Research article

Further irreducibility criteria for polynomials associated with the complete residue systems in any imaginary quadratic field

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Abstract: Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $O_K$ its ring of integers. Let $\pi$ and $\beta$ be an irreducible element and a nonzero element, respectively, in $O_K$. In the authors’ earlier work, it was proved for the cases, $m \not\equiv 1 \pmod{4}$ and $m \equiv 1 \pmod{4}$ that if $\pi = \alpha_n \beta^n + \alpha_{n-1} \beta^{n-1} + \cdots + \alpha_1 \beta + \alpha_0 =: f(\beta)$, where $n \geq 1$, $\alpha_n \in O_K \setminus \{0\}$, $\alpha_0, \ldots, \alpha_{n-1}$ belong to a complete residue system modulo $\beta$, and the digits $\alpha_{n-1}$ and $\alpha_n$ satisfy certain restrictions, then the polynomial $f(x)$ is irreducible in $O_K[x]$. In this paper, we extend these results by establishing further irreducibility criteria for polynomials in $O_K[x]$. In addition, we provide elements of $\beta$ that can be applied to the new criteria but not to the previous ones.

Keywords: imaginary quadratic field; ring of integers; complete residue system; irreducible element; irreducible polynomial

Mathematics Subject Classification: 11R04, 11R09, 11R11

1. Introduction

Determining the irreducibility of a polynomial has been one of the most intensively studied problems in mathematics. Among many irreducibility criteria for polynomials in $\mathbb{Z}[x]$, a classical result of A. Cohn [1] states that if we express a prime $p$ in the decimal representation as

$$p = a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_1 10 + a_0,$$

then the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is irreducible in $\mathbb{Z}[x]$. This result was subsequently generalized to any base $b$ by Brillhart et al. [2]. In 2002, Murty [3] gave another proof of this fact that was conceptually simpler than the one in [2].

In the present work, we are interested in studying the result of A. Cohn in any imaginary quadratic field. Let $K = \mathbb{Q}(\sqrt{m})$ with a unique squarefree integer $m \neq 1$, be a quadratic field. We have seen that the quadratic field $K$ is said to be real if $m > 0$ and imaginary if $m < 0$. The set of algebraic integers
that lie in \( K \) is denoted by \( O_K \). Indeed,

\[
O_K = \{ a + b\sigma_m \mid a, b \in \mathbb{Z} \},
\]

where

\[
\sigma_m := \begin{cases} 
\sqrt{m} & \text{if } m \not\equiv 1 \pmod{4}, \\
1 + \sqrt{m} & \text{if } m \equiv 1 \pmod{4}
\end{cases}
\]

Clearly, \( O_{\mathbb{Q}(i)} = \mathbb{Z}[i] \), the ring of Gaussian integers, where \( i = \sqrt{-1} \). It is well known that \( O_K \) is an integral domain and \( K \) is its quotient field. Then the set of units in \( O_K[x] \) is \( U(O_K) \), the group of units in \( O_K \).

In general, we know that a prime element in \( O_K \) is an irreducible element and the converse holds if \( O_K \) is a unique factorization domain. A nonzero polynomial \( p(x) \in O_K[x] \) is said to be irreducible in \( O_K[x] \) if \( p(x) \) is not a unit and if \( p(x) = f(x)g(x) \) in \( O_K[x] \), then either \( f(x) \) or \( g(x) \) is a unit in \( O_K \).

Polynomials that are not irreducible are called reducible. For \( \beta = a + b\sigma_m \in O_K \), we denote the norm of \( \beta \) by

\[
N(\beta) = \begin{cases} 
a^2 - mb^2 & \text{if } m \not\equiv 1 \pmod{4}, \\
a^2 + ab + b^2 \left( \frac{1-m}{4} \right) & \text{if } m \equiv 1 \pmod{4}.
\end{cases}
\]

Clearly, \( N(\beta) \in \mathbb{Z} \) for all \( \beta \in O_K \). To determine whether \( \alpha \in O_K \) is an irreducible element, we often use the fact that if \( N(\alpha) = \pm p \), where \( p \) is a rational prime, then \( \alpha \) is an irreducible element [4].

For \( \alpha, \beta \in O_K \) with \( \alpha \neq 0 \), we say that \( \alpha \) divides \( \beta \), denoted by \( \alpha \mid \beta \), if there exists \( \delta \in O_K \) such that \( \beta = \alpha \delta \). For \( \alpha, \beta, \gamma \in O_K \) with \( \gamma \neq 0 \), we say that \( \alpha \) is congruent to \( \beta \) modulo \( \gamma \) and we write \( \alpha \equiv \beta \) (mod \( \gamma \)), if \( \gamma \mid (\alpha - \beta) \). By a complete residue system modulo \( \beta \) in \( O_K \), abbreviated by CRS (\( \beta \)) [5], we mean a set of \( |N(\beta)| \) elements \( C = \{ \alpha_1, \alpha_2, \ldots, \alpha_{|N(\beta)|} \} \) in \( O_K \), which satisfies the following.

(i) For each \( \alpha \in O_K \), there exists \( \alpha_i \in C \) such that \( \alpha \equiv \alpha_i \) (mod \( \beta \)).

(ii) For all \( i, j \in \{1, 2, \ldots, |N(\beta)|\} \) with \( i \neq j \), we have \( \alpha_i \neq \alpha_j \) (mod \( \beta \)).

We have seen from [6] that

\[
C = \left\{ x + yi \mid x = 0, 1, \ldots, \frac{a^2 + b^2}{d} - 1 \text{ and } y = 0, 1, \ldots, d-1 \right\}
\]

(1.1)

is a CRS (\( \beta \)), where \( \beta = a + bi \in \mathbb{Z}[i] \) and \( d = \gcd(a, b) \). It is clear that

\[
C' := \{ x + yi \mid x = 0, 1, \ldots, \max\{|a|, |b|\} - 1 \text{ and } y = 0, 1, \ldots, d-1 \} \subseteq C.
\]

In 1977, Singthongla et al. [7] established the result of A. Cohn in \( O_K[x] \), where \( K \) is an imaginary quadratic field such that \( O_K \) is a Euclidean domain, namely \( m = -1, -2, -3, -7, \) and \(-11 \) [4]. Regarding the complete residue system (1.1), they established irreducibility criteria for polynomials in \( \mathbb{Z}[i][x] \) as the following results.

**Theorem A.** [7] Let \( \beta \in \{2 \pm 2i, 1 \pm 3i, 3 \pm i\} \) or \( \beta = a + bi \in \mathbb{Z}[i] \) be such that \(|\beta| \geq 2 + \sqrt{2}\) and \( a \geq 1 \). For a Gaussian prime \( \pi \), if

\[
\pi = \alpha_0 \beta^0 + \alpha_{-1} \beta^{-1} + \cdots + \alpha_1 \beta + \alpha_0 =: f(\beta),
\]

with \( n \geq 1 \), \( \Re(\alpha_n) \geq 1 \), and \( \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in C' \) satisfying \( \Re(\alpha_{n-1}) \Im(\alpha_n) \geq \Re(\alpha_n) \Im(\alpha_{n-1}) \), then \( f(x) \) is irreducible in \( \mathbb{Z}[i][x] \).
In the proof of Theorem A in [7], the inequality
\[ |\beta| \geq \frac{3 + \sqrt{1 + 4M}}{2}, \]  
(1.2)
where \( M = \sqrt{\max\{|a, |b|\} - 1}^2 + (d - 1)^2 \) is necessary. It can be verified that for \( \beta = a + bi \in \mathbb{Z}[i] \), if \(|\beta| < 2 + \sqrt{2}\) and \( a \geq 1 \), then \( \beta \in \{3 \pm i, 2 \pm 2i, 2 \pm i, 1 \pm 3i, 1 \pm 2i, 1 \pm i, 3, 2, 1\} \). It is clear that the Gaussian integers \( 2 \pm 2i, 1 \pm 3i, \) and \( 3 \pm i \) satisfy (1.2), while \( 2 \pm i, 1 \pm 2i, 1 \pm i, 3, 2, 1 \) do not. Consequently, we cannot apply Theorem A for these numbers. However, there is an irreducibility criterion for polynomials in \( \mathbb{Z}[i][x] \) using \( \beta = 3 \) in [7].

**Theorem B.** [7] If \( \pi \) is a Gaussian prime such that
\[ \pi = \alpha_n 3^n + \alpha_{n-1} 3^{n-1} + \cdots + \alpha_1 3 + \alpha_0, \]
where \( n \geq 3 \), \( \text{Re}(\alpha_n) \geq 1 \), and \( \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C} \) satisfying the conditions
\[ \text{Re}(\alpha_{n-1}) \geq \text{Re}(\alpha_n) \text{ Im}(\alpha_{n-1}) \]
\[ \text{Re}(\alpha_{n-2}) \geq \text{Re}(\alpha_n) \text{ Im}(\alpha_{n-2}) \]
then the polynomial \( f(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \cdots + \alpha_1 x + \alpha_0 \) is irreducible in \( \mathbb{Z}[i][x] \).

