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Research article

Geometric inequalities of $\mathcal{PR}$-warped product submanifold in para-Kenmotsu manifold

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Abstract: The main purpose of this paper is to study the properties of $\mathcal{PR}$-semi-invariant submanifold of para-Kenmotsu manifold. We obtain the integrability conditions for the invariant distribution and anti-invariant distribution. We obtain some existence and non-existence results of $\mathcal{PR}$-semi-invariant warped product submanifolds. We provide some necessary and sufficient conditions for $\mathcal{PR}$-semi-invariant submanifold to be a $\mathcal{PR}$-semi-invariant warped product submanifold in para-Kenmotsu manifold. We also derive some sharp inequalities for $\mathcal{PR}$-semi-invariant warped product submanifold in para-Kenmotsu manifolds.

Keywords: para contact manifold; submanifold; warped product
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1. Introduction

Warped product manifold is one of the most significant generalization of Cartesian product of Riemannian manifolds (or pseudo-Riemannian manifolds). This fruitful generalization was initiated by R. L. Bishop and B. O'Neill in 1969 (see [1]). But warped products viewed in the physical and mathematical literature before 1969. For instance, semi-reducible space which is used for warped product by Kruchkovich in 1957 [2]. It has been successfully utilized in general theory of relativity, black holes and string theory.

Warped product geometry was taking more attention in 2002 when Chen studied $\mathcal{CR}$-warped product in K*ä*hlerian manifolds and derived several non-existence results for such warped product manifold of type $M_\perp \times_f M_T$, where $M_\perp$ and M_T stands for anti-invariant and invariant submanifold (see, [3, 4]). Thereafter, authors of [5, 6] studied $\mathcal{CR}$-warped product submanifolds in Sasakian and Kenmotsu manifold and also derived some useful optimal inequalities for such warped products. After that numerous author studied the same (c.f., [7–9]). In 2018, Siraj Uddin derived some useful optimal inequalities for semi-slant warped product submanifolds of Kenmotsu manifold [10].

In 2014, B. Y. Chen explored new class of warped product termed as a $\mathcal{PR}$-warped product and find the exact solutions of the system of partial differential equations associated with $\mathcal{PR}$-warped products [11]. Then after, the authors of [12, 13] studied different classes of $\mathcal{PR}$-warped products in paraCosymplectic manifold. Recently, the authors of [14, 15] studied $\mathcal{PR}$-semi-slant and $\mathcal{PR}$-pseudo-slant warped product submanifold of para-Kenmotsu manifold. Motivated by them, we studied $\mathcal{PR}$-semi-invariant warped product submanifolds in para-Kenmotsu manifold and proved some existence and characterization results and also obtain some optimal inequalities.

This paper is formulated as follows: Section 2 includes some necessary information related to para-contact and para-Kenmotsu manifold. And also contains some important information about the submanifolds theory of para-Kenmotsu manifold. We provide basic information related to warped product submanifold and obtain some existence conditions and non-existence conditions for warped product submanifold of para-Kenmotsu manifold into the section 3. Section 4 includes some results related to integrability conditions of $\mathcal{PR}$-semi invariant submanifold in para-Kenmotsu manifold. In section 5, we provide some characterization results allied to $\mathcal{PR}$-semi-invariant warped product submanifolds. We obtain optimal inequalities for $F \times_f B$ and $B \times_f F$ into the section 6. Lastly, we give some examples of these warped products in section 7.

2. Preliminaries

From the literature, a smooth manifold $\widetilde{M}^{2n+1}$ of dimension $(2n+1)$ furnished an almost paracontact structure (φ, ξ, η) which includes a $(1,1)$-type tensor field φ, a vector field ξ and a 1-form η globally defined on $\widetilde{M}^{2n+1}$ which satisfies the accompanying relation for all $U \in \Gamma(TM^{2n+1})$ [12, 16]:

$$\varphi^2 U = U - \eta(U)\xi, \ \eta(\xi) = 1. \tag{2.1}$$

The tensor field φ induces an almost paracomplex structure $\mathcal{J}$ on a $2n$-dimensional horizontal distribution $\mathfrak{D}$ described as the kernel of 1-form η i.e. $\mathfrak{D} = ker(\eta)$. The horizontal distribution $\mathfrak{D}$ can be expressed as an orthogonal direct sum of the two eigen distribution $\mathfrak{D}^+$ and $\mathfrak{D}^-$, the eigen distributions $\mathfrak{D}^+$ and $\mathfrak{D}^-$ having eigenvalue $+1$ and -1, respectively and each has dimension n. Moreover, $\mathfrak{D}$ is invariant distribution, therefore $T\widetilde{M}^{2n+1}$ can be expressed in the following form;

$$T\widetilde{M}^{2n+1} = \mathfrak{D} \oplus \langle \xi \rangle. \tag{2.2}$$

If $\widetilde{M}^{2n+1}$ admits an almost paracontact structure (φ, ξ, η) then it is said to be an almost paracontact manifold [12, 16]. In view of (2.1), we obtain

$$\eta \circ \varphi = 0, \ \varphi \circ \xi = 0 \ and \ rank(\varphi) = 2n. \tag{2.3}$$

An almost paracontact manifold $\widetilde{M}^{2n+1}$ is called an almost paracontact metric manifold if it admits a pseudo-Riemannian metric of index n compatible with the triplet (φ, ξ, η) by the following relation:

$$g(\varphi U, \varphi V) = \eta(U)\eta(V) - g(U, V), \tag{2.4}$$

for all $U, V \in \Gamma(T\widetilde{M}^{2n+1})$; $\Gamma(T\widetilde{M}^{2n+1})$ denotes the Lie algebra on $\widetilde{M}^{2n+1}$. The dual of the unitary structural vector field ξ allied to g is η i.e.

$$\eta(U) = g(U, \xi). \tag{2.5}$$

By the utilization of (2.1)–(2.4), we attain

$$g(U, \varphi V) + g(\varphi U, V) = 0. \tag{2.6}$$

Definition 2.1. An almost paracontact manifold $\widetilde{M}^{2n+1}$ is said to be a para-Kenmotsu manifold [17] if it fulfills one additional condition

$$(\widetilde{\nabla}_U \varphi)V = \eta(V)\varphi U + g(U, \varphi V)\xi. \tag{2.7}$$

In the relation (2.7); symbol $\widetilde{\nabla}$ indicates Levi-Civita connection with respect to above defined metric g.

Replace V by ξ into (2.7) then applying (2.1), we achieve that

$$\widetilde{\nabla}_U \xi = -\varphi^2 U. \tag{2.8}$$

Proposition 2.2. *On para-Kenmotsu pseudo Riemannian manifold the following relations hold*

$$\eta(\widetilde{\nabla}_U \xi) = 0,\ \widetilde{\nabla}\eta = -\eta \otimes \eta + g, \tag{2.9}$$
$$\mathcal{L}_\xi \varphi = 0,\ \mathcal{L}_\xi \eta = 0, \mathcal{L}_\xi g = -2(g - \eta \otimes \eta), \tag{2.10}$$

where $\mathcal{L}$ denotes the Lie differentiation.

Geometry of submanifolds

Let M be a paracompact and connected smooth pseudo-Riemannian manifold of dimension m and $\widetilde{M}^{2n+1}$ be a para-Kenmotsu manifold. Let $\psi : M \longrightarrow \widetilde{M}^{2n+1}$ be an isometric immersion. Then $\psi(M)$ is known as an isometrically immersed submanifold of a para-Kenmotsu manifold. Let us denote ψ_* for the differential map (or push forward map) of immersion ψ is characterized by $\psi_* : T_pM \longrightarrow T_{\psi(p)}\widetilde{M}^{2n+1}$. Therefore, the induced pseudo-Riemannian metric $\mathfrak{g}$ on $\psi(M)$ is defined as follows $\mathfrak{g}(U, V)_p = g(\psi_* U, \psi_* V)$, for all $U, V \in T_pM$. For our convenience we use M and p on the place of $\psi(M)$ and $\psi(p)$. Now, denoting $\Gamma(TM)$ for set of all vector fields on M, $\Gamma(TM^\perp)$ for the set of all normal vector fields of M, ∇ for induced Levi-Civita connection on TM and $\nabla^\perp$ for normal connection on the normal bundle $\Gamma(TM^\perp)$. Then the Gauss and Weingarten formulas are characterized by the following relations:

$$\widetilde{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.11}$$
$$\widetilde{\nabla}_U \zeta = -A_\zeta U + \nabla^\perp_U \zeta, \tag{2.12}$$

for any $U, V \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^\perp)$, where A_ζ be a shape operator and h is a second fundamental form which are allied to the normal section ζ by the following relation:

$$g(h(U, V), \zeta) = g(A_\zeta U, V). \tag{2.13}$$

The mean curvature vector H on M is described by $H = \frac{1}{m}trace(h)$. Let $p \in M$ and $\{U_1, U_2, \cdots, U_m, U_{m+1}, \cdots, U_{2n+1}\}$ be an orthonormal basis of the $T_p\widetilde{M}^{2n+1}$ in which $\{U_1, U_2, \cdots, U_m\}$ are tangent to M and $\{U_{m+1}, U_{m+2}, \cdots, U_{2n+1}\}$ are normal to M. Now, we set

$$h_{ij}^k = g(h(U_i, U_j), U_k), \tag{2.14}$$

for $i, j \in \{1, 2, \cdots, m\}$ and $k \in \{m+1, m+2, \cdots, 2n+1\}$. The norm of h is defined by the following relation:

$$||h|| = \sqrt{\left(\sum_{i,j=1}^{m} g(h(U_i, U_j), h(U_i, U_j))\right)}. \tag{2.15}$$

An isometrically immersed pseudo-Riemannian submanifold M in a para-Kenmotsu manifold $\widetilde{M}^{2n+1}$ is said to be

- *totally geodesic* if h is vanishes identically i.e., $h \equiv 0$;
- *umbilical* if for a normal vector field ζ, shape operator A_ζ is proportional to identity transformation;
- *totally umbilical* if for any tangent vectors U and V, M satisfies:

$$h(U, V) = g(U, V)H; \tag{2.16}$$

- *minimal* if trace of h (or H) vanishes identically;
- *extrinsic sphere* if M satisfies (2.16) and H is parallel with respect to $\nabla^\perp$ (for more details see, [12, 16]).

From now and in all we will denote the para-Kenmotsu manifold by $\mathcal{K}^{2n+1}$ and its pseudo-Riemannian submanifold by $\mathcal{N}$. Let $tan : T_p\mathcal{K}^{2n+1} \longrightarrow T_p\mathcal{N}$ and $nor : T_p\mathcal{K}^{2n+1} \longrightarrow T_p\mathcal{N}^\perp$ be two endomorphism. Then for any $U \in \Gamma(T\mathcal{N})$, we can write

$$\varphi U = tU + nU, \tag{2.17}$$

where $tU = tan(\varphi U)$ and $nU = nor(\varphi U)$. Similarly, for any $\zeta \in \Gamma(T\mathcal{N}^\perp)$, we have

$$\varphi\zeta = t'\zeta + n'\zeta, \tag{2.18}$$

where $t'\zeta = tan(\varphi\zeta)$ and $n'\zeta = nor(\varphi\zeta)$. In view of (2.6) and (2.16)–(2.18), we attain for any $U, V \in \Gamma(T\mathcal{N})$ and $\forall \zeta_1, \zeta_2 \in \Gamma(T\mathcal{N}^\perp)$ that

$$g(n'\zeta_1, \zeta_2) = -g(\zeta_1, n'\zeta_2), \; g(tU, V) = -g(U, tV). \tag{2.19}$$

Moreover, by the consequences of (2.6), (2.17) and (2.18), we have

$$g(nU, \zeta) = -g(U, t'\zeta). \tag{2.20}$$

Moreover, the covariant derivative of φ, n and t are charcterized by

$$(\widetilde{\nabla}_U\varphi)V = \widetilde{\nabla}_U\varphi V - \varphi\widetilde{\nabla}_U V, \tag{2.21}$$

$$(\nabla_U n)V = \nabla^{\perp}_U nV - n\nabla_U V, \tag{2.22}$$
$$(\nabla_U t)V = \nabla_U tV - t\nabla_U V, \tag{2.23}$$

for some $U, V \in \Gamma(T\mathcal{N})$.

