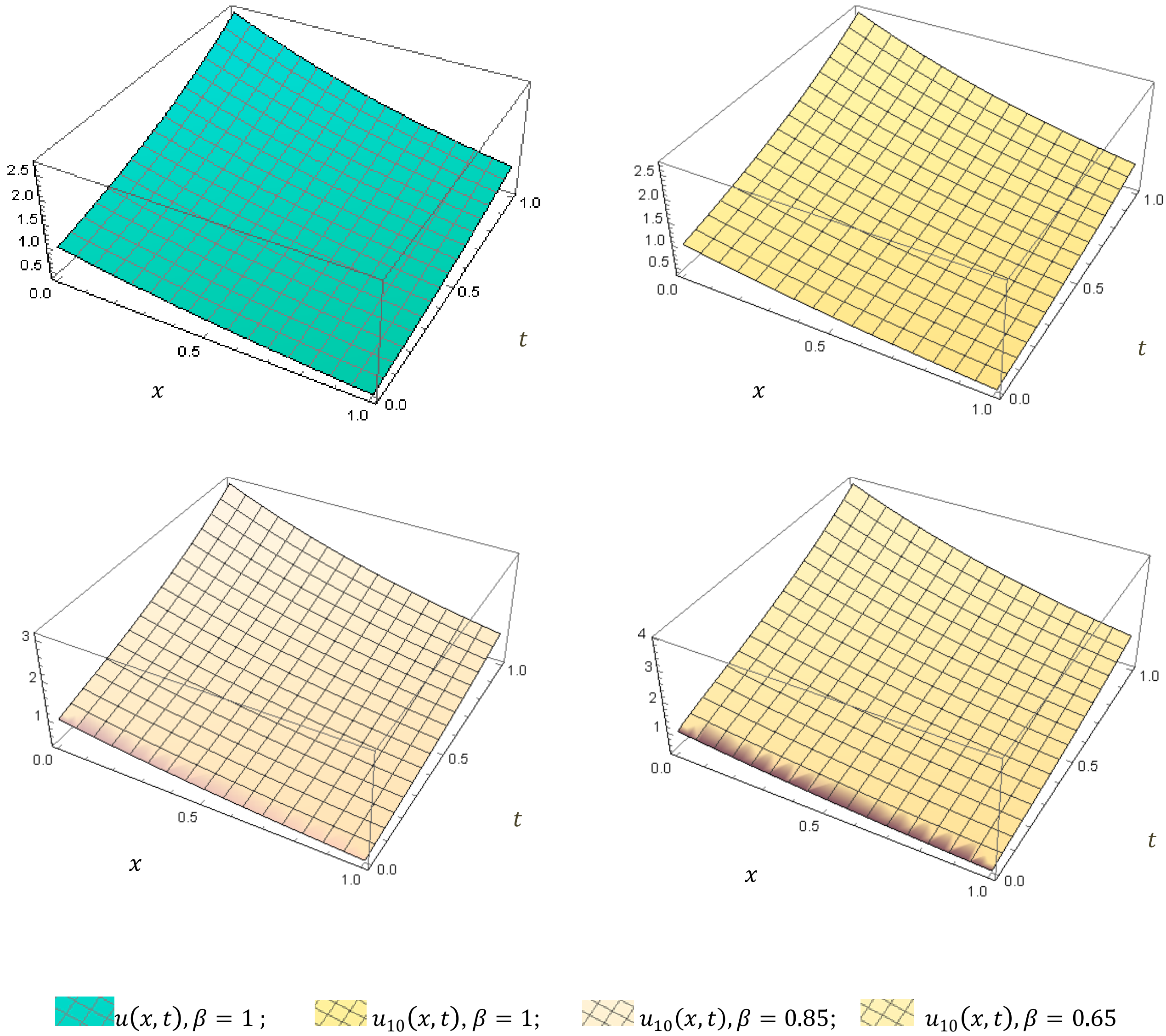

Figure 2. 3D-Surface plots of exact solution $u(x,t)$ and the 10th MFPS approximate solution $u_{10}(x,t)$ for FGDE (4.1) and (4.2) with $t \in [0,1]$ and $x \in [0,1]$ at various values of β.