In 2017, Tadee et al. [8] derived three explicit representations for a complete residue system in a general quadratic field \( K = \mathbb{Q}(\sqrt{m}) \). We are interested in the first one and only the case \( m \equiv 1 \pmod{4} \) because the complete residue system in another case, \( m \equiv 1 \pmod{4} \) is inapplicable for our work. The \( CRS(\beta) \) for \( m \equiv 1 \pmod{4} \) in [8] is the set
\[ C := \{ x + y \sqrt{m} \mid x = 0, 1, \ldots, \frac{|N(\beta)|}{d} - 1 \text{ and } y = 0, 1, \ldots, d - 1 \}, \]
(1.3)
where \( \beta = a + b \sqrt{m} \) and \( d = \gcd(a, b) \).

Recently, Phetnun et al. [9] constructed a complete residue system in a general quadratic field \( K = \mathbb{Q}(\sqrt{m}) \) for the case \( m \equiv 1 \pmod{4} \), which is similar to that in (1.3). They then determined the so-called base-\( \beta(C) \) representation in \( O_K \) and generalized Theorem A for any imaginary quadratic field by using such representation. These results are as the following.

**Theorem C.** [9] Let \( K = \mathbb{Q}(\sqrt{m}) \) be a quadratic field with \( m \equiv 1 \pmod{4} \). If \( \beta = a + b\sigma_m \in O_K \setminus \{0\} \) with \( d = \gcd(a, b) \), then the set
\[ C = \{ x + y\sigma_m \mid x = 0, 1, \ldots, \frac{|N(\beta)|}{d} - 1 \text{ and } y = 0, 1, \ldots, d - 1 \} \]
(1.4)
is a \( CRS(\beta) \).

From (1.3) and (1.4), we have shown in [9] for any \( m < 0 \), that the set
\[ C' := \{ x + y\sigma_m \mid x = 0, 1, \ldots, \max\{|a, |b|\} - 1 \text{ and } y = 0, 1, \ldots, d - 1 \} \subseteq C. \]
(1.5)
Moreover, if \( d = 1 \), then \( C' = \{0, 1, \ldots, \max\{|a, |b|\} - 1\} \), while \( b = 0 \) implies \( C' = \{ x + y\sigma_m \mid x, y = 0, 1, \ldots, |a| - 1 \} = C \).
Definition A. [9] Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field. Let $\beta \in O_K \setminus \{0\}$ and $C$ be a CRS($\beta$). We say that $\eta \in O_K \setminus \{0\}$ has a base-$\beta$(C) representation if

$$\eta = \alpha_n \beta^n + \alpha_{n-1} \beta^{n-1} + \cdots + \alpha_1 \beta + \alpha_0,$$

(1.6)

where $n \geq 1$, $\alpha_n \in O_K \setminus \{0\}$, and $\alpha_i \in \mathbb{C}$ $(i = 0, 1, \ldots, n - 1)$. If $\beta \in \mathbb{C}$ $(i = 0, 1, \ldots, n - 1)$, then (1.6) is called a base-$\beta$(C') representation of $\eta$.

Theorem D. [9] Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $m \equiv 1 \pmod{4}$. Let $\beta = a + b \sqrt{m} \in O_K$ be such that $|\beta| \geq 2 + \sqrt{1 - m}$ and $a \geq 1 + \sqrt{1 - m}$. For an irreducible element $\pi$ in $O_K$ with $|\pi| \geq |\beta|$, if

$$\pi = \alpha_n \beta^n + \alpha_{n-1} \beta^{n-1} + \cdots + \alpha_1 \beta + \alpha_0 =: f(\beta)$$

is a base-$\beta$(C') representation with $\text{Re}(\alpha_n) \geq 1$ satisfying $\text{Re}(\alpha_{n-1}) \text{Im}(\alpha_n) \geq \text{Re}(\alpha_n) \text{Im}(\alpha_{n-1})$, then $f(x)$ is irreducible in $O_K[x]$.

Theorem E. [9] Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $m \equiv 1 \pmod{4}$. Let $\beta = a + b\sqrt{m} \in O_K$ be such that $|\beta| \geq 2 + \sqrt{(9 - m)/4}$, $a \geq 1$, and $a + (b/2) \geq 1$. For an irreducible element $\pi$ in $O_K$ with $|\pi| > \sqrt{(9 - m)/4} (|\beta| - 1)$, if

$$\pi = \alpha_n \beta^n + \alpha_{n-1} \beta^{n-1} + \cdots + \alpha_1 \beta + \alpha_0 =: f(\beta)$$

is a base-$\beta$(C') representation with $\text{Re}(\alpha_n) \geq 1$ satisfying $\text{Re}(\alpha_{n-1}) \text{Im}(\alpha_n) \geq \text{Re}(\alpha_n) \text{Im}(\alpha_{n-1})$, then $f(x)$ is irreducible in $O_K[x]$.

In this work, we first establish further irreducibility criteria for polynomials in $O_K[x]$, where $K = \mathbb{Q}(\sqrt{m})$ is an imaginary quadratic field, which extend Theorem D and Theorem E. We observe that the result for the case $m \equiv 1 \pmod{4}$ is a generalization of Theorem B. Furthermore, we provide elements of $\beta$ that can be applied to the new criteria but not to the previous ones.

2. Further irreducibility criteria

In this section, we establish irreducibility criteria for polynomials in $O_K[x]$, where $K$ is an imaginary quadratic field. To prove this, we first recall the essential lemmas in [7, 10] as the following.

Lemma 1. [10] Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field. Then $|\beta| \geq 1$ for all $\beta \in O_K \setminus \{0\}$.

We note for an imaginary quadratic field $K$ that $|\alpha| = 1$ for all $\alpha \in U(O_K)$.

Lemma 2. [7] Let $f(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \cdots + \alpha_1 x + \alpha_0 \in \mathbb{C}[x]$ be such that $n \geq 3$ and $|\alpha_i| \leq M$ $(0 \leq i \leq n - 2)$ for some real number $M \geq 1$. If $f(x)$ satisfies the following:

(i) $\text{Re}(\alpha_n) \geq 1$, $\text{Re}(\alpha_{n-1}) \geq 0$, $\text{Im}(\alpha_{n-1}) \geq 0$, $\text{Re}(\alpha_{n-2}) \geq 0$, and $\text{Im}(\alpha_{n-2}) \geq 0$,
(ii) $\text{Re}(\alpha_{n-1}) \text{Im}(\alpha_n) \geq \text{Re}(\alpha_n) \text{Im}(\alpha_{n-1})$,
(iii) $\text{Re}(\alpha_{n-2}) \text{Im}(\alpha_n) \geq \text{Re}(\alpha_n) \text{Im}(\alpha_{n-2})$, and
(iv) $\text{Re}(\alpha_{n-2}) \text{Im}(\alpha_{n-1}) \geq \text{Re}(\alpha_{n-1}) \text{Im}(\alpha_{n-2})$,

then any complex zero $\alpha$ of $f(x)$ satisfies $|\alpha| < M^{1/3} + 0.465572$ if $|\arg \alpha| \leq \pi/6$; otherwise

$$\text{Re}(\alpha) < \frac{\sqrt{3}}{2} \left( \frac{1 + \sqrt{1 + 4M}}{2} \right).$$
We note that the inequality $|\alpha| < M^{1/3} + 0.465572$ appears in Lemma 2 follows from the proof of the lemma in [7] as follows: It was shown in [7] that

$$0 = \left| \frac{f(\alpha)}{\alpha^n} \right| > \frac{|\alpha|^3 - |\alpha|^2 - M}{|\alpha|^2(|\alpha| - 1)} = \frac{h(|\alpha|)}{|\alpha|^2(|\alpha| - 1)}, \quad (2.1)$$

where $h(x) = x^3 - x^2 - M$. To obtain such inequality, the authors suppose to the contrary that $|\alpha| \geq M^{1/3} + 0.465572$. One can show that $h(\alpha)$ is increasing on $(-\infty, 0) \cup (2/3, \infty)$. Since $M^{1/3} + 0.465572 > 2/3$, it follows that

$$h(|\alpha|) \geq h(M^{1/3} + 0.465572)$$

$$= 0.396716M^{2/3} - 0.280872138448M^{1/3} - 0.115841163475170752$$

$$> 0.396716M^{2/3} - 0.280873M^{1/3} - 0.115842$$

$$= M^{1/3}(0.396716M^{1/3} - 0.280873) - 0.115842$$

$$\geq 0.000001, \text{ since } M \geq 1$$

$$> 0,$$

which contradicts to (2.1).

Now, we proceed to our first main results. To obtain an irreducibility criterion for the case $m \not\equiv 1 \pmod{4}$, we begin with the following lemma.

**Lemma 3.** Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $m \not\equiv 1 \pmod{4}$. Let $\beta = a + b\sqrt{m} \in O_K$ be such that $a > 1$ and

$$M := \sqrt{(\max\{a, |b|\}^2 - m(d - 1)^2,} \quad (2.2)$$

where $d = \gcd(a, b)$. Then $M \geq 1$.

**Proof.** If $b = 0$, then $M = \sqrt{(a - 1)^2 - m(a - 1)^2} = \sqrt{1 - m(a - 1)} > 1$. Now, assume that $b \neq 0$ and we treat two separate cases.

**Case 1:** $|b| \geq a$. Then $M = \sqrt{(|b| - 1)^2 - m(d - 1)^2} \geq \sqrt{(|b| - 1)^2} = |b| - 1 \geq 1$.

**Case 2:** $|b| < a$. Then $M = \sqrt{(a - 1)^2 - m(d - 1)^2} \geq \sqrt{(a - 1)^2} = a - 1 \geq 1$.

From every case, we conclude that $M \geq 1$. \hfill \Box

By applying Lemmas 1–3, we have the following.