Proposition 2.3. *If $\mathcal{N}$ is tangent to ξ in a $\mathcal{K}^{2n+1}$. Thus, we have*

$$(\nabla_U t)V = A_{nV}U + t' h(U, V) + \eta(V)tU - g(tU, V)\xi, \tag{2.24}$$
$$(\nabla_U n)V = n' h(U, V) + \eta(V)nU - h(U, tV), \tag{2.25}$$

for every $U, V \in \Gamma(T\mathcal{N})$.

Proof. By the consequence of (2.11), (2.12), (2.17) and (2.21)–(2.23), we have

$$(\widetilde{\nabla}_U\varphi)V + A_{nV}U = -t' h(U, V) + (\nabla_U t)V - n' h(U, V) + h(U, tV) + (\nabla_U n)V,$$

for any $U \in \Gamma(T\mathcal{N})$. Employing (2.7) and (2.17) into above expression then considering tangential part and normal part of obtained expression, we have (2.24) and (2.25), respectively. □

Proposition 2.4. *Let $\mathcal{N}$ is normal to ξ in a $\mathcal{K}^{2n+1}$. Then, we have*

$$(\nabla_U t)V = t' h(U, V) + A_{nV}U, \tag{2.26}$$
$$(\nabla_U n)V = n' h(U, V) + g(U, tV)\xi - h(U, tV), \tag{2.27}$$

for every $U, V \in \Gamma(T\mathcal{N})$.

Proof. Immediately, form (2.7), (2.11), (2.12), (2.17), (2.21) and (2.22), we derive (2.26) and (2.27). □

Proposition 2.5. *If ξ is tangent to $\mathcal{N}$ in $\mathcal{K}^{2n+1}$. Thus, we obtain*

$$(\nabla_U t')\zeta = A_{n'\zeta}U - g(nU, \zeta)\xi - tA_\zeta U, \tag{2.28}$$
$$(\nabla_U n')\zeta = -h(U, t'\zeta) - nA_\zeta U, \tag{2.29}$$

for any $U \in \Gamma(T\mathcal{N})$ and $\zeta \in \Gamma(T\mathcal{N}^\perp)$.

Proof. Employing (2.11), (2.12), (2.18), (2.22) and (2.23) into (2.21), we achieve that

$$(\widetilde{\nabla}_U\varphi)\zeta = (\nabla_U n')\zeta - A_{n'\zeta}U + tA_\zeta U + nA_\zeta U + h(U, t'\zeta) + (\nabla_U t')\zeta,$$

for any $U \in \Gamma(TM)$. Utilizing (2.7) and (2.17) into above expression, we achieve (2.28) and (2.29). □

Proposition 2.6. *If $\mathcal{N}$ is normal to ξ in $\mathcal{K}^{2n+1}$. Then we achieve for any $U \in \Gamma(T\mathcal{N})$ and $\zeta \in \Gamma(T\mathcal{N}^\perp)$ that*

$$(\nabla_U t')\zeta = A_{n'\zeta}U - tA_\zeta U - \eta(\zeta)tU, \tag{2.30}$$
$$(\nabla_U n')\zeta = -nA_\zeta U + \eta(\zeta)nU + g(U, t'\zeta)\xi - h(U, tV). \tag{2.31}$$

Proof. From (2.7), (2.11), (2.12), (2.17), (2.18) and (2.21), we achieve

$$\begin{aligned}&\nabla_U t'\zeta + h(U, t'\zeta) - A_{n'\zeta}U + \nabla^{\perp}_U n'\zeta \\ &= t\nabla^{\perp}_U\zeta + n\nabla^{\perp}_U\zeta - t'A_\zeta U - n'A_\zeta U + \eta(V)\varphi U + g(U, \varphi\zeta)\xi.\end{aligned} \tag{2.32}$$

Now, employing (2.22) and (2.23) into (2.32) then comparing tangential and normal parts, we have (2.30) and (2.31). □

Let us consider $U, \xi \in \Gamma(T\mathcal{N})$, thus by the direct application of (2.8) and (2.11), we have

$$\nabla_U\xi = -\varphi^2 U,\ h(U,\xi) = 0. \tag{2.33}$$

If $\xi \in \Gamma(T\mathcal{N}^{\perp})$, then by the consequence of (2.8) and (2.12), we have

$$A_\xi U = U,\ \nabla^{\perp}_U\xi = 0. \tag{2.34}$$

From above we conclude the following remarks

Remark 2.7. Let M is tangent to ξ in $\mathcal{K}^{2n+1}$, then the relation (2.33) holds on $\mathcal{N}$.

Remark 2.8. Let M is normal to ξ in $\mathcal{K}^{2n+1}$, then the Eq (2.34) holds in $\mathcal{N}$.

Lemma 2.9. *If $\mathcal{N}$ is tangent to ξ in $\mathcal{K}^{2n+1}$, then the endomorphism t and bundle 1-form n satisfies:*

$$t^2 + t'n = \mathcal{I} - \eta\otimes\xi, \tag{2.35}$$

$$nt + n'n = 0. \tag{2.36}$$

Proof. Operating φ on (2.17), we have

$$\varphi^2 U = \varphi(tU) + \varphi(nU).$$

Employing (2.1) and (2.17) into above expression, we achieve

$$U - \eta(U)\xi = t^2U + ntU + t'nU + n'nU.$$

Comparing tangential and normal part of above expression, we get (2.35) and (2.36). □

In similar way we prove the following result:

Lemma 2.10. *If ξ is normal to $\mathcal{N}$ in $\mathcal{K}^{2n+1}$. Then, we obtain*

$$tt' + t'n' = 0, \tag{2.37}$$

$$nt' + n'^2 = \mathcal{I}. \tag{2.38}$$

3. Warped product submanifold

Let (B, g_B) and (F, g_F) be two pseudo-Riemannian manifolds with pseudo-Riemannian metric g_B and g_F, respectively and $f : B \longrightarrow (0, \infty)$ is a C^∞-function. Then, $\mathcal{N} = B \times_f F$ is called warped product manifold [1, 12] if the pseudo-Riemannian product manifold $B \times F$ furnished a pseudo-Riemannian warping metric g fulfill the accompanying condition:

$$g = g_B + f^2 g_F, \tag{3.1}$$

where f is called warping function. If f is constant on B then M is called trivial warped product manifold. For a warped product $\mathcal{N} = B \times_f F$, F is called a fiber and B is called a base. The leaves $B \times \{p\} = \sigma_2^{-1}(q)$ and $\{p\} \times F = \sigma_1^{-1}(p)$, for $(p, q) \in \mathcal{N}$ are pseudo-Riemannian manifolds. Now, recall the following proposition.

Proposition 3.1. *For all $V_1, V_2 \in \Gamma(TF)$ and $U_1, U_2 \in \Gamma(TB)$, we obtain on $\mathcal{N} = B \times_f F$ that*

$$\nabla_{U_1} U_2 \in \Gamma(TB), \tag{3.2}$$

$$\nabla_{U_1} Z = \nabla_Z U_1 = (U_1 \ln f) Z, \tag{3.3}$$

$$\nabla'_{V_1} V_1 = \nabla_{V_1} V_2 + g(V_1, V_2) \nabla(\ln f), \tag{3.4}$$

where grad(f) indicates the gradient of positive function f characterized by $g(\nabla f, U_1) = U_1(f)$.

Remark 3.2. If $\mathcal{N} = B \times_f F$ is a warped product manifold, then the pseudo-Riemannian submanifold F is totally umbilical and the pseudo-Riemannian submanifold B is totally geodesic in $\mathcal{N}$.

In 1956, J. F. Nash derived a very useful theorem in Riemannian geometry which is known as Nash embedding theorem. The theorem states "every Riemannian manifold can be isometrically embedded in some Euclidean spaces" (see, [18]). This theorem shows that any warped product of Riemannian (or pseudo-Riemannian) manifold can be realized (or embedded) as a Riemannian (or pseudo-Riemannian) submanifolds in Euclidean space. Due to this fact, B. Y. Chen asked a very interesting question in 2002. The question is "What can we conclude from an isometric immersion of an arbitrary warped product into a Euclidean space or into a space form with arbitrary codimension" (see, [19]). Thereafter several geometers studied warped product submanifold into a different ambient manifolds. After that the warped product was become very active and populor research area among the geometers. Due to this fact, we have studied the warped product submanifold in para-Kenmotsu manifold which is not studied yet. Now, we prove some results related to existence of warped product submanifolds in para-Kenmotsu manifold.

Proposition 3.3. *There does not exists a non-trivial warped product submanifold $\mathcal{N} = B \times_f F$ in $\mathcal{K}^{2n+1}$ such that $\xi \in \Gamma(TF)$.*

Proof. If there exist a non-trivial warped product $\mathcal{N} = B \times_f F$, then by the consequence of (3.3), we attain $\nabla_{U_1} V_1 = U_1(\ln f) V_1$ for all $U_1 \in \Gamma(TB)$ and $V_1 \in \Gamma(TF)$. Now taking $V_1 = \xi$ then applying (2.1) and (2.33) into above expression, we obtain $U_1(\ln f) g(\xi, \xi) = -g(U_1, \xi) = 0$. This shows that f is constant function, contradiction. □

Proposition 3.4. *There exists a non-trivial pseudo-Riemannian warped product submanifold $\mathcal{N} = B \times_f F$ in $\mathcal{K}^{2n+1}$ such that $\xi \in \Gamma(TB)$.*

Proof. If there exists a non-trivial warped product submanifold then by the utilization of relation (3.3), we obtain $\xi(\ln f)|V_1|^2 = -|V_1|^2$ for any space-like or time-like vector field $V_1 \in \Gamma(TF)$. Now using Proposition 3.3 into above expression, we have $\nabla f = -\xi$, which is first order partial differential equation, in this case f is not constant. □

Above Propositions shows that the warped product submanifold in $\mathcal{K}^{2n+1}$ exists if ξ is tangent to the first factor. Thus by the direct consequence of (2.8) and (3.3), we have

$$\xi(\ln f) = -1, \ h(U_1, \xi) = 0. \tag{3.5}$$

4. $\mathcal{PR}$-semi invariant submanifolds

Definition 4.1. Let $\mathcal{N}$ be tangent to ξ in $\mathcal{K}^{2n+1}$. Then $\mathcal{N}$ is said to be a $\mathcal{PR}$-semi-invariant [12] if there exists a φ-invariant distribution $\mathfrak{D}_T$ and a φ-anti-invariant distribution $\mathfrak{D}_\perp$ satisfying

$$T\mathcal{N} = \mathfrak{D}_T \oplus \mathfrak{D}_\perp \oplus \langle \xi \rangle. \tag{4.1}$$

Let P and Q are two orthogonal projections on $\mathfrak{D}_T$ and $\mathfrak{D}_\perp$, respectively. Thus for $U \in \Gamma(T\mathcal{N})$ can be expressed as follows:

$$U = PU + QU + \eta(U)\xi. \tag{4.2}$$

From (4.2), we have

$$P^2 - P = 0, \ Q^2 - Q = 0, \ PQ = QP = 0. \tag{4.3}$$

By the application of (2.17) and (4.2), we achieve that,

$$\varphi U = tPU + nPU + tQU + nQU,$$

using the fact $\mathcal{N}$ is $\mathcal{PR}$-semi invariant, we acquire that

$$\varphi U = tPU + nQU, \ nPU = 0, \ tQU = 0. \tag{4.4}$$

In the light of (2.18), we attain that $\varphi(tU + nU) = U - \eta(U)\xi$. Reuse of (2.18) and (4.4) gives $t^2 = U - \eta(U)\xi$. In view of last expression and (2.19), we conclude that the paracontact structure admits on $\mathfrak{D}_T \oplus \langle \xi \rangle$. Therefore, the dimension of the distribution $\mathfrak{D}_T$ must be even. Despite if we denotes ν for φ-invariant subspace of $T\mathcal{N}^\perp$, thus the normal bundle $T\mathcal{N}^\perp$ can be expressed as follows:

$$T\mathcal{N}^\perp = n\mathfrak{D}_\perp \oplus \nu. \tag{4.5}$$

Theorem 4.2. *Let $\psi : \mathcal{N} \longrightarrow \mathcal{K}^{2n+1}$ be an isometric immersion. Then necessary and sufficient condition for $\mathcal{N}$ to be a $\mathcal{PR}$-semi-invariant submanifold in $\mathcal{K}^{2n+1}$ is $n \circ t = 0$.*

Proof. Firstly, suppose that $\mathcal{N}$ is immersed as a $\mathcal{PR}$-semi invariant submanifold in para-Kenmotsu manifold $\mathcal{K}^{2n+1}$, then by application of (2.17), (4.2) and (4.4), we achieve

$$ntU = nQtPU = nt(QP)U = 0, \ \forall U \in \Gamma(T\mathcal{N}). \tag{4.6}$$

Thus the relation (4.6) shows that the $n \circ t = 0$.