Problem 4.2. Consider the following homogeneous nonlinear fractional gas model [32,34]:

$$\mathcal{D}_t^{\beta} u(x,t) - \log(q)\, u(x,t) + u(x,t)\frac{\partial}{\partial x}u(x,t) - \log(q)\, u^2(x,t) = 0, \tag{4.9}$$

with the initial data

$$u(x,0) = q^{-x}, \tag{4.10}$$

where $q > 0$, $0 < \beta \le 1$, and $(x,t) \in [0,1] \times \mathbb{R}$. For the integer case $\beta = 1$, the exact solution of (4.9) and (4.10) is $u(x,t) = q^{t-x}$.

Employing the LRPS technique, we start by applying the LT on both sides of (4.9) and using the initial data (4.10) to get

$$\begin{aligned} \mathcal{U}(x,s) = \frac{q^{-x}}{s} + \frac{\log q}{s^{\beta}}\mathcal{U}(x,s) - \frac{1}{s^{\beta}}\mathcal{L}\left\{\mathcal{L}^{-1}\left[\mathcal{U}(x,s)\frac{\partial}{\partial x}\mathcal{U}(x,s)\right]\right\} \\ - \frac{\log q}{s^{\beta}}\mathcal{L}\{[\mathcal{L}^{-1}\mathcal{U}(x,s)]^2\}. \end{aligned} \tag{4.11}$$

Thus, the LFSE of (4.11) has the following shape

$$\mathcal{U}(x,s) = \frac{u_0(x)}{s} + \sum_{m=1}^{\infty}\frac{u_m(x)}{s^{m\beta+1}}. \tag{4.12}$$

Obviously $u_0(x) = \lim_{s\to\infty} s\,\mathcal{U}(x,s) = u(x,0) = q^{-x}$. So, the j-th LFSE of (4.11) will be written as

$$\mathcal{U}_j(x,s) = \frac{q^{-x}}{s} + \sum_{m=1}^{j}\frac{u_m(x)}{s^{m\beta+1}}. \tag{4.13}$$

Next, the j-th L-REF $\mathcal{L}\left\{Res^j\left(\mathcal{U}_j(x,s)\right)\right\}$ of (4.11) is defined as

$$\begin{aligned} &\mathcal{L}\left\{Res^j\left(\mathcal{U}_j(x,s)\right)\right\} \\ &\quad= \sum_{m=1}^{j}\frac{u_m(x)}{s^{m\beta+1}} - \frac{\log q}{s^{\beta}}\left(\frac{q^{-x}}{s} + \sum_{m=1}^{j}\frac{u_m(x)}{s^{m\beta+1}}\right) \\ &\quad+ \frac{1}{s^{\beta}}\mathcal{L}\left\{\mathcal{L}^{-1}\left[\frac{q^{-x}}{s} + \sum_{m=1}^{j}\frac{u_m(x)}{s^{m\beta+1}}\right]\mathcal{L}^{-1}\left[-\log q\,\frac{q^{-x}}{s} + \sum_{m=1}^{j}\frac{u'_m(x)}{s^{m\beta+1}}\right]\right\} \\ &\quad+ \frac{\log q}{s^{\beta}}\mathcal{L}\left\{\left[\mathcal{L}^{-1}\left(\frac{q^{-x}}{s} + \sum_{m=1}^{j}\frac{u_m(x)}{s^{m\beta+1}}\right)\right]^2\right\}. \end{aligned} \tag{4.14}$$

Hence, the unknown functions $u_m(x)$ can be computed via writing the j-th LFSE of (4.13) into the j-th L-RFE $\mathcal{L}\left\{Res^j\left(\mathcal{U}_j(x,s)\right)\right\}$ of (4.14), multiplying the resultant equation by $s^{j\beta+1}$ and taking into account the result $\lim_{s\to\infty} s^{j\beta+1}\mathcal{L}\left\{Res^j\left(\mathcal{U}_j(x,s)\right)\right\} = 0$ for $m = 1,2,\dots,j$. In this way, one can determine the forms of the functions $u_m(x)$, respectively, as

$$u_1(x) = q^{-x}\log(q),$$
$$u_2(x) = q^{-x}(\log(q))^2,$$
$$u_3(x) = q^{-x}(\log(q))^3,$$
$$u_4(x) = q^{-x}(\log(q))^4,$$
$$\vdots.$$

