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Research article

Existence and uniqueness of a positive solutions for the product of operators

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Abstract: In this paper, we prove the existence of a positive solution for some equations involving multiplication of concave (possibly nonlinear) operators. Also, we provide a successively sequence to approximate the solution for such equations. This kind of the solution is necessary for quadratic differential and integral equations.

Keywords: fixed point; product of operator; quadratic integral equation; positive solution; concave operator
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1. Introduction

In this paper we prove some fixed point theorems for the problem

$$x = A(x) + B(x) \cdot C(x). \tag{1.1}$$

However, these kind of theorems are related to some "quadratic" problems. Let us mention the quadratic integral equation

$$x(t) = g(t, x(t)) + u(t, x(t)) \int_0^1 K(t, s) f(s, x(s)) ds. \tag{1.2}$$

Special cases of Eq (1.2) were investigated in connection with the applications of some kind of problems in the theories of radiative transfer, neutron transport, and the kinetic theory of gases [4]. A more general problem (motivated by some practical interests in plasma physics) was investigated in [21]. See [11, 24] for other applications.

So far, two methods have been proposed to solve Eq (1.1). In the former, the measure of non-compactness technique (see [2, 8–10, 14, 17]) is used to prove the existence of a solution for

Eq (1.1), and in the latter, Dhage [13] used the combining Schauder's fixed point theorem and Banach's contraction principle to prove the existence of a solution for Eq (1.1), also see [5]. Some authors have combined the two methods and have proved the existence of a solution for Eq (1.1).

In this paper, we prove the existence of a positive solution for the Eq (1.1) in which the operators A, B, and C are concave (or convex) or monotone. Also, we give a successively sequence to approximate it. But what is our motivation to prove the existence of a positive solution for the Eq (1.1) when the operators A, B, and C are concave (or convex) or monotone? The mentioned methods have not provided a way to approximate the solution for the Eq (1.1). Also, in the case that the operators A, B, and C are concave (or convex) or monotone, we do not have to suppose the continuity, compactness, and upper-lower assumptions for the operators A, B, and C. These assumptions play an important roles in order to prove the existence of positive solutions for nonlinear differential and integral equations and they are difficult to verify for real problems. Furthermore, there exist more extensively applied of the positive solution of nonlinear differential and integral equations in practical issues (see [3,12,28–31]).

The start of proving the existence of a positive solution for differential and integral equations can be found in the Picard investigation (see [25], p.129–138). Authors in [18–20] generalized theorems for abstract operator equations with special positive operators called u_0-concave. After that, ordered concavity (convexity) and α-concavity (convexity) were introduced by Amann [1] in 1976 and Potter [26] in 1977. In [7,22,23,32–34], some others type of concave operators were investigated.

The paper is organized as follows: In Section 2, we introduce some of the preliminaries needed for the next sections. In Section 3, we prove some existential results for the Eq (1.1). Furthermore, we provide some examples that satisfy the main results. In Section 4, we prove the existence of positive solutions for nonlinear quadratic integral equations by theorems provided in the main results section. Section 5 is devoted to concluding and proposing new ideas.

2. Some basic definitions and notations

Throughout this paper, we assume that E is a real Banach algebra. Which means, E is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in E, \lambda \in \mathbb{R}$):

(1) $(xy)z = x(yz)$;
(2) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
(3) $\lambda(xy) = (\lambda x)y = x(\lambda y)$;
(4) $\|xy\| \leq \|x\|\|y\|$.

Now let us recall the concepts of cone and partial order for a Banach algebra. A subset P of E is called a cone of E if

(1) P is a non-empty closed and $\theta \in P$;
(2) $\lambda P + \gamma P \subseteq P$ for all non-negative real numbers λ, γ;
(3) $P^2 = P \cdot P \subseteq P$;
(4) $P \cap (-P) = \{\theta\}$,

where θ denotes the null of E. For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$. The cone P is called normal if there is a number $M > m_0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \Rightarrow \|x\| \leq M\|y\|.$$

The last positive number satisfying the above inequality is called the normal constant of P. In the following, we always assume that P is a cone in E and, $\leq$ is the partial ordering with respect to P. We call such space ordered Banach algebra and denote it by (OBA).

If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 \leqslant x \leqslant x_2\}$ is called the order interval between x_1 and x_2. An operator $A : E \to E$ is called increasing (decreasing) if $x \leqslant y$ implies $Ax \leqslant Ay$ ($Ax \geqslant Ay$) where $x, y \in E$. For $h > \theta$ (i.e. $h \geqslant \theta$ and $h \neq \theta$), set

$$P_h = \{x \mid x \in E,\ \exists\, \lambda(x) > 0,\ \mu(x) > 0,\ s.t.\ \lambda(x)h \leqslant x \leqslant \mu(x)h\}.$$

It is easy to notice that $P_h \subseteq P$.

Lemma 2.1. *([15]) The two following assumptions are equivalent:*
(1) *P is a normal cone,*
(2) $x_n \leqslant z_n \leqslant y_n$ *($n = 1, 2, 3, ...$) and* $\|x_n - x\| \to 0$, $\|y_n - x\| \to 0$, *imply that* $\|z_n - x\| \to 0$.

Definition 2.1. *([16]) Let α be a real number such that $0 \leq \alpha < 1$. An operator $A : P \to P$ is called an α-concave ((-α)-convex) if it satisfies,*

$$A(tx) \geqslant t^{\alpha} Ax \ \ (A(tx) \leqslant t^{-\alpha} Ax), \quad \forall t \in (0, 1), x \in P. \tag{2.1}$$

Theorem 2.2. *([6]) Assume that P is a normal cone and the operator T satisfies:*
(D1) $T : P_h \to P_h$ is an increasing self-map in P_h;
(D2) For any $x \in P_h$ and $t \in (0, 1)$, there exists $\beta(t) \in (0, 1)$ such that $T(tx) \geqslant t^{\beta(t)} Tx$;
(D3) For every $x_0 \in P$, there is a constant $l \geq 0$ such that $x_0 \in [\theta, lh]$.
Then, operator equation $x = Tx + x_0$ has a unique solution in P_h.

3. Main results

Now the main results could be stated and proved.