Conversely, let M be a pseudo-Riemannian submanifold tangent to ξ in a para-Kenmotsu manifold $\mathcal{K}^{2n+1}$ satisfying $n \circ t = 0$. Therefore, by the consequences of (2.6) and (2.35), we attain that $g(U, tt'\zeta) = 0$, $\forall U \in \Gamma(T\mathcal{N})$ and $\zeta \in \Gamma(T\mathcal{N}^{\perp})$. In view of (2.37), we have $t'n' = 0$. From (2.35) and (2.38), we find

$$t^3 = t, \ n'^3 = n'. \tag{4.7}$$

If we put $t^2 = P$ and $\mathcal{I} - t^2 = Q$, we have (4.4). This implies that P and Q are orthogonal projections on distributions D_T and $D_{\perp}$, respectively. Further, from (4.7) we achieve $t = tP$, $n = nQ$, $nP = 0$ and $tQ = 0$. This shows that $\mathfrak{D}_T$ and $\mathfrak{D}_{\perp}$ are invariant and anti-invariant, respectively. Hence completes the proof of theorem. □

Lemma 4.3. *Let $\mathcal{N}$ be a $\mathcal{PR}$-semi-invariant submanifold in $\mathcal{K}^{2n+1}$. Then we obtain*

$$g(h(U_1, \varphi U_2), \varphi\zeta) = -g(h(U_1, U_2), \zeta), \tag{4.8}$$

$$g(h(\varphi U_1, V_1), \varphi\zeta) = -g(h(U_1, V_1), \zeta), \tag{4.9}$$

$$g(h(U_1, V_1), \zeta) = g(h(V_1, \varphi U_1), \zeta), \tag{4.10}$$

for every $V_1 \in \Gamma(\mathfrak{D}_{\perp})$ and $U_1, U_2 \in \Gamma(\mathfrak{D}_T)$.

Proof. By the consequence (2.7), (2.11) and (2.21), we obtain $g(h(U_1, \varphi U_2), \varphi\zeta) = g(\varphi\widetilde{\nabla}_{U_1} U_2, \varphi\zeta)$. By direct use of (2.4) into above expression gives Eq (4.8). Similarly, we obtain (4.9) and (4.10). □

Lemma 4.4. *Let $\mathcal{N}$ be a $\mathcal{PR}$-semi-invariant submanifold in $\mathcal{K}^{2n+1}$. Then we have*

$$g(h(U_1, V_1), \varphi\zeta) = -\ g(\nabla^{\perp}_{U_1}\varphi V_1, \zeta), \tag{4.11}$$

$$g(h(V_1, V_2), \varphi\zeta) = -\ g(\nabla^{\perp}_{V_1}\varphi V_2, \zeta), \tag{4.12}$$

for any $V_1 \in \Gamma(\mathfrak{D}_{\perp})$ and $U_1, U_2 \in \Gamma(\mathfrak{D}_T)$.

Proof. By the direct consequence of (2.4), (2.11)–(2.13), (2.22) and (4.4), we achieve (4.11) and (4.12). □

Theorem 4.5. *Let $\mathcal{N}$ be a $\mathcal{PR}$-semi-invariant submanifold in $\mathcal{K}^{2n+1}$. Then, the invariant distribution $\mathfrak{D}_T$ is integrable if and only if h fulfills*

$$h(U_1, \varphi U_2) = h(\varphi U_1, U_2), \tag{4.13}$$

for any $U_1, U_2 \in \Gamma(\mathfrak{D}_T)$.

Proof. By the consequence of (2.11), we obtain for any $V_1 \in \Gamma(\mathfrak{D}_{\perp})$ that;

$$g(\nabla_{U_1} U_2, V_1) = \eta(V_1)\eta(\nabla_{U_1} U_2) - g(\varphi\widetilde{\nabla}_{U_1} U_2, \varphi V_1).$$

Form (2.7) and (2.21), above equation reduces into the following form

$$g(\nabla_{U_1} U_2, V_1) = -g(\widetilde{\nabla}_{U_1} \varphi U_2, \varphi V_1).$$

By the application of Eqs (2.11), (2.17) and (4.4) above expression reduces into the following form

$$g(\nabla_{U_1} U_2, V_1) = -g(h(U_1, \varphi U_2), \varphi V_1). \tag{4.14}$$

Interchange the role of U_1 and U_2 into the above expression, we obtain

$$g(\nabla_{U_2} U_1, V_1) = -g(h(\varphi U_1, U_2), \varphi V_1). \tag{4.15}$$

In view of (4.14) and (4.15), we get the result. □

Theorem 4.6. *Let $\mathcal{N}$ be a $\mathcal{PR}$-semi-invariant submanifold in $\mathcal{K}^{2n+1}$. Then the distribution $\mathfrak{D}_\perp$ is integrable if shape operator satisfying:*

$$A_{\varphi V_1} V_2 = A_{\varphi V_2} V_1, \tag{4.16}$$

for any $V_1, V_2 \in \Gamma(\mathfrak{D}_\perp)$.

Proof. Let us consider any $U_1 \in \Gamma(\mathfrak{D}_T)$, then by the consequence of Eq (2.11), we enlist

$$g(\nabla_{V_1} V_2, U_1) = \eta(\nabla_{V_1} V_2)\eta(U_1) - g(\varphi\widetilde{\nabla}_{V_1} V_2, \varphi U_1).$$

Now employing Eqs (2.7) and (2.21) into above relation then we attain

$$g(\nabla_{V_1} V_2, U_1) = -g(\widetilde{\nabla}_{V_1} \varphi V_2, \varphi U_1).$$

Applying (2.12) and (4.4) into above expression, we achieve

$$g(\nabla_{V_1} V_2, U_1) = g(A_{\varphi V_2} V_1, \varphi U_1). \tag{4.17}$$

Interchange the role of U_1 and U_2 into (4.17), we have

$$g(\nabla_{V_2} V_1, U_1) = g(A_{\varphi V_1} V_2, \varphi U_1). \tag{4.18}$$

From (4.17) and (4.18), we get (4.16). This completes the proof. □

Lemma 4.7. *Let $\mathcal{N}$ be a $\mathcal{PR}$-semi-invariant submanifold in $\mathcal{K}^{2n+1}$. Then, we conclude that*

$$(\nabla_{U_1} t)\xi = tU_1, \ (\nabla_{V_1} t)\xi = 0, \tag{4.19}$$

$$(\nabla_{U_1} n)\xi = 0, \ (\nabla_{V_1} n)\xi = nV_1, \tag{4.20}$$

for any $U_1 \in \Gamma(\mathfrak{D}_T)$ and $V_1 \in \Gamma(\mathfrak{D}_\perp)$.

Proof. By the direct consequence of (2.24), (2.25), (4.3) and (4.4). □

5. $\mathcal{PR}$-semi invariant warped product

A $\mathcal{PR}$-semi-invariant warped product submanifold was studied in [12]. In this section, we study $\mathcal{PR}$-semi-invariant warped products into a para-Kenmotsu manifold. It is defined as $B \times_f F$ or $F \times_f B$, where B is a φ-anti-invariant submanifold of $\mathcal{N}$ and F be a φ-invariant submanifold of $\mathcal{N}$ which are the integral manifolds of anti-invariant distributions $\mathfrak{D}_\perp$ and invariant distributions $\mathfrak{D}_T$, respectively (see, [12]). If f is a constant function, then $\mathcal{N}$ is called $\mathcal{PR}$-semi-invariant product (or trivial or proper warped product).

Proposition 5.1. *There does not exists a $\mathcal{PR}$-semi-invariant non-trivial warped product submanifold of the form $\mathcal{N} = F \times_f B$ in $\mathcal{K}^{2n+1}$ such that $\xi \in \Gamma(TB)$.*

Proof. If there exists a proper warped product then $\nabla_{U_1}\xi = U_1(\ln f)\xi$, for all tangent vector field U_1 in $\Gamma(TF)$. In view of (2.1) and (2.33), we have $U_1(\ln f) = 0$. This means f can't be non-constant, contradiction. □

Proposition 5.2. *There does not exists a $\mathcal{PR}$-semi-invariant non-trivial warped product submanifold of the form $\mathcal{N} = B \times_f F$ in $\mathcal{K}^{2n+1}$ such that $\xi \in \Gamma(TF)$.*

Proof. If there exists a proper warped product then $\nabla_{U_1}\xi = U_1(\ln f)\xi$, for all $U_1 \in \Gamma(TF)$. In view of (2.1) and (2.33), we have $U_1(\ln f) = 0$. This relation shows that the function f is constant, contradiction. □

Proposition 5.3. *There does not exists a $\mathcal{PR}$-semi-invariant non-trivial warped product submanifold of the form $\mathcal{N} = B \times_f F$ in $\mathcal{K}^{2n+1}$ such that $\xi \in \Gamma(T\mathcal{N}^\perp)$.*

Proof. Let $\mathcal{N}$ be a non-trivial $\mathcal{PR}$-semi-invariant warped product with $\xi \in \Gamma(TM^\perp)$. Then by the consequence of (2.4), (2.7), (2.11), (2.21) and (3.3), we obtain $\forall U_1 \in \Gamma(TF)$ and $V_1 \in \Gamma(TB)$ that;

$$g(h(V_1, \varphi U_1), \varphi V_1) = -U_1(\ln f)\|V_1\|^2.$$

Now, we replace V_1 by φV_1 into above relation and applying Eq (2.1), we obtain

$$0 = U_1(\ln f)\|V_1\|^2,$$

since V_1 is not a lightlike vector, therefore, f is constant on F, contradiction. Hence complete the proof. □

5.1. $\mathcal{PR}$-semi-invariant warped product of the form $\boldsymbol{B} \times_f \boldsymbol{F}$

In this section, we analyze the geometry of $\mathcal{PR}$-semi-invariant warped product submanifold of type $B \times_f F$. By the virtue of Proposition 5.2 implies for above mention warped product ξ is tangent to B. We derive some important results for $\mathcal{PR}$-semi-invariant warped product submanifold of type $B \times_f F$:

Lemma 5.4. *Let $\mathcal{N} = B \times_f F$ be a non-trivial $\mathcal{PR}$-semi invariant warped product submanifold in $\mathcal{K}^{2n+1}$. Then, we achieve for all $U_1, U_2 \in \Gamma(TF)$ and $V_1, V_2 \in \Gamma(TB)$ that:*

$$A_{nV_1}U_1 = -\eta(V_1)\varphi U_1 - V_1(\ln f)tU_1, \tag{5.1}$$

$$A_{nV_1}V_2 = A_{nV_2}V_1 = t' h(V_1, V_2) = 0, \tag{5.2}$$

$$h(tU_1, U_2) = h(U_1, tU_2) = -g(U_1, U_2)\nabla f + n' h(U_1, U_2). \tag{5.3}$$

Proof. From the Eqs (2.4), (2.11), (2.13) and (4.4), we obtain $g(A_{nV_1}U_1, U_2) = -g(\widetilde{\nabla}_{U_1}U_2, \varphi V_1)$. Now applying (2.6) and (2.21) into last expression, we arrive that

$$g(A_{nV_1}U_1, U_2) = -g(\widetilde{\nabla}_{U_1}\varphi U_2, V_1) + g((\widetilde{\nabla}_{U_1}\varphi)U_2, V_1).$$

