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Research article

Generalized version of Jensen and Hermite-Hadamard inequalities for interval-valued (h_1, h_2)-Godunova-Levin functions

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Abstract: Interval analysis distinguishes between inclusion relation and order relation. Under the inclusion relation, convexity and nonconvexity contribute to different kinds of inequalities. The construction and refinement of classical inequalities have received a great deal of attention for many classes of convex as well as nonconvex functions. Convex theory, however, is commonly known to rely on Godunova-Levin functions because their properties enable us to determine inequality terms more precisely than those obtained from convex functions. The purpose of this study was to introduce a $(\subseteq)$ relation to established Jensen-type and Hermite-Hadamard inequalities using (h_1, h_2)-Godunova-Levin interval-valued functions. To strengthen the validity of our results, we provide several examples and obtain some new and previously unknown results.

Keywords: Hermite-Hadamard inequality; Jensen type inequality; interval (h_1, h_2)-Godunova-Levin function
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1. Introduction

There has been much debate in the biography of interval analysis, but the key research outcomes, intervals, and interval-valued functions were first introduced by Moore [1], in 1950. The mathematical community has been paying close attention to this research field since its inception. Experts believe interval analysis is useful in global optimization and constraint solving algorithms. In scientific computation, interval analysis is useful, particularly for accuracy, round-off errors, and automatic validation. For the last five decades, there has been a lot of curiosity about it and it has been used in many areas, such as differential equations with intervals [2], aerodynamic load analysis [3], aeroelasticity [4], computer graphics [5], and so on. We recommend readers in addition to other interval analysis results and applications, see e.g., [6–12].

It is well known that the convexity of functions is important in mathematics, economics, probability theory, optimal control theory, and other scientific disciplines. According to various scholars, function convexity is based on inequality. Among elementary mathematics, the Hermite-Hadamard inequality is a popular subject since it offers the first geometrical interpretation of convex mappings. There has been extensive research on the Hermite-Hadamard inequality for various classes of convexity due to its importance. The following is the classical Hermite-Hadamard ($\mathcal{H}-\mathcal{H}$) inequality:

$$\chi\Big(\frac{\Upsilon+\Omega}{2}\Big) \leq \frac{1}{\Omega-\Upsilon}\int_{\Upsilon}^{\Omega}\chi(\nu)d\nu \leq \frac{\chi(\Upsilon)+\chi(\Omega)}{2},$$

where $\chi : S \subseteq \mathcal{R} \to \mathcal{R}$ be a convex on interval S of real numbers and $\Upsilon, \Omega \in S$ with $\Upsilon < \Omega$. A number of convexity classes have been considered in developing this inequality, see e.g., [13–16]. Since Varoşanec [17], introduced the notion of h-convex function in 2007, different authors have developed more refined Hermite-Hadamard inequalities related to h-convex functions, see e.g., [18–21]. This inequality was proved in 2018 by Awan et al. [22], using (h_1, h_2)-convex functions.

Theorem 1.1. *Let* $\chi : [\Upsilon, \Omega] \to \mathcal{R}^+$. *If* χ *is* (h_1, h_2)*-convex and* $h_1(\frac{1}{2})h_2(\frac{1}{2}) \neq 0$. *Then*

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\chi\Big(\frac{\Upsilon+\Omega}{2}\Big) \leq \frac{1}{\Omega-\Upsilon}\int_{\Upsilon}^{\Omega}\chi(\nu)d\nu \leq [\chi(\Upsilon)+\chi(\Omega)]\int_0^1 h_1(\sigma)h_2(1-\sigma)d\sigma. \tag{1.1}$$

Later, An et al. [23], introduced the concept of (h_1, h_2)-convex interval-valued functions (in short I-V-Fs) and prove the above inequality in that generalization. Further Nwaeze et al. [24] developed the $\mathcal{H}-\mathcal{H}$ inequality for n-polynomial for convex I-V-Fs; Ali et al. [25] and Kalsoom et al. [26] applied quantum calculus to refine this concept. Moreover, by Khan et al. [27–32] this concept has been generalized to convex fuzzy I-V-Fs as well. For some recent results related to these inequalities for interval-valued functions, see e.g., [33–36]. In 2019, Ohud Almutairi and his co-author proved the following inequality using the h-Godunova-Levin function [37].

Theorem 1.2. *Let* $\chi : [\Upsilon, \Omega] \to R$. *If* χ *is h-Godunova-Levin convex function and* $h(\frac{1}{2}) \neq 0$. *Then*

$$\frac{h(\frac{1}{2})}{2}\varphi\Big(\frac{\Upsilon+\Omega}{2}\Big) \leq \frac{1}{\Omega-\Upsilon}\int_{\Upsilon}^{\Omega}\chi(\nu)d\nu \leq [\chi(\Upsilon)+\chi(\Omega)]\int_0^1 \frac{d\sigma}{h(\sigma)}.$$

Further, Costa et al. [38], present a fuzzy Jensen-type inequality for I-V-Fs while Hongxin Bai, et al. [39], develop a Jensen-type inequality for interval nonconvex (h_1, h_2) functions. Motivated by Ohud Almutairi [37], An et al. [23], and Hongxin Bai, et al. [39], we introduce the notion of interval-valued (h_1, h_2)-Godunova-Levin functions and develop Jensen and $\mathcal{H}-\mathcal{H}$ inequalities for this newly introduced class of functions. The article is divided into the following sections. The necessary mathematical background is provided in Section 2. Section 3 presents the problem description as well as our key findings. Section 4 provides conclusions.

2. Preliminaries

To begin, a short overview of terms, notations, and properties used in this paper is necessary [13]. Consider $I \subseteq \mathcal{R}$, where I is closed as well as bounded. For any $[\Upsilon] \in I$ is defined by

$$[\Upsilon] = [\underline{\Upsilon}, \overline{\Upsilon}] = \{x \in \mathcal{R} | \underline{\Upsilon} \leq x \leq \overline{\Upsilon}\}, (\underline{\Upsilon}, \overline{\Upsilon} \in \mathcal{R}).$$

The interval $[\Upsilon]$ shows degeneration when $\underline{\Upsilon}$=$\overline{\Upsilon}$. We state $[\Upsilon]$ is positive when $\underline{\Upsilon} > 0$ or negative when $\overline{\Upsilon} < 0$. Assume that the bundle of all and positive intervals consists of the following $\mathcal{R}_{\mathcal{I}}$, $\mathcal{R}_{\mathcal{I}}^{+}$, respectively. Consider any real number ν and $[\Upsilon]$, the interval $\nu[\Upsilon]$ is given as:

$$\nu.[\underline{\Upsilon}, \overline{\Upsilon}] = \begin{cases} [\nu\underline{\Upsilon}, \nu\overline{\Upsilon}], if \nu > 0, \\ \{0\}, if \nu = 0, \\ [\nu\overline{\Upsilon}, \nu\underline{\Upsilon}], if \nu < 0. \end{cases}$$