Theorem 3.1. *Let P be a normal cone, $A : P \to P$ is an increasing α-concave operator, $B : P \to P$ is an increasing γ_1-concave operator, and $C : P \to P$ is an increasing γ_2-concave operator such that $\gamma_1 + \gamma_2 = \gamma \leqslant 1$. Also, suppose that*
(i) there exists $h > \theta$ such that $h \cdot h \in P_h$, and $Ah, Bh, Ch \in P_h$;
(ii) there exists a constant $\delta_0 > 0$ such that for all $x \in P$, we have $Ax \geqslant \delta_0 Bx \cdot Cx$.
Then, the operator Eq (1.1) has a unique solution x^* in P_h. Moreover, for the constructing successively sequence $y_n = Ay_{n-1} + By_{n-1} \cdot Cy_{n-1} (n = 1, 2, \cdots)$ and for any initial value $y_0 \in P_h$, we have $y_n \to x^*$ as $n \to \infty$.

Proof. Since $Ah, Bh, Ch \in P_h$, there exist constants $\lambda_1, \lambda_2, \mu_1, \mu_2, \upsilon_1, \upsilon_2 > 0$ such that $\lambda_1 h \leqslant Ah \leqslant \lambda_2 h, \mu_1 h \leqslant Bh \leqslant \mu_2 h, \upsilon_1 h \leqslant Ch \leqslant \upsilon_2 h$. We have

$$\lambda_1 h + \mu_1 \upsilon_1 h \cdot h \leqslant Ah + Bh \cdot Ch \leqslant \lambda_2 h + \mu_2 \upsilon_2 h \cdot h.$$

By (i), there exist $r, s > 0$ such that $rh \leqslant h \cdot h \leqslant sh$. We get

$$(\lambda_1 + \mu_1 \upsilon_1 r)h \leqslant Ah + Bh \cdot Ch \leqslant (\lambda_2 + \mu_2 \upsilon_2 s)h.$$

Hence, we can write $K_1 h \leqslant Ah + Bh \cdot Ch \leqslant J_1 h$, where $K_1 = \lambda_1 + \mu_1 \upsilon_1 r$ and $J_1 = \lambda_2 + \mu_2 \upsilon_2 s$. Thus, $Ah + Bh \cdot Ch \in Ph$. Define the operator $T = A + B \cdot C$ by $Tx = Ax + Bx \cdot Cx$. Then $T : P \to P$ and $Th \in P_h$. Next, we show that $T : P_h \to P_h$. By (2.1), for any $t \in (0, 1)$ and $x \in P$, we have

$$A(\frac{1}{t}x) \leqslant \frac{1}{t^\alpha}Ax, \quad B(\frac{1}{t}x) \leqslant \frac{1}{t^{\gamma_1}}Bx, \quad C(\frac{1}{t}x) \leqslant \frac{1}{t^{\gamma_2}}Cx.$$

For any $x \in P_h$, we can choose a sufficiently small number $t_0 \in (0, 1)$ such that

$$t_0 h \leqslant x \leqslant \frac{1}{t_0}h. \tag{3.1}$$

Note that $T : P \to P$ is an increasing self-map and by (3.1),

$$\begin{aligned} Tx = Ax + Bx \cdot Cx &\leqslant A(\frac{1}{t_0}h) + B(\frac{1}{t_0}h) \cdot C(\frac{1}{t_0}h) \\ &\leqslant \frac{1}{t_0^\alpha}Ah + \frac{1}{t_0^\gamma}Bh \cdot Ch \leqslant \frac{\lambda_2}{t_0^\alpha}h + \frac{\mu_2 \upsilon_2}{t_0^\gamma}h \cdot h = J_2 h, \end{aligned}$$

where $J_2 = \frac{\lambda_2}{t_0^\alpha} + \frac{\mu_2 \upsilon_2}{t_0^\gamma}s$. Also,

$$\begin{aligned} Tx = Ax + Bx \cdot Cx &\geqslant A(t_0 h) + B(t_0 h) \cdot C(t_0 h) \\ &\geqslant t_0^\alpha Ah + t_0^\gamma Bh \cdot Ch \geqslant \lambda_1 t_0^\alpha h + \mu_1 \upsilon_1 t_0^\gamma h \cdot h = K_2 h, \end{aligned}$$

where $K_2 = \lambda_1 t_0^\alpha + \mu_1 \upsilon_1 t_0^\gamma r$. Thus $Tx \in P_h$. Henece, $T : P_h \to P_h$. Moreover, $A : P_h \to P_h$, $B : P_h \to P_h$ and $C : P_h \to P_h$. In the following, we show that for any $t \in (0, 1)$, there exists $\beta_0(t) \in (\alpha, 1)$ with respect to t, such that for all $x \in P_h$,

$$T(tx) \geq t^{\beta_0(t)} Tx, \quad \forall t \in (0, 1). \tag{3.2}$$

By (ii), there exists $\delta_0 > 0$ such that $Ax \geqslant \delta_0 Bx \cdot Cx$. Consider the following function:

$$f(t) = \frac{t^\beta - t}{t^\alpha - t^\beta}, \quad \forall t \in (0, 1), \ where\, \beta \in (\alpha, 1).$$

It is easy to prove that f is non-negative in $(0, 1)$. Especially, for any $t \in (0, 1)$ we have $t^\beta > t$ and $t^\alpha > t^\beta$. Furthermore, for fixing $t \in (0, 1)$, we have $lim_{\beta \to 1^-} f(t) = 0$. So, there exists $\beta_0(t) \in (\alpha, 1)$ with respect to t such that

$$\frac{t^{\beta_0(t)} - t}{t^\alpha - t^{\beta_0(t)}} \leqslant \delta_0, \quad t \in (0, 1).$$

Hence, we have

$$Ax \geqslant \delta_0 Bx \cdot Cx \geqslant \frac{t^{\beta_0(t)} - t}{t^\alpha - t^{\beta_0(t)}} Bx \cdot Cx, \quad \forall t \in (0, 1), \quad x \in P_h.$$

Then, we can get

$$t^{\alpha}Ax + tBx \cdot Cx \geqslant t^{\beta_0(t)}Ax + t^{\beta_0(t)}Bx \cdot Cx, \quad \forall t \in (0,1), x \in P_h.$$

Consequently, for any $t \in (0,1)$ and $x \in P_h$ we have

$$\begin{aligned} T(tx) &= A(tx) + B(tx) \cdot C(tx) \geqslant t^{\alpha}Ax + t^{\gamma}Bx \cdot Cx \geqslant t^{\alpha}Ax + tBx \cdot Cx \\ &\geqslant t^{\beta_0(t)}Ax + t^{\beta_0(t)}Bx \cdot Cx, \quad \forall t \in (0,1), x \in P_h. \end{aligned}$$