**Theorem 1.** Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $m \not\equiv 1 \pmod{4}$. Let $\beta = a + b\sqrt{m} \in O_K$ be such that $|\beta| \geq M^{1/3} + 1.465572$ and $a \geq 1 + (\sqrt{3}/2)(1 + \sqrt{1 + 4M})/2$, where $M$ is defined as in (2.2). For an irreducible element $\pi$ in $O_K$, if

$$\pi = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1\beta + \alpha_0 =: f(\beta)$$

is a base-$\beta(C')$ representation with $n \geq 3$ and $\Re(\alpha_n) \geq 1$ satisfying conditions (ii)–(iv) of Lemma 2, then $f(x)$ is irreducible in $O_K[x]$.

**Proof.** Suppose to the contrary that $f(x)$ is reducible in $O_K[x]$. Then $f(x) = g(x)h(x)$ with $g(x)$ and $h(x)$ in $O_K[x] \setminus U(O_K)$. We first show that either $\deg g(x) \geq 1$ and $\deg h(x) \geq 1$ or $\deg h(x) = 1$ and $|h(\beta)| = 1$. It follows from $\deg f(x) \geq 3$ that $g(x)$ or $h(x)$ is a positive degree polynomial. If either $\deg g(x) = 0$
where \( \pi = \alpha \in O_K \). Then \( \deg g(x) = \deg f(x) \) and \( f(x) = \alpha g(x) \) so that \( \pi = \alpha \beta \). Since \( \pi \) is an irreducible element and \( \alpha \notin U(O_K) \), we obtain \( g(\beta) \in U(O_K) \) and thus, \( |g(\beta)| = 1 \). Otherwise, both \( \deg g(x) \geq 1 \) and \( \deg h(x) \geq 1 \), we have that \( \pi = g(\beta)h(\beta) \). Using the irreducibility of \( \pi \) again, we deduce that either \( g(\beta) \) or \( h(\beta) \) is a unit and hence, either \( |g(\beta)| = 1 \) or \( |h(\beta)| = 1 \), as desired.

We now assume without loss of generality that \( \deg g(x) \geq 1 \) and \( |g(\beta)| = 1 \). We will show that this cannot happen. Note that \( M \geq 1 \) by Lemma 3. Moreover, since \( \alpha_i \in C' \) for all \( i \in \{0, 1, \ldots, n-1\} \), where \( C' \) is defined as in (1.5), we have

\[
|\alpha_i| \leq (\max\{|a, |b|\} - 1) + (d - 1) \sqrt{m} = \sqrt{(\max\{|a, |b|\} - 1)^2 - m(d - 1)^2} = M
\]

for all \( i \in \{0, 1, \ldots, n-1\} \). Since \( \deg g(x) \geq 1 \), \( g(x) \) can be expressed in the form

\[
g(x) = \varepsilon \prod_i (x - \gamma_i),
\]

where \( \varepsilon \in O_K \) is the leading coefficient of \( g(x) \) and the product is over the set of complex zeros of \( g(x) \). It follows from Lemma 2 that any complex zero \( \gamma \) of \( g(x) \) satisfies either

\[
|\gamma| < M^{1/3} + 0.465572 \text{ or } \text{Re}(\gamma) < \frac{\sqrt{3}}{2} \left( \frac{1 + \sqrt{1 + 4M}}{2} \right).
\]

In the first case, it follows from \( |\beta| \geq M^{1/3} + 1.465572 \) that

\[
|\beta - \gamma| \geq |\beta| - |\gamma| > |\beta| - \left( M^{1/3} + 0.465572 \right) \geq 1.
\]

In the latter case, it follows from \( a \geq 1 + (\sqrt{3}/2) \left( (1 + \sqrt{1 + 4M})/2 \right) \) that

\[
|\beta - \gamma| \geq \text{Re}(\beta - \gamma) = \text{Re}(\beta) - \text{Re}(\gamma) = a - \text{Re}(\gamma) > a - \frac{\sqrt{3}}{2} \left( \frac{1 + \sqrt{1 + 4M}}{2} \right) \geq 1.
\]

From both cases, by using Lemma 1, we obtain

\[
1 = |g(\beta)| = |\varepsilon| \prod_i |\beta - \gamma_i| \geq \prod_i |\beta - \gamma_i| > 1,
\]

which is a contradiction. This completes the proof.

By taking \( \beta = 3 \) together with \( m = -1 \) in Theorem 1, we obtain Theorem B. This shows that Theorem 1 is a generalization of Theorem B. We will show in the next section that if \( \beta = a + bi \in \mathbb{Z}[i] \setminus \{0\} \) with \( b = 0 \), then \( \beta = 3 \) is the only element that can be applied to Theorem 1.

Next, we illustrate the use of Theorem 1 by the following example.

Example 1. Let \( K = \mathbb{Q}(\sqrt{-5}) \), \( \beta = 3 + \sqrt{-5} \in O_K \), and \( \pi = -9069 - 5968 \sqrt{-5} \). Then \( d = 1 \) and so \( C' = \{0, 1, 2\} \). Note that \( M = \sqrt{(3 - 1)^2 + 5(1 - 1)^2} = 2, |\beta| = \sqrt{14} > M^{1/3} + 1.465572, a = 3 > 1 + (\sqrt{3}/2) \left( (1 + \sqrt{1 + 4M})/2 \right), \) and \( \pi \) is an irreducible element because \( N(\pi) = (-9069)^2 + 5(-5968)^2 = 260331881 \) is a rational prime. Now, we have

\[
\pi = (13 + 8 \sqrt{-5})\beta^5 + 2\beta^4 + 2\beta^3 + \beta^2 + 2\beta + 1
\]
Theorem 2. The following theorem.

is irreducible in $O_K[x]$. 

Note from Example 1 that we cannot apply Theorem D to conclude the irreducibility of the polynomial $f(x)$ because $|\beta| = |3 + \sqrt{-5}| < 2 + \sqrt{6} = 2 + \sqrt{1 - m}$. Moreover, we see that $a = 3 < 1 + \sqrt{6} = 1 + \sqrt{1 - m}$.

For the case $m \equiv 1 \pmod{4}$, we start with the following lemma.

Lemma 4. Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $m \equiv 1 \pmod{4}$. Let $\beta = a + b\sigma_m \in O_K$ be such that $a + (b/2) \geq 1 + (\sqrt{3}/2)(1 + \sqrt{1 + 4M}/2)$ and

$$M := \sqrt{(\max(|a|, |b|) - 1)^2 + (\max(|a|, |b|) - 1)(d - 1) + (d - 1)^2 \left(\frac{1 - m}{4}\right)},$$ \hspace{1cm} (2.3)

where $d = \gcd(a, b)$. Then $M \geq 1$.

Proof. If $b = 0$, then $a \geq 1 + (\sqrt{3}/2)(1 + \sqrt{1 + 4M}/2) > 1$. If $a = 0$, then $b/2 \geq 1 + (\sqrt{3}/2)(1 + \sqrt{1 + 4M}/2) > 1$. Thus, $b > 2$ and so

$$M = \sqrt{(b - 1)^2 + (b - 1)^2 + (b - 1)^2 \left(\frac{1 - m}{4}\right)} > \sqrt{(b - 1)^2} = b - 1 > 1.$$ 

Now, assume that $|a| \geq 1$ and $|b| \geq 1$. If $|a| = 1$ and $|b| = 1$, then $M = 0$, yielding a contradiction because $a + (b/2) \geq 1 + (\sqrt{3}/2)(1 + \sqrt{1 + 4M}/2)$. Then $|a| > 1$ or $|b| > 1$. It follows from $d \geq 1$ that

$$M \geq \sqrt{(2 - 1)^2 + (2 - 1)(d - 1) + (d - 1)^2 \left(\frac{1 - m}{4}\right)} \geq \sqrt{(2 - 1)^2} = 1.$$ 

By applying Lemmas 1, 2 and 4, we obtain an irreducibility criterion for the case $m \equiv 1 \pmod{4}$ as the following theorem.

Theorem 2. Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $m \equiv 1 \pmod{4}$. Let $\beta = a + b\sigma_m \in O_K$ be such that $|\beta| \geq M^{1/3} + 1.465572$ and $a + (b/2) \geq 1 + (\sqrt{3}/2)(1 + \sqrt{1 + 4M}/2)$, where $M$ is defined as in (2.3). For an irreducible element $\pi \in O_K$, if

$$\pi = \alpha_n \beta^n + \alpha_{n-1} \beta^{n-1} + \cdots + \alpha_1 \beta + \alpha_0 =: f(\beta)$$

is a base-$\beta(C')$ representation with $n \geq 3$ and $\text{Re}(\alpha_n) \geq 1$ satisfying conditions (ii)–(iv) of Lemma 2, then $f(x)$ is irreducible in $O_K[x]$. 