For $[\Upsilon] = [\underline{\Upsilon}, \overline{\Upsilon}]$, and $[\Omega] = [\underline{\Omega}, \overline{\Omega}]$, algebraic operations are defined as:

$$[\Upsilon] + [\Omega] = [\underline{\Upsilon} + \underline{\Omega}, \overline{\Upsilon} + \overline{\Omega}],$$

$$[\Upsilon] - [\Omega] = [\underline{\Upsilon} - \underline{\Omega}, \overline{\Upsilon} - \overline{\Omega}],$$

$$[\Upsilon].[\Omega] = [min\{\underline{\Upsilon}, \underline{\Upsilon}\overline{\Omega}, \overline{\Upsilon}\underline{\Omega}, \overline{\Upsilon}\}, max\{\underline{\Upsilon}, \underline{\Upsilon}\overline{\Omega}, \overline{\Upsilon}, \overline{\Upsilon}\underline{\Omega}\}],$$

$$[\Upsilon]/[\Omega] = [min\{\underline{\Upsilon}/\underline{\Omega}, \underline{\Upsilon}/\overline{\Omega}, \overline{\Upsilon}/\underline{\Omega}, \overline{\Upsilon}/\overline{\Omega}\}, max\{\underline{\Upsilon}/\underline{\Omega}, \underline{\Upsilon}/\overline{\Omega}, \overline{\Upsilon}/\overline{\Omega}, \overline{\Upsilon}/\overline{\Omega}\}],$$

where

$$0 \notin [\underline{\Upsilon}, \overline{\Omega}].$$

The Hausdorff-Pompeiu distance for intervals is established as:

$$d([\underline{\Upsilon}, \overline{\Upsilon}], [\underline{\Omega}, \overline{\Omega}]) = max\{|\underline{\Upsilon} - \underline{\Omega}|, |\overline{\Upsilon} - \overline{\Omega}|\}.$$

The metric space $(\mathcal{R}_I, d)$ is often complete. An explanation of how operations are defined on $\mathcal{R}_{\mathcal{I}}$ give rise to a number of algebraic features that make it quasilinear space, see [41]. As follows, they can be categorized

- (Associativity of addition) $(\Upsilon + \Omega) + \beta = \Upsilon + (\Omega + \beta) \ \forall \ \Upsilon, \Omega, \beta \in \mathcal{R}_{\mathcal{I}}$,
- (Commutativity of addition) $\Upsilon + \Omega = \Omega + \Upsilon \ \forall \ \Upsilon, \Omega \in \mathcal{R}_{\mathcal{I}}$,
- (Additivity element) $\Upsilon + 0 = 0 + \Upsilon \ \forall \ \Upsilon \in \mathcal{R}_{\mathcal{I}}$,
- (Cancellation of law) $\beta + \Upsilon = \beta + \Omega \Rightarrow \Upsilon = \Omega \ \forall \ \Upsilon, \Omega, \beta \in \mathcal{R}_{\mathcal{I}}$,
- (Associativity of multiplication) $(\Upsilon \cdot \Omega) \cdot \beta = \Upsilon \cdot (\Omega \cdot \beta) \ \forall \ \Upsilon, \Omega, \beta \in \mathcal{R}_{\mathcal{I}}$,
- (Commutativity of multiplication) $\Upsilon \cdot \Omega = \Omega \cdot \Upsilon \ \forall \ \Upsilon, \Omega \in \mathcal{R}_{\mathcal{I}}$,
- (Unity element) $\Upsilon \cdot 1 = 1 \cdot \Upsilon \ \forall \ \Upsilon \in \mathcal{R}_{\mathcal{I}}$,
- (Cancellation of law) $\Upsilon + \beta = \Omega + \beta \Rightarrow \Upsilon = \Omega \ \forall \ \Upsilon, \Omega, \beta \in \mathcal{R}_{\mathcal{I}}$.

In addition, inclusion $\subseteq$ is one of the set property is defined as:

$$[\Upsilon] \subseteq [\Omega] \Longleftrightarrow [\underline{\Upsilon}, \overline{\Upsilon}] \subseteq [\underline{\Omega}, \overline{\Omega}] \Longleftrightarrow \underline{\Omega} \leq \underline{\Upsilon}, \overline{\Upsilon} \leq \overline{\Omega}.$$

A definition of Riemannian integrability of interval valued functions is given in [40] by Dinghas. Let $\mathcal{IR}_{([\Upsilon,\Omega])}$ denote the pack of all $\mathcal{IR}$-integrable functions on $[\Upsilon, \Omega]$.

Definition 2.1. ([40]) Let $\chi : [\Upsilon, \Omega] \rightarrow \mathcal{R}_I$ be such that $\chi(\nu) = [\underline{\chi}(\nu), \overline{\chi}(\nu)]$ for each $\nu \in [\Upsilon, \Omega]$ and $\underline{\chi}, \overline{\chi} \in \mathcal{IR}_{([\Upsilon,\Omega])}$. Thus, we say $\chi \in \mathcal{IR}_{([\Upsilon,\Omega])}$ and represented by

$$\int_{\Upsilon}^{\Omega} \chi(\nu)d\nu = \Big[\int_{\Upsilon}^{\Omega} \underline{\chi}(\nu)d\nu, \int_{\Upsilon}^{\Omega} \overline{\chi}(\nu)d\nu \Big].$$

Definition 2.2. ([37]) A positive function $\chi : \tau \subseteq \mathcal{R} \rightarrow \mathcal{R}$ is known as Godunova-Levin convex, if $\forall$ $\Upsilon, \Omega \in \tau$ and $x \in (0, 1)$, we have

$$\chi(\nu(\Upsilon) + (1 - \nu)\Omega) \leq \frac{\chi(\Upsilon)}{\nu} + \frac{\chi(\Omega)}{(1 - \nu)}.$$

Definition 2.3. ([37]) Let $h : (0, 1) \rightarrow \mathcal{R}^+$, $h \neq 0$. A function $\chi : \tau \rightarrow \mathcal{R}^+$ is known as h-Godunova-Levin convex, if $\forall$ $\Upsilon, \Omega \in \tau$ and $x \in (0, 1)$, we have

$$\chi(\nu\Upsilon + (1 - \nu)\Omega) \leq \frac{\chi(\Upsilon)}{h(\nu)} + \frac{\chi(\Omega)}{h(1 - \nu)}.$$

Definition 2.4. ([23, 37]) Suppose $h_1, h_2 : (0, 1) \subseteq \tau \rightarrow \mathcal{R}^+$, $h_1, h_2 \neq 0$. A function $\chi : \tau \rightarrow \mathcal{R}^+$ is known as (h_1, h_2)- Godunova-Levin convex, if $\forall$ $\Upsilon, \Omega \in \tau$ and $x \in (0, 1)$, one has

$$\chi(\nu\Upsilon + (1 - \nu)\Omega) \leq \frac{\chi(\Upsilon)}{h_1(\nu)h_2(1 - \nu)} + \frac{\chi(\Omega)}{h_1(1 - \nu)h_2(\nu)}. \tag{2.1}$$

Remark 2.1.
- *If $h_2 = 1$, then above Eq (2.1) provides a result for the h-Godunova-Levin function [37].*
- *If $h_1 = h_2 = 1$, then above Eq (2.1) provides a result for the P-function [17].*
- *If $h_1(\nu) = (\nu)^s$, $h_2 = 1$, then above Eq (2.1) provides a result for the s-Godunoava-Levin function [42].*
- *If $h_1 = \frac{1}{h(\nu)}$, $h_2 = 1$, then above Eq (2.1) provides a result for the h-convex function [17].*

3. Main results

As a closing to the current part of the preliminaries, we introduced a new concept of interval-valued (h_1, h_2)-Godunova-Levin convexity.