Therefore, the LFSE $\mathcal{U}(x,s)$ of the Laplace algebraic equation (4.12) takes the form

$$\mathcal{U}(x,s) = q^{-x}\left(\frac{1}{s} + \frac{\log(q)}{s^{\beta+1}} + \frac{(\log(q))^2}{s^{2\beta+1}} + \frac{(\log(q))^3}{s^{3\beta+1}} + \cdots + \frac{(\log(q))^j}{s^{j\beta+1}} + \cdots\right). \tag{4.15}$$

Lastly, by taking the inverse LT of both sides of the achieved LFSE (4.15), one could conclude that the MFPS approximate solution of the nonlinear time-fractional gas model (4.9) and (4.10) has the following infinite series formula:

$$u(x,t) = q^{-x}\sum_{m=0}^{\infty}\frac{\left(t^\beta\log(q)\right)^m}{\Gamma(m\beta+1)}, \tag{4.16}$$

which coincides with the obtained results in [32,34]. Moreover, when $\beta = 1$, the MFPS approximate solution is entirely in harmony with the exact solution $u(x,t) = q^{t-x}$.

Table 2 shows the accuracy of the proposed approach via computed numerical results for the tenth MFPS approximate solutions of Problem 4.2 at fixed values of x and some node points of t, with step size 0.25 for different values of β. The geometric behaviors of the tenth-MFPS approximate solutions against the exact solution are drawn in 3D surface plots for $t \in [0,1]$ and $x \in [0,1]$ at various values of β, as shown in Figure 3. It is manifest from this figure that the obtained approximate solution converges continuously to the standard-case $\beta = 1$ as β moves over (0, 1). Also, the graphs of the behaviors of the obtained tenth-MFPS approximate solutions are consistent with each other, especially when considering the standard derivative.

Table 2. Comparison of numerical tenth MFPS approximate solutions for Problem 4.2.

x_i	t_i	$u_{10}(x,t)$			
		$\beta = 1$	$\beta = 0.80$	$\beta = 0.60$	$\beta = 0.40$
-1	0.25	0.42044821	0.39369508	0.36549146	0.33892131
	0.50	0.35355339	0.33256158	0.31643312	0.30560356
	0.75	0.29730178	0.28745081	0.28378632	0.28490461
	1.0	0.25000002	0.25217222	0.25942999	0.26992039
1	0.25	1.68179283	1.57478033	1.46196585	1.35568524
	0.50	1.41421356	1.33024631	1.26573247	1.22241427
	0.75	1.18920712	1.14980321	1.13514527	1.13961843
	1.0	1.00000008	1.00868888	1.03771999	1.07968155