Proof. Suppose to the contrary that \( f(x) \) is reducible in \( O_K[x] \). Then \( f(x) = g(x)h(x) \) with \( g(x) \) and \( h(x) \) in \( O_K[x] \setminus U(O_K) \). It can be proved similarly to the proof of Theorem 1 that either \( \deg g(x) \geq 1 \) and \( |g(\beta)| = 1 \) or \( \deg h(x) \geq 1 \) and \( |h(\beta)| = 1 \). We may assume without loss of generality that \( \deg g(x) \geq 1 \) and \( |g(\beta)| = 1 \). We will show that this cannot happen. By Lemma 4, we have \( M \geq 1 \). For \( i \in \{0, 1, \ldots, n - 1\} \), since \( \alpha_i \in C' \), it follows from the definition of \( C' \) in (1.5) that

\[
|\alpha_i| \leq \left( \max\{|a|, |b|\} - 1 \right) + (d - 1) \left( \frac{1 + \sqrt{m}}{2} \right)
\]

\[
= \left( \max\{|a|, |b|\} - 1 \right) + \frac{(d - 1)}{2} \left( \sqrt{m} \right)
\]

\[
= \sqrt{\left( \max\{|a|, |b|\} - 1 \right)^2 + \left( \max\{|a|, |b|\} - 1 \right)(d - 1) + (d - 1)^2 \left( \frac{1 - m}{4} \right)}
\]

\[
= M.
\]

The remaining proof is again similar to that of Theorem 1 by using Lemmas 1, 2 and \( \Re(\beta) = a + (b/2) \).

We illustrate the use of Theorem 2 by the following example.

**Example 2.** Let \( K = \mathbb{Q}(\sqrt{-3}) \), \( \beta = 4 - \sigma_3 \), and \( \pi = 359 - 278\sigma_3 \). Then \( d = 1 \) and so \( C' = \{0, 1, 2, 3\} \). Note that \( M = \sqrt{(4 - 1)^2 + (4 - 1)(1 - 1) + (1 - 1)^2} = 3 \), \( |\beta| = \sqrt{3} > M^{1/3} + 1.465572 \), \( a + (b/2) = 3.5 > 1 + (\sqrt{3}/2)((1 + \sqrt{1 + 4M}/2) \), and \( \pi \) is an irreducible element because \( N(\pi) = 359^2 - 359 \cdot 278 + (-278)^2 = 106363 \) is a rational prime. Now, we have

\[
\pi = \beta^4 + 3\beta^3 + \beta^2 + 2\beta + 1
\]

is its base-\( \beta \) \( (C') \) representation with \( n = 4 \) and \( \Re(\alpha_n) = 1 \) satisfying conditions (ii)-(iv) of Lemma 2. By using Theorem 2, we obtain that

\[
f(x) = x^4 + 3x^3 + x^2 + 2x + 1
\]

is irreducible in \( O_K[x] \).

From Example 2, we emphasize that we cannot apply Theorem E to conclude the irreducibility of the polynomial \( f(x) \) because \( |\beta| = |4 - \sigma_3| < 2 + \sqrt{3} = 2 + \sqrt{(9 - m)/4} \), although \( a = 4 > 1 \) and \( a + (b/2) = 4 - (1/2) > 1 \).

3. **Comparison of the criteria**

Let \( K = \mathbb{Q}(\sqrt{m}) \) be an imaginary quadratic field. In this section, we will try to find elements of \( \beta = a + b\sigma_m \in O_K \setminus \{0\} \) that can be applied to Theorem 1, respectively, Theorem 2 but not to Theorem D, respectively, Theorem E. We are only interested in two cases, namely \( b = 0 \) and \( b \neq 0 \) with \( d = \gcd(a, b) = 1 \) because the remaining case, \( b \neq 0 \) with \( d > 1 \) requires us to solve a multi-variable system of inequalities, which is more complicated. To proceed with this objective, we begin with the following remarks.
Remark 1. Let \( a \) and \( m \) be integers with \( m < 0 \). Then the following statements hold.

(i) \( a \geq 1 + \frac{\sqrt{3}}{2} \left( \frac{1 + \sqrt{1 + 4(a-1)}}{2} \right) \) if and only if \( a \geq 3. \)

(ii) \( a \geq 1 + \frac{\sqrt{3}}{2} \left( \frac{1 + \sqrt{1 + 4 \sqrt{1-m}(a-1)}}{2} \right) \) if and only if \( a \geq \frac{4 + 2 \sqrt{3} + 3 \sqrt{1-m}}{4} \).

(iii) \( a \geq 1 + \frac{\sqrt{3}}{2} \left( \frac{1 + \sqrt{1 + 4 \sqrt{(9-m)/4}(a-1)}}{2} \right) \) if and only if \( a \geq \frac{4 + 2 \sqrt{3} + 3 \sqrt{9-m}/4}{4} \).

Proof. For convenience, we let \( A = a - 1 \). We have for any real number \( x > 0 \) that

\[
a \geq 1 + \frac{\sqrt{3}}{2} \left( \frac{1 + \sqrt{1 + 4x(a-1)}}{2} \right) \text{ if and only if } A \geq \frac{\sqrt{3}}{4} \left( 1 + \sqrt{1 + 4xA} \right),
\]

if and only if \( \left( \frac{4 \sqrt{3}A}{3} - 1 \right)^2 \geq 1 + 4xA, \)

if and only if \( \frac{16A^2}{3} - \frac{8 \sqrt{3} + 12x} \geq 0, \)

if and only if \( A[4A - (2 \sqrt{3} + 3x)] \geq 0, \)

if and only if \( 4A - (2 \sqrt{3} + 3x) \geq 0, \)

if and only if \( A \geq \frac{2 \sqrt{3} + 3x}{4}, \)

if and only if \( a \geq \frac{4 + 2 \sqrt{3} + 3x}{4}. \) (3.1)

Substituting \( x = 1, x = \sqrt{1-m} \), and \( x = \sqrt{(9-m)/4} \) in (3.1) lead to (i)–(iii), respectively, as desired. \( \square \)

To compare Theorem 1 with Theorem D and to compare Theorem 2 with Theorem E, we require the following remark.

Remark 2. For any real number \( x \), the following statements hold.

(i) \( \frac{4 + 2 \sqrt{3} + 3 \sqrt{x}}{4} \geq \left( x + \sqrt{x} \right)^{1/3} + 1.465572 \) for all \( x \in [3, \infty) \).

(ii) \( \sqrt{x^2 + 5} \geq (x - 1)^{1/3} + 1.465572 \) for all \( x \in [1, \infty) \).

(iii) \( \sqrt{3x + 1} \geq (x - 1)^{1/3} + 1.465572 \) for all \( x \in [1, \infty) \).

(iv) \( \sqrt{\frac{x^2}{2} + 1} \geq (x - 1)^{1/3} + 1.465572 \) for all \( x \in [4, \infty) \).

(v) \( x \geq \left( \sqrt{2}(x - 1) \right)^{1/3} + 1.465572 \) for all \( x \in [2.85, \infty) \).

(vi) \( \sqrt{x^2 + 1} \geq (x - 1)^{1/3} + 1.465572 \) for all \( x \in [3, \infty) \).

(vii) \( \sqrt{-73 - 121x} > 4 + \sqrt{9} - x \) for all \( x \in (-\infty, -2) \).

(viii) \( \sqrt{29 - 9x} > 4 + \sqrt{9} - x \) for all \( x \in (-\infty, -3) \).
Proof of Remark 2. By using the WolframAlpha computational intelligence (www.wolframalpha.com), it can be verified by considering the graphs of both left and right functions of each inequality. □

3.1. Comparison of the criteria for \( m \neq 1 \) (mod 4)

Let \( K = \mathbb{Q}(\sqrt{m}) \) be an imaginary quadratic field with \( m \neq 1 \) (mod 4). In this subsection, we will find elements of \( \beta \in O_K \setminus \{0\} \) that can be applied to Theorem 1 but not to Theorem D. Now, let \( \beta = a + b\sqrt{m} \) be a nonzero element in \( O_K \) that can be applied to Theorem 1 but not to Theorem D. Then \( |\beta| \geq M^{1/3} + 1.465572 \) and \( a \geq 1 + (\sqrt{3}/2)(1 + \sqrt{1 + 4M}/2) \), where \( M \) is defined as in (2.2). Since \( \beta \) cannot be applied to Theorem D, one can consider two possible cases, namely, \( |\beta| < 2 + \sqrt{1 - m} \) or \( |\beta| \geq 2 + \sqrt{1 - m} \) as follows:

Case A: \( |\beta| < 2 + \sqrt{1 - m} \). Then, we now try to find elements of \( \beta \) that satisfy the following inequality system:

\[
|\beta| < 2 + \sqrt{1 - m} \\
|\beta| \geq M^{1/3} + 1.465572 \\
a \geq 1 + \frac{\sqrt{3}}{2} \left( 1 + \sqrt{1 + 4 \sqrt{1 - m}} \right). (3.2)
\]

We consider two cases as follows:

Case 1: \( b = 0 \). Then \( \beta = a \) and \( M = \sqrt{(a - 1)^2 - m(a - 1)^2} = \sqrt{1 - m(a - 1)} \). Thus, the system (3.2) becomes

\[
a < 2 + \sqrt{1 - m} \quad (3.3) \\
a \geq \left( \sqrt{1 - m(a - 1)} \right)^{1/3} + 1.465572 \quad (3.4) \\
a \geq 1 + \frac{\sqrt{3}}{2} \left( 1 + \frac{\sqrt{1 + 4 \sqrt{1 - m(a - 1)}}}{2} \right). (3.5)
\]

By (3.5) and Remark 1(ii), we have \( a \geq \frac{4 + 2\sqrt{3} + 3\sqrt{1 - m}}{4} \), which together with (3.3) yield

\[
\frac{4 + 2\sqrt{3} + 3\sqrt{1 - m}}{4} \leq a < 2 + \sqrt{1 - m}. (3.6)
\]

To show that the integers \( \beta = a \) satisfying (3.6) are solutions of the system above, we must show that they also satisfy (3.4). If \( m = -1 \), then \( a \geq \frac{4 + 2\sqrt{3} + 3\sqrt{2}}{4} \approx 2.93 \). It follows from Remark 2(v) with \( x = a \) that \( a \geq \left( \sqrt{2(a - 1)} \right)^{1/3} + 1.465572 = \left( \sqrt{1 - m(a - 1)} \right)^{1/3} + 1.465572 \). Assume that \( m \leq -2 \). By taking \( x = 1 - m \) in Remark 2(i), we obtain that

\[
\frac{4 + 2\sqrt{3} + 3\sqrt{1 - m}}{4} \geq (1 - m + \sqrt{1 - m})^{1/3} + 1.465572 \\
= \left( \sqrt{1 - m(2 + \sqrt{1 - m - 1})} \right)^{1/3} + 1.465572 \\
> \left( \sqrt{1 - m(a - 1)} \right)^{1/3} + 1.465572, \text{ by } (3.3),
\]

implying (3.4).