Definition 3.1. Suppose $h_1, h_2 : (0, 1) \subseteq \tau \rightarrow \mathcal{R}^+$ are non-negative functions where $h_1, h_2 \neq 0$. A function $\chi : \tau \rightarrow {\mathcal{R}_I}^+$ is known as interval-valued (h_1, h_2) Godunova-Levin convex if $\forall$ $\Upsilon, \Omega \in \tau$ and $x \in (0, 1)$, we have

$$\frac{\chi(\Upsilon)}{h_1(\nu)h_2(1 - \nu)} + \frac{\chi(\Omega)}{h_1(1 - \nu)h_2(\nu)} \supseteq \chi(\nu\Upsilon + (1 - \nu)\Omega). \tag{3.1}$$

The collection of all interval-valued (h_1, h_2)-Godunova-Levin convex function is denoted by $\chi \in SGX((h_1, h_2), \tau, {\mathcal{R}_I}^+)$. If the inequality (3.1) is altered then χ is known as interval (h_1, h_2)-Godunova-Levin Concave, i.e., $\chi \in SGV((h_1, h_2), \tau, {\mathcal{R}_I}^+)$.

Proposition 3.1. *Let $\chi : [\Upsilon, \Omega] \rightarrow {\mathcal{R}_I}^+$ be interval valued function such that $\chi(\nu) = [\underline{\chi}(\nu), \overline{\chi}(\nu)]$. Then $\chi \in SGX((h_1, h_2), [\Upsilon, \Omega], {\mathcal{R}_I}^+)$ iff $\underline{\chi} \in SGX((h_1, h_2), [\Upsilon, \Omega], \mathcal{R}^+)$ and $\overline{\chi} \in SGV((h_1, h_2), [\Upsilon, \Omega], \mathcal{R}^+)$.*

Proof. Let χ be interval valued (h_1, h_2)-Godunova-Levin convex function and suppose that $x, y \in [\Upsilon, \Omega]$, $\nu \in (0, 1)$, then

$$\frac{\chi(\Upsilon)}{h_1(\nu)h_2(1-\nu)} + \frac{\chi(\Omega)}{h_1(1-\nu)h_2(\nu)} \subseteq \chi(\nu x + (1-\nu)y),$$

that is,

$$\Big[\frac{\underline{\chi}(\Upsilon)}{h_1(\nu)h_2(1-\nu)} + \frac{\underline{\chi}(\Omega)}{h_1(1-\nu)h_2(\nu)}, \frac{\overline{\chi}(\Upsilon)}{h_1(\nu)h_2(1-\nu)} + \frac{\overline{\chi}(\Omega)}{h_1(1-\nu)h_2(\nu)}\Big] \subseteq \chi(\nu x + (1-\nu)y).$$

It follows that we have

$$\frac{\underline{\chi}(\Upsilon)}{h_1(\nu)h_2(1-\nu)} + \frac{\underline{\chi}(\Omega)}{h_1(1-\nu)h_2(\nu)} \geq \underline{\chi}(\nu x + (1-\nu)y)$$

and

$$\frac{\overline{\chi}(\Upsilon)}{h_1(\nu)h_2(1-\nu)} + \frac{\overline{\chi}(\Omega)}{h_1(1-\nu)h_2(\nu)} \leq \overline{\chi}(\nu x + (1-\nu)y).$$

This shows that $\underline{\chi} \in SGX((h_1, h_2), [\Upsilon, \Omega], \mathcal{R}^+)$ and $\overline{\chi} \in SGV((h_1, h_2), [\Upsilon, \Omega], \mathcal{R}^+)$. Conversely suppose that if $\underline{\chi} \in SGX((h_1, h_2), [\Upsilon, \Omega], \mathcal{R}^+)$ and $\overline{\chi} \in SGV((h_1, h_2), [\Upsilon, \Omega], \mathcal{R}^+)$. Then from above definition and set inclusion we have $\chi \in SGX((h_1, h_2), [\Upsilon, \Omega], \mathcal{R}_I{}^+)$. This completes the proof. □

Proposition 3.2. *Let $\chi : [\Upsilon, \Omega] \to \mathcal{R}_I{}^+$ be interval valued function such that $\chi(\nu) = [\underline{\chi}(\nu), \overline{\chi}(\nu)]$. Then $\chi \in SGV((h_1, h_2), [\Upsilon, \Omega], \mathcal{R}_I{}^+)$ iff $\underline{\chi} \in SGV((h_1, h_2), [\Upsilon, \Omega], \mathcal{R}^+)$ and $\overline{\chi} \in SGX((h_1, h_2), [\Upsilon, \Omega], \mathcal{R}^+)$. This can be similar to a Proposition 3.1.*

3.1. *In this section, we can be established some variants of Hermite-Hadamard inequality by using the definition of (h_1, h_2)-Godunova-Levin I-V-Fs.*

Throughout the article $H(x, y) = h_1(x)h_2(y)\ \forall\ x, y \in (0, 1)$.

Theorem 3.1. *Let $\chi : [\Upsilon, \Omega] \to \mathcal{R}_I^+$, $h_1, h_2 : (0, 1) \to \mathcal{R}^+$ and $H(\frac{1}{2}, \frac{1}{2}) \neq 0$. If $\chi \in SGX((h_1, h_2), [\Upsilon, \Omega], \mathcal{R}_I{}^+)$ and $\chi \in \mathcal{IR}_{[\Upsilon,\Omega]}$, then*

$$\frac{[H(\frac{1}{2}, \frac{1}{2})]}{2}\chi\Big(\frac{\Upsilon+\Omega}{2}\Big) \supseteq \frac{1}{\Omega - \Upsilon}\int_{\Upsilon}^{\Omega} \chi(\nu)d\nu \supseteq [\chi(\Upsilon) + \chi(\Omega)]\int_0^1 \frac{dx}{H(x, 1-x)}. \tag{3.2}$$

Proof. According to our hypothesis,

$$\frac{\chi(x\Upsilon + (1-x)\Omega)}{H\Big(\frac{1}{2}, \frac{1}{2}\Big)} + \frac{\chi((1-x)\Upsilon + x\Omega)}{H\Big(\frac{1}{2}, \frac{1}{2}\Big)} \subseteq \chi\Big(\frac{\Upsilon+\Omega}{2}\Big).$$

As a result of integrating above inequality over (0, 1), we obtain

$$\int_0^1 \underline{\chi}(x\Upsilon + (1-x)\Omega)dx + \int_0^1 \underline{\chi}((1-x)\Upsilon + x\Omega)dx \geq H\Big(\frac{1}{2}, \frac{1}{2}\Big)\int_0^1 \underline{\chi}\Big(\frac{\Upsilon+\Omega}{2}\Big)dx,$$

$$\int_0^1 \overline{\chi}(x\Upsilon + (1-x)\Omega)dx + \int_0^1 \overline{\chi}((1-x)\Upsilon + x\Omega)dx \leq H\Big(\frac{1}{2}, \frac{1}{2}\Big)\int_0^1 \overline{\chi}\Big(\frac{\Upsilon+\Omega}{2}\Big)dx.$$

It follows that we have

$$\frac{2}{\Omega-\Upsilon}\int_{\Upsilon}^{\Omega}\underline{\chi}(\nu)d\nu \geq H\Big(\frac{1}{2},\frac{1}{2}\Big)\int_0^1 \underline{\chi}\Big(\frac{\Upsilon+\Omega}{2}\Big)dx = H\Big(\frac{1}{2},\frac{1}{2}\Big)\underline{\chi}\Big(\frac{\Upsilon+\Omega}{2}\Big).$$