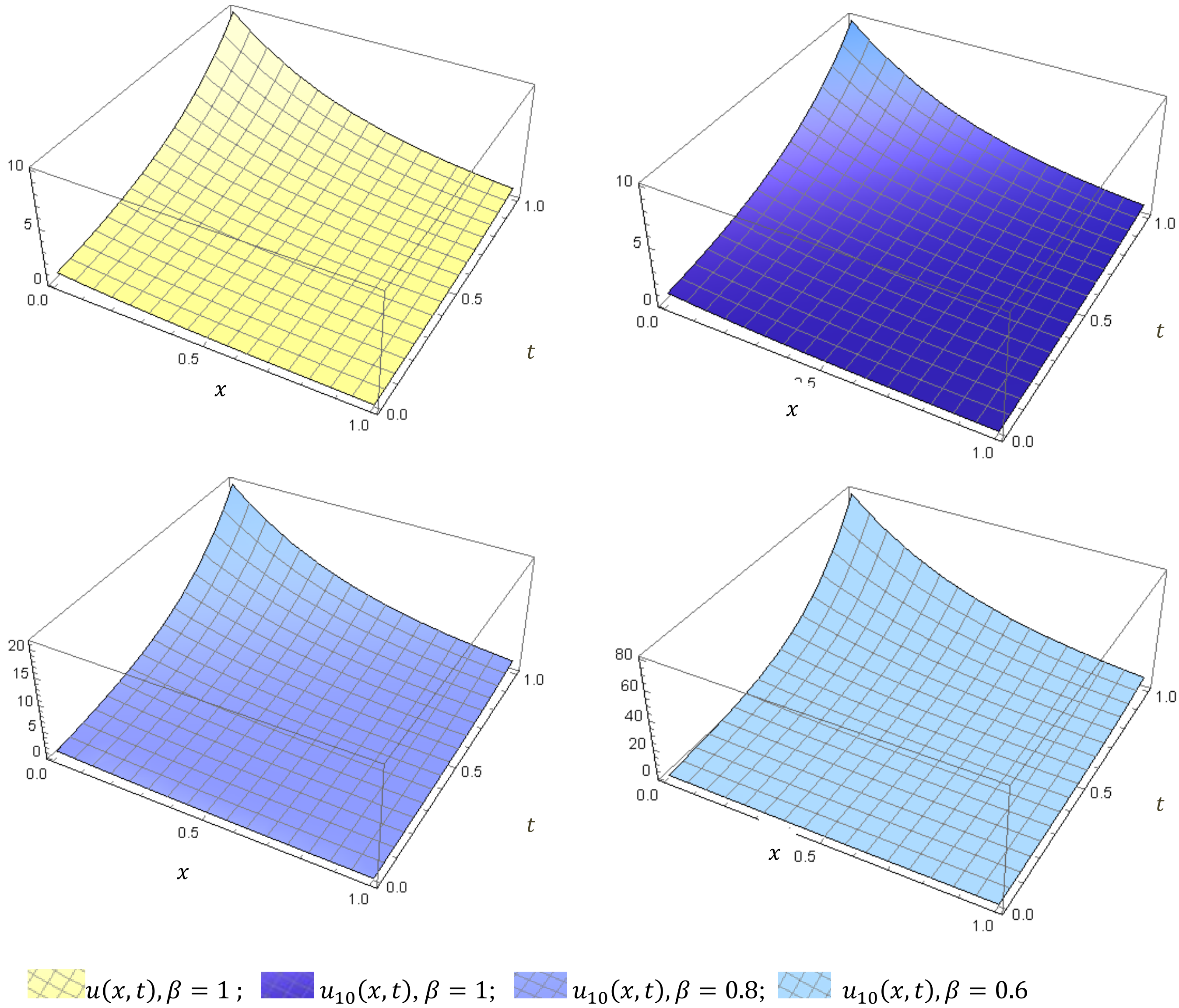

Figure 3. 3D surface plots of exact solution $u(x,t)$ and the 10th MFPS approximate solution $u_{10}(x,t)$ for FGDE (4.9) and (4.10) with $t \in [0,1]$, $x \in [0,1]$ and $q = 10$ at various values of β.

Problem 4.3. Consider the following non-homogeneous fractional gas model [32,34]

$$\mathcal{D}_t^{\beta} u(x,t) + u(x,t)\frac{\partial}{\partial x}u(x,t) + (1+t)^2\, u^2(x,t) = x^2, \tag{4.17}$$

with the initial data

$$u(x,0) = x, \tag{4.18}$$

where $0 < \beta \leq 1$, and $(x,t) \in [0,1] \times \mathbb{R}$. For the integer case $\beta = 1$, the exact solution of (4.17) and (4.18) is $u(x,t) = \frac{x}{1+t}$.

Now, to perform the process of our proposed method, we start with the initial data (4.18), and applying the LT on both sides of (4.18) yields

$$\mathcal{U}(x,s)=\frac{x}{s}-\frac{1}{s^{\beta}}\mathcal{L}\left\{\mathcal{L}^{-1}\left[\mathcal{U}(x,s)\frac{\partial}{\partial x}\mathcal{U}(x,s)\right]\right\}-\frac{1}{s^{\beta}}\mathcal{L}\{[\mathcal{L}^{-1}\mathcal{U}(x,s)]^{2}\}+\frac{1}{s^{\beta}}\mathcal{L}\{x^{2}\}. \tag{4.19}$$

As in the manner of the LRPS algorithm, we construct the j-th L-REF of (4.19) as

$$\begin{aligned}\mathcal{L}&\left\{Res^{j}\left(\mathcal{U}_{j}(x,s)\right)\right\}\\ &=\sum_{m=1}^{j}\frac{u_{m}(x)}{s^{m\beta+1}}-\frac{1}{s^{\beta}}\mathcal{L}\{x^{2}\}\\ &+\frac{1}{s^{\beta}}\mathcal{L}\left\{\mathcal{L}^{-1}\left[\frac{x}{s}+\sum_{m=1}^{j}\frac{u_{m}(x)}{s^{m\beta+1}}\right]\mathcal{L}^{-1}\left[\frac{1}{s}+\sum_{m=1}^{j}\frac{u'_{m}(x)}{s^{m\beta+1}}\right]\right\}\\ &+\frac{1}{s^{\beta}}\mathcal{L}\left\{\mathcal{L}^{-1}\left[\frac{1}{s}+\frac{2}{s^{2}}+\frac{2}{s^{3}}\right]\left[\mathcal{L}^{-1}\left(\frac{x}{s}+\sum_{m=1}^{j}\frac{u_{m}(x)}{s^{m\beta+1}}\right)\right]^{2}\right\}.\end{aligned} \tag{4.20}$$