Therefore,

$$T(tx) \geqslant t^{\beta_0(t)}T(x), \quad \forall t \in (0,1), x \in P_h.$$

Let $x_0 = \theta$. Application of Theorem 2.2 implies that the equation $Tx = x$ has a unique solution x^* in P_h. It can be concluded that the operator Eq (1.1) has a unique solution x^* in P_h. Now we can construct the successively sequence $y_n = Ay_{n-1} + By_{n-1} \cdot Cy_{n-1} (n = 1, 2, \cdots)$ for any initial point $y_0 \in P_h$. Since $y_0 \in P_h$ and $Ty_0 \in P_h$, we can choose a sufficiently small number $t_0 \in (0,1)$ such that

$$t_0 y_0 \leqslant Ty_0 \leqslant \frac{1}{t_0} y_0. \tag{3.3}$$

Note that $0 < \beta_0(t_0) < 1$, and we can also take a positive integer k such that

$$k > \frac{1}{1 - \beta_0(t_0)}. \tag{3.4}$$

Put $u_0 = t_0^k y_0, v_0 = \frac{1}{t_0^k} y_0$. Evidently, $u_0, v_0 \in P_h$ and $u_0 \leqslant y_0 \leqslant v_0$. By the monotonicity of T, we have $Tu_0 \leqslant Tv_0$. Furthermore, by combining (3.2) and (3.3) we have

$$\begin{aligned} Tu_0 &= T(t_0^k y_0) = T(t_0 t_0^{k-1} y_0) \\ &\geqslant t_0^{\beta_0(t_0)} T(t_0 t_0^{k-2} y_0) \geqslant t_0^{\beta_0(t_0)} t_0^{\beta_0(t_0)} T(t_0^{k-2} y_0) \\ &\geqslant \cdots \geqslant (t_0^{\beta_0(t_0)})^k T y_0 \geqslant (t_0^{\beta_0(t_0)})^k t_0 y_0 = t_0^{k\beta_0(t_0)+1} y_0. \end{aligned} \tag{3.5}$$

By (3.4), one can obtains that $k\beta_0(t_0) + 1 < k$. Thus

$$t_0^{k\beta_0(t_0)+1} > t_0^k. \tag{3.6}$$

Therefore, $Tu_0 \geqslant t_0^{k\beta_0(t_0)+1} y_0 > t_0^k y_0 = u_0$. By (3.2),

$$T(\frac{1}{t}x) \leqslant \frac{1}{t^{\beta_0(t)}} T(x), \quad \forall t \in (0,1), x \in P_h.$$

Thus,

$$\begin{aligned} Tv_0 &= T(\frac{1}{t_0^k} y_0) = T(\frac{1}{t_0} \frac{1}{t_0^k} y_0) \leqslant \frac{1}{t_0^{\beta_0(t_0)}} T(\frac{1}{t_0^{k-1}} y_0) = \frac{1}{t_0^{\beta_0(t_0)}} T(\frac{1}{t_0} \frac{1}{t_0^{k-2}} y_0) \\ &\leqslant \frac{1}{t_0^{\beta_0(t_0)}} \frac{1}{t_0^{\beta_0(t_0)}} T(\frac{1}{t_0^{k-2}} y_0) \leqslant \cdots \leqslant \frac{1}{(t_0^{\beta_0(t_0)})^k} T(y_0) \leqslant \frac{1}{t_0^{k\beta_0(t_0)+1}} y_0. \end{aligned} \tag{3.7}$$

The application of (3.6) implies $Tv_0 \leqslant \frac{1}{t_0^{k\beta_0(t_0)+1}} y_0 \leqslant \frac{1}{t_0^k} y_0 = v_0$. Thus, $u_0 \leqslant Tu_0 \leqslant Tv_0 \leqslant v_0$. For $n = 1, 2, \cdots$, let $u_n = Tu_{n-1}, v_n = Tv_{n-1}$. Then, $u_n \leqslant y_n \leqslant v_n$ $(n = 1, 2, \cdots)$. Similar to the proof of Theorem 1.3 of [35], there exists $y^* \in P_h$ such that $Ty^* = y^*$ and $\lim_{n\to\infty} u_n = \lim v_n = y^*$. Then, by Lemma 2.1, $y_n \to y^*(n \to \infty)$. Since the fixed point of the operator T in P_h is unique, we have $x^* = y^*$. □

Example 3.1. *Consider the Banach space $E = C[0, 1]$ with the supremum norm. E is a Banach algebra by the multiplication $(x \cdot y)(t) = x(t) \cdot y(t)$. Assume that $P = \{x(t) \in C[0, 1] \mid x(t) \geqslant 0,\ \forall t \in [0, 1]\}$. Then E is an (OBA) by the cone P. Let us define operators $A, B, C : P \longrightarrow P$ as the following*

$$A(x) = \sqrt{x+1}, \quad B(x) = \frac{1}{\sqrt{x+1}}, \quad C(x) = 1.$$

Assume that $h(t) = 1$. We can prove that all of the assumptions of Theorem 3.1 are satisfied and the operator $T = A + B \cdot C$ has a unique positive solution.

Example 3.2. *Consider the Banach space $E = L_\infty[0, 1]$ with the L_∞ norm. E is a Banach algebra by the multiplication $(x \cdot y)(t) = x(t) \cdot y(t)$. Assume that $P = \{x(t) \in L_\infty[0, 1] \mid x(t) \geqslant 0,\ \forall t \in [0, 1]\}$. Then E is an (OBA) by the cone P. For any $x \in P$ and $t \in [0, 1]$, let us define*

$$A(x) = \frac{1}{\sqrt{x+1}}, \quad B(x) = \sqrt{x+1}, \quad C(x) = 1, \quad h(t) = 1.$$

It is easy to prove that for any $x \in P$, $A(x), B(x), C(x) \in P$. Also we have $h > 0$. Therefore, $A, B, C, h : P \longrightarrow P$. It is easily noticed that, A is an α-concave operator (for some $0 < \alpha < 1$), B is a γ_1 concave operator (for some $0 \leqslant \gamma_1 < 1$), and C is a γ_2 concave operator (for some $0 \leqslant \gamma_2 < 1$) such that $\gamma_1 + \gamma_1 \leqslant 1$. Now let us consider that

$$x(t) = \begin{cases} n & t = \frac{1}{n}, \\ 0 & t \in [0, 1] \backslash \{\frac{1}{n}\}_n. \end{cases}$$

Then, $x(t) \in P$ and for $t = \frac{1}{n}$ we have

$$A(x(\frac{1}{n})) = \frac{1}{\sqrt{x(\frac{1}{n}) + 1}} = \frac{1}{\sqrt{n+1}}, \quad B(x(\frac{1}{n})) = \sqrt{x(\frac{1}{n}) + 1} = \sqrt{n+1}.$$

Therefore, there is no $\delta_0 > 0$ such that for all $t \in [0, 1]$ we have $Ax(t) \geqslant \delta_0 Bx(t) \cdot Cx(t)$. Hence, the assumption (ii) of Theorem 3.1 is not satisfied. It is easy to prove that the rest of the assumptions of Theorem 3.1 are satisfied and $T = A + B \cdot C$ has a unique positive solution.