We note for \( m = -1 \) that the inequality (3.6) implies \( a = 3 \). Hence, \( \beta = 3 \in \mathbb{Z}[i] \) is the only element that can be applied to Theorem 1 but not to Theorem D.

**Case 2**: \( b \neq 0 \) and \( d = 1 \). There are two further subcases:

**Subcase 2.1**: \( |b| \geq a \). Then \( |\beta| = \sqrt{a^2 - mb^2} \) and \( M = \sqrt{|b| - 1}^2 = |b| - 1 \). Thus, the system (3.2) becomes

\[
\begin{align*}
\sqrt{a^2 - mb^2} &< 2 + \sqrt{1 - m} \\
\sqrt{a^2 - mb^2} &\geq (|b| - 1)^{1/3} + 1.465572
\end{align*}
\]

(3.7)

\[
a \geq 1 + \frac{\sqrt{3}}{2} \left(1 + \sqrt{1 + 4(|b| - 1)}\right).
\]

(3.8)

Since \( |b| \geq a \), we obtain from (3.8) that \( a \geq 1 + (\sqrt{3}/2)\left(1 + \sqrt{1 + 4(a - 1)}\right)/2 \). Using Remark 1(i), we have that \( a \geq 3 \). It follows from \( |b| \geq a \), \( a \geq 3 \), and \( m \leq -1 \) that

\[
\sqrt{a^2 - mb^2} \geq \sqrt{a^2 - ma^2} = \sqrt{a^2(1 - m)} \geq \sqrt{9(1 - m)} = 3 \sqrt{1 - m} > 2 + \sqrt{1 - m},
\]

which is contrary to (3.7). Thus, the system above has no integer solution \((a, b)\). This means that the assumptions in the system generate no pairs \((a, b)\) that are solutions to Theorem 1 and that are also not solutions to Theorem D.

**Subcase 2.2**: \( |b| < a \). Then \( |\beta| = \sqrt{a^2 - mb^2} \) and \( M = \sqrt{(a - 1)^2} = a - 1 \). Thus, the system (3.2) becomes

\[
\begin{align*}
\sqrt{a^2 - mb^2} &< 2 + \sqrt{1 - m} \\
\sqrt{a^2 - mb^2} &\geq (a - 1)^{1/3} + 1.465572
\end{align*}
\]

(3.9)

\[
a \geq 1 + \frac{\sqrt{3}}{2} \left(1 + \sqrt{1 + 4(a - 1)}\right).
\]

(3.10)

Using Remark 1(i) and (3.11), we have \( a \geq 3 \). Since \( m \leq -1 \), we obtain \((6 - 5m)^2 = 25m^2 - 60m + 36 > 16m^2 - 52m + 36 = 4(9 - 4m)(1 - m)\), yielding \( 6 - 5m > 2 \sqrt{9 - 4m}(1 - m) \). It follows that

\[
(\sqrt{9 - 4m} - \sqrt{1 - m})^2 = 10 - 5m - 2(\sqrt{9 - 4m}(1 - m)) > 4
\]

and so \( \sqrt{9 - 4m} - \sqrt{1 - m} > 2 \). If \( |b| \geq 2 \), then \( \sqrt{a^2 - mb^2} \geq \sqrt{9 - 4m} \geq 2 + \sqrt{1 - m} \), which is contrary to (3.9). Thus, \( |b| = 1 \). Using (3.9) and \( a \geq 3 \), we have \( \sqrt{9 - m} \leq \sqrt{a^2 - m} < 2 + \sqrt{1 - m} \) and so \( 9 \leq a^2 < 5 + 4 \sqrt{1 - m} \), i.e., \( 3 \leq a < \sqrt{5 + 4 \sqrt{1 - m}} \). We next show that the pairs \((a, b)\) with

\[
3 \leq a < \sqrt{5 + 4 \sqrt{1 - m}} \text{ and } b = \pm 1
\]

(3.12)

also satisfy (3.10). Since \( |b| = 1 \), \( a \geq 3 \), and Remark 2(vi) with \( x = a \), we have

\[
\sqrt{a^2 - mb^2} = \sqrt{a^2 - m} \geq \sqrt{a^2 + 1} \geq (a - 1)^{1/3} + 1.465572,
\]
yielding (3.10). Thus, we conclude that the pairs \((a, b)\) satisfying (3.12) are solutions of the system above.

**Case B:** \(|\beta| \geq 2 + \sqrt{1 - m}\). Since we cannot apply the element \(\beta\) to Theorem D, we have \(a < 1 + \sqrt{1 - m}\). Now, we try again to find elements of \(\beta\) that satisfy the following inequality system:

\[
\begin{align*}
|\beta| & \geq 2 + \sqrt{1 - m} \\
|\beta| & \geq M^{1/3} + 1.465572 \\
a & < 1 + \sqrt{1 - m} \\
a & \geq 1 + \frac{\sqrt{3}}{2} \left(1 + \frac{\sqrt{1 + 4M}}{2}\right). 
\end{align*}
\]

(3.13)

We consider two cases as follows:

**Case 1:** \(b = 0\). Then \(a < 1 + \sqrt{1 - m} < 2 + \sqrt{1 - m} \leq |\beta| = a\), which is a contradiction. Hence, the system (3.13) has no integer solution \(\beta = a\). In other words, the assumptions in the system generate no pairs \((a, b)\) that are solutions to Theorem 1 and that are also not solutions to Theorem D.

**Case 2:** \(b \neq 0\) and \(d = 1\). There are two further subcases:

**Subcase 2.1:** \(|b| \geq a\). Then \(|\beta| = \sqrt{a^2 - mb^2}\) and \(M = \sqrt{(|b| - 1)^2} = |b| - 1\). Thus, the system (3.13) becomes

\[
\begin{align*}
\sqrt{a^2 - mb^2} & \geq 2 + \sqrt{1 - m} \\
a & < 1 + \sqrt{1 - m} \\
\sqrt{a^2 - mb^2} & \geq (|b| - 1)^{1/3} + 1.465572 \\
a & \geq 1 + \frac{\sqrt{3}}{2} \left(1 + \frac{\sqrt{1 + 4(|b| - 1)}}{2}\right). 
\end{align*}
\]

(3.14) (3.15) (3.16) (3.17)

Since \(|b| \geq a\), we obtain from (3.17) that \(a \geq 1 + (\sqrt{3}/2) \left(1 + \sqrt{1 + 4(a - 1)}\right)/2\). It follows from Remark 1(i) that \(a \geq 3\). Since \(d = 1\), we have \(|b| > a\). By using (3.15) together with \(a \geq 3\), we have \(3 \leq a < 1 + \sqrt{1 - m}\), implying \(m \leq -5\). It can be verified by using (3.17) that \(|b| \leq \left(4 \sqrt{3}(a - 1) - 3\right)^2 + 27)/36\). Now, we have that

\[
3 \leq a < 1 + \sqrt{1 - m} \quad \text{and} \quad a < |b| \leq \frac{4 \sqrt{3}(a - 1) - 3}{36}. 
\]

(3.18)

To show that the pairs \((a, b)\) satisfying (3.18) are solutions of the system, it remains to show that they also satisfy (3.14) and (3.16). Since \(|b| > a \geq 3\) and \(m < 0\), we obtain

\[
\sqrt{a^2 - mb^2} > \sqrt{a^2 - ma^2} = a \sqrt{1 - m} \geq 3 \sqrt{1 - m} > 2 + \sqrt{1 - m},
\]
yielding (3.14). From Remark 2(ii) with \(x = |b|\), we have

\[
\sqrt{a^2 - mb^2} > \sqrt{5 + b^2} \geq (|b| - 1)^{1/3} + 1.465572,
\]
showing (3.16).