To find out $u_1(x)$, we consider $k=1$ in (4.20), that is,

$$\begin{aligned}&\mathcal{L}\left\{Res^{1}\left(\mathcal{U}_{1}(x,s)\right)\right\}\\ &\quad=\frac{u_{1}(x)}{s^{\beta+1}}-\frac{1}{s^{\beta}}\mathcal{L}\{x^{2}\}+\frac{1}{s^{\beta}}\mathcal{L}\left\{\mathcal{L}^{-1}\left[\frac{x}{s}+\frac{u_{1}(x)}{s^{\beta+1}}\right]\mathcal{L}^{-1}\left[\frac{1}{s}+\frac{u'_{1}(x)}{s^{\beta+1}}\right]\right\}\\ &\quad+\frac{1}{s^{\beta}}\mathcal{L}\left\{\mathcal{L}^{-1}\left[\frac{1}{s}+\frac{2}{s^{2}}+\frac{2}{s^{3}}\right]\left[\mathcal{L}^{-1}\left(\frac{x}{s}+\frac{u_{1}(x)}{s^{\beta+1}}\right)\right]^{2}\right\}\end{aligned}$$

$$\begin{aligned}&=-\frac{x^{2}}{s^{1+\beta}}+\frac{u_{1}(x)}{s^{\beta+1}}+\frac{2u_{0}^{2}(x)}{s^{3+\beta}}+\frac{2u_{0}^{2}(x)}{s^{2+\beta}}+\frac{u_{0}^{2}(x)}{s^{1+\beta}}+2\frac{u_{0}(x)u_{1}(x)}{s^{1+2\beta}}\\ &\quad+\frac{4\Gamma(\beta+2)u_{0}(x)u_{1}(x)}{s^{2+2\beta}\Gamma(\beta+1)}+\frac{2\Gamma(\beta+2)u_{0}(x)u_{1}(x)}{s^{3+2\beta}\Gamma(\beta+1)}\\ &\quad+\frac{\Gamma(2\beta+1)u_{1}^{2}(x)}{s^{1+3\beta}\Gamma^{2}(\beta+1)}++\frac{2\Gamma(2\beta+2)u_{1}^{2}(x)}{s^{2+3\beta}\Gamma^{2}(\beta+1)}\\ &\quad+\frac{\Gamma(2\beta+3)u_{1}^{2}(x)}{s^{3+3\beta}\Gamma^{2}(\beta+1)}+\frac{u_{0}(x)u'_{0}(x)}{s^{1+\beta}}+\frac{u_{1}(x)u'_{0}(x)}{s^{1+2\beta}}\\ &\quad+\frac{u_{0}(x)u'_{1}(x)}{s^{1+2\beta}}+\frac{\Gamma(2\beta+1)u_{1}(x)u'_{1}(x)}{s^{1+3\beta}\Gamma^{2}(\beta+1)}.\end{aligned} \tag{4.21}$$