Example 3.3. *Consider E, that is defined in Example 3.1. Let us define operators $A, B, C : P \longrightarrow P$ as the following*

$$A(x) = 11 + \sin(x), \quad B(x) = 2 + \sin(x), \quad C(x) = 2 + \cos(x).$$

Suppose that $h(t) = \frac{\pi}{4}$. We have $A, B, C : P \longrightarrow P$ and $h \cdot h > 0$. Also, for all $x \in P$, $Ax \geqslant Bx \cdot Cx$. None of the functions A, B, and C are concave. But the operator $T = A + B \cdot C$ has a unique positive solution.

Example 3.4. *Consider the Banach space $E = C[0,1]$ with the supremum norm. Let us consider the multiplication*

$$(f \cdot g)(t) = \int_0^x f(t)g(x-t)dt$$

for any $x, y \in E$. E is a Banach algebra([27]). Assume that, $P = \{x(t) \in C[0,1] \mid x(t) \geqslant 0,\ \forall t \in [0,1]\}$. Then, E is an (OBA) by the cone P. Now suppose that, $h(t) = 1 > 0$. We have $h \cdot h(x) = x$. Then, $h \cdot h \notin P_h$.

Example 3.5. *Consider E that is defined in Example 3.4. Let $h(t) = 1 > 0$. Let us define the operators $A, B, C : P \longrightarrow P$ as the following*

$$A(x) = \sqrt{x+1}, \quad B(x) = \frac{1}{\sqrt{x+1}} + 1, \quad C(x) = 1.$$

We can prove that all of the assumptions of Theorem 3.1, except $h \cdot h > 0$, are satisfied. Operator $T = A + B \cdot C$ has no positive solution.

Let the operators $A, B, C : P \longrightarrow P$ be defined as the following

$$A(x) = \sqrt{x+1}, \quad B(x) = 1, \quad C(x) = \frac{1}{2}.$$

It is easy to prove that all of the assumptions of the Theorem 3.1, except the assumption $h \cdot h > 0$, are satisfied. The equation $T = A + B \cdot C$ has a unique positive solution.

Lemma 3.2. *Assume that, P is a normal cone and the operator A satisfies the following conditions:*
(D1) $A : P_h \to P_h$ is decreasing in P_h;
(D2) For any $x \in P_h$ and $t \in (0,1)$, there exists $\alpha(t) \in (0,1)$ such that $A(tx) \leqslant t^{-\alpha(t)}Ax$.

Then, there exist $u_0, v_0 \in P_h$ such that $u_0 < v_0$, $u_0 \leqslant Av_0 \leqslant Au_0 \leq v_0$.

Proof. Since $Ah \in P_h$, we can select a sufficiently small number t_0 such that

$$t_0 h \leqslant Ah \leqslant \frac{1}{t_0} h. \tag{3.8}$$

Note that $0 < \alpha(t_0) < 1$, and we can choose a positive integer k such that

$$k > \frac{1}{1 - \alpha(t_0)}. \tag{3.9}$$

Set $u_0 = t_0^k h$ and $v_0 = \frac{1}{t_0^k} h$. Evidently, $u_0, v_0 \in P_h$ and $u_0 < v_0$. By the monotonicity of A, $Au_0 \geqslant Av_0$. Furthermore, by (D2) and (3.8) we have,

$$\begin{aligned} Au_0 &= A(t_0^k h) = A(t_0 t_0^{k-1} h) \\ &\leqslant t_0^{-\alpha(t_0)} A(t_0^{k-1} h) \leqslant \cdots \leqslant t_0^{-k\alpha(t_0)} A(t_0 h) \\ &\leqslant t_0^{-k\alpha(t_0)} \frac{1}{t_0} h = t_0^{-k\alpha(t_0)-1} h. \end{aligned} \tag{3.10}$$

By (3.10), we get that $k\alpha(t_0) + 1 < k$. Thus,

$$t_0^{-k\alpha(t_0)-1} < t_0^{-k}. \tag{3.11}$$

Hence,

$$Au_0 \leqslant t_0^{-k\alpha(t_0)-1}h < t_0^{-k}h < v_0.$$

By (D2),

$$t^{\alpha(t)}A(x) \leqslant A(\frac{x}{t}) \qquad \forall t \in (0,t), x \in P_h. \tag{3.12}$$

Thus,

$$\begin{aligned} Av_0 = A(\frac{1}{t_0^k}h) &= A(\frac{1}{t_0}\frac{1}{t_0^{k-1}}h) \geqslant t_0^{\alpha(t_0)}A(\frac{1}{t_0^{k-1}}h) = t_0^{\alpha(t_0)}A(\frac{1}{t_0}\frac{1}{t_0^{k-2}}h) \\ &\geqslant t_0^{\alpha(t_0)}t_0^{\alpha(t_0)}A(\frac{1}{t_0^{k-2}}h) \geqslant \cdots \geqslant t_0^{k\alpha(t_0)}A(\frac{1}{t_0}h) = t_0^{k\alpha(t_0)+1}h. \end{aligned}$$

Application of (3.9) implies that $Av_0 \geqslant t_0^{k\alpha(t_0)+1}h \geqslant t_0^k h = u_0$. So we have

$$u_0 \leqslant Av_0 \leqslant Au_0 \leqslant v_0. \tag{3.13}$$

□

Theorem 3.3. *Assume that P is normal cone, the operator T satisfies (D1) and (D2) of Lemma 3.2, and there is a constant $l \geqslant 0$ such that $x_0 \in [\theta, lh]$.*

Then, the operator equation $Tx + x_0 = x$ has a unique solution in P_h.