**Subcase 2.2:** \(|b| < a\). Then \(|\beta| = \sqrt{a^2 - mb^2}\) and \(M = \sqrt{(a - 1)^2} = a - 1\). Thus, the system (3.13) becomes

\[
\sqrt{a^2 - mb^2} \geq 2 + \sqrt{1 - m}
\]

(3.19)
\[ a < 1 + \sqrt{1 - m} \]  
(3.20)

\[ \sqrt{a^2 - mb^2} \geq (a - 1)^{1/3} + 1.465572 \]  
(3.21)

\[ a \geq 1 + \frac{\sqrt{3}}{2} \left( 1 + \sqrt{1 + 4(a - 1)} \right). \]  
(3.22)

Again, using Remark 1(i) and (3.22), we obtain \( a \geq 3 \). By using (3.20) together with \( a \geq 3 \), we have \( 3 \leq a < 1 + \sqrt{1 - m} \), implying \( m \leq -5 \). Using (3.19), we can verify that \(|b| \geq \sqrt{(5 - m + 4 \sqrt{1 - m - a^2})/(-m)}\). Now, we have that

\[ 3 \leq a < 1 + \sqrt{1 - m} \text{ and } \sqrt{5 - m + 4 \sqrt{1 - m - a^2}} \leq |b| < a. \]  
(3.23)

To show that the pairs \((a, b)\) satisfying (3.23) are solutions of the system, it remains to show that they also satisfy (3.21). It follows from \( b^2 \geq 1, m \leq -5 \), and Remark 2(ii) with \( x = a \) that

\[ \sqrt{a^2 - mb^2} \geq \sqrt{a^2 + 5} \geq (a - 1)^{1/3} + 1.465572, \]

yielding (3.21).

From every case, we conclude that elements of \( \beta = a + b \sqrt{m} \in O_K \setminus \{0\} \) with \( m \neq 1 \) (mod 4) that can be applied to Theorem 1 but not to Theorem D are shown in the following tables.

We note from Subcase 2.2 in Table 1 that the number of \( a \) roughly grows as \( 2 \sqrt{1 - m} \). To see this, since \( 8 \sqrt{1 - m} > 1 \), we have

\[ 5 + 4 \sqrt{1 - m} < 4 \sqrt{1 - m} + 8 \sqrt{1 - m} + 4 = (2 \sqrt{1 - m} + 2)^2 \]

and so \( 3 \leq a < \sqrt{5 + 4 \sqrt{1 - m}} < 2 \sqrt{1 - m} + 2 \). This means that the number of such \( a \) is approximately \( 2 \sqrt{1 - m} \).

### Table 1. Case A: \(|\beta| < 2 + \sqrt{1 - m}\).

<table>
<thead>
<tr>
<th>\beta = a + b \sqrt{m}, d = \gcd(a, b)</th>
<th>Integer solutions ((a, b))</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Case 1: (b = 0)</strong></td>
<td>[ \frac{4 + 2 \sqrt{3} + 3 \sqrt{1 - m}}{4} \leq a &lt; 2 + \sqrt{1 - m} \text{ and } b = 0 ]</td>
</tr>
<tr>
<td><strong>Case 2: (b \neq 0) and (d = 1)</strong></td>
<td></td>
</tr>
<tr>
<td>**Subcase 2.1: (</td>
<td>b</td>
</tr>
<tr>
<td>**Subcase 2.2: (</td>
<td>b</td>
</tr>
</tbody>
</table>
We note from Table 2 that the complicated lower bound in Subcase 2.2 is actually very close to 1. Indeed, we show that

\[ \sqrt{\frac{5 - m + 4 \sqrt{1 - m}}{m} - a^2} < 2. \]

Since \( m \leq -1 \), it follows that

\( (4 - 3m)^2 - 16(1 - m) = (9m^2 - 24m + 16) - 16 + 16m = 9m^2 - 8m = m(9m - 8) > 0, \)

showing \( (4 - 3m)^2 > 16(1 - m) \) and so \( 4 - 3m > 4 \sqrt{1 - m} \). Using \( 3 \leq a < 1 + \sqrt{1 - m} \), we have that

\[ -2 + m - 2 \sqrt{1 - m} < -a^2 \leq -9. \]

It follows that

\[
0 < \frac{3 + 2 \sqrt{1 - m}}{-m} = \frac{(5 - m + 4 \sqrt{1 - m}) + (-2 + m - 2 \sqrt{1 - m})}{-m}
\leq \frac{(5 - m + 4 \sqrt{1 - m}) - 9}{-m}

\leq \frac{-4 - m + 4 \sqrt{1 - m}}{-m}
< \frac{-4 + m + 4(3m)}{-m}
= 4.
\]

This shows that \( \sqrt{(5 - m + 4 \sqrt{1 - m} - a^2)/(-m)} < \sqrt{4} = 2 \), as desired.

| Table 2. Case B: \( |\beta| \geq 2 + \sqrt{1 - m} \).  |
|---------------------------------------------------|
| \( \beta = a + b \sqrt{m}, \ d = \gcd(a, b) \)      |
| Integer solutions \( (a, b) \)                     |
| Case 1: \( b = 0 \)                                |
| none                                              |
| Case 2: \( b \neq 0 \) and \( d = 1 \)            |
| Subcase 2.1: \( |b| \geq a \) \hspace{1cm} 3 \leq a < 1 + \sqrt{1 - m} \text{ and } a < |b| \leq \frac{(4 \sqrt{3(a - 1)} - 3)^2 + 27}{36} |
| Subcase 2.2: \( |b| < a \) \hspace{1cm} 3 \leq a < 1 + \sqrt{1 - m} \text{ and } \sqrt{\frac{5 - m + 4 \sqrt{1 - m} - a^2}{-m}} \leq |b| < a |

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3.2. Comparison of the criteria for $m \equiv 1 \pmod{4}$

Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $m \equiv 1 \pmod{4}$. In this subsection, we find elements of $\beta \in O_K \setminus \{0\}$ that can be applied to Theorem 2 but not to Theorem E. Now, let $\beta = a + b\sigma_m$ be a nonzero element in $O_K$ that can be applied to Theorem 2 but not to Theorem E. Then $|\beta| \geq M^{1/3} + 1.465572$ and $a + (b/2) \geq 1 + (\sqrt{3}/2)((1 + \sqrt{1+4M})/2)$, where $M$ is defined as in (2.3). Since $\beta$ cannot be applied to Theorem E, one can consider two possible cases, namely, $|\beta| < 2 + \sqrt{(9-m)/4}$ or $|\beta| \geq 2 + \sqrt{(9-m)/4}$ as follows:

**Case A:** $|\beta| < 2 + \sqrt{(9-m)/4}$. Then we will find elements of $\beta$ that satisfy the inequality system:

$$
|\beta| < 2 + \sqrt{\frac{9-m}{4}}
$$

$$
|\beta| \geq M^{1/3} + 1.465572
$$

$$
a + \frac{b}{2} \geq 1 + \frac{\sqrt{3}}{2} \left(1 + \frac{1 + \sqrt{1+4M}}{2}\right).
$$

(3.24)

We consider two cases as follows:

**Case 1:** $b = 0$. Then $\beta = a$ and $M = \sqrt{(a-1)^2 + (a-1)(a-1) + (a-1)^2(1-m)/4} = \sqrt{(9-m)/4(a-1)}$. Thus, the system (3.24) becomes

$$
a < 2 + \sqrt{\frac{9-m}{4}}
$$

(3.25)

$$
a \geq \left(\frac{9-m}{4}(a-1)\right)^{1/3} + 1.465572
$$

(3.26)

$$
a \geq 1 + \frac{\sqrt{3}}{2} \left(1 + \frac{1 + 4 \sqrt{(9-m)/4(a-1)}}{2}\right).
$$

(3.27)

By (3.27) and Remark 1(iii), we have that $a \geq (4 + 2 \sqrt{3} + 3 \sqrt{(9-m)/4})/4$, which together with (3.25) yield

$$
4 + 2 \sqrt{3} + 3 \sqrt{(9-m)/4}/4 \leq a < 2 + \sqrt{\frac{9-m}{4}}.
$$

(3.28)

To show that the integers $\beta = a$ satisfying (3.28) are solutions of the system above, we must show that they also satisfy (3.26). By taking $x = (9-m)/4$ in Remark 2(i) and using (3.25), we obtain that

$$
4 + 2 \sqrt{3} + 3 \sqrt{(9-m)/4}/4 \geq \left(\frac{9-m}{4} + \sqrt{\frac{9-m}{4}}\right)^{1/3} + 1.465572
$$

$$
= \left[\sqrt{\frac{9-m}{4}}(2 + \sqrt{\frac{9-m}{4}} - 1)\right]^{1/3} + 1.465572
$$

(3.29)

$$
> \left(\frac{9-m}{4}(a-1)\right)^{1/3} + 1.465572.
$$
It follows from (3.28) and (3.29) that $a > \left( \sqrt{(9-m)/4} (a-1) \right)^{1/3} + 1.465572$, yielding (3.26).