Next, we multiply both sides of (4.21) by $s^{\beta+1}$ to get

$$
\begin{aligned}
&s^{\beta+1}\mathcal{L}\{Res^1(\mathcal{U}_1(x,s))\} \\
&\quad= -x^2 + u_0^2(x) + \frac{2u_0^2(x)}{s^2} + \frac{2u_0^2(x)}{s} + u_1(x) + \frac{4u_0(x)u_1(x)}{s^{2+\beta}} + \frac{4u_0(x)u_1(x)}{s^{1+\beta}} \\
&\quad+ \frac{2u_0(x)u_1(x)}{s^{\beta}} + \frac{6\beta u_0(x)u_1(x)}{s^{2+\beta}} + \frac{4\beta u_0(x)u_1(x)}{s^{1+\beta}} + \frac{2\beta^2 u_0(x)u_1(x)}{s^{2+\beta}} \\
&\quad+ \frac{2\Gamma(2\beta+1)u_1^2(x)}{s^{2+2\beta}\Gamma^2(\beta+1)} + \frac{2\Gamma(2\beta+1)u_1^2(x)}{s^{1+2\beta}\Gamma^2(\beta+1)} + \frac{\Gamma(2\beta+1)u_1^2(x)}{s^{2\beta}\Gamma^2(\beta+1)} \\
&\quad+ \frac{6\beta\Gamma(2\beta+1)u_1^2(x)}{s^{2+2\beta}\Gamma^2(\beta+1)} + \frac{4\beta\Gamma(2\beta+1)u_1^2(x)}{s^{1+2\beta}\Gamma^2(\beta+1)} + \frac{4\beta^2\Gamma(2\beta+1)u_1^2(x)}{s^{2+2\beta}\Gamma^2(\beta+1)} \\
&\quad+ u_0(x)u_0'(x) + \frac{u_1(x)u_0'(x)}{s^{\beta}} + \frac{u_0(x)u_1'(x)}{s^{\beta}} + \frac{\Gamma(2\beta+1)u_1(x)u_1'(x)}{s^{2\beta}\Gamma^2(\beta+1)}.
\end{aligned} \tag{4.22}
$$

Thereafter, by solving $\lim_{s\to\infty} s^{\beta+1}\mathcal{L}\{Res^1(\mathcal{U}_1(x,s))\} = 0$, one can get $u_1(x) = -x$. So, the first LFSE of (4.19) could be written as $\mathcal{U}_1(x,s) = \frac{x}{s} - \frac{x}{s^{\beta+1}}$.

For construction of the second LFSE of (4.19), we should substitute $\mathcal{U}_2(x,s) = \frac{x}{s} - \frac{x}{s^{1+\beta}} + \frac{u_2(x)}{s^{1+2\beta}}$ into the second L-REF of (4.19). By multiplying the resultant equation by the factor $s^{1+2\beta}$, the second unknown coefficient will be obtained such that $u_2(x) = 2x$. Thus, the second LFSE of (4.19) could be written as $\mathcal{U}_2(x,s) = \frac{x}{s} - \frac{x}{s^{\beta+1}} + \frac{2x}{s^{1+2\beta}}$.

Following the procedure of the LRPS algorithm, the forms of the unknown functions u_m, $m = 1,2,3,\dots,j$, could be found by solving the system $\lim_{s\to\infty} s^{j\beta+1}\mathcal{L}\left\{Res^j\left(\mathcal{U}_j(x,s)\right)\right\} = 0$ for $u_m(x)$, and hence the LFSE of (4.19) can be formulated as

$$
\mathcal{U}(x,s) = \frac{x}{s} - \frac{x}{s^{\beta+1}} + \frac{2x}{s^{2\beta+1}} - \frac{6x}{s^{3\beta+1}} + \frac{24x}{s^{4\beta+1}} - \frac{120x}{s^{5\beta+1}} + \frac{720x}{s^{6\beta+1}} + \cdots = x\sum_{m=0}^{\infty} \frac{(-1)^m m!}{s^{m\beta+1}}. \tag{4.23}
$$

As the last phase in finding the approximate solution of the main problem, we apply the inverse LT on both sides of LFSE (4.23) to get the following MFPS approximate solution of non-homogeneous FGDEs (4.17) and (4.18):

$$
\begin{aligned}
u(x,t) &= x\left(1 + \frac{1}{\Gamma(\beta+1)}t^{\beta} - \frac{2}{\Gamma(2\beta+1)}t^{2\beta} + \frac{6}{\Gamma(3\beta+1)}t^{3\beta} + \frac{24}{\Gamma(4\beta+1)}t^{4\beta}\right. \\
&\qquad \left. + \cdots\right) = x\sum_{m=0}^{\infty} \frac{(-1)^m m!}{\Gamma(m\beta+1)} t^{m\beta}.
\end{aligned} \tag{4.24}
$$

Particularly, when $\beta = 1$, the MFPS approximate solution (4.24) reduces to the closed-form $u(x,t) = \frac{x}{1+t}$, which agrees with the exact solution of the classical form of (4.17) and (4.18) as in [32,34].

The harmony between the exact and obtained approximate solutions is illustrated via computing the absolute errors $|u(x,t) - u_{10}(x,t)|$ of the IVP (4.17) and (4.18) for some selected grid points $t_i = 0.15i, i = 1,2,3,4$, with fixed values of x, as summarized in Table 3. To explain the geometric behaviors of the obtained solutions by the LRPS approach, Figure 4 demonstrates the effect of the fractional order derivative on the pattern of the obtained solutions via the LRPS method and the full compatibility between the LRPS curves and the exact solution. Also, one can see the convergence between the exact solution and the obtained approximate solution for non-linear fractional gas model (4.17) and (4.18) at different β values, especially at the standard order derivative, as in Figure 5.