In view of (2.7), above equation reduces into the following form

$$g(A_{nV_1}U_1, U_2) = g(\varphi U_1, U_2)\eta(V_1) - g(\widetilde{\nabla}_{U_1}\varphi U_2, V_1). \tag{5.4}$$

By virtue of (2.11), (2.17), (5.4) and property of Riemannian connection gives

$$g(A_{nV_1}U_1, U_2) = -g(\varphi U_1, U_2)\eta(V_1) + g(tU_2, \nabla_{U_1}V_1)). \tag{5.5}$$

In light of (2.19) and (3.3), we get (5.1). Employing (2.7), (2.11), (2.12), (2.17), (2.18) and (4.4) into (2.21), we achieve that

$$-g(\varphi V_1, V_2)\xi - t^{'}h(V_1, V_2) + A_{nV_2}V_1 = t\nabla_{V_1}V_2 - \eta(V_2)nV_1 - n\nabla_{V_1}V_2 - n^{'}h(V_1, V_2).$$

Beacause $\mathcal{N}$ is warped product then B is totally geodesic then considering the tangential part of obtained expression

$$-g(\varphi V_1, V_2)\xi = -A_{nV_2}V_1 - t\nabla_{V_1}V_2.$$

Using (3.2) and (4.4), we arrive at Eq (5.2). Using (2.7), (2.11), (2.17), (2.18) and (4.4) into (2.21), then we find

$$g(U_1, \varphi U_2)\xi - h(U_1, tU_2) + n\nabla_{U_1}U_2 = \nabla_{U_1}tU_2 - t^{'}h(U_1, U_2) - t\nabla_{U_1}U_2 - n^{'}h(U_1, U_2).$$

Now using (3.3) into above expression then after taking normal part of above expression, we have

$$h(U_1, tU_2) = -n^{'}h(U_1, U_2) - g(U_1, U_2)n(\nabla(\ln f)).$$

Interchange the role of U_1 and U_2 into above relation gives (5.3). □

Theorem 5.5. *Let $\mathcal{N} = B \times_f F$ be a $\mathcal{PR}$-semi invariant warped product submanifold in $\mathcal{K}^{2n+1}$. Then the both distributions $\mathfrak{D}_\perp$ and $\mathfrak{D}_T$ are integrable.*

Proof. By direct consequence of (4.4), Theorem 4.5, Theorem 4.6 and Lemma 5.4, we easily achieve the result. □

Lemma 5.6. *If $\mathcal{N} = B \times_f F$ be a non-trivial $\mathcal{PR}$-semi-invariant warped product submanifold in $\mathcal{K}^{2n+1}$. Then, we have*

$$g(h(U_1, U_2), nV_1) = (V_1 \ln f + \eta(V_1))g(U_1, tU_2), \tag{5.6}$$
$$g(h(U_1, V_1), nV_1) = g(h(U_1, V_2), nV_1), \tag{5.7}$$
$$g(h(V_1, V_2), nV_1') = g(h(V_1, V_1'), nV_2), \tag{5.8}$$

for all $V_1, V_2, V_1' \in \Gamma(TB)$ and $U_1, U_2 \in \Gamma(TF)$.

Proof. By the application of Eqs (2.6), (2.11), (2.17) and (4.4), we have

$$g(h(U_1, U_2), nV_1) = -g(\varphi\widetilde{\nabla}_{U_1} U_2, V_1).$$

Now utilizing (2.7) and (2.21) into above expression, we achieve

$$g(h(U_1, U_2), nV_1) = -g(\widetilde{\nabla}_{U_1}\varphi U_2, V_1) - g(\varphi U_1, U_2)\eta(V_1). \tag{5.9}$$

By the consequence of (2.11), (2.17), (4.4) and (5.9), the above relation reduces into the following form

$$g(h(U_1, U_2), nV_1) = -g(tU_1, U_2)\eta(V_1) + g(tU_2, \nabla_{U_1} V_1). \tag{5.10}$$

By the virtue of (2.19), (3.3) and (5.10), we achieve (5.6). By the consequence of (2.6), (2.11), (2.17) and (4.4), we obtain $g(h(U_1, V_1), nV_2) = -g(\varphi\widetilde{\nabla}_{U_1} V_1, V_2)$. In light of (2.7) and (2.21), we get (5.7). Proceed similar process for accomplish the Eq (5.8). □

Lemma 5.7. *Let $\mathcal{N} = B\times_f F$ be a non-trivial $\mathcal{PR}$-semi invariant warped product submanifold in $\mathcal{K}^{2n+1}$. Then for all $U_1, U_2 \in \Gamma(TF)$ and $V_1, V_2 \in \Gamma(TB)$, we obtain*

$$(\nabla_{U_1} t)V_1 = -V_1(\ln f)tU_1, \tag{5.11}$$

$$(\nabla_{U_1} t)U_2 = -g(U_1, tU_2)\nabla \ln f, \tag{5.12}$$

$$(\nabla_{V_1} t)U_1 = 0, \tag{5.13}$$

$$(\nabla_{V_1} t)V_2 = 0. \tag{5.14}$$

Proof. By the direct use of (3.3), (3.4) and (4.4), we obtain (5.11)–(5.14). □

Now, we prove some results related to characterization of $\mathcal{PR}$-semi invariant warped product submanifold:

Theorem 5.8. *Let $\mathcal{N}$ be a $\mathcal{PR}$-semi-invariant submanifold in $\mathcal{K}^{2n+1}$. Then $\mathcal{N}$ is form a $\mathcal{PR}$-semi-invariant warped product submanifold if and only if the shape operator A satisfies:*

$$A_{\varphi V_1} U_1 = (\eta(V_1) + V_1(\mu))\varphi U_1,\ V_1 \in \Gamma(\mathfrak{D}_\perp \oplus \langle\xi\rangle) \text{ and } U_1 \in \Gamma(\mathfrak{D}_T), \tag{5.15}$$

for any C^∞-function μ over $\mathcal{N}$ which satisfying $U_2(\mu) = 0$, $\forall U_2 \in \Gamma(\mathfrak{D}_T)$.

Proof. Let $\mathcal{N} = B \times_f F$ be a non-trivial warped product submanifold in $\mathcal{K}^{2n+1}$. Thus, by the direct application of Eqs (2.17), (4.4) and (5.1), we attain for any $V_1 \in \Gamma(TB)$ and $U_1 \in \Gamma(TF)$ that

$$A_{\varphi V_1} U_1 = (\eta(V_1) + V_1(\ln f))\,\varphi U_1.$$

If we take $\mu = \ln f$ in above expression and use fact that $\mathcal{N}$ is warped product, then we accomplished (5.15).

Conversely, let $\mathcal{N}$ be a $\mathcal{PR}$-semi-invariant submanifold in $\mathcal{K}^{2n+1}$ satisfying (5.15). Thus, by the utilization of (2.7), (2.11) and (2.16), we obtain for all $U_1 \in \Gamma(\mathfrak{D}_T)$ and $V_1 \in \Gamma(\mathfrak{D}_\perp \oplus \langle\xi\rangle)$ that

$$g(\nabla_{V_1} V_2, \varphi U_1) = -g(\widetilde{\nabla}_{V_1}\varphi V_2, U_1).$$

In light of (2.12) and (5.15), the above equation taking into the form

$$g(\nabla_{V_1}V_2, \varphi U_1) = -(V_1(\mu) - \eta(V_1))g(\varphi U_1, V_1) = 0.$$

Above discussion demonstrate that the distribution $\mathfrak{D}_\perp \oplus \langle\xi\rangle$ is form totally geodesic foliation. Furthermore, by the consequence of (2.4), (2.11) and (2.12), we obtain $\forall U_1, U_2 \in \Gamma(\mathfrak{D}_T)$ and $V_1 \in \Gamma(\mathfrak{D}_\perp \oplus \langle\xi\rangle)$ that

$$g(\nabla_{U_1}U_2, V_1) = \eta(\widetilde{\nabla}_{U_1}U_2)\eta(V_1) - g(\varphi\widetilde{\nabla}_{U_1}U_2, \varphi V_1).$$

Now employing (2.7), (2.8), (2.21) and (4.4) into above relation:

$$g(\nabla_{U_1}U_2, V_1) = \eta(V_1)g(U_1, U_2) + g(\varphi U_2, \widetilde{\nabla}_{U_1}\varphi V_1).$$

In view of (2.4), (2.12) and (5.15) above relation reduces into the following form

$$g(\nabla_{U_1}U_2, V_1) = \eta(V_1)g(U_1, U_2) - g(A_{\varphi V_1}U_1, \varphi U_2) = -V_1(\mu)g(U_1, U_2).$$

Above calculation shows that the distribution $\mathfrak{D}_\perp$ defines spherical foliation. After using Hiepko Theorem [20], we can conclude that $B \times_f F$ is a warped product. □

Theorem 5.9. *Let $\mathcal{N}$ be a $\mathcal{PR}$-semi-invariant submanifold in $\mathcal{K}^{2n+1}$. Then $\mathcal{N}$ is form a non-trivial $\mathcal{PR}$-semi-invariant warped product $\mathcal{N} = B \times_f F$ if and only if the endomorphism t satisfies the following condition:*

$$(\nabla_U t)V = \eta(V)tU - g(PU, PV)\nabla(\mu), \ \forall U, V \in \Gamma(T\mathcal{N}), \tag{5.16}$$

for any C^∞-function μ over $\mathcal{N}$ fulfilling $U_2(\mu) = 0$, $\forall U_2 \in \Gamma(\mathfrak{D}_T)$.

Proof. Let $\mathcal{N} = B \times_f F$ be a non-trivial warped product submanifold in $\mathcal{K}^{2n+1}$, then by the application of (4.2), we attain that

$$(\nabla_U t)V = (\nabla_{QU}t)QV + (\nabla_{QU}t)PV + (\nabla_{PU}t)QV + \eta(V)(\nabla_U t)\xi + (\nabla_{PU}t)PV.$$

We obtain by the utilization of (2.24) and Lemma 5.7 that

$$(\nabla_U t)V = \eta(V)tU - g(PU, PV)\nabla(\ln f),$$

Then, by taking $\mu = \ln f$ and using the fact that $\mathcal{N}$ is warped product, we accomplished (5.16).

Conversely, let $\mathcal{N}$ be a $\mathcal{PR}$-semi-invariant submanifold in $\mathcal{K}^{2n+1}$ satisfying (5.16). If we interchange U with V_1 and V with V_2 into the Eq (5.16), then we obtain for any $V_1, V_2 \in \Gamma(\mathfrak{D}_\perp \oplus \langle\xi\rangle$ and $U_1 \in \Gamma(\mathcal{D}_T)$ that;

$$g(\nabla_{V_1}t)V_2, U_1) = g(\nabla_{V_1}V_2, tU_1) = 0. \tag{5.17}$$

On the other hand, by the use Eqs (2.23), (3.4) and (5.16), we compute

$$g((\nabla_{U_1}t)U_2, V_1) = g(\nabla_{U_1}U_2, tV_1) + g(\mathfrak{h}(U_1, tU_2), V_1) = -g(V_1, \nabla\mu)g(U_1, tU_2).$$

By above relation, we accomplish that

$$\mathfrak{h}(U_1, U_2) = -\nabla\mu g(U_1, U_2). \tag{5.18}$$

Above calculation proves that the $\mathfrak{D}_T$ defines spherical foliation i.e., the distribution is totally umbilical and has a parallel mean curvature. And the distribution $\mathfrak{D}_T$ always integrable. The relation (5.17) shows that the the distribution $\mathfrak{D}_\perp \oplus \langle\xi\rangle$ is defines totally geodesic foliation. After using Hiepko Theorem [20], we conclude that $\mathcal{N}$ is form a $\mathcal{PR}$-semi-invariant warped product of type $B \times_f F$, where F is leaf of the distribution $\mathfrak{D}_T$ and B is leaf of the distribution $\mathfrak{D}_\perp$. □

Theorem 5.10. *Let $\mathcal{N}$ be a $\mathcal{PR}$-semi-invariant submanifold in $\mathcal{K}^{2n+1}$. Then $\mathcal{N}$ is form a non-trivial $\mathcal{PR}$-semi-invariant warped product $\mathcal{N} = B \times_f F$ if and only if the endomorphism n satisfies:*

$$(\nabla_U n)V = \eta(V)nU - g(PU, PV)n(\nabla\mu) + n'h(U, QV), \tag{5.19}$$

for all $U, V \in \Gamma(T\mathcal{N})$ and a smooth function μ over $\mathcal{N}$ satisfying $U_2(\mu) = 0$, $\forall U_2 \in \Gamma(\mathfrak{D}_T)$ and $\xi(\mu) = -1$.