**Case 2:** $b \neq 0$ and $d = 1$. There are two further subcases:

**Subcase 2.1:** $|b| \geq |a|$. Then $|\beta| = \sqrt{a^2 + ab + b^2(1-m)/4}$ and $M = \sqrt{|b|-1}^2 = |b| - 1$. Thus, the system (3.24) becomes

$$\sqrt{a^2 + ab + b^2 \left( \frac{1-m}{4} \right)} < 2 + \sqrt{\frac{9-m}{4}}, \quad (3.30)$$

$$\sqrt{a^2 + ab + b^2 \left( \frac{1-m}{4} \right)} \geq (|b|-1)^{1/3} + 1.465572$$

$$a + \frac{b}{2} \geq 1 + \frac{\sqrt{3}}{2} \left( \frac{1 + \sqrt{1+4(|b|-1)}}{2} \right). \quad (3.31)$$

In this subcase, we now show that the system has no integer solution $(a, b)$. If $a < 0$, then it follows from (3.31) that $b > 0$ and so $(b/2) - 1 \geq 1 + (\sqrt{3}/2) \left( 1 + \sqrt{1+4(b-1)/2} \right)$. Then $b^2 - (11 + \sqrt{3})b + (19 + 4 \sqrt{3}) \geq 0$, implying $b \geq 11$. It follows from $a^2 \geq 1$, $a > 1 - (b/2)$, $b \geq 11$, and Remark 2(vii) with $x = m$ that

$$\sqrt{a^2 + ab + b^2 \left( \frac{1-m}{4} \right)} > \sqrt{1 + \left( \frac{1-b}{2} \right) b + b^2 \left( \frac{1-m}{4} \right)}$$

$$= \sqrt{b^2 \left( \frac{1-b}{4} \right)} + b + 1$$

$$\geq \sqrt{-121 - 121m + 48}$$

$$= \frac{1}{2} \sqrt{-73 - 121m}$$

$$> \frac{1}{2} (4 + \sqrt{9-m})$$

$$= 2 + \sqrt{\frac{9-m}{4}}, \quad (3.32)$$

which is contrary to (3.30). Thus, $a \geq 0$. If $a = 0$, then $|b| = 1$ because $d = 1$. This contradicts to (3.31), so $a \geq 1$. If $|b| = 1$, then $a = 1$ and so (3.31) is false. Thus, $|b| \geq 2$ and so $|b| > a$ because $d = 1$.

It follows from (3.31) and $|b| \geq 2$ that $a + (b/2) > 2.4$ and so $|b| + (b/2) > 2.4$. This implies that $b \geq 2$ or $b \leq -5$. If $b = 2$, then we obtain that $2 = |b| > a \geq (\sqrt{3}/2) \left( 1 + \sqrt{1+4(2-1)/2} \right) > 1.4$, which is a contradiction. If $b = 3$, then we obtain that $3 = |b| > a \geq (\sqrt{3}/2) \left( 1 + \sqrt{1+4(3-1)/2} \right) - (1/2) > 1.2$, which implies that $a = 2$. It follows that

$$\sqrt{a^2 + ab + b^2 \left( \frac{1-m}{4} \right)} \leq \sqrt{2^2 + 2 \cdot 3 + 3^2 \left( \frac{1-m}{4} \right)}$$

$$= \frac{1}{4} (\sqrt{49 - 9m} + \sqrt{49 - 9m})$$
\[ > \frac{1}{4}(8 + \sqrt{36 - 4m}) \]
\[ = 2 + \sqrt{\frac{9 - m}{4}}, \]
which is contrary to (3.30). If \( b \geq 4 \), then
\[ \sqrt{a^2 + ab + b^2 \left( \frac{1 - m}{4} \right)} \geq \sqrt{1 + 4 + 16 \left( \frac{1 - m}{4} \right)} \]
\[ = \frac{1}{2} \left( \sqrt{9 - 4m} + \sqrt{9 - 4m} \right) \]
\[ > \frac{1}{2} \left( 4 + \sqrt{9 - m} \right) \]
\[ = 2 + \sqrt{\frac{9 - m}{4}}, \]
which is contrary to (3.30). If \( b \leq -5 \), then
\[ a - \frac{5}{2} \geq a + \frac{b}{2} \geq 1 + \frac{\sqrt{3}}{2} \left( \frac{1 + \sqrt{1 + 4(|b| - 1)}}{2} \right) \geq 1 + \frac{\sqrt{3}}{2} \left( \frac{1 + \sqrt{1 + 4(5 - 1)}}{2} \right) > 3.22, \]
showing \( a \geq 6 \). Since \( b \leq -5 \) and \( a \geq 6 \), it follows from \( -b = |b| > a \) that
\[ \sqrt{a^2 + ab + b^2 \left( \frac{1 - m}{4} \right)} > \sqrt{a^2 - b^2 + b^2 \left( \frac{1 - m}{4} \right)} \]
\[ = \sqrt{b^2 \left( \frac{1 - m}{4} - 1 \right) + a^2} \]
\[ \geq \sqrt{25 \left( \frac{1 - m}{4} - 1 \right) + 36} \]
\[ = \frac{1}{4} \left( \sqrt{69 - 25m} + \sqrt{69 - 25m} \right) \]
\[ > \frac{1}{4} \left( 8 + \sqrt{36 - 4m} \right) \]
\[ = 2 + \sqrt{\frac{9 - m}{4}}, \]
which is contrary to (3.30).

Thus, in this subcase, we conclude that the assumptions in the system generate no pairs \((a, b)\) that are solutions to Theorem 2 and that are also not solutions to E.

Subcase 2.2: \(|b| < |a|\). Then \( |\beta| = \sqrt{a^2 + ab + b^2(1 - m)/4} \) and \( M = \sqrt{|a| - 1} = |a| - 1 \). Thus, the system (3.24) becomes
\[ \sqrt{a^2 + ab + b^2 \left( \frac{1 - m}{4} \right)} < 2 + \sqrt{\frac{9 - m}{4}} \] (3.33)
\[
\sqrt{a^2 + ab + b^2 \left( \frac{1-m}{4} \right)} \geq (|a| - 1)^{1/3} + 1.465572 \tag{3.34}
\]

\[
a + \frac{b}{2} \geq 1 + \frac{\sqrt{3}}{2} \left( 1 + \frac{\sqrt{1+4(|a|-1)}}{2} \right). \tag{3.35}
\]

If \( a < 0 \), then it follows from \( a + (b/2) > 1 \) that \( b > 0 \). Since \( |a| > |b| = b \) and (3.35), we obtain \((b/2) - 1 \geq a + (b/2) > 1 + (\sqrt{3}/2) \left( 1 + \frac{\sqrt{1+4(b-1)}}{2} \right)\), implying \( b \geq 11 \). Now, we have that \( a^2 > 1, a > 1 - (b/2) \), and \( b \geq 11 \). It can be proved similarly to (3.32) that

\[
\sqrt{a^2 + ab + b^2 \left( \frac{1-m}{4} \right)} > 2 + \sqrt{\frac{9-m}{4}},
\]

which is contrary to (3.33). Thus, \( a \geq 0 \). If \( a = 0 \) or \( a = 1 \), then \( 0 < |b| < |a| \leq 1 \), which is impossible so that \( a \geq 2 \). If \( b = -1 \), then it follows from (3.35) that \( a - (1/2) \geq 1 + (\sqrt{3}/2) \left( 1 + \frac{\sqrt{1+4(a-1)}}{2} \right) \), implying \( a \geq 4 \). By taking \( x = a \) in Remark 2(iii), we have

\[
\sqrt{a^2 - a + \frac{1-m}{4}} = \sqrt{a(a-1) + \frac{1-m}{4}} \geq \sqrt{3a+1} \geq (a - 1)^{1/3} + 1.465572,
\]

yielding (3.34). It can be verified by (3.33) with \( b = -1 \) that \( a < \left( \sqrt{8 \sqrt{9-m} + 25 + 1} \right)/2 \). This shows that

\[
4 \leq a < \frac{\sqrt{8 \sqrt{9-m} + 25 + 1}}{2}, \text{ when } b = -1. \tag{3.36}
\]

If \( b = 1 \), then it follows from (3.35) that \( a + (1/2) \geq 1 + (\sqrt{3}/2) \left( 1 + \frac{\sqrt{1+4(a-1)}}{2} \right) \), implying \( a \geq 2 \). By taking \( x = a \) in Remark 2(iii), we have that

\[
\sqrt{a^2 + a + \frac{1-m}{4}} = \sqrt{a(a+1) + \frac{1-m}{4}} \geq \sqrt{3a+1} \geq (a - 1)^{1/3} + 1.465572,
\]

yielding (3.34). It can be verified by (3.33) with \( b = 1 \) that \( a < \left( \sqrt{8 \sqrt{9-m} + 25 - 1} \right)/2 \) and thus

\[
2 \leq a < \frac{\sqrt{8 \sqrt{9-m} + 25 - 1}}{2}, \text{ when } b = 1. \tag{3.37}
\]

We next show for \( b \geq 2 \) or \( b \leq -2 \) that the system above has no integer solution \((a,b)\). If \( b \geq 2 \), then \( a = |a| > |b| = b \geq 2 \) and so \( a \geq 3 \). It follows that

\[
\sqrt{a^2 + ab + b^2 \left( \frac{1-m}{4} \right)} \geq \sqrt{3^2 + 3 \cdot 2 + 2^2 \left( \frac{1-m}{4} \right)} = \frac{1}{2} \left( \sqrt{16-m} + \sqrt{16-m} \right)
\]
which is contrary to (3.33). If \( b = -2 \), then we obtain from (3.35) that \( a - 1 \geq 1 + (\sqrt{3}/2)((1 + \sqrt{1 + 4(a-1)})/2) \), implying \( a \geq 4 \). Since \( d = 1 \) and \( b = -2 \), we have that \( a \geq 5 \). Hence,

\[
\sqrt{a^2 - 2a + 1 - m} = \sqrt{a(a - 2) + 1 - m} \\
\geq \sqrt{5(3) + 1 - m} \\
= \frac{1}{2}(\sqrt{16 - m} + \sqrt{16 - m}) \\
> \frac{1}{2}(4 + \sqrt{9 - m}) \\
= 2 + \sqrt{\frac{9 - m}{4}},
\]

which is contrary to (3.33). If \( b \leq -3 \), then we have \( a - (3/2) \geq a + (b/2) \geq 1 + (\sqrt{3}/2)((1 + \sqrt{1 + 4(a-1)})/2) \), this implies that \( a \geq 5 \). Since \( a > |b| = -b \), we obtain that \( -b \leq a - 1 \) and so \( ab \geq -a^2 + a \). It follows from \( b \leq -3 \), \( a \geq 5 \), \( ab \geq -a^2 + a \), and Remark 2(viii) with \( x = m \) that

\[
\sqrt{a^2 + ab + b^2\left(\frac{1-m}{4}\right)} \geq \sqrt{a^2 - a^2 + a + b^2\left(\frac{1-m}{4}\right)} \\
\geq \sqrt{9\left(\frac{1-m}{4}\right)} + 5 \\
= \frac{1}{2}\sqrt{29 - 9m} \\
> \frac{1}{2}(4 + \sqrt{9 - m}) \\
= 2 + \sqrt{\frac{9 - m}{4}},
\]

which is contrary to (3.33).