Table 3. Numerical simulation for fractional gas model (4.17) and (4.18) at $\beta = 1$ and $n = 10$ with different values of t.

x_i	t_i	$u(x,t)$	$u_{10}(x,t)$	$\|u(x,t) - u_{10}(x,t)\|$
0.2	0.15	0.17391304	0.17391304	$1.5043053 \times 10^{-10}$
	0.30	0.15384615	0.15384642	2.7253385×10^{-7}
	0.45	0.13793103	0.13795216	2.1134873×10^{-5}
	0.60	0.12500000	0.12545349	4.5349632×10^{-4}
0.4	0.15	0.34782609	0.347826087	$3.0086106 \times 10^{-10}$
	0.30	0.30769238	0.307692853	5.4506769×10^{-7}
	0.45	0.27586207	0.275904339	4.2269746×10^{-5}
	0.60	0.25000000	0.250906992	9.0699264×10^{-5}

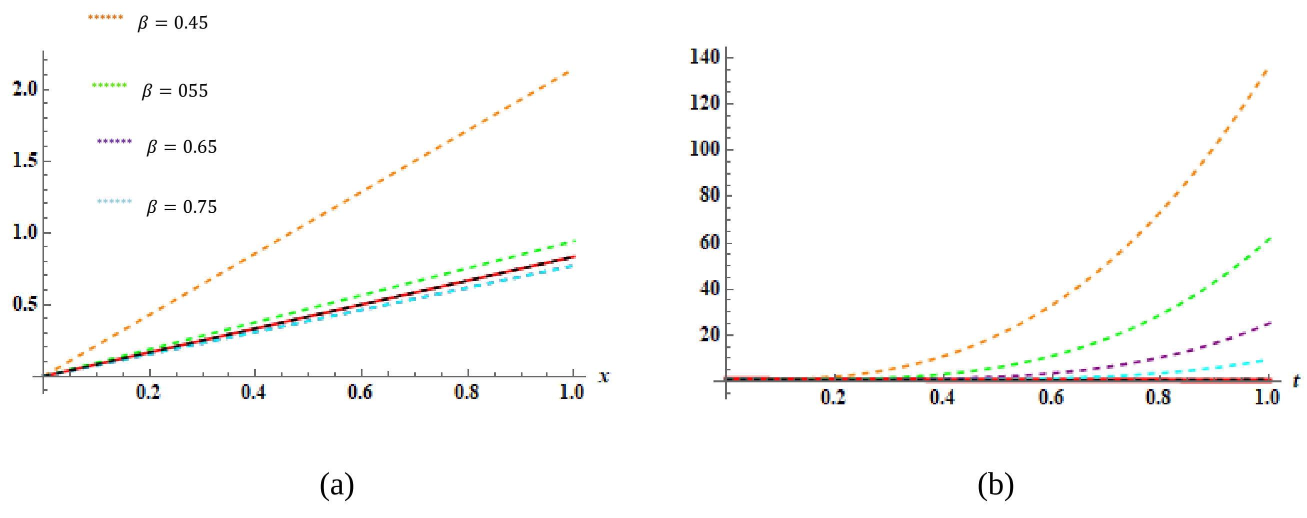

Figure 4. (a) Plots of exact and tenth MFPS solutions at various β values and $t = 0.2$. (b) Plots of exact and tenth MFPS solutions at various β values and $x = 1$, for Problem 4.3.