Proof. For all $x \in P_h$, we have $Tx \in P_h$. Then, there exist real numbers $\lambda, \mu > 0$ such that $\lambda h \leqslant Tx \leqslant \mu h$. Thus,

$$\lambda h \leqslant Tx + x_0 \leqslant (\mu + l)h. \tag{3.14}$$

Hence,

$$Tx + x_0 \in P_h, \quad \forall x \in P_h. \tag{3.15}$$

Define the operator F by,

$$Fx = Tx + x_0, \quad \forall x \in P_h. \tag{3.16}$$

By (3.15), and considering the monotonicity of the operator T, the operator $F : P_h \to P_h$ is decreasing. Furthermore, for all $x \in P_h$ and $t \in (0,1)$, we have

$$F(tx) = T(tx) + x_0 \leqslant t^{-\alpha(t)}T(x) + t^{-\alpha(t)}x_0 \leqslant t^{-\alpha(t)}F(x). \tag{3.17}$$

Lemma 3.2 implies that, there exist $u_0, v_0 \in P_h$ such that

$$u_0 < v_0, \quad u_0 \leqslant Fv_0 \leqslant Fu_0 \leqslant v_0. \tag{3.18}$$

Construct the successively sequences

$$u_n = Fu_{n-1}, v_n = Fv_{n-1}, n = 1, 2, \cdots .$$

By the monotonicity of F, we have $v_1 = Fv_0 \leqslant Fu_0 = u_1$. Similarly, we have

$$u_0 \leqslant v_1 \leqslant u_2 \leqslant v_2 \leqslant u_1 \leqslant v_0. \tag{3.19}$$

By continuing this process, for $n = 1, 2, \cdots$, we get

$$u_0 \leqslant v_1 \leqslant u_2 \leqslant \cdots \leqslant u_{2n} \leqslant v_{2n+1} \leqslant u_{2n+1} \leqslant v_{2n} \leqslant \cdots \leqslant v_2 \leqslant u_1 \leqslant v_0. \tag{3.20}$$

Therefore, $\{u_{2n}\}$, $\{v_{2n+1}\}$ are the increasing and $\{u_{2n+1}\}$, $\{v_{2n}\}$ are decreasing sequences. By (3.20), for $n = 1, 2, \cdots$, we have

$$u_{2n} \leqslant v_{2n}, \qquad v_{2n+1} \leqslant u_{2n+1}. \tag{3.21}$$

Assume that,

$$t_{2n} = \sup\{t \mid tv_{2n} \leqslant u_{2n}\}, \qquad t_{2n+1} = \sup\{t \mid tu_{2n+1} \leqslant v_{2n+1}\}.$$

Thus, for $n = 1, 2, \cdots$, we have $u_{2n} \geqslant t_{2n}v_{2n}$ and $v_{2n+1} \geqslant t_{2n+1}v_{2n+1}$. Then,

$$u_{2n+1} \geqslant u_{2n} \geqslant t_{2n}v_{2n} \geqslant t_{2n}v_{2n+1} \quad n = 1, 2, \cdots,$$

$$v_{2m} \geqslant v_{2m+1} \geqslant t_{2m+1}u_{2m+1} \geqslant t_{2m+1}u_{2m} \quad m = 1, 2, \cdots.$$

Therefore, $t_{n+1} \geqslant t_n$, i.e. t_n is an increasing sequence such that $t_n \subseteq (0, 1]$. If $t_n \to t^*$ as $n \to \infty$, then $t^* = 1$. Otherwise, $0 < t^* < 1$. We distinguish two cases:
Case (i): There exists an integer N such that $t_N = t^*$. In this case, we know $t_n = t^*$, for all $n \geqslant N$. So for $n \geqslant N$, we have

$$u_{2n+1} = Fu_{2n} \leqslant F(t^*v_{2n}) \leqslant (t^*)^{-\alpha(t^*)}F(v_{2n}) = (t^*)^{-\alpha(t^*)}v_{2n+1},$$

i.e. $(t^*)^{\alpha(t^*)}u_{2n+1} \leqslant v_{2n+1}$. By the definition of t_{2n+1}, we have $t_{2n+1} = t^* \geqslant (t^*)^{\alpha(t^*)} > t^*$, which is a contradiction.
Case (ii): For all integer n, $t_n < t^*$. Then,

$$\begin{aligned} v_{2n+1} &= F(v_{2n}) \geqslant F(t_{2n}^{-1}u_{2n}) = F(\frac{t^*}{t_{2n}}\frac{u_{2n}}{t^*}) \\ &= F(\frac{\frac{t^*}{t_{2n}}u_{2n}}{t^*}) \geqslant (t^*)^{\alpha(t^*)}F(\frac{t^*}{t_{2n}}u_{2n}) = (t^*)^{\alpha(t^*)}F(\frac{u_{2n}}{\frac{t_{2n}}{t^*}}) \\ &\geqslant (t^*)^{\alpha(t^*)}(\frac{t_{2n}}{t^*})^{(\alpha(\frac{t_{2n}}{t^*}))}Fu_{2n} \geqslant (t^*)^{\alpha(t^*)}(\frac{t_{2n}}{t^*})u_{2n+1} \\ &= t_{2n}(t^*)^{\alpha(t^*)-1}u_{2n+1}. \end{aligned} \tag{3.22}$$

By the definition of t_n, we have $t_{2n+1} \geqslant t_{2n}(t^*)^{\alpha(t^*)-1}$. If $n \to \infty$, we get $t^* \geqslant (t^*)^{\alpha(t^*)} > t^*$, which is a contradiction. Thus, $lim_{n\to\infty}t_n = 1$. For any natural number p, we have

$$\theta \leqslant u_{2(n+p)} - u_{2n} \leqslant v_{2(n+p)} - t_{2n}v_{2n} \leqslant v_{2n} - t_{2n}v_{2n} = (1 - t_{2n})v_{2n} \leqslant (1 - t_{2n})v_0,$$

$$\theta \leqslant v_{2n} - v_{2(n+p)} \leqslant v_{2n} - u_{2n} \leqslant v_0 - u_{2n} \leqslant v_0 - t_{2n}v_0 \leqslant (1 - t_{2n})v_0.$$

Since P is normal, we have

$$\|u_{2(n+p)} - u_{2n}\| \leqslant N(1 - t_{2n})\|v_0\| \to 0 \quad (n \to \infty),$$

$$\|v_{2n} - v_{2(n+p)}\| \leq N(1 - t_{2n})\|v_0\| \to 0 \quad (n \to \infty),$$

where N is the normal constant. Hence, we can claim that u_{2n} and v_{2n} are Cauchy sequences. Since E is a complete space, there exist u^* and v^* such that $u_{2n} \to u^*$, $v_{2n} \to v^*$ as $n \to \infty$. By (3.20), we know that $u_{2n} \leqslant u^* \leqslant v^* \leqslant v_{2n}$ where $u^*, v^* \in P_h$. Then

$$\theta \leqslant v^* - u^* \leqslant v_{2n} - u_{2n} \leqslant (1 - t_{2n})v_0.$$