Proof. Let $\mathcal{N} = F \times_f B$ be a non-trivial $\mathcal{PR}$-semi invariant warped product submanifold in $\mathcal{K}^{2n+1}$ such that F is a φ-invariant submanifold and B is a φ-anti-invariant submanifold, then from (4.2) we have

$$\begin{aligned}(\nabla_U n)V =& (\nabla_{PU} n)QV + (\nabla_{PU} n)PV + (\nabla_{QU} n)PV + (\nabla_{QU} n)QV \\ &+ \eta(V)(\nabla_U n)\xi.\end{aligned}$$

By the utilization of (2.24) and (3.4), we attain

$$(\nabla_U n)V = \eta(V)nU - g(PU, PV)n(\nabla \ln f) + n'h(U, QV),$$

taking $\mu = \ln f$ in above equation to obtain (5.19).

Conversely, let $\mathcal{N}$ be a $\mathcal{PR}$-semi-invariant submanifold in $\mathcal{K}^{2n+1}$ satisfying (5.19). Then, if we replace U by U_1 and V by U_2 into the relation (5.19), we acquire that

$$(\nabla_{U_1} n)U_2 = -g(U_1, U_2)n(\nabla\mu),$$

for $U_1, U_2 \in \Gamma(\mathfrak{D}_T)$. Now applying Eq (2.22) into above expression, we obtain

$$n'h(U_1, U_2) = h(U_1, tU_2) - g(U_1, U_2)n(\nabla\mu). \tag{5.20}$$

Above discussion show that the distribution $\mathfrak{D}_T$ is integrable. By the consequence of (2.4), (2.17) and (4.4), we have

$$g(\nabla_{U_1} U_2, V_1) = g(n\nabla_{U_1} U_2, nV_1) - \eta(V_1)\eta(\nabla_{U_1} U_2).$$

In view of (2.8), (2.22) and (2.25), we attain

$$g(\nabla_{U_1} U_2, V_1) = -g(U_1, U_2)\left(\eta(V_1) + g(n(\nabla\mu), nV_1)\right).$$

Now employing (2.4) and (2.17) into above relation, we acquire that

$$g(\mathfrak{h}(U_1, U_2), V_1) = g(\nabla\mu, V_1)g(U_1, U_2). \tag{5.21}$$

This proves that the $\mathfrak{D}_T$ defines spherical foliation. Despite that, if we replace U by V_1 and V by U_1 into the Eq (5.19), then we get

$$(\nabla_{V_1} n)U_1 = 0 \tag{5.22}$$

for $U_1 \in \mathfrak{D}_T$ and $V_1 \in \Gamma(\mathfrak{D}_\perp \oplus \langle\xi\rangle)$. Thus by the consequence of (2.4), (2.8) and (2.17), we obtain

$$g(\nabla_{V_1} V_2, U_1) = -\eta(\nabla_{V_1} U_1)\eta(V_2) + g(\varphi\nabla_{V_1} U_1, \varphi V_2) = g(n\nabla_{V_1} U_1, nV_2),$$

for any $U_1 \in \mathfrak{D}_T$ and $V_1, V_2 \in \Gamma(\mathfrak{D}_\perp \oplus \langle\xi\rangle)$. Now utilizing (2.22) and (5.22) into above expression, we arrive $g(\nabla_{V_1} V_2, U_1) = 0$. Above calculation shows that the distribution $\mathfrak{D}_\perp \oplus \langle\xi\rangle$ defines totally geodesic foliation. Therefore, by an application of Hiepko Theorem [20], we can conclude that $\mathcal{N}$ is a $\mathcal{PR}$-semi invariant warped product of type $B \times_f F$, where B is leaf of the distribution $\mathfrak{D}_\perp \oplus \langle\xi\rangle$ and F is leaf of the distribution $\mathfrak{D}_T$. Hence completes the proof. □

5.2. *$\mathcal{PR}$-semi-invariant warped product submanifold of the form $\boldsymbol{F} \times_f \boldsymbol{B}$*

In this section, we analyze the geometry of $\mathcal{PR}$-semi-invariant warped product submanifold of type $F \times_f B$. We derive some important results for such warped products:

Lemma 5.11. *Let $\mathcal{N} = F \times_f B$ be a non-trivial $\mathcal{PR}$-semi-invariant warped product submanifold in $\mathcal{K}^{2n+1}$, thus we obtain for any $U_1 \in \Gamma(TF)$ and $V_1, V_2 \in \Gamma(TB)$ that*

$$g(h(U_1, V_1), nV_2) = -\varphi U_1(\ln f)g(V_1, V_2), \tag{5.23}$$
$$g(h(\varphi U_1, V_1), nV_2) = -(\eta(U_1) + U_1(\ln f))\, g(V_1, V_2). \tag{5.24}$$

Proof. In view of (2.6), (2.7) and (2.11), we obtain

$$g(h(U_1, V_1), nV_2) = -g(\widetilde{\nabla}_{V_1}\varphi U_1, V_2).$$

Now, we just use use (3.3) into above expression to obtain (5.23). We achieve (5.24) if we replace U_1 by φU_1 into (5.23). □

Lemma 5.12. *For a non-trivial $\mathcal{PR}$-semi invariant warped product submanifold $\mathcal{N} = F \times_f B$ in $\mathcal{K}^{2n+1}$, we obtain*

$$(\nabla_V t)U_1 = \eta(U_1)tV + g(tV, U_1)\xi + tU_1(\ln f)QV, \tag{5.25}$$
$$(\nabla_V t)V_1 = g(QV, V_1)t\nabla \ln f, \tag{5.26}$$
$$(\nabla_V n)U_1 = U_1(\ln f)nQV, \tag{5.27}$$
$$(\nabla_V n)V_1 = n'h(V, V_1), \tag{5.28}$$

for all $V_1 \in \Gamma(TB)$, $V \in \Gamma(T\mathcal{N})$ and $U_1 \in \Gamma(TF)$.

Proof. In view of (2.23) and (4.2), we obtain

$$(\nabla_V t)U_1 = \nabla_{QV} tU_1 - t\nabla_{QV} U_1 + \eta(V)(\nabla_\xi t)U_1 + (\nabla_{PV} t)U_1.$$

Now using (2.24), (3.3) and (4.4) into above expression, we obtain (5.25). For the proof of (5.26), we consider

$$(\nabla_V t)V_1 = (\nabla_{PV} t)V_1 + \nabla_{QV} tV_1 - t\nabla_{QV} V_1 + \eta(V)(\nabla_\xi t)V_1. \tag{5.29}$$

In above expression we just utilize (2.23), (2.24) and (3.4) to achieve (5.26). Similarly, by the use of (2.23), (2.25), (3.3), (3.4), (4.2) and (4.4), we easily achieve (5.27) and (5.28). □

Theorem 5.13. *Let $\psi : \mathcal{N} \longrightarrow \mathcal{K}^{2n+1}$ be an isometric immersion and $\mathcal{N}$ be a $\mathcal{PR}$-semi-invariant submanifold. Then $\mathcal{N}$ is a $\mathcal{PR}$-semi-invariant warped product submanifold $\mathcal{N} = F \times_f B$ if and only if A satisfies:*

$$A_{\varphi V_1} U_1 = -\varphi U_1(\mu) V_1, \ \forall V_1 \in \Gamma(\mathfrak{D}_\perp), \ U_1 \in \Gamma(\mathfrak{D}_T \oplus \langle \xi \rangle), \tag{5.30}$$

where μ is a smooth function on $\mathcal{N}$ satisfying $V_2(\mu) = 0$, for all $V_2 \in \Gamma(\mathfrak{D}_\perp)$.

Proof. Let $\mathcal{N} = F \times_f B$ be a non-trivial warped product submanifold $\mathcal{K}^{2n+1}$, then, in view of (2.6), (2.7), (2.11) and (4.4), we have

$$A_{\varphi V_1} U_1 = -\varphi U_1(\ln f) V_1,$$

for any $V_1 \in \Gamma(TB)$ and $U_1 \in \Gamma(TF)$. By using the fact that $\mathcal{N}$ is a warped product and taking $\mu = \ln f$, then we acquire that $V_2(\mu) = 0$, for $V_2 \in \Gamma(TB)$.

Conversely, let $\mathcal{N}$ be a $\mathcal{PR}$-semi-invariant submanifold in $\mathcal{K}^{2n+1}$ satisfying (5.30). Therefore, in light of Eqs (2.6) and (2.11), we attain that for all $U_1, U_2 \in \Gamma(\mathfrak{D}_T \oplus \langle \xi \rangle)$ and $V_1 \in \Gamma(\mathfrak{D}_\perp)$ that;

$$g(h(U_1, U_2), \varphi V_1) = g(\widetilde{\nabla}_{U_1} U_2, \varphi V_1) = -g(U_2, \widetilde{\nabla}_{U_1} \varphi V_1).$$

In view of (2.12) and (5.30) above expression reduces into the following form

$$g(h(U_1, U_2), \varphi V_1) = \varphi U_1(\mu) g(U_2, V_1) = 0.$$

Above expression shows that the anti-invariant distribution $\mathfrak{D}_\perp$ defines totally geodesic foliation. Further, by the consequence of (2.6), (2.7), (2.12), (2.21) and (4.4)), we achieve

$$g(\nabla_{V_1} V_2, \varphi U_1) = g(A_{\varphi V_1} V_2, U_1) + \eta(V_2) g(\varphi V_1, U_1),$$

for all $U_1 \in \Gamma(\mathfrak{D}_T \oplus \langle \xi \rangle)$ and $V_1 \in \Gamma(\mathfrak{D}_\perp)$. At this moment we applying (5.30) into above relation, then we accomplish that

$$g(\nabla_{V_1} V_2, \varphi U_1) = -\varphi U_1(\mu) g(V_1, V_2) = \nabla \mu g(V_1, V_2). \tag{5.31}$$

Above discussion proves that the $\mathfrak{D}_T$ defines spherical foliation. By using Hiepko Theorem [20], we can conclude that $\mathcal{N}$ is a $\mathcal{PR}$-semi-invariant warped product of type $F \times_f B$, where F is leaf of the distribution $\mathfrak{D}_T \oplus \langle \xi \rangle$ and B is leaf of the distribution $\mathfrak{D}_\perp$. This finishes the proof of theorem. □

Theorem 5.14. *Let $\psi : \mathcal{N} \longrightarrow \mathcal{K}^{2n+1}$ be an isometric immersion. Then necessary and sufficient condition for a $\mathcal{PR}$-semi-invariant submanifold in $\mathcal{K}^{2n+1}$ be a non-trivial $\mathcal{PR}$-semi-invariant warped product $\mathcal{N} = F \times_f B$ if and only if endomorphism t satisfies*

$$(\nabla_U t)V = tV(\mu)QU + g(QU, QV)t\nabla(\mu) + \eta(PV)tU + g(tU, PV)\xi, \tag{5.32}$$

for every $U, V \in \Gamma(T\mathcal{N})$, where μ is smooth function on $\mathcal{N}$ satisfying $V_1(\mu) = 0$, $\forall V_1 \in \Gamma(\mathfrak{D}_\perp)$.

Proof. Let $\mathcal{N} = F \times_f B$ be a m-dimensional non-trivial $\mathcal{PR}$-semi-invariant warped product such that B is a φ-anti-invariant submanifold and F is a φ-invariant submanifold, then we attain by the use of (4.2) that

$$(\nabla_U t)V = (\nabla_U t)QV + \eta(V)(\nabla_U t)\xi + (\nabla_U t)PV.$$

By the utilization of (2.24), (5.25) and (5.26), we have

$$(\nabla_U t)V = tV(\ln f)QU + g(QU, QV)t\nabla(\ln f) + g(tU, PV)\xi + \eta(PV)tU,$$

taking $\mu = \ln f$ in above equation to obtain (5.32).