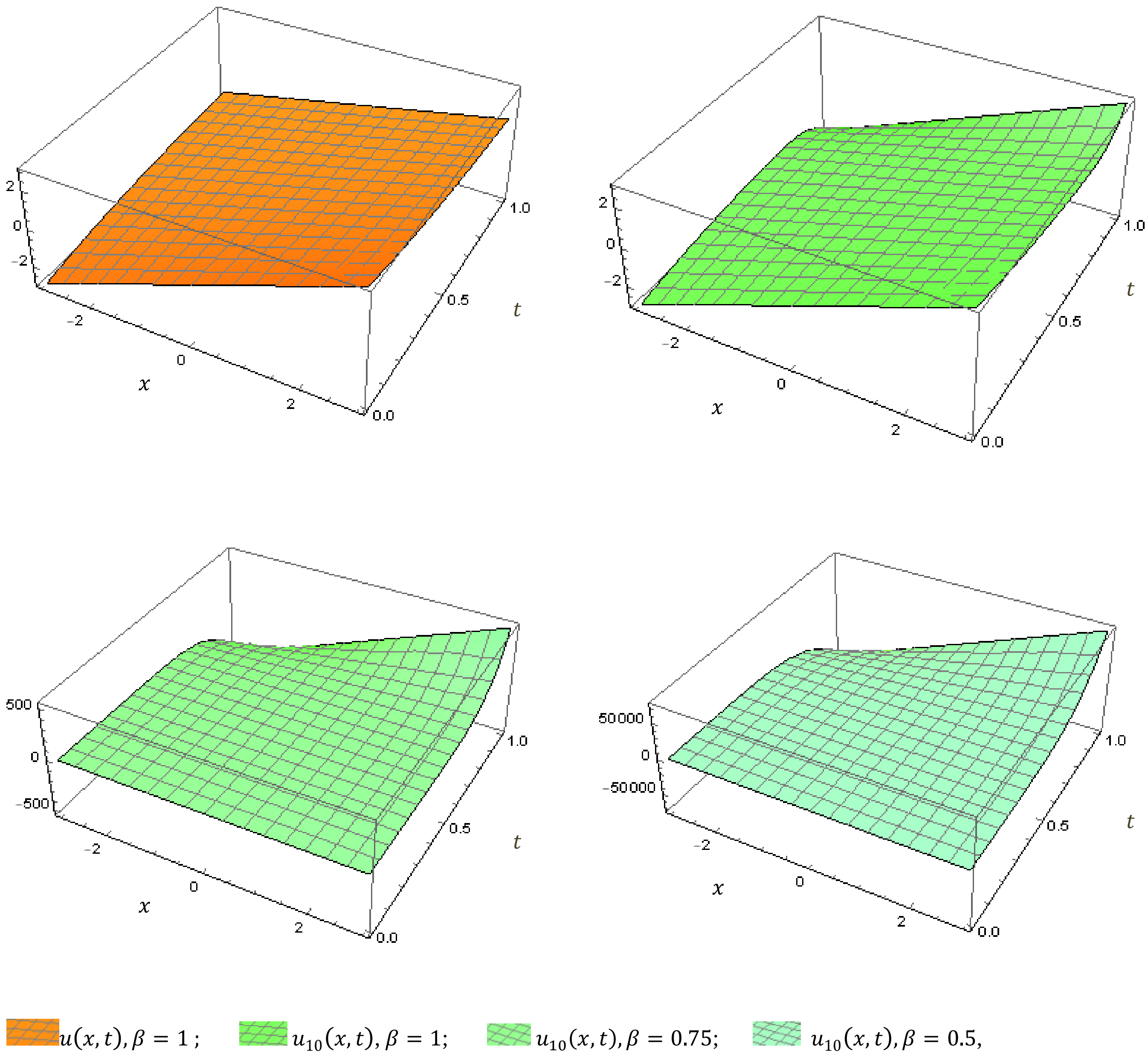

Figure 5. 3D surface plots of exact solution $u(x,t)$ and the tenth MFPS approximate solution $u_{10}(x,t)$ for Problem 4.3 with $t \in [0,1]$ and $x \in [0,1]$ at various values of β.

5. Conclusions

In this work, the LRPS technique was profitably used to create the analytical approximate solution for both homogeneous and non-homogeneous non-linear time-FGDEs along with appropriate initial data. The main idea of the proposed technique is to determine the unknown coefficients of LFSE for the new equation in the Laplace space by using the limit concept. The analytical approximate solutions for the solved gas fractional initial value problems are achieved in rapidly convergent MFPS formulas with fast, more accurate computations with no perturbation, discretization or physical hypotheses. The performance and reliability of the LRPS technique have been studied by carrying out three illustrative examples. The obtained results via our technique are compatible with results obtained by the homotopy perturbation transform technique [32] and the reduced differential transform technique [34]. Consequently, the LRPS technique is a direct, easy and convenient tool to treat a various range of non-linear time-fractional PDEs that arise in engineering and science problems.

Conflict of interest

The authors declare no conflict of interest.

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