Similarly,

$$\frac{2}{\Omega-\Upsilon}\int_{\Upsilon}^{\Omega}\overline{\chi}(\nu)d\nu \leq H\Big(\frac{1}{2},\frac{1}{2}\Big)\int_0^1 \overline{\chi}\Big(\frac{\Upsilon+\Omega}{2}\Big)dx = H\Big(\frac{1}{2},\frac{1}{2}\Big)\overline{\chi}\Big(\frac{\Upsilon+\Omega}{2}\Big).$$

This implies

$$\frac{[H(\frac{1}{2},\frac{1}{2})]}{2}\Big[\underline{\chi}\Big(\frac{\Upsilon+\Omega}{2}\Big),\overline{\chi}\Big(\frac{\Upsilon+\Omega}{2}\Big)\Big] \supseteq \frac{1}{\Omega-\Upsilon}\int_{\Upsilon}^{\Omega}\chi(\nu)d\nu. \tag{3.3}$$

Now by Definition 3.1,

$$\frac{\chi(\Upsilon)}{h_1(\nu)h_2(1-\nu)} + \frac{\chi(\Omega)}{h_1(1-\nu)h_2(\nu)} \subseteq \chi(\nu x+(1-\nu)y).$$

As a result of integrating above inequality over (0, 1), we obtain

$$\chi(\Upsilon)\int_0^1 \frac{dx}{h_1(x)h_2(1-x)} + \chi(\Omega)\int_0^1 \frac{dx}{h_1(1-x)h_2(x)} \subseteq \int_0^1 \chi(x\Upsilon+(1-x)\Omega)dx. \tag{3.4}$$

It implies that

$$[\chi(\Upsilon)+\chi(\Omega)]\int_0^1 \frac{dx}{h_1(x)h_2(1-x)} \subseteq \frac{1}{\Omega-\Upsilon}\int_{\Upsilon}^{\Omega}\chi(\nu)d\nu. \tag{3.5}$$

Combinig Eqs (3.4) and (3.5), we get required result

$$\frac{[H(\frac{1}{2},\frac{1}{2})]}{4}f\Big(\frac{\Upsilon+\Omega}{2}\Big) \supseteq \frac{1}{\Omega-\Upsilon}\int_{\Upsilon}^{\Omega}\chi(\nu)d\nu \supseteq [\chi(\Upsilon)+\chi(\Omega)]\int_0^1 \frac{dx}{H(x,1-x)}.$$

□

Example 3.1. Let $h_1(x) = \frac{1}{x}$, $h_2(x) = 1$ for $x \in (0,1)$, $[\Upsilon,\Omega] = [-1,1]$, and $\chi : [\Upsilon,\Omega] \to \mathcal{R}_I^+$ be defined by $\chi(\nu) = [\nu^2, 4-e^\nu]$, then

$$\frac{[H(\frac{1}{2},\frac{1}{2})]}{2}\chi\Big(\frac{\Upsilon+\Omega}{2}\Big) = \chi(0) = [0,3]$$

$$\frac{1}{\Omega-\Upsilon}\int_{\Upsilon}^{\Omega}\chi(\nu)d\nu = \frac{1}{2}\Big[\int_{-1}^1 \nu^2 d\nu, \int_{-1}^1 (4-e^\nu)d\nu\Big]$$

$$= \frac{1}{2}\Big[\int_{-1}^1 \nu^2 d\nu, \int_{-1}^1 (4-e^\nu)d\nu\Big] = \Big[\frac{1}{3}, 4-\frac{e-\frac{1}{e}}{2}\Big]$$

$$[\chi(\Upsilon)+\chi(\Omega)]\int_0^1 \frac{dx}{H(x,1-x)} = \Big[1, 4-\frac{e+\frac{1}{e}}{2}\Big].$$

Thus we obtain

$$[0,3] \supseteq \Big[\frac{1}{3}, 4-\frac{e-\frac{1}{e}}{2}\Big] \supseteq \Big[1, 4-\frac{e+\frac{1}{e}}{2}\Big].$$

Consequently, the above theorem is verified.

Theorem 3.2. *Let* $\chi : [\Upsilon, \Omega] \rightarrow \mathcal{R}_I^+$, $h_1, h_2 : (0,1) \rightarrow \mathcal{R}^+$ *and* $H(\frac{1}{2}, \frac{1}{2}) \neq 0$. *If* $\chi \in SGX((h_1, h_2), [\Upsilon, \Omega], \mathcal{R}_I{}^+)$ *and* $\chi \in \mathcal{IR}_{[\Upsilon,\Omega]}$, *then*

$$\frac{[H(\frac{1}{2}, \frac{1}{2})]^2}{4}\chi\Big(\frac{\Upsilon + \Omega}{2}\Big) \supseteq \triangle_1 \supseteq \frac{1}{\Omega - \Upsilon}\int_{\Upsilon}^{\Omega} \chi(\nu)d\nu \supseteq \triangle_2 \supseteq \bigg\{[\chi(\Upsilon)+\chi(\Omega)]\Big[\frac{1}{2}+\frac{1}{H(\frac{1}{2}, \frac{1}{2})}\Big]\bigg\}\int_0^1 \frac{dx}{H(x, 1-x)},$$

where

$$\triangle_1 = \frac{[H(\frac{1}{2}, \frac{1}{2})]}{4}\Big[\chi\Big(\frac{3\Upsilon + \Omega}{4}\Big) + \chi\Big(\frac{\Upsilon + 3\Omega}{4}\Big)\Big],$$

$$\triangle_2 = \Big[\chi\Big(\frac{\Upsilon + \Omega}{2}\Big) + \Big(\frac{\chi(\Upsilon) + \chi(\Omega)}{2}\Big)\Big]\int_0^1 \frac{dx}{H(x, 1-x)}.$$

Proof. Consider $[\Upsilon, \frac{\Upsilon+\Omega}{2}]$, we have

$$\frac{\chi\Big(x\Upsilon + (1-x)\frac{\Upsilon+\Omega}{2}\Big)}{[H(\frac{1}{2}, \frac{1}{2})]} + \frac{\chi\Big((1-x)\Upsilon + x\frac{\Upsilon+\Omega}{2}\Big)}{[H(\frac{1}{2}, \frac{1}{2})]} \subseteq \chi\left(\frac{(x\Upsilon + (1-x)\frac{\Upsilon+\Omega}{2} + x\frac{\Upsilon+\Omega}{2} + (1-x)\Upsilon}{2}\right)$$

$$= \chi\left(\frac{\Upsilon + \frac{\Upsilon+\Omega}{2}}{2}\right) = \chi\left(\frac{3\Upsilon + \Omega}{2}\right).$$

As a result of integrating above inequality over $(0, 1)$, we obtain

$$\frac{1}{H(\frac{1}{2}, \frac{1}{2})}\Big[\int_0^1 \chi(x\Upsilon + (1-x)\frac{\Upsilon + \Omega}{2})dx + \int_0^1 \chi(x\frac{\Upsilon + \Omega}{2} + (1-x)\Upsilon)dx\Big] \subseteq \chi\Big(\frac{3\Upsilon + \Omega}{2}\Big).$$