Conversely, let $\mathcal{N}$ be a $\mathcal{PR}$-semi-invariant submanifold in $\mathcal{K}^{2n+1}$ satisfying (5.32). Then, with the help of (4.4) and (5.32), we obtain

$$g((\nabla_{U_1} t)U_2, V_1) = g(\nabla_{U_1} tU_2, V_1) = 0. \tag{5.33}$$

for any $U_1, U_2 \in \Gamma(\mathfrak{D}_T \oplus \langle\xi\rangle)$ and $V_1 \in \Gamma(\mathfrak{D}_\perp)$. The Eq (5.33) implies that $D_T \oplus \langle\xi\rangle$ defines totally geodesic foliation. However, if we substitute V_1 at the place U and U_2 at the place V into the Eq (5.32), then we get

$$(\nabla_{V_1} t)U_2 = tU_2(\mu)V_1 \tag{5.34}$$

for any $V_1 \in \Gamma(\mathfrak{D}_\perp)$ and $U_2 \in \Gamma(\mathfrak{D}_T \oplus \langle\xi\rangle)$. Now taking inner product with $V_2 \in \Gamma(\mathfrak{D}_\perp)$ into (5.34), we achieve that

$$\begin{aligned}
&g((\nabla_{V_1} t)U_2, V_2) = -g(tU_2, \nabla\mu)g(V_1, V_2),\\
&g(\mathfrak{h}(V_1, V_2), tU_2) = -g(tU_2, \nabla\mu)g(V_1, V_2)\\
&\mathfrak{h}(V_1, V_2) = -\nabla\mu g(V_1, V_2).
\end{aligned}$$

Above calculation demonstrate that the distribution $\mathfrak{D}_\perp$ defines spherical foliation. After using the Hiepko theorem [20], we can conclude that $\mathcal{N}$ is form a $\mathcal{PR}$-semi-invariant warped product of type $F \times_f B$, where F is leaf of $\mathfrak{D}_T$ and B is leaf of $\mathfrak{D}_\perp$. This accomplished the proof of theorem. □

Theorem 5.15. *Let $\mathcal{N}$ be a $\mathcal{PR}$-semi-invariant submanifold $\mathcal{K}^{2n+1}$. Then $\mathcal{N}$ is a non-trivial $\mathcal{PR}$-semi-invariant warped product of the form $F \times_f B$ if and only if*

$$(\nabla_U n)V = PV(\mu)nU + \eta(V)nU + n'h(U, QV), \tag{5.35}$$

for any $U, V \in \Gamma(T\mathcal{N})$ and $V_1 \in \Gamma(\mathfrak{D}_\perp)$, where μ is smooth function on M satisfying $V_1(\mu) = 0$.

Proof. Let $F \times_f B$ be a proper $\mathcal{PR}$-semi invariant warped product submanifold of dimension m in $\mathcal{K}^{2n+1}$ such that B is a φ-anti-invariant submanifold and F is a φ-invariant submanifold, subsequently we obtain from (4.2) that

$$(\nabla_U n)V = (\nabla_U n)PV + \eta(V)(\nabla_U n)\xi + (\nabla_U n)QV.$$

By the utilization of (2.24), (5.27) and (5.28), we have

$$(\nabla_U n)V = PV(\ln f)nU + \eta(V)nU + n'h(U, QV)$$

taking $\mu = \ln f$ in above equation to obtain (5.35).

Conversely, let $\mathcal{N}$ be a $\mathcal{PR}$-semi-invariant submanifold in $\mathcal{K}^{2n+1}$ satisfying (5.32). Replace U with U_1 and V with U_2 in the relation (5.35), we have

$$(\nabla_{U_1} n)U_2 = 0, \tag{5.36}$$

$U_1, U_2 \in \Gamma(\mathfrak{D}_T)$. In light of (2.4), (2.8) and (2.17), we attain

$$g(\nabla_{U_1} U_2, V_1) = -g(n\nabla_{U_1} U_2, nV_1),$$

for any $V_1 \in \Gamma(\mathfrak{D}_\perp)$. Now utilize (2.22) and (5.36) into above expression, we achieve

$$g(\nabla_{U_1} U_2, V_1) = 0. \tag{5.37}$$

The relation (5.37) implies that the distribution $\mathfrak{D}_T \oplus \langle\xi\rangle$ defines totally geodesic foliation. Moreover, if we replace U by V_1 and V by U_2 into the Eq (5.32), then we get

$$(\nabla_{V_1} n)U_2 = (tU_2(\mu) + \eta(U_1))\, nV_1, \tag{5.38}$$

for any $U_2 \in \Gamma(\mathfrak{D}_T)$ and $V_1 \in \Gamma(\mathfrak{D}_\perp)$. By the utilization of (2.4), (2.17) and (4.4) into above expression, we obtain

$$g(\nabla_{V_1} V_2, U_1) = -\eta(\nabla_{V_1} V_2)\eta(U_1) + g(n\nabla_{V_1} U_1, nV_2),$$

for any $U_2 \in \Gamma(\mathfrak{D}_T)$ and $V_1 \in \Gamma(\mathfrak{D}_\perp)$. Now employing (2.4), (2.17), (2.22) and (5.35) into above relation

$$g(\nabla_{V_1} V_2, U_1) = -\,(\eta(U_1) + tU_2(\mu))\, g(V_1, V_2). \tag{5.39}$$

Above calculation demonstrate that the distribution $\mathfrak{D}_\perp$ defines spherical foliation. After using Hiepko Theorem [20], we can deduce that $\mathcal{N}$ is a $\mathcal{PR}$-semi invariant warped product of type $F \times_f B$, where B is leaf of the distribution $\mathfrak{D}_\perp$ and F is leaf of the distribution $\mathfrak{D}_T$. This proves the the result. □

6. Optimal inequalities

6.1. Inequalities $\mathcal{PR}$-semi invariant warped product submanifold of type $\boldsymbol{B} \times_f \boldsymbol{F}$

In this section, we derive some inequalities for warped product submanifold of type $\mathcal{N} = B \times_f F$ into para-Kenmotsu manifold. We assume the dimension of B is $s + 1$ and the dimension of F is $2r$. Now, we construct a frame field for $F \times_f B$ as follows:

- An orthonormal basis $\{U_i, U_i^* = \varphi U_i\}$ for $\mathfrak{D}_T$, where $i = 1, 2, ...r$. Further, one can suppose $\epsilon_i = g(U_i, U_i) = 1$ and $\epsilon_i^* = g(U_i^*, U_i^*) = -1$.
- An orthonormal basis $\{V_a, \xi\}$ for $\mathfrak{D}_\perp$, where $i = 1, 2, \cdot, s$ and $\epsilon_a = g(V_a, V_a)$.
- An orthonormal basis $\{V_a^* = \varphi V_a, \zeta_b, \zeta_b^* = \varphi \zeta_b\}$ for $TM^\perp$, $b = 1, 2, ...t$. Moreover, one can assume $\epsilon_\alpha = g(\zeta_\alpha, \zeta_\alpha)$ and $\epsilon_\alpha^* = g(\zeta_\alpha^*, \zeta_\alpha^*)$.

Theorem 6.1. *Let $\mathcal{N} = B \times_f F$ be a non-trivial $\mathcal{PR}$-semi invariant warped product submanifold in $\mathcal{N}$ such that B is a time-like submanifold. Then, h satisfies the following relation:*

$$||h||^2 \geq r\left(||\nabla^B \ln f||^2 - 1\right) + ||h_\nu||^2, \tag{6.1}$$

where $||h_\nu^{\mathfrak{D}_T}||^2 = g(h_\nu(\mathfrak{D}_T, \mathfrak{D}_T), h_\nu(\mathfrak{D}_T, \mathfrak{D}_T))$. Moreover, if equality holds then M is mixed totally geodesic and if $h(\mathfrak{D}_\perp, \mathfrak{D}_\perp) \perp \nu$, then we have

$$||h||^2 \geq r\left(||\nabla^B \ln f||^2 - 1\right). \tag{6.2}$$

Proof. If $\mathfrak{D}_\perp$ is a time-like distribution then $\epsilon_a = -1$. Now the square norm of h is given by:

$$||h||^2 = ||h(\mathfrak{D}_T, \mathfrak{D}_T)||^2 + 2||(h(D_T, D_\perp)||^2 + ||h(\mathfrak{D}_\perp, \mathfrak{D}_\perp)||^2. \tag{6.3}$$

First, we consider

$$\begin{aligned} ||h(\mathfrak{D}_T, \mathfrak{D}_T)||^2 = &\sum_{i,j=1}^r \epsilon_i^* \epsilon_j g(h(U_i^*, U_j), h(U_i^*, U_j)) \\ &+ \sum_{i,j=1}^r \epsilon_i^* \epsilon_j^* g(h(U_i^*, U_j^*), h(U_i^*, U_j^*)) \\ &+ \sum_{i,j=1}^r \epsilon_i \epsilon_j g(h(U_i, U_j), h(U_i, U_j)) \\ &+ \sum_{i,j=1}^r \epsilon_i \epsilon_j^* g(h(U_i, U_j^*), h(U_i, U_j^*)). \end{aligned} \tag{6.4}$$

Now using the fact that $\mathfrak{D}_T$ is φ-invariant distribution, then we have

$$\begin{aligned} h(U_i, U_j) &= h_{ij}^c n V_c + h_{ij}^\alpha \zeta_\alpha + h_{ij}^{\alpha^*} \zeta_{\alpha^*}, \\ h(U_i^*, U_j) &= h_{i^*j}^c n V_c + h_{i^*j}^\alpha \zeta_\alpha + h_{i^*j}^{\alpha^*} \zeta_{\alpha^*}, \\ h(U_i, U_j^*) &= h_{ij^*}^c n V_c + h_{ij^*}^\alpha \zeta_\alpha + h_{ij^*}^{\alpha^*} \zeta_{\alpha^*}, \\ h(U_i^*, U_j^*) &= h_{i^*j^*}^c n V_c + h_{i^*j^*}^\alpha \zeta_\alpha + h_{i^*j^*}^{\alpha^*} \zeta_{\alpha^*}. \end{aligned} \tag{6.5}$$

With the help of (6.5) and Lemma 4.3, the Eq (6.4) reduces into the following form

$$\begin{aligned}||h(\mathfrak{D}_T,\mathfrak{D}_T)||^2 =& \sum_{i,j=1}^{r}\sum_{\alpha=1}^{t}(h_{ij}^{\alpha})^2-(h_{ij}^{\alpha^*})^2-(h_{ij^*}^{\alpha})^2+(h_{i,j^*}^{\alpha^*})^2\\&-\sum_{i,j=1}^{r}\sum_{\alpha=1}^{s}(h_{i^*j^*}^{\alpha^*})^2-(h_{ij^*}^{\alpha})^2+(h_{i^*j}^{\alpha^*})^2+(h_{i^*j^*}^{\alpha})^2\\&+\sum_{i,j=1}^{r}\sum_{c=1}^{s}(h_{ij}^{c})^2-(h_{i^*j}^{c})^2+(h_{i^*j^*}^{c})^2-(h_{ij^*}^{c})^2.\end{aligned}$$

Now using (4.8) and the intergrability condition into above expression, we compute

$$\begin{aligned}||h(\mathfrak{D}_T,\mathfrak{D}_T)||^2 =& \sum_{i,j=1}^{r}\sum_{c=1}^{s}(h_{ij}^{c})^2-2(h_{i^*j}^{c})^2+(h_{i^*j^*}^{c})^2\\&+4\sum_{i,j=1}^{r}\sum_{\alpha=1}^{t}(h_{ij}^{\alpha})^2-(h_{ij}^{\alpha^*})^2.\end{aligned}\tag{6.6}$$