Furthermore,

$$\|v^* - u^*\| \leqslant N(1 - t_{2n})\|v_0\| \to 0 \quad (n \to \infty).$$

Thus, $u^* = v^*$. Let $x^* = u^* = v^*$. Also, by (3.20), we have

$$\theta \leqslant v_{2n+1} - u_{2n} \leqslant v_{2n} - u_{2n},$$

$$\theta \leqslant u_{2n+1} - v_{2n+1} \leqslant v_{2n} - v_{2n+1}.$$

Then $v_{2n+1} \to x^*$ and $u_{2n+1} \to x^*$ as $n \to \infty$. By the inequality $u_{2n} \leqslant x^* \leqslant v_{2n}$ for $n = 1, 2, \cdots$, we have

$$v_{2n+1} = Fv_{2n} \geqslant Fx^* \geqslant Fu_{2n} = u_{2n+1}.$$

If $n \to \infty$, we get $x^* = Fx^*$. That is, x^* is a fixed point of F in P_h. In the following, we prove that x^* is the unique fixed point of F in P_h. Let $\bar{x}$ be any fixed point of F in P_h. Set $r_1 = \sup\{r > 0 \mid rx^* \leqslant \bar{x} \leqslant \frac{1}{r}x^*\}$. Evidently, $0 < r_1 < \infty$ and $r_1 x^* \leqslant \bar{x} \leqslant \frac{1}{r_1}x^*$. Next, we prove that $r_1 \geqslant 1$. If $0 < r_1 < 1$,

$$r_1^{\alpha(r_1)}x^* \leqslant F(\frac{x^*}{r_1}) \leqslant \bar{x} = F\bar{x} \leqslant F(r_1 x^*) \leqslant r_1^{-\alpha(r_1)}x^*.$$

However, by $r_1^{\alpha(r_1)}x^* \leqslant \bar{x}$, $r_1^{\alpha(r_1)} \leqslant r_1$. Since $r_1^{\alpha(r_1)} > r_1$, we get a contradiction. Hence, $r_1 \geqslant 1$ and we get $\bar{x} \geqslant r_1 x^* \geqslant x^*$. Similarly, we can prove that $x^* \geqslant \bar{x}$, and $x^* = \bar{x}$. Therefore, F has a unique fixed point x^* in P_h. That is to say, $Tx + x_0 = x$ has a unique solution in P_h. □

Comment 3.4. *In [26], Theorem 3.3 is proved by Hilbert's projective metric method where, α is constant function. But, the successively sequence that converges to the fixed point has not been obtained.*

Theorem 3.5. *Consider that P is a normal cone, $A, B : P \to P$ are decreasing operators, and $C : P \to P$ is a decreasing (-α)-convex operator. Assume that*
(i) there exists $h > \theta$ such that $Ah \in P_h, Bh \in P_h$ and $Ch \in P_h$;
(ii) $h \cdot h \in P_h$.

Then, the operator Eq (1.1) has a unique solution x^* in P_h. Moreover, for the constructing successively sequence $y_n = Ay_{n-1} + By_{n-1} \cdot Cy_{n-1}, n = 1, 2, ...$ and for any initial value $y_0 \in P_h$, we have $y_n \to x^*$ as $n \to \infty$.

Proof. Since $Ah, Bh, Ch \in P_h$, there exist constants $\lambda_1, \lambda_2, \mu_1, \mu_2, \upsilon_1, \upsilon_2 > 0$ such that $\lambda_1 h \leqslant Ah \leqslant \lambda_2 h, \mu_1 h \leqslant Bh \leqslant \mu_2 h, \upsilon_1 h \leqslant Ch \leqslant \upsilon_2 h$. Similar to the proof of the Theorem 3.1, we can prove that, if the operator $T = A + B \cdot C$ is defined by $Tx = Ax + Bx \cdot Cx$. Then, $T : P \to P$ and $Th \in P_h$. Next, we show that $T : P_h \to P_h$. By (2.1) and the monotonicity of A, B, we have

$$A(\frac{1}{t}x) \geqslant Ax, \quad B(\frac{1}{t}x) \geqslant Bx, \quad C(\frac{1}{t}x) \geq t^{\alpha}Ax, \qquad t \in (0, 1), x \in P.$$

For any $x \in P_h$, we can select a sufficiently small number $t_0 \in (0, 1)$ such that

$$t_0 h \leqslant x \leqslant \frac{1}{t_0} h. \tag{3.23}$$

$T : P \to P$ is decreasing and by (3.23) we have,

$$\begin{aligned} Tx = Ax + Bx \cdot Cx &\geqslant A(\frac{1}{t_0}h) + B(\frac{1}{t_0}h) \cdot C(\frac{1}{t_0}h) \\ &\geqslant Ah + t_0^{\alpha} Bh \cdot Ch \geqslant \lambda_2 h + \mu_2 \upsilon_2 t_0^{\alpha} h \cdot h. \end{aligned}$$

By (ii), there exist $s > 0$ such that $\frac{1}{s}h \leqslant h \cdot h \leqslant sh$. So, we have $Tx \geqslant J_2 h$ where $J_2 = \lambda_2 + \mu_2 \upsilon_2 \frac{1}{s} t_0^{\alpha}$. Also,

$$\begin{aligned} Tx = Ax + Bx \cdot Cx &\leqslant A(t_0 h) + B(t_0 h) \cdot C(t_0 h) \\ &\leqslant Ah + t_0^{-\alpha} Bh \cdot Ch \leqslant \lambda_1 h + \mu_1 \upsilon_1 t_0^{-\alpha} h \cdot h. \end{aligned}$$

Hence, we have $Tx \leqslant K_2 h$, where $K_2 = \lambda_1 + \mu_1 \upsilon_1 t_0^{-\alpha} s$. Thus, $Tx \in P_h$. So, $T : P_h \to P_h$. Moreover, $A : P_h \to P_h, B : P_h \to P_h$, and $C : P_h \to P_h$. On the other hand, for any $t \in (0, 1)$ and $x \in P_h$,

$$Ax + t^{-\alpha} Bx \cdot Cx \leqslant t^{-\alpha}(Ax + Bx \cdot Cx).$$

Then,

$$T(tx) = A(tx) + B(tx) \cdot C(tx) \leqslant t^{-\alpha} T(x) \quad \forall t \in (0, 1), x \in P_h.$$