Then above inequality become as

$$\frac{1}{H(\frac{1}{2}, \frac{1}{2})}\Big[\frac{2}{\Upsilon - \Omega}\int_{\frac{\Upsilon+\Omega}{2}}^{\Upsilon} \chi(\nu)d\nu + \frac{2}{\Omega - \Upsilon}\int_{\Upsilon}^{\frac{\Upsilon+\Omega}{2}} \chi(\nu)d\nu\Big] \subseteq \chi\Big(\frac{3\Upsilon + \Omega}{2}\Big).$$

This implies that

$$\frac{1}{H(\frac{1}{2}, \frac{1}{2})}\Big[\frac{2}{\Omega - \Upsilon}\int_{\Upsilon}^{\frac{\Upsilon+\Omega}{2}} \chi(\nu)d\nu + \frac{2}{\Omega - \Upsilon}\int_{\Upsilon}^{\frac{\Upsilon+\Omega}{2}} \chi(\nu)d\nu\Big] \subseteq \chi\Big(\frac{3\Upsilon + \Omega}{2}\Big),$$

$$\frac{4}{H(\frac{1}{2}, \frac{1}{2})}\Big[\frac{1}{\Omega - \Upsilon}\int_{\Upsilon}^{\frac{\Upsilon+\Omega}{2}} \chi(\nu)d\nu\Big] \subseteq \chi\Big(\frac{3\Upsilon + \Omega}{2}\Big),$$

$$\frac{1}{\Omega - \Upsilon}\int_{\Upsilon}^{\frac{\Upsilon+\Omega}{2}} \chi(\nu)d\nu \subseteq \frac{[H(\frac{1}{2}, \frac{1}{2})]}{4}f\Big(\frac{3\Upsilon + \Omega}{2}\Big). \tag{3.6}$$

Similarly for interval $[\frac{\Upsilon+\Omega}{2}, \Omega]$, we have

$$\frac{1}{\Omega - \Upsilon}\int_{\frac{\Upsilon+\Omega}{2}}^{\Omega} \chi(\nu)d\nu \subseteq \frac{[H(\frac{1}{2}, \frac{1}{2})]}{4}f\Big(\frac{3\Upsilon + \Omega}{2}\Big). \tag{3.7}$$

Adding the inclusions (3.6) and (3.7), we get

$$\triangle_1 = \frac{[H(\frac{1}{2},\frac{1}{2})]}{4}\Big[\chi\Big(\frac{3\Upsilon+\Omega}{4}\Big)+\chi\Big(\frac{\Upsilon+3\Omega}{4}\Big)\Big] \supseteq \Big[\frac{1}{\Omega-\Upsilon}\int_{\Upsilon}^{\Omega}\chi(\nu)d\nu\Big]$$

$$= \frac{1}{2}\Big[\frac{2}{\Omega-\Upsilon}\int_{\Upsilon}^{\frac{\Upsilon+\Omega}{2}}\chi(\nu)d\nu+\frac{2}{\Omega-\Upsilon}\int_{\frac{\Upsilon+\Omega}{2}}^{\Omega}\chi(\nu)d\nu\Big]$$

$$\supseteq \frac{1}{2}\Big[\Big[\chi(\Upsilon)+\chi\Big(\frac{\Upsilon+\Omega}{2}\Big)\Big]\int_0^1\frac{dx}{H(x,1-x)}\Big]+\frac{1}{2}\Big[\Big[\chi(\Omega)+\chi\Big(\frac{\Upsilon+\Omega}{2}\Big)\Big]\int_0^1\frac{dx}{H(x,1-x)}\Big]$$

$$= \frac{1}{2}\Big[\Big\{\chi(\Upsilon)+\chi(\Omega)+2\chi\Big(\frac{\Upsilon+\Omega}{2}\Big)\Big\}\int_0^1\frac{dx}{H(x,1-x)}\Big]$$

$$= \Big[\frac{\chi(\Upsilon)+\chi(\Omega)}{2}+\chi\Big(\frac{\Upsilon+\Omega}{2}\Big)\Big]\int_0^1\frac{dx}{H(x,1-x)} = \triangle_2.$$

Now consider

$$\frac{[H(\frac{1}{2},\frac{1}{2})]^2}{4}\chi\Big(\frac{\Upsilon+\Omega}{2}\Big) = \frac{[H(\frac{1}{2},\frac{1}{2})]^2}{4}\chi\Big(\frac{1}{2}\Big(\frac{3\Upsilon+\Omega}{4}\Big)+\frac{1}{2}\Big(\frac{\Upsilon+3\Omega}{4}\Big)\Big)$$

$$\supseteq \frac{[H(\frac{1}{2},\frac{1}{2})]^2}{4}\Bigg[\frac{\chi\big(\frac{3\Upsilon+\Omega}{4}\big)}{H(\frac{1}{2},\frac{1}{2})}+\frac{\chi\big(\frac{\Upsilon+3\Omega}{4}\big)}{H(\frac{1}{2},\frac{1}{2})}\Bigg] = \frac{[H(\frac{1}{2},\frac{1}{2})]^2}{4H(\frac{1}{2},\frac{1}{2})}\Big[\chi\Big(\frac{3\Upsilon+\Omega}{4}\Big)+\chi\Big(\frac{3\Upsilon+\Omega}{4}\Big)\Big]$$

$$= \frac{[H(\frac{1}{2},\frac{1}{2})]}{4}\Big[\chi\Big(\frac{3\Upsilon+\Omega}{4}\Big)+\chi\Big(\frac{3\Upsilon+\Omega}{4}\Big)\Big] = \triangle_1$$

$$\supseteq \frac{[H(\frac{1}{2},\frac{1}{2})]}{4}\Big\{\frac{1}{H(\frac{1}{2},\frac{1}{2})}\Big[\chi(\Upsilon)+\chi\Big(\frac{\Upsilon+\Omega}{2}\Big)\Big]+\frac{1}{H(\frac{1}{2},\frac{1}{2})}\Big[\chi(\Omega)+\chi\Big(\frac{\Upsilon+\Omega}{2}\Big)\Big]\Big\}$$

$$= \frac{[H(\frac{1}{2},\frac{1}{2})]}{4}\Big\{\frac{1}{H(\frac{1}{2},\frac{1}{2})}\Big[\chi(\Upsilon)+\chi(\Omega)+2\chi\Big(\frac{\Upsilon+\Omega}{2}\Big)\Big]\Big\}$$

$$= \frac{1}{4}\Big\{\chi(\Upsilon)+\chi(\Omega)+2\chi\Big(\frac{\Upsilon+\Omega}{2}\Big)\Big\} = \frac{1}{2}\Big[\frac{\chi(\Upsilon)+\chi(\Omega)}{2}+\chi\Big(\frac{\Upsilon+\Omega}{2}\Big)\Big]$$

$$\supseteq \Big[\frac{\chi(\Upsilon)+\chi(\Omega)}{2}+\chi\Big(\frac{\Upsilon+\Omega}{2}\Big)\Big]\int_0^1\frac{dx}{H(x,1-x)} = \triangle_2$$

$$\supseteq \Big[\frac{\chi(\Upsilon)+\chi(\Omega)}{2}+\frac{\chi(\Upsilon)}{H(\frac{1}{2},\frac{1}{2})}+\frac{\chi(\Omega)}{H(\frac{1}{2},\frac{1}{2})}\Big]\int_0^1\frac{dx}{H(x,1-x)}$$

$$= \Big[\frac{\chi(\Upsilon)+\chi(\Omega)}{2}+\frac{1}{H(\frac{1}{2},\frac{1}{2})}\big[\chi(\Upsilon)+\chi(\Omega)\big]\Big]\int_0^1\frac{dx}{H(x,1-x)}$$

$$= \Big\{[\chi(\Upsilon)+\chi(\Omega)]\Big[\frac{1}{2}+\frac{1}{H(\frac{1}{2},\frac{1}{2})}\Big]\Big\}\int_0^1\frac{dx}{H(x,1-x)}.$$