Now, applying (5.6), (5.12) and (5.25) into above relation

$$\begin{aligned}||h(\mathfrak{D}_T,\mathfrak{D}_T)||^2 =& \sum_{i,j=1}^{r}\sum_{a=1}^{s}(V_a(\ln f)+\eta(V_a))^2\,g(U_i,U_{j^*})^2\\&+\sum_{i,j=1}^{r}\sum_{a=1}^{s}(\eta(V_a)+V_a(\ln f))^2\,g(U_{i^*},U_j)^2\\&+2\sum_{i,j=1}^{r}\sum_{a=1}^{s}(\eta(V_a)+V_a(\ln f))^2\,g(U_i,U_j)^2\\&+4\sum_{i,j=1}^{r}\sum_{\alpha=1}^{t}(h_{ij}^{\alpha})^2-(h_{ij}^{\alpha^*})^2.\end{aligned}\tag{6.7}$$

Above expression reduces into the following form

$$\begin{aligned}||h(\mathfrak{D}_T,\mathfrak{D}_T)||^2 =& \sum_{i,j=1}^{r}\sum_{a=1}^{s}(\eta(V_a))^2+(V_a(\ln f))^2+2\eta(V_a)V_a(\ln f))g(U_i,U_j)^2\\&+4\sum_{i,j=1}^{r}\sum_{\alpha=1}^{t}(h_{ij}^{\alpha})^2-(h_{ij}^{\alpha^*})^2.\end{aligned}\tag{6.8}$$

By adding and subtracting same quantity into above expression

$$\begin{aligned}||h(\mathfrak{D}_T,\mathfrak{D}_T)||^2 =& 2\sum_{i,j=1}^{r}\sum_{a=1}^{s+1}(V_a(\ln f))^2\,g(U_i,U_j)^2-2\sum_{i,j=1}^{r}(\xi(\ln f))^2 g(U_i,U_j)^2\\&+4\sum_{i,j=1}^{r}\sum_{\alpha=1}^{t}(h_{ij}^{\alpha})^2-(h_{ij}^{\alpha^*})^2.\end{aligned}$$

By the definition of gradient above relation reduces into the following form

$$\|h(\mathfrak{D}_T,\mathfrak{D}_T)\|^2 = r\left(\|\nabla^B \ln f\|^2 - 1\right) + 4\sum_{i,j=1}^{r}\sum_{\alpha=1}^{t}(h_{ij}^{\alpha})^2 - (h_{ij}^{\alpha^*})^2. \tag{6.9}$$

Now consider

$$\|h(\mathfrak{D}_\perp,\mathfrak{D}_\perp)\|^2 = \sum_{a=1}^{s}\epsilon_a\epsilon_b g(h(V_a,V_b),h(V_a,V_b)).$$

By the utilization of $h(V_a,V_b) = h_{ab}^{c} nV_c + h_{ab}^{\alpha}\zeta_\alpha + h_{ab}^{\alpha^*}\zeta_{\alpha^*}$ into above relation, we achieve,

$$\|h(\mathfrak{D}_\perp,\mathfrak{D}_\perp)\|^2 = \sum_{a,b=1}^{s}\sum_{\alpha=1}^{t}(h_{ab}^{\alpha})^2 - (h_{ab}^{\alpha^*}\zeta_{\alpha^*})^2 + \sum_{a,b,c=1}^{s}(h_{ab}^{c})^2.$$

Last term is vanishes since $(h_{ab}^{\alpha})^2 = (h_{ab}^{\alpha^*})^2$. Therefore, above expression reduces into following form,

$$\|h(\mathfrak{D}_\perp,\mathfrak{D}_\perp)\|^2 = \sum_{a,b,c=1}^{s}(h_{ab}^{c})^2. \tag{6.10}$$

Lastly, consider

$$\begin{aligned}\|(h(\mathfrak{D}_T,\mathfrak{D}_\perp)\|^2 &= \sum_{i=1}^{r}\sum_{a=1}^{s}\epsilon_i\epsilon_a g(h(U_i,V_a),h(U_i,V_a))\\ &+ \sum_{i=1}^{r}\sum_{a=1}^{s}\epsilon_{i^*}\epsilon_a g(h(U_i^*,V_a),h(U_i^*,V_a)).\end{aligned} \tag{6.11}$$

Now using

$$\begin{aligned}h(U_i,V_a) &= h_{ia}^{b} nV_b + h_{ia}^{\alpha}\zeta_\alpha + h_{ia}^{\alpha^*}\zeta_{\alpha^*}\\ h(U_{i^*},V_a) &= h_{i^*a}^{b} nV_b + h_{i^*a}^{\alpha}\zeta_\alpha + h_{i^*a}^{\alpha^*}\zeta_{\alpha^*}.\end{aligned} \tag{6.12}$$

These expressions employing into (6.11), we have

$$\|(h(\mathfrak{D}_T,\mathfrak{D}_\perp)\|^2 = \sum_{i=1}^{r}\sum_{a,c=1}^{s}(h_{i^*a}^{c})^2 - (h_{ia}^{c})^2 + \sum_{i=1}^{r}\sum_{a=1}^{s}\sum_{\alpha=1}^{t}(h_{i^*a}^{\alpha})^2 - (h_{ia}^{\alpha})^2 + (h_{ia}^{\alpha^*})^2 - (h_{i^*a}^{\alpha^*})^2.$$

Since $\mathfrak{D}_\perp$ is totally geodesic then $h_{i^*a}^{c} = h_{ia}^{c}$. So, above relation become

$$\|h(\mathfrak{D}_T,\mathfrak{D}_\perp)\|^2 = \sum_{i=1}^{r}\sum_{a=1}^{s}\sum_{\alpha=1}^{t}(h_{i^*a}^{\alpha})^2 - (h_{ia}^{\alpha})^2 + (h_{ia}^{\alpha^*})^2 - (h_{i^*a}^{\alpha^*})^2.$$

By the virtue of Lemma 4.3 and Lemma 4.4, we obtain

$$\|h(\mathfrak{D}_T,\mathfrak{D}_\perp)\|^2 = \sum_{i=1}^{r}\sum_{a=1}^{s}\sum_{\alpha=1}^{t}(h_{ia}^{\alpha^*})^2 - (h_{i^*a}^{\alpha^*})^2. \tag{6.13}$$

By utilization of (6.9), (6.10) and (6.13), we obtain (6.1). So, equality holds if and only if $\|h(\mathfrak{D}_T,\mathfrak{D}_\perp)\| = 0$, this follows (2). The statement (3) directly follows from (6.1) and (6.9). □

Theorem 6.2. *Let $\mathcal{N} = B \times_f F$ be a non-trivial $\mathcal{PR}$-semi invariant warped product submanifold in $\mathcal{K}^{2n+1}$ such that B is a space-like submanifold then h satisfying the following relation*

$$\|h\|^2 \leq \|h_\nu\|^2 + r\left(\|\nabla^F \ln f\|^2 - 1\right). \tag{6.14}$$

Moreover, if equality holds then $\mathcal{N}$ is mixed totally geodesic and if $h(\mathfrak{D}_\perp, \mathfrak{D}_\perp) \perp \nu$, then we receive that

$$\|h\|^2 \leq r\left(\|\nabla^B \ln f\|^2 - 1\right). \tag{6.15}$$

6.2. *Inequalities $\mathcal{PR}$-semi invariant warped product submanifold of type $\boldsymbol{F} \times_f \boldsymbol{B}$*

In this section, we derive some inequalities for warped product submanifold of type $F \times_f B$ into para-Kenmotsu manifold. We assume the dimension of B is s and the dimension of F is $2r + 1$. Now, we construct a frame field for $F \times_f B$ as follows:

- An orthonormal basis $\{U_i, X^* = \varphi U_i, \xi\}$ for $\mathfrak{D}_T$, where $i = 1, 2, ...r$. Further, one can suppose $\epsilon_i = g(U_i, U_i) = 1$ and $\epsilon_i^* = g(U_i^*, U_i^*) = -1$.
- An orthonormal basis $\{V_a\}$ for $\mathfrak{D}_\perp$, where $i = 1, 2, \cdot, s$ and $\epsilon_a = g(V_a, V_a)$.
- An orthonormal basis $\{Z_a^* = \varphi V_a, \zeta_b, \zeta_b^* = \varphi \zeta_b\}$ for $TM^\perp$, $b = 1, 2, ...t$. Moreover, one can assume $\epsilon_\alpha = g(\zeta_\alpha, \zeta_\alpha)$ and $\epsilon_\alpha^* = g(\zeta_\alpha^*, \zeta_\alpha^*)$.

Theorem 6.3. *Let $\mathcal{N} = F \times_f B$ be a non-trivial $\mathcal{PR}$-semi-invariant warped product submanifold in $\mathcal{K}^{2n+1}$ such that ξ is tangent to B and B is a time-like submanifold with $\nabla^\perp(\varphi F) \subset \varphi(F)$. Then, h satisfies*

$$\|h\|^2 \geq \|h_\nu\|^2 + 2s(\|\nabla^F \ln f\|^2 - 1), \tag{6.16}$$

where $\nabla^F \ln f$ is gradient of F. Moreover, equality holds if $\|h(\mathfrak{D}_\perp, \mathfrak{D}_\perp)\| = 0$ and if M is mixed totally geodesic submanifold then the inequality (6.16) *taking the following form*

$$\|h\|^2 \geq \|h_\nu^{\mathfrak{D}_T}\|^2. \tag{6.17}$$

Proof. Now the square norm of h is given by:

$$\|h\|^2 = \|h(\mathfrak{D}_T, \mathfrak{D}_T)\|^2 + 2\|(h(D_T, D_\perp)\|^2 + \|h(\mathfrak{D}_\perp, \mathfrak{D}_\perp)\|^2. \tag{6.18}$$

First, we consider

$$\begin{aligned}\|h(\mathfrak{D}_T, \mathfrak{D}_T)\|^2 = &\sum_{i,j=1}^{r} \epsilon_i^* \epsilon_j^* g(h(U_i^*, U_j^*), h(U_i^*, U_j^*)) \\ &+ \sum_{i,j=1}^{r} \epsilon_i \epsilon_j g(h(U_i, U_j), h(U_i, U_j)) \\ &+ \sum_{i,j=1}^{r} \epsilon_i^* \epsilon_j g(h(U_i^*, U_j), h(U_i^*, U_j)) \\ &+ \sum_{i,j=1}^{r} \epsilon_i \epsilon_j^* g(h(U_i, U_j^*), h(U_i, U_j^*)).\end{aligned} \tag{6.19}$$

Now using the fact that $\mathfrak{D}_T$ is φ-invariant distribution, then we have

$$
\begin{aligned}
h(U_i, U_j) &= h_{ij}^{\alpha}\zeta_\alpha + h_{ij}^{\alpha^*}\zeta_{\alpha^*},\\
h(U_i^*, U_j) &= h_{i^*j}^{\alpha}\zeta_\alpha + h_{i^*j}^{\alpha^*}\zeta_{\alpha^*},\\
h(U_i, U_j^*) &= h_{ij^*}^{\alpha}\zeta_\alpha + h_{ij^*}^{\alpha^*}\zeta_{\alpha^*},\\
h(U_i^*, U_j^*) &= h_{i^*j^*}^{\alpha}\zeta_\alpha + h_{i^*j^*}^{\alpha^*}\zeta_{\alpha^*}.
\end{aligned} \tag{6.20}
$$

With the help of (6.20) and Lemma 4.3, the Eq (6.19) reduces into following form

$$
\begin{aligned}
\|h(\mathfrak{D}_T, \mathfrak{D}_T)\|^2 = &\sum_{i,j=1}^{r}\sum_{\alpha=1}^{t}(h_{ij}^{\alpha})^2 - (h_{ij}^{\alpha^*})^2 - (h_{ij^*}^{\alpha})^2 + (h_{i,j^*}^{\alpha^*})^2 + (h_{i^*j^*}^{\alpha})^2\\
&- \sum_{i,j=1}^{r}\sum_{\alpha=1}^{t}(h_{i^*j^*}^{\alpha^*})^2 - (h_{ij^*}^{\alpha})^2 + (h_{i^*j}^{\alpha^*})^2.
\end{aligned}
$$

Now using (4.8) and the intergrability condition into above expression, we calculate

$$
\|h(\mathfrak{D}_T, \mathfrak{D}_T)\|^2 = \sum_{i,j=1}^{r}\sum_{\alpha=1}^{t} 4[(h_{ij}^{\alpha})^2 - (h_{ij}^{\alpha^*})^2]. \tag{6.21}
$$