Therefore, T is the ($-\alpha$)-convex operator. Let, $x_0 = \theta$. Application of Theorem 3.3 implies that the equation $Tx = x$ has a unique solution x^* in P_h. That is, the operator Eq (1.1) has a unique solution x^* in P_h. Now, we construct successively the sequence $y_n = Ay_{n-1} + By_{n-1} \cdot Cy_{n-1} (n = 1, 2, \cdots)$ for any initial point $y_0 \in P_h$. Since $y_0 \in P_h$ and $Ty_0 \in P_h$, we can choose a sufficiently small number $t_0 \in (0, 1)$ such that

$$t_0 y_0 \leqslant Ty_0 \leqslant \frac{1}{t_0} y_0. \tag{3.24}$$

Since $0 < \alpha(t_0) < 1$, we can also take a positive integer k such that

$$k > \frac{1}{1 - \alpha(t_0)}. \tag{3.25}$$

Set $u_0 = t_0^k y_0, v_0 = \frac{1}{t_0^k} y_0$. Let $u_{n+1} = Tu_n$, $v_{n+1} = Tv_n (n = 1, 2, \cdots)$. By Theorem 3.3, $u_n \to x^*$ and $v_n \to x^*$(as $n \to \infty$). By (3.25), we have $u_0 \leqslant y_1 \leqslant v_0$. Let us define $y_{n+1} = Ty_n$. Since T is monotone decreasing and by (3.20), we get

$$v_{2n-1} \leqslant y_{2n} \leqslant u_{2n-1}, \quad v_{2n} \leqslant y_{2n+1} \leqslant u_{2n}, \quad n = 1, 2, \cdots . \tag{3.26}$$

Then by Lemma 2.1, $y_{2n} \to x^*$, $y_{2n+1} \to x^*$ as $n \to \infty$. Thus, for $\varepsilon > 0$, there exists an integer N such that for $n \geqslant N$,

$$\|y_{2n} - x^*\| < \varepsilon, \quad \|y_{2n+1} - x^*\| < \varepsilon. \tag{3.27}$$

Therefore, (3.27) show that $y_n \to x^*$ as $n \to \infty$. □

4. Applications to nonlinear integral equation

Theorem 4.1. *Assume that* $E = C[0,1]$, $P = \{x(t) \in C[0,1] \mid x(t) \geqslant 0,\ \forall t \in [0,1]\}$,

(H1) $g(t,x) : [0,1]\times[0,\infty) \to [0,\infty)$, $u(t,x) : [0,1]\times[0,\infty) \to [0,\infty)$ *and* $f(t,x) : [0,1]\times[0,\infty) \to [0,\infty)$ *are increasing operators with respect to x;*

(H2) there exists $h > \theta$ *in P such that* $g(t,h), u(t,h) \in P_h$ *and* $h \cdot h \in P_h$*;*

(H3) $g(t,x)$ *is an* α*-concave operator,* $f(t,x)$ *is a* γ_1*-concave operator, and* $u(t,x)$ *is a* γ_2*-concave operator with respect to x such that* $\gamma_1 + \gamma_2 = \gamma \leqslant 1$*;*

(H4) $G(t,s)$ *is non-negative for any* $t, s \in [0,1]$. *Also, for any fixed* $t \in [0,1]$, *the function* $G(t,s)$ *is bounded in* $[0,1]$ *and for any fixed* $s \in [0,1]$, *we have* $G(t,s) \in P_h$*;*

(H5) there exists $\delta_0 > 0$ *such that for any* $t \in [0,1]$ *and any* $y \in [0,\infty)$ *we have*

$$g(t,y) \geqslant \delta_0 u(t,y) \cdot \int_0^1 G(t,s) f(s,y) ds.$$

Then the problem

$$x(t) = g(t,x(t)) + u(t,x(t)) \int_0^1 G(t,s) f(s,x(s)) ds$$

has a unique positive solution x^* *in* P_h. *Moreover, for any* $x_0 \in P_h$ *and for the constructing successively sequence*

$$x_{n+1}(t) = g(t,x_n(t)) + u(t,x_n(t)) \int_0^1 G(t,s) f(s,x_n(s)) ds, \quad n = 0,1,2,\cdots,$$

we have $\|x_n - x^*\| \to 0$ *as* $n \to \infty$.

Proof. Let us define the operators $A : P \to E$, $B : P \to E$, and $C : P \to E$ as the following:

$$(Ax)(t) = g(t,x(t)), \quad (Bx)(t) = u(t,x(t)),$$
$$(Cx)(t) = \int_0^1 G(t,s) f(s,x(s)) ds.$$

It can easily be noticed that x^* is a solution of the problem (4.1) if $x^* = Ax^* + Bx^* \cdot Cx^*$. By $(H1)$ and $(H4)$, we notice that $A : P \to P$, $B : P \to P$ and $C : P \to P$. By $(H3)$, for any $\lambda \in (0,1)$ and $x \in P$, we have

$$\begin{aligned} C(\lambda x)(t) &= \int_0^1 G(t,s) f(s,\lambda x(s)) ds \\ &\geqslant \lambda^{\gamma_1} \int_0^1 G(t,s) f(s,x(s)) ds = \lambda^{\gamma_1} C(x)(t). \end{aligned}$$

Then, C is a γ_1-concave operator with respect to x. By $(H2)$, $Ah \in P_h$ and $Bh \in P_h$. By $(H4)$, for any $s \in [0,1]$ there exist $\lambda(s), \mu(s) > 0$ such that

$$\lambda(s)h(t) \leqslant G(t,s) \leqslant \mu(s)h(t).$$

Since $G(t,s)$ is bounded, $\lambda(s), \mu(s)$ are bounded positive real numbers(for any $s \in [0,1]$). Therefore,

$$Ch(t) = \int_0^1 G(t,s) f(s,h(s)) ds \leqslant \int_0^1 \mu(s)h(t) f(s,1) ds = h(t) \int_0^1 \mu(s) f(s,1) ds$$

and

$$Ch(t) = \int_0^1 G(t,s)f(s,h(s))ds \geqslant \int_0^1 \lambda(s)h(t)f(s,0)ds = h(t)\int_0^1 \lambda(s)f(s,0)ds.$$

Thus, $Ch \in P_h$. Hence, the condition (i) of Theorem 3.1 is satisfied. By $(H2)$, the condition (iii) of Theorem 3.1 is satisfied and by $(H5)$ the condition (ii) of Theorem 3.1 is satisfied. Then, Theorem 4.1 follows from Theorem 3.1. □

Example 4.1. *Assume that $E = C[0,1]$, $P = \{x(t) \in C[0,1] \mid x(t) \geqslant 0,\ \forall t \in [0,1]\}$. Let us define*

$$g(t,x(t)) = \frac{\sqrt{x(t)}}{1+\sqrt{x(t)}},\ u(t,x(t)) = \frac{x(t)}{\sqrt{1+x^2(t)}},\ f(t,x(t)) = \sqrt{1+x(t)}$$

for $t \in [0,1]$ and $x \in P$. It is easy to prove that g is $\alpha-$concave (for $\alpha = \frac{1}{2}$). Also, f is a γ_1-concave operator (for $\gamma_1 = 0$) and u is a γ_2-concave operator (for $\gamma_2 = 1$) with respect to x. Suppose that $G(t,s) = \frac{e^{-ts}}{1+ts}$ (for $t, s \in [0,1]$). It is easy to see that g, u, f, and G are satisfied in all assumptions of Theorem 4.1 for $h(t) = 1$. Hence, the problem

$$x(t) = \frac{\sqrt{x(t)}}{1+\sqrt{x(t)}} + \frac{x(t)}{\sqrt{1+x^2(t)}}\int_0^1 \frac{e^{-ts}\sqrt{1+x(s)}}{1+ts}ds$$

has a unique positive solution.