□

Example 3.2. Let $h_1(x) = \frac{1}{x}$, $h_2(x) = 1$ for $x \in (0,1)$, $[\Upsilon, \Omega] = [-1, 1]$, and $\chi : [\Upsilon, \Omega] \to \mathcal{R}_I{}^+$ be defined by $\chi(\nu) = [\nu^2, 4 - e^{\nu}]$, then

$$\frac{[H(\frac{1}{2},\frac{1}{2})]^2}{4}\chi\Big(\frac{\Omega+\Upsilon}{2}\Big) = \chi(0) = [0,3],$$

$$\triangle_1 = \frac{[H(\frac{1}{2},\frac{1}{2})]}{4}\Big[\chi\Big(\frac{\Omega+3\Upsilon}{4}\Big)+\chi\Big(\frac{\Omega+3\Upsilon}{4}\Big)\Big],$$

$$\triangle_1 = \frac{1}{2}\Big[\chi\Big(\frac{-1}{2}\Big),\chi\Big(\frac{1}{2}\Big)\Big] = \Big[\frac{1}{4}, 4 - \frac{\sqrt{e}+\frac{1}{\sqrt{e}}}{2}\Big].$$

$$\frac{1}{\Omega-\Upsilon}\int_{\Upsilon}^{\Omega}\chi(\nu)d\nu = \frac{1}{2}\Big[\int_{-1}^{1}\nu^2 d\nu, \int_{-1}^{1}(4-e^{\nu})d\nu\Big],$$

$$= \frac{1}{2}\Big[\int_{-1}^{1}\nu^2 d\nu, \int_{-1}^{1}(4-e^{\nu})d\nu\Big] = \Big[\frac{1}{3}, 4 - \frac{e-\frac{1}{e}}{2}\Big].$$

$$\triangle_2 = \Big[\frac{\chi(\Upsilon)+\chi(\Omega)}{2} + \chi\Big(\frac{\Omega+\Upsilon}{2}\Big)\Big]\int_0^1 \frac{dx}{H(x,1-x)},$$

$$\triangle_2 = \frac{1}{2}\bigg(\Big[1, 4 - \frac{e+\frac{1}{e}}{2}\Big] + [0,3]\bigg),$$

$$\triangle_2 = \Big[\frac{1}{2}, \frac{7}{2} - \frac{e+\frac{1}{e}}{4}\Big].$$

$$\bigg\{[\chi(\Upsilon)+\chi(\Omega)]\Big[\frac{1}{2} + \frac{1}{H(\frac{1}{2},\frac{1}{2})}\Big]\bigg\}\int_0^1 \frac{dx}{H(x,1-x)}$$

$$= \Big[1, 4 - \frac{e+\frac{1}{e}}{2}\Big].$$

Thus we obtain

$$[0,3] \supseteq \Big[\frac{1}{4}, 4 - \frac{\sqrt{e}+\frac{1}{\sqrt{e}}}{2}\Big] \supseteq \Big[\frac{1}{3}, 4 - \frac{e-\frac{1}{e}}{2}\Big]$$

$$\supseteq \Big[\frac{1}{2}, \frac{7}{2} - \frac{e+\frac{1}{e}}{4}\Big] \supseteq \Big[1, 4 - \frac{e+\frac{1}{e}}{2}\Big].$$

As a result, the preceding theorem is confirmed.

Theorem 3.3. *Let* $\chi, \rho : [\Upsilon, \Omega] \to \mathcal{R}_I^+$, $h_1, h_2 : (0,1) \to \mathcal{R}^+$ *and* $H(\frac{1}{2},\frac{1}{2}) \neq 0$. *If* $\chi, \rho \in SGX((h_1,h_2),[\Upsilon,\Omega],\mathcal{R}_I{}^+)$ *and* $\chi,\rho \in \mathcal{IR}_{[\Upsilon,\Omega]}$, *then*

$$\frac{1}{\Omega-\Upsilon}\int_{\Upsilon}^{\Omega}\chi(\nu)\rho(\nu)d\nu \supseteq M(\Upsilon,\Omega)\int_0^1 \frac{1}{H^2(x,1-x)}dx + N(\Upsilon,\Omega)\int_0^1 \frac{1}{H(x,x)H(1-x,1-x)}dx,$$

where

$$M(\Upsilon,\Omega) = \chi(\Upsilon)\rho(\Upsilon)+\chi(\Omega)\rho(\Omega), N(\Upsilon,\Omega) = \chi(\Upsilon)\rho(\Omega)+\chi(\Omega)\rho(\Upsilon).$$

Proof. We assume that $\chi, \rho \in SGX((h_1, h_2), [\Upsilon, \Omega], \mathcal{R}_I^+)$ then, we have

$$\frac{\chi(\Upsilon)}{h_1(x)h_2(1-x)} + \frac{\chi(\Omega)}{h_1(1-x)h_2(x)} \subseteq \chi\Big(\Upsilon x + (1-x)\Omega\Big),$$

$$\frac{\rho(\Upsilon)}{h_1(x)h_2(1-x)} + \frac{\rho(\Omega)}{h_1(1-x)h_2(x)} \subseteq \rho\Big(\Upsilon x + (1-x)\Omega\Big).$$

Then,

$$\chi\Big(\Upsilon x + (1-x)\Omega\Big)\rho\Big(\Upsilon x + (1-x)\Omega\Big) \supseteq$$

$$\frac{\chi(\Upsilon)\rho(\Upsilon)}{H^2(x, 1-x)} + \frac{\chi(\Omega)\rho(\Omega)}{H^2(1-x, x)} + \frac{\chi(\Upsilon)\rho(\Omega) + \chi(\Omega)\rho(\Upsilon)}{H(x, x)H(1-x, 1-x)}.$$

As a result of integrating above inequality over $(0, 1)$, we obtain

$$\int_0^1 \chi\Big(\Upsilon x + (1-x)\Omega\Big)\rho\Big(\Upsilon x + (1-x)\Omega\Big)dx$$

$$= \Bigg[\int_0^1 \underline{\chi}\Big(\Upsilon x + (1-x)\Omega\Big)\underline{\rho}\Big(\Upsilon x + (1-x)\Omega\Big)dx, \int_0^1 \overline{\chi}\Big(\Upsilon x + (1-x)\Omega\Big)\overline{\rho}\Big(\Upsilon x + (1-x)\Omega\Big)dx\Bigg]$$

$$= \left[\frac{1}{\Omega-\Upsilon}\int_\Upsilon^\Omega \underline{\chi}(\nu)\underline{\rho}(\nu)d\nu, \frac{1}{\Omega-\Upsilon}\int_\Upsilon^\Omega \overline{\chi}(\nu)\overline{\rho}(\nu)d\nu\right] = \frac{1}{\Omega-\Upsilon}\int_\Upsilon^\Omega \chi(\nu)\rho(\nu)d\nu \supseteq$$

$$\int_0^1 \frac{[\chi(\Upsilon)\rho(\Upsilon) + \chi(\Omega)\rho(\Omega)]}{H^2(x, 1-x)}dx + \int_0^1 \frac{[\chi(\Upsilon)\rho(\Omega) + \chi(\Omega)\rho(\Upsilon)]}{H(x, x)H(1-x, 1-x)}dx$$

$$= M(\Upsilon, \Omega)\int_0^1 \frac{1}{H^2(x, 1-x)}dx + N(\Upsilon, \Omega)\int_0^1 \frac{1}{H(x, x)H(1-x, 1-x)}dx.$$