Now, consider

$$
\|h(\mathfrak{D}_\perp, \mathfrak{D}_\perp)\|^2 = \sum_{a=1}^{s+1}\epsilon_a\epsilon_b g(h(V_a, V_b), h(V_a, V_b)). \tag{6.22}
$$

By the utilization of $h(V_a, V_b) = h_{ab}^{c} nV_c + h_{ab}^{\alpha}\zeta_\alpha + h_{ab}^{\alpha^*}\zeta_{\alpha^*}$ into (6.22), we achieve,

$$
\|h(\mathfrak{D}_\perp, \mathfrak{D}_\perp)\|^2 = \sum_{a,b=1}^{s}\sum_{\alpha=1}^{t}(h_{ab}^{\alpha})^2 - (h_{ab}^{\alpha^*}\zeta_{\alpha^*})^2 + \sum_{a,b,c=1}^{s}(h_{ab}^{c})^2.
$$

Last term is vanishes, since $(h_{ab}^{\alpha})^2 = (h_{ab}^{\alpha^*})^2$. Therefore, above expression reduces into following form,

$$
\|h(\mathfrak{D}_\perp, \mathfrak{D}_\perp)\|^2 = \sum_{a,b,c=1}^{s}(h_{ab}^{c})^2. \tag{6.23}
$$

Lastly, taking

$$
\begin{aligned}
\|(h(\mathfrak{D}_T, \mathfrak{D}_\perp)\|^2 = &\sum_{i=1}^{r}\sum_{a=1}^{s+1}\epsilon_i\epsilon_a g(h(U_i, V_a), h(U_i, V_a))\\
&+ \sum_{i=1}^{r}\sum_{a=1}^{s+1}\epsilon_{i^*}\epsilon_a g(h(U_i^*, V_a), h(U_i^*, V_a)).
\end{aligned} \tag{6.24}
$$

In this case $h(U_i, V_a) = h_{ia}^{b} nV_b + h_{ia}^{\alpha}\zeta_\alpha + h_{ia}^{\alpha^*}\zeta_{\alpha^*}$ and $h(U_{i^*}, V_a) = h_{i^*a}^{b} nZ_b + h_{i^*a}^{\alpha}\zeta_\alpha + h_{i^*a}^{\alpha^*}\zeta_{\alpha^*}$. Last expressions employing into (6.24) then after applying (5.6), we obtain

$$g((h(\mathfrak{D}_T, \mathfrak{D}_\perp), h(\mathfrak{D}_T, \mathfrak{D}_\perp))$$
$$= \sum_{i}^{r} \sum_{a,c=1}^{s} (h^c_{i^*a})^2 - (h^c_{ia})^2 + 2 \sum_{i=1}^{r} \sum_{a=1}^{s} \sum_{\alpha=1}^{t} (h^\alpha_{i^*a})^2 - (h^\alpha_{ia})^2.$$

Now employing Lemma 5.11 into above expression, then we find

$$\|(h(\mathfrak{D}_T, \mathfrak{D}_\perp)\|^2 = s(\|\nabla^F\|^2 - 1). \tag{6.25}$$

By utilization of (6.21), (6.23) and (6.25), we obtain (6.16). □

Theorem 6.4. *Let $\mathcal{N} = F \times_f B$ be a non-trivial $\mathcal{PR}$-semi-invariant warped product submanifold in $\mathcal{K}^{2n+1}$ such that ξ is tangent to B and B is a space-like submanifold with $\nabla^\perp(\varphi F) \subset \varphi(F)$. Then the following results holds:*

(1) h is fulfills

$$\|h\|^2 \leq \|h_\nu\|^2 + 2s(\|\nabla^F \ln f\|^2 - 1), \tag{6.26}$$

where $\|h^{\mathfrak{D}_T}_\nu\|^2 = g(h_\nu(\mathfrak{D}_T, \mathfrak{D}_T), h_\nu(\mathfrak{D}_T, \mathfrak{D}_T))$.

(2) If $\|h(\mathfrak{D}_\perp, \mathfrak{D}_\perp)\| = 0$, then equality holds in (6.26).

(3) If $\mathcal{N}$ is mixed totally geodesic submanifold then the inequality (6.26) *taking the following form*

$$\|h\|^2 \leq \|h^{\mathfrak{D}_T}_\nu\|^2. \tag{6.27}$$

7. Examples

Example 7.1. Choose $\widetilde{M} = \mathbb{R}^4 \times \mathbb{R}^+$ together the the usual Cartesian coordinates (x_1, x_2, x_3, x_4, s). Then the structure (φ, ξ, η) over $\widetilde{M}$ is defined by

$$\varphi e_1 = e_3,\ \varphi e_2 = e_4, \varphi e_3 = e_1, \tag{7.1}$$
$$\varphi e_4 = e_2,\ \xi = e_9,\ \eta = ds, \tag{7.2}$$

where $e_i = \frac{\partial}{\partial x_i}$, for $i \in \{1, 2, 3, 4, 5\}$ and $e_6 = \frac{\partial}{\partial s}$, and the pseudo-Riemannian metric g is defined as

$$g = -e^{-2s} \sum_{i=1}^{2} dx_i^2 + e^{-2s} \sum_{i=3}^{4} dx_i^2 + \eta \otimes \eta. \tag{7.3}$$

Then by the simple computation, we can easily see that $\widetilde{M}$ is para-Kenmotsu manifold. Suppose M be an immersed submanifold into $\widetilde{M}$ by an immersion σ which is given by

$$x_1 = \alpha \cos\theta,\ x_2 = \alpha \sin\theta,\ x_3 = 2\alpha,\ x_4 = \alpha^2,\ x_5 = s.$$

So, we can easily provide the generating set for the tangent bundle of submanifold as follows:

$$Z_1 = -\alpha \sin\theta e_1 + \alpha \cos\theta e_2,$$
$$Z_3 = \cos\theta e_1 + \sin\theta e_2 + 2e_3 + 2\alpha e_4,$$
$$Z_3 = \xi.$$

The space $\varphi(TM)$ with respect to φ is spanned by the following vectors

$$\varphi Z_1 = -\alpha \sin\theta e_3 + \alpha \cos\theta e_4,$$
$$\varphi Z_2 = 2e_1 + 2\alpha e_2 + \cos\theta e_3 + \sin\theta e_4,$$
$$\varphi Z_3 = 0.$$

Clearly, the vectors φZ_1 is orthogonal to the tangent bundle TM. Therefore, the distribution $span\{Z_1\}$ is anti-invariant under φ and the distribution $span\{Z_2\}$ is invariant under φ. The induced metric g_M on M is given by:

$$g_M = ds^2 + e^{-2s}(4\alpha^2 + 3)d\alpha^2 + e^{-2z}\alpha^2 d\theta^2. \tag{7.4}$$

Above discussion demonstrate that M is a non-trivial $\mathcal{PR}$-semi-invariant warped product submanifold of type $M_T \times_f M_\perp$ with warping function $f = \alpha e^{-s}$.

Example 7.2. Let us consider $\widetilde{M} = \mathbb{R}^{10} \times \mathbb{R}^+$ together the the usual Cartesian coordinates $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, s)$. Then the structure (φ, ξ, η) over $\widetilde{M}$ is defined by

$$\varphi e_1 = e_3,\ \varphi e_2 = e_4,\ \varphi e_3 = e_1,\ \varphi e_4 = e_2, \tag{7.5}$$
$$\varphi e_5 = e_6,\ \varphi e_6 = e_5,\ \varphi e_7 = e_9,\ \varphi e_8 = e_{10}, \tag{7.6}$$
$$\varphi e_9 = e_7,\ \varphi e_{10} = e_8,\ \xi = e_{11} = \frac{\partial}{\partial s},\ \eta = ds, \tag{7.7}$$

where $e_i = \frac{\partial}{\partial x_i}$, for $i \in \{1, 3, ..., 9\}$ and $i \in \{2, 4, ..., 10\}$, and the pseudo-Riemannian metric tensor g is defined as

$$g = e^{-2s}\sum_{i=1}^{5} dx_i^2 - e^{-2s}\sum_{i=6}^{10} dx_i^2 + \eta \otimes \eta. \tag{7.8}$$

Then by simple computation, we can easily see that $\widetilde{M}$ is para-Kenmotsu manifold. Suppose M be an immersed submanifold into $\widetilde{M}$ by an immersion σ which is defined by

$$x_1 = \alpha \sinh\theta,\ x_2 = \beta \cosh\phi,\ x_3 = \alpha \sinh\phi,\ x_4 = \beta \cosh\theta,\ x_5 = \alpha + \beta,\ x_6 = \frac{\alpha - \beta}{2},$$
$$x_7 = \alpha \cosh\theta,\ x_8 = \beta \sinh\phi,\ x_9 = \alpha \cosh\phi,\ x_{10} = \beta \sinh\theta,\ x_{11} = s.$$

So, we can easily provide the generating set for the tangent bundle of submanifold as follows:

$$Z_\theta = \alpha \cosh\theta e_1 + \beta \sinh\theta e_4 + \alpha \sinh\theta e_7 + \beta \cosh\theta e_{10},$$
$$Z_\phi = \beta \sinh\phi e_2 + \alpha \cosh\phi e_3 + \beta \cosh\phi e_8 + \alpha \sinh\phi e_9,$$

$$Z_\alpha = \sinh\theta e_1 + \sinh\phi e_3 + e_5 + \frac{1}{2}e_6 + \cosh\theta e_7 + \cosh\phi e_9,$$
$$Z_\beta = \cosh\phi e_2 + \cosh\theta e_4 + e_5 - \frac{1}{2}e_6 + \sinh\phi e_8 + \sinh\theta e_{10},$$
$$Z_s = \xi.$$

The space $\varphi(TM)$ is spanned by the following vectors

$$\varphi Z_\theta = \beta\sinh\theta e_2 + \alpha\cosh\theta e_3 + \beta\cosh\theta e_8 + \alpha\sinh\theta e_9,$$
$$\varphi Z_\phi = \alpha\cosh\phi e_1 + \beta\sinh\phi e_4 + \alpha\sinh\phi e_7 + \beta\cosh\phi e_{10},$$
$$\varphi Z_\alpha = \sinh\phi e_1 + \sinh\theta e_3 + \frac{1}{2}e_5 + e_6 + \cosh\phi e_7 + \cosh\theta e_9,$$
$$\varphi Z_\beta = \cosh\theta e_2 + \cosh\phi e_4 - \frac{1}{2}e_5 + e_6 + \sinh\theta e_8 + \sinh\phi e_{10}.$$

Clearly, the vectors φZ_θ and φZ_ϕ is orthogonal to TM. Therefore, the distribution $\mathcal{D}_\perp = span\{Z_\theta, Z_\phi\}$ is anti-invariant under φ and the distribution $\mathcal{D}_T = span\{Z_\alpha, Z_\beta\}$ is invariant under φ. The induced metric tensor g_M on $M = M_T \times_f M_\perp$ is given by:

$$g_M = \frac{1}{4}(4ds^2 + 3e^{-2s}(d\alpha^2 - 13\mathrm{d}\beta^2)) + e^{-2s}(\beta^2 - \alpha^2)(d\theta^2 + \mathrm{d}\phi^2). \tag{7.9}$$

Above calculation manifest that M is a form a non-trivial $\mathcal{PR}$-semi-invariant warped product submanifold of $\widetilde{M}$ such that the warping function $f = (\beta^2 - \alpha^2)^{\frac{1}{2}}e^{-s}$.

The authors confirm that the data supporting the findings of this study are available within the article (and/or) its supplementary materials.

8. Conclusions

In this paper, we obtained existence and non-existence conditions for warped product submanifold of para-Kenmotsu manifold, derived results related to integrability conditions of $\mathcal{PR}$-semi invariant submanifold in para-Kenmotsu manifold. We established characterization results allied to $\mathcal{PR}$-semi-invariant warped product submanifolds. Furthermore, optimal inequalities for $F \times_f B$ and $B \times_f F$ are obtained and examples of these warped products are illustrated.

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Conflict of interest

The authors declare no conflicts of interest.

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