Thus, in this subcase, we obtain that the pairs \((a, b)\) with \( b \neq 0 \) and \( d = 1 \) satisfying (3.36) or (3.37) are integer solutions of the system (3.24).

**Case B:** \(|\beta| \geq 2 + \sqrt{(9 - m)/4}\). Since \( a + (b/2) > 1 \) and we cannot apply \( \beta \) to Theorem E, it follows that \( a < 1 \). Thus, we have to find elements of \( \beta \) that satisfy the following inequality system:

\[
|\beta| \geq 2 + \sqrt{\frac{9 - m}{4}}, \quad a < 1 \\
|\beta| \geq M^{1/3} + 1.465572 \quad \text{(3.38)} \\
a + \frac{b}{2} \geq 1 + \frac{\sqrt{3}}{2}\left(\frac{1 + \sqrt{1 + 4M}}{2}\right).
\]
Note that $M \geq 1$ by Lemma 4. Then $b/2 \geq 1 + (\sqrt{3}/2)(1 + \sqrt{5}/2) > 2.4$ and so $b \geq 5$. If $b < |a|$, then $a \leq -6$ and so $a + (b/2) < a + b < a + |a| = 0$, which is a contradiction. Thus, $b \geq |a| = -a$ and so $M = \sqrt{(b-1)^2} = b - 1$. Hence, the system (3.38) becomes

$$\sqrt{a^2 + ab + b^2 \left(\frac{1-m}{4}\right)} \geq 2 + \sqrt{\frac{9-m}{4}}, \quad a < 1$$  \quad \tag{3.39}$$

$$\sqrt{a^2 + ab + b^2 \left(\frac{1-m}{4}\right)} \geq (b-1)^{1/3} + 1.465572$$

$$a + b/2 \geq 1 + \frac{\sqrt{3}}{2} \left(\frac{1 + \sqrt{1+4(b-1)}}{2}\right).$$  \quad \tag{3.41}$$

Since $b \geq 5$ and $d = 1$, we have $a \leq -1$. It follows by (3.41) that $(b/2) - 1 \geq 1 + (\sqrt{3}/2)\left((1 + \sqrt{1+4(b-1)})/2\right)$, implying $b \geq 11$. Note that $b \geq -a$, $b \geq 11$, and $d = 1$ imply $b > -a$. That is, $-b < a \leq -1$. Now, we have that

$$b \geq 11 \text{ and } 1 + \frac{\sqrt{3}}{2} \left(\frac{1 + \sqrt{1+4(b-1)}}{2}\right) - \frac{b}{2} \leq a \leq -1$$  \quad \tag{3.42}$$

To show that the pairs $(a, b)$ satisfying (3.42) are solutions of the system, it remains to show that they also satisfy (3.39) and (3.40). Since $a^2 \geq 1$, $a > 1 - (b/2)$, and $b \geq 11$, we obtain by Remark 2(vii) with $x = m$ that

$$\sqrt{a^2 + ab + b^2 \left(\frac{1-m}{4}\right)} > \sqrt{1 + \left(\frac{1-b}{2}\right)b + b^2 \left(\frac{1-m}{4}\right)}$$

$$= \sqrt{b^2 \left(\frac{1-b}{2}\right) + b + 1}$$

$$\geq \sqrt{121 \left(\frac{1-b}{2}\right) + 12}$$

$$= \frac{1}{2} \sqrt{-73 - 121m}$$

$$> \frac{1}{2} \left(4 + \sqrt{9-m}\right)$$

$$= 2 + \sqrt{\frac{9-m}{4}},$$

showing (3.39). It follows from $a^2 \geq 1$, $a > 1 - (b/2)$, $m \leq -3$, and Remark 2(iv) with $x = b$ that

$$\sqrt{a^2 + ab + b^2 \left(\frac{1-m}{4}\right)} > \sqrt{1 + \left(\frac{1-b}{2}\right)b + b^2} > \sqrt{\frac{b^2}{2} + 1} \geq (b-1)^{1/3} + 1.465572,$$

yielding (3.40), as desired.

From every case, we conclude that elements of $\beta = a + b\sigma_m \in O_k \backslash \{0\}$ with $m \equiv 1 \pmod{4}$ that can be applied to Theorem 2 but not to Theorem E are shown in the following tables.
We note from Subcase 2.2 in Table 3 that when $b = -1$, the number of $a$ roughly grows as $\sqrt[4]{4(9 - m)}$. Otherwise, $b = 1$ implies that the number of $a$ roughly grows as $\sqrt[4]{4(9 - m)} + 1$. To see these, one can see that

$$8 \sqrt{9 - m} + 25 < 8 \sqrt{9 - m} + 20 \sqrt[4]{4(9 - m)} + 25 = (2 \sqrt[4]{4(9 - m)} + 5)^2$$

and so $\sqrt{8 \sqrt{9 - m} + 25} < 2 \sqrt[4]{4(9 - m)} + 5$. If $b = -1$, then

$$4 \leq a < \frac{\sqrt{8 \sqrt{9 - m} + 25 + 1}}{2} < \frac{2 \sqrt[4]{4(9 - m)} + 6}{2} = \sqrt[4]{4(9 - m)} + 3,$$

showing that the number of such $a$ is approximately $\sqrt[4]{4(9 - m)}$. If $b = 1$, we obtain

$$2 \leq a < \frac{\sqrt{8 \sqrt{9 - m} + 25 - 1}}{2} < \frac{2 \sqrt[4]{4(9 - m)} + 4}{2} = \sqrt[4]{4(9 - m)} + 2,$$

showing that the number of such $a$ is approximately $\sqrt[4]{4(9 - m)} + 1$.

**Table 3.** Case A: $|\beta| < 2 + \sqrt[4]{\frac{9 - m}{4}}$.

<table>
<thead>
<tr>
<th>$\beta = a + b\sigma_m$, $d = \gcd(a, b)$</th>
<th>Integer solutions $(a, b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1: $b = 0$</td>
<td>$\frac{4 + 2 \sqrt[4]{3} + 3 \sqrt[4]{(9 - m)/4}}{4} \leq a &lt; 2 + \sqrt[4]{\frac{9 - m}{4}}$ and $b = 0$</td>
</tr>
<tr>
<td>Case 2: $b \neq 0$ and $d = 1$</td>
<td>none</td>
</tr>
<tr>
<td>Subcase 2.1: $</td>
<td>b</td>
</tr>
<tr>
<td></td>
<td>$4 \leq a &lt; \frac{\sqrt{8 \sqrt{9 - m} + 25 + 1}}{2}$, when $b = -1$,</td>
</tr>
<tr>
<td>Subcase 2.2: $</td>
<td>b</td>
</tr>
</tbody>
</table>

From Table 4, one can verify that if $b \geq |a|$ and $d = 1$, then $b \geq 11$ and

$$4.2 - \frac{b}{2} \approx 1 + \frac{\sqrt{3}}{2} \left(1 + \frac{\sqrt{1 + 4(b - 1)}}{2}\right) - \frac{b}{2} \leq 1 + \frac{\sqrt{3}}{2} \left(1 + \frac{\sqrt{1 + 4(b - 1)}}{2}\right) - \frac{b}{2} \leq a \leq -1.$$
This implies that the number of possible values of \(a\) is at most \(\lfloor \frac{b}{2} - 4.2 \rfloor\), the greatest integer less than or equal to \((b/2) - 4.2\).

### Table 4. Case B: \(|\beta| \geq 2 + \sqrt{\frac{9-m}{4}}

<table>
<thead>
<tr>
<th>\beta = a + b\sigma_m, d = \gcd(a, b)</th>
<th>Integer solutions ((a, b))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b &lt;</td>
<td>a</td>
</tr>
<tr>
<td>(b \geq</td>
<td>a</td>
</tr>
</tbody>
</table>

### 4. Conclusions

Let \(K = \mathbb{Q}(\sqrt{m})\) be an imaginary quadratic field with \(O_K\) its ring of integers. In this paper, further irreducibility criteria for polynomials in \(O_K[x]\) are established which extend the authors’ earlier works (Theorems D and E). Moreover, elements of \(\beta \in O_K\) that can be applied to the new criteria but not to the previous ones are also provided.

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### Conflicts of interest

All authors declare no conflicts of interest in this paper.

### References