Theorem is proved. □

Example 3.3. Let $h_1(x) = \frac{1}{x}$, $h_2(x) = 1$ for $x \in (0, 1)$, $[\Upsilon, \Omega] = [-1, 1]$, and $\chi, \rho : [\Upsilon, \Omega] \to \mathcal{R}_I^+$ and

$$\chi(\nu) = [\nu^2, 4 - e^\nu], \rho(\nu) = [\nu, 3 - \nu^2].$$

Then

$$\frac{1}{\Omega-\Upsilon}\int_\Upsilon^\Omega \chi(\nu)\rho(\nu)d\nu = \left[\int_0^1 \nu^3 d\nu, \int_0^1 (4 - e^\nu)(3 - \nu^2)d\nu\right] = \left[\frac{1}{4}, \frac{61}{3} - 3e\right],$$

$$M(\Upsilon, \Omega)\int_0^1 \frac{1}{H^2(x, 1-x)}dx = M(0, 1)\int_0^1 x^2 dx = \Big[\frac{1}{3}, \frac{31}{3} - e\Big],$$

$$N(\Upsilon, \Omega)\int_0^1 \frac{1}{H(x, x)H(1-x, 1-x)}dx = N(0, 1)\int_0^1 (x - x^2)dx = \Big[0, \frac{32}{6} - \frac{4e}{6}\Big].$$

It follows that

$$\left[\frac{1}{4}, \frac{61}{3} - 3e\right] \supseteq \Big[\frac{1}{3}, \frac{31}{3} - e\Big] + \Big[0, \frac{32}{6} - \frac{4e}{6}\Big] = \Big[\frac{1}{3}, \frac{31}{3} + \frac{32 - 4e}{6} - e\Big].$$

As a result, the preceding theorem is confirmed.

Theorem 3.4. *Let* $\chi, \rho : [\Upsilon, \Omega] \to \mathcal{R}_I^+$, $h_1, h_2 : (0,1) \to \mathcal{R}^+$ *and* $H(\frac{1}{2}, \frac{1}{2}) \neq 0$. *If* $\chi, \rho \in SGX((h_1, h_2), [\Upsilon, \Omega], \mathcal{R}_I{}^+)$ *and* $\chi, \rho \in \mathcal{IR}_{[\Upsilon,\Omega]}$, *then*

$$\frac{[H(\frac{1}{2}, \frac{1}{2})]^2}{2}\chi\Big(\frac{\Upsilon+\Omega}{2}\Big)\rho\Big(\frac{\Upsilon+\Omega}{2}\Big) \supseteq \frac{1}{\Omega-\Upsilon}\int_{\Upsilon}^{\Omega}\chi(\nu)\rho(\nu)d\nu$$

$$+M(\Upsilon,\Omega)\int_0^1 \frac{1}{H(x,x)H(1-x,1-x)}dx + N(\Upsilon,\Omega)\int_0^1 \frac{1}{H^2(x,1-x)}dx.$$

Proof. By hypothesis, one has

$$\chi\Big(\frac{\Omega+\Upsilon}{2}\Big) \supseteq \frac{\chi\Big(\Upsilon x+(1-x)\Omega\Big)}{H(\frac{1}{2},\frac{1}{2})} + \frac{\chi\Big(\Upsilon(1-x)+x\Omega\Big)}{H(\frac{1}{2},\frac{1}{2})},$$

$$g\Big(\frac{\Omega+\Upsilon}{2}\Big) \supseteq \frac{\rho\Big(\Upsilon x+(1-x)\Omega\Big)}{H(\frac{1}{2},\frac{1}{2})} + \frac{\rho\Big(\Upsilon(1-x)+x\Omega\Big)}{H(\frac{1}{2},\frac{1}{2})}.$$

Then

$$\chi\Big(\frac{\Omega+\Upsilon}{2}\Big)\rho\Big(\frac{\Upsilon+\Omega}{2}\Big)$$

$$\supseteq \frac{1}{[H(\frac{1}{2},\frac{1}{2})]^2}\Big[\chi\Big(\Upsilon x+(1-x)\Omega\Big)\rho\Big(\Upsilon x+(1-x)\Omega\Big)+\chi\Big(\Upsilon(1-x)+x\Omega\Big)\rho\Big(\Upsilon(1-x)+x\Omega\Big)\Big]$$

$$+\frac{1}{[H(\frac{1}{2},\frac{1}{2})]^2}\Big[\chi\Big(\Upsilon x+(1-x)\Omega\Big)\rho\Big(\Upsilon(1-x)+x\Omega\Big)+\chi\Big(\Upsilon(1-x)+x\Omega\Big)\rho\Big(\Upsilon x+(1-x)\Omega\Big)\Big]$$

$$\supseteq \frac{1}{[H(\frac{1}{2},\frac{1}{2})]^2}\Big[\chi\Big(\Upsilon x+(1-x)\Omega\Big)\rho\Big(\Upsilon x+(1-x)\Omega\Big)+\chi\Big(\Upsilon(1-x)+(x\Omega\Big)\rho\Big(\Upsilon(1-x)+x\Omega\Big)\Big]$$

$$+\frac{1}{[H(\frac{1}{2},\frac{1}{2})]^2}\Big[\Big(\frac{\chi(\Upsilon)}{H(x,1-x)}+\frac{\chi(\Omega)}{H(1-x,x)}\Big)\Big(\frac{\rho(\Upsilon)}{H(1-x,x)}+\frac{\rho(\Omega)}{H(x,1-x)}\Big)$$

$$+\Big(\frac{\chi(\Upsilon)}{H(1-x,x)}+\frac{\chi(\Omega)}{H(x,1-x)}\Big)\Big(\frac{\rho(\Upsilon)}{H(x,1-x)}+\frac{\rho(\Omega)}{H(1-x,x)}\Big)\Big]$$

$$= \frac{1}{[H(\frac{1}{2},\frac{1}{2})]^2}\Big[\chi\Big(\Upsilon x+(1-x)\Omega\Big)\rho\Big(\Upsilon x+(1-x)\Omega\Big)+\chi\Big(\Upsilon(1-x)+\Omega x\Big)\rho\Big(\Upsilon(1-x)+\Omega x\Big)\Big]$$

$$+\frac{1}{[H(\frac{1}{2},\frac{1}{2})]^2}\Big[\Big(\frac{2}{H(x,x)H(1-x,1-x)}\Big)M(\Upsilon,\Omega)+\Big(\frac{1}{H^2(x,1-x)}+\frac{1}{H^2(1-x,x)}\Big)N(\Upsilon,\Omega)\Big].$$