Theorem 4.2. *Assume that $E = C[0,1]$, $P = \{x(t) \in C[0,1] \mid x(t) \geqslant 0,\ \forall t \in [0,1]\}$,*

(H1) $g(t,x) : [0,1]\times[0,\infty) \to [0,\infty)$, $u(t,x) : [0,1]\times[0,\infty) \to [0,\infty)$ and $f(t,x) : [0,1]\times[0,\infty) \to [0,\infty)$ are decreasing with respect to x;

(H2) there exists $h > \theta$ in P such that $g(t,h), u(t,h) \in P_h$, and $h \cdot h \in P_h$;

(H3) $f(t,x)$ is (-α)-convex with respect to x;

(H4) $G(t,s)$ is non-negative for any $t, s \in [0,1]$, for any fixed $t \in [0,1]$, the function $G(t,s)$ is bounded in $[0,1]$, and for any fixed $s \in [0,1]$, we have $G(t,s) \in P_h$.

Then, the problem

$$x(t) = g(t,x(t)) + u(t,x(t))\int_0^1 G(t,s)f(s,x(s))ds,$$

has a unique positive solution x^* in P_h. Moreover, for any $x_0 \in P_h$ and for the constructing successively sequence

$$x_{n+1}(t) = g(t,x_n(t)) + u(t,x_n(t))\int_0^1 G(t,s)f(s,x_n(s))ds, \quad n = 0,1,2,\cdots$$

we have $\|x_n - x^*\| \to 0$ as $n \to \infty$.

Proof. Let us define the operators $A : P \to E$, $B : P \to E$ and $C : P \to E$ as the following:

$$(Ax)(t) = g(t,x(t)),\ \ (Bx)(t) = u(t,x(t)),$$
$$(Cx)(t) = \int_0^1 G(t,s)f(s,x(s))ds.$$

It is easily noticed that if $x^* = Ax^* + Bx^* \cdot Cx^*$, x^* is a solution of the problem (4.1). By ($H1$) and ($H4$), we have $A : P \to P$, $B : P \to P$ and $C : P \to P$. By ($H3$), for any $\lambda \in (0, 1)$ and $x \in P$ we have,

$$C(\lambda x)(t) = \int_0^1 G(t,s)f(s,\lambda x(s))ds \geqslant \lambda^{-\alpha}\int_0^1 G(t,s)f(s,x(s))ds = \lambda^{-\alpha}C(x)(t).$$

Hence, C is an (-α)-convex operator. By ($H2$), we have $Ah \in P_h$ and $Bh \in P_h$. By ($H4$), for any $s \in [0, 1]$, there exist $\lambda(s)$ and $\mu(s) > 0$ such that

$$\lambda(s)h(t) \leqslant G(t,s) \leqslant \mu(s)h(t).$$

Since $G(t, s)$ is bounded, $\lambda(s)$ and $\mu(s)$ are bounded positive real numbers (for any $s \in [0, 1]$). Therefore, we have

$$Ch(t) = \int_0^1 G(t,s)f(s,h(s))ds \leqslant \int_0^1 \mu(s)h(t)f(s,0)ds$$

$$= h(t)\int_0^1 \mu(s)f(s,0)ds$$

and

$$Ch(t) = \int_0^1 G(t,s)f(s,h(s))ds \geqslant \int_0^1 \lambda(s)h(t)f(s,1)ds$$

$$= h(t)\int_0^1 \lambda(s)f(s,1)ds.$$

Thus, $Ch \in P_h$. Hence, the assumption (i) of Theorem 3.5 is satisfied. By ($H2$), the assumption (iii) of Theorem 3.5 is satisfied. Then, Theorem 4.2 follows from Theorem 3.5. □

Example 4.2. *Assume that* $E = C[0, 1]$, $P = \{x(t) \in C[0, 1] \mid x(t) \geqslant 0,\ \forall t \in [0, 1]\}$. *Let us define,*

$$g(t,x(t)) = \frac{1}{1+x^2(t)},\ u(t,x(t)) = arccot(x(t)-4),\ f(t,x(t)) = \frac{1}{\sqrt{1+x(t)}},$$

for $t \in [0, 1]$ *and* $x \in P$. *It can be proved that* f *is (-*α*)-convex (for* $\alpha = \frac{1}{2}$ *). Also* u, g *are convex sub-homogeneous in* x*. Suppose that* $G(t, s) = \frac{e^{-ts}}{1+ts}$*(for* $t, s \in [0, 1]$*). We can see that* g*,* u*,* f*, and* G *are satisfied in all assumptions of Theorem 4.2 for* $h(t) = 1$*. Then the problem*

$$x(t) = \frac{1}{1+x^2(t)} + arccot(x(t)-4)\int_0^1 \frac{e^{-ts}}{1+ts}\frac{1}{\sqrt{1+x(t)}}ds$$

has a unique positive solution.

5. Conclusions

In this paper, we firstly proved the existence of a positive solution for the Eq (1.1) and approximated it by the constructing successively sequence, where A is an α-concave operator, $B : P \to P$ is an increasing γ_1-concave operator and $C : P \to P$ is an increasing γ_2-concave operator such that $\gamma_1 + \gamma_2 = \gamma \leqslant 1$.

Secondly, we proved the existence of a positive solution for the Eq (1.1) and approximated it by the constructing successively sequence, where A, C are decreasing operators and C is a (-α)-convex operator.

Thirdly, we proved the existence a positive solution for some nonlinear integral equations and approximated it by the constructing successively sequence(especially in the case of quadratic integral equation).

Remark 5.1. *It is suggested that the Theorems 3.1 and 3.5 be proved without the assumption $h \cdot h > \theta$, and also the Theorem 3.1 be proved without assumption (ii). Another interesting topic can be the comparison of the results of Theorems 3.1 and 3.5 with the results of theorems that are proven by the measure of non-compactness [2] and Dhage's techniques [13].*

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Conflict of interest

Authors state no conflicts of interest.

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