As a result of integrating above inequality over $(0,1)$, we obtain

$$\int_0^1 \chi\Big(\frac{\Upsilon+\Omega}{2}\Big)\rho\Big(\frac{\Upsilon+\Omega}{2}\Big)dx = \Big[\int_0^1 \underline{\chi}\Big(\frac{\Upsilon+\Omega}{2}\Big)\rho\Big(\frac{\Upsilon+\Omega}{2}\Big)dx, \int_0^1 \overline{\chi}\Big(\frac{\Upsilon+\Omega}{2}\Big)\rho\Big(\frac{\Upsilon+\Omega}{2}\Big)dx\Big] = \chi\Big(\frac{\Upsilon+\Omega}{2}\Big)\rho\Big(\frac{\Upsilon+\Omega}{2}\Big)dx$$

$$\supseteq \frac{2}{[H(\frac{1}{2},\frac{1}{2})]^2}\left[\frac{1}{\Omega-\Upsilon}\int_{\Upsilon}^{\Omega}\chi(\nu)\rho(\nu)d\nu\right]+\frac{2}{[H(\frac{1}{2},\frac{1}{2})]^2}\left[M(\Upsilon,\Omega)\int_0^1\frac{1}{H(x,x)H(1-x,1-x)}dx\right.$$
$$\left.+N(\Upsilon,\Omega)\int_0^1\frac{1}{H^2(x,1-x)}dx\right].$$

Multiply both sides by $\frac{\left[H(\frac{1}{2},\frac{1}{2})\right]^2}{2}$, we get

$$\frac{[H(\frac{1}{2},\frac{1}{2})]^2}{2}\chi\Big(\frac{\Omega+\Upsilon}{2}\Big)g\Big(\frac{\Omega+\Upsilon}{2}\Big)\supseteq\frac{1}{\Omega-\Upsilon}\int_{\Upsilon}^{\Omega}\chi(\nu)\rho(\nu)d\nu+M(\Upsilon,\Omega)\int_0^1\frac{1}{H(x,x)H(1-x,1-x)}dx$$
$$+N(\Upsilon,\Omega)\int_0^1\frac{1}{H^2(x,1-x)}dx.$$

This completes the proof.

□

Example 3.4. Let $h_1(x)=\frac{1}{x}$, $h_2(x)=2$ for $x\in(0,1)$, $[\Upsilon,\Omega]=[-1,1]$, and $\chi,\rho:[\Upsilon,\Omega]\to\mathcal{R}_I{}^+$ and

$$\chi(\nu)=[\nu^2,5-e^{\nu}],\rho(\nu)=[\nu,4-\nu^2].$$

Then

$$\frac{[H(\frac{1}{2},\frac{1}{2})]^2}{2}\chi\Big(\frac{\Upsilon+\Omega}{2}\Big)\phi\Big(\frac{\Upsilon+\Omega}{2}\Big)=8\chi\Big(\frac{1}{2}\Big)\rho\Big(\frac{1}{2}\Big)=\left[1,150-30\sqrt{e}\right]$$
$$\frac{1}{\Omega-\Upsilon}\int_{\Upsilon}^{\Omega}\chi(\nu)\rho(\nu)d\nu=\left[\int_0^1\nu^3d\nu,\int_0^1(5-e^{\nu})(4-\nu^2)d\nu\right]=\left[\frac{1}{4},\frac{61}{3}-3e\right]$$
$$M(\Upsilon,\Omega)\int_0^1\frac{1}{H(x,x)H(1-x,1-x)}dx=\frac{M(0,1)}{4}\int_0^1(x-x^2)dx=\Big[\frac{17}{24},\frac{31}{24}-\frac{3e}{24}\Big]$$
$$N(\Upsilon,\Omega)\int_0^1\frac{1}{H^2(x,1-x)}dx=\frac{N(0,1)}{4}\int_0^1x^2dx=\Big[0,\frac{32}{12}-\frac{4e}{12}\Big].$$

It follows that

$$\left[1,150-30\sqrt{e}\right]\supseteq\left[\frac{1}{4},\frac{61}{3}-3e\right]+\Big[\frac{17}{24},\frac{31}{24}-\frac{3e}{24}\Big]+\Big[0,\frac{32}{12}-\frac{4e}{12}\Big]=\Big[\frac{23}{24},\frac{-83e+583}{24}\Big].$$

As a result, the preceding theorem is confirmed.

3.2. In this section, we can be established Jensen-type inequality by using the definition of (h_1,h_2)-Godunova-Levin I-V-Fs.

Theorem 3.5. *([13,39]) Let $e_1,e_2,e_3...e_k\in R^+$ with $k\geq 2$. If h_1,h_2 is super multiplicative non-negative functions and if $\chi\in SGX((h_1,h_2),[\Upsilon,\Omega],R_I{}^+)$ with $\nu_1,\nu_2,\nu_3,...,\nu_k\in I$. Then the inequality become as:*

$$\chi\bigg(\frac{1}{E_k}\sum_{i=1}^{k}e_i\nu_i\bigg)\supseteq\sum_{i=1}^{k}\left[\frac{\chi(\nu_i)}{H\Big(\frac{e_i}{E_k},\frac{E_k-1}{E_k}\Big)}\right],$$

where $E_k=\sum_{i=1}^{k}e_i$.

Proof. When $k = 2$ the above inequality is true. Now we suppose that inequality is true for $k - 1$. Consider

$$\chi\left(\frac{1}{E_k}\sum_{i=1}^{k} e_i v_i\right) = \chi\left(\frac{e_k}{E_k}v_k + \sum_{i=1}^{k-1}\frac{e_i}{E_k}v_i\right)$$

$$\supseteq \frac{\chi(v_k)}{h_1(\frac{e_k}{E_k})h_2(\frac{E_k-1}{E_k})} + \frac{\chi(\sum_{i=1}^{k-1}\frac{e_i}{E_k}v_i)}{h_1(\frac{E_k-1}{E_k})h_2(\frac{e_k}{E_k})}$$

$$\supseteq \frac{\chi(v_k)}{h_1(\frac{e_k}{E_k})h_2(\frac{E_k-1}{E_k})}$$

$$+\sum_{i=1}^{k-1}\left[\frac{\chi(v_i)}{H\left(\frac{e_i}{E_k}, \frac{E_k-2}{E_k-1}\right)}\right]\frac{1}{h_1(\frac{E_k-1}{E_k})h_2(\frac{e_k}{E_k})}$$

$$\supseteq \frac{\chi(v_k)}{h_1(\frac{e_k}{E_k})h_2(\frac{E_k-1}{E_k})} + \sum_{i=1}^{k-1}\left[\frac{\chi(v_i)}{H\left(\frac{e_i}{E_k}, \frac{E_k-2}{E_k-1}\right)}\right]$$

$$\supseteq \sum_{i=1}^{k}\left[\frac{\chi(v_i)}{H\left(\frac{e_i}{E_k}, \frac{E_k-1}{E_k}\right)}\right].$$

As a result, Mathematical induction confirms the conclusion. □

4. Conclusions

In this paper, we introduce the (h_1, h_2)-Godunova-Levin concept for I-V-Fs. The purpose of the above concept was to study Jensen and Hermite-Hadamrd inequalities using I-V-Fs. The inequalities previously developed by An et al. [23], and Hongxin Bai, et al. [39] are generalized in our results. In addition, some useful examples are provided to support our main conclusions. Continuing this research direction, we will investigate Jensen and Hermite-Hadamrd type inequalities for I-V-Fs and fuzzy-valued functions over time scales. We think that this is an intriguing topic that can be explored in the future to find equivalent inequalities depending on the type of convexity. By utilizing these concepts, a new direction for convex optimization can be developed. By embracing this concept, we hope to support other authors in securing their roles in various fields of science.

Conflict of interest

The authors declare that there is no conflict of interest in publishing this paper.

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