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Research article

Gap solitons in periodic difference equations with sign-changing saturable nonlinearity

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Abstract: In this paper, we consider the existence of gap solitons for a class of difference equations:

$$Lu_n - \omega u_n = f_n(u_n), n \in \mathbb{Z},$$

where $Lu_n = a_n u_{n+1} + a_{n-1}u_{n-1} + b_n u_n$ is the discrete difference operator in one spatial dimension, $\{a_n\}$ and $\{b_n\}$ are real valued T-periodic sequences, $\omega \in \mathbb{R}$, $f_n(\cdot) \in C(\mathbb{R}, \mathbb{R})$ and $f_{n+T}(\cdot) = f_n(\cdot)$ for each $n \in \mathbb{Z}$. Under general asymptotically linear conditions on the nonlinearity $f_n(\cdot)$, we establish the existence of gap solitons for the above equation via variational methods when $tf_n(t)$ is allowed to be sign-changing. Our methods further extend and improve the existing results.

Keywords: Schrödinger equations; gap solitons; critical point theory; approximation
Mathematics Subject Classification: 65Q10, 39A60

1. Introduction

Difference equations represent the discrete counterpart of ordinary differential equations, and they are applied in various research fields, such as computer science, economics and biology [1–4]. Many authors have obtained excellent results for difference equations [5–8]. In this paper, we focus on the discrete nonlinear Schrödinger (DNLS) equations which are widely utilized to describe a multitude of physical and biological phenomena, such as nonlinear optics [9, 10], biomolecular chains [11], the lattice dynamics of solids and the localization of electromagnetic waves in photonic crystals with a nonlinear response [12]. For more reviews on this theme, we refer to [13–20].

In this paper, we consider the existence of discrete gap solitons for the following periodic (DNLS) equation

$$Lu_n - \omega u_n = f_n(u_n), n \in \mathbb{Z}, \tag{1.1}$$

where L is a second order difference operator given by

$$Lu_n = a_n u_{n+1} + a_{n-1}u_{n-1} + b_n u_n,\ n \in \mathbb{Z}, \tag{1.2}$$

$\{a_n\}$ and $\{b_n\}$ are real valued T-periodic sequences and $\omega \in \mathbb{R}$. $f_n(\cdot) \in C(\mathbb{R},\mathbb{R})$, $f_n(0) = 0$ and $f_{n+T}(t) = f_n(t)$.

We are interested in the existence of gap solitons. This problem appears when we consider the standing wave solutions for the following DNLS equation

$$i\dot{\psi}_n = -\Delta\psi_n + V_n\psi_n - f_n(\psi_n), n \in \mathbb{Z}, \tag{1.3}$$

where $\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$ is the discrete Laplacian in one spatial dimension, $f_n(\cdot)$ is continuous from $\mathbb{C}$ into $\mathbb{C}$ and $f_n(0) = 0$ for every $n \in \mathbb{Z}$. The sequence $\{V_n\}$ of real numbers and the sequence $\{f_n(\cdot)\}$ of functions are assumed to be T-periodic, i.e., $V_{n+T} = V_n$, $f_{n+T}(\cdot) = f_n(\cdot)$. The typical saturable nonlinearities are as follows:

$$f_n(t) = \frac{t^3}{1 + l_n t^2}$$

and

$$f_n(t) = (1 - e^{-l_n t^2})t,$$

where l_n is a real valued T-periodic sequence. The saturable nonlinearities can describe optical pulse propagation in various doped fibers [21, 22]. In Eq (1.3), we suppose the nonlinearity f_n is gauge invariant, i.e.,

$$f_n(e^{i\theta}t) = e^{i\theta}f_n(t),\ \theta \in \mathbb{R}. \tag{1.4}$$

Spatially localized standing waves are often called breathers or solitons. Since solitons are spatially localized time-periodic solutions and decay to zero at infinity, ψ_n has the following form:

$$\psi_n = u_n e^{-i\omega t},\quad \lim_{n\to\pm\infty} u_n = 0, \tag{1.5}$$

where $\{u_n\}$ is a real valued sequence and $\omega \in \mathbb{R}$ is the temporal frequency. Then, Eq (1.3) becomes

$$-\Delta u_n + V_n u_n - \omega u_n = f_n(u_n),\quad n \in \mathbb{Z},\quad \lim_{n\to\pm\infty} u_n = 0. \tag{1.6}$$

In Eq (1.1), the operator L is a bounded and self-adjoint operator in the space l^2. The spectrum $\sigma(L)$ has a band structure, i.e., $\sigma(L)$ is a union of a finite number of closed intervals [23]. Thus the complement $\mathbb{R}\backslash\sigma(L)$ consists of a finite number of open intervals called spectral gaps, two of which are semi-infinite. Generally, a soliton for Eq (1.3) with the temporal frequency ω belonging to a spectral gap is called a gap soliton. In this work, we fix one spectral gap by $(-\infty,\beta)$ and consider exploring nontrivial solutions that are not equal to 0 identically for the case $\omega \in (-\infty,\beta)$.

In 2006, Pankov [24] first studied the gap solitons for the periodic DNLS equation, Eq (1.3) with the function $f_n(t) = l_n|t^2|t$ given by the linking theorem in combination with periodic approximations. Since then, the DNLS equations with saturable nonlinearities have also been studied very intensively. These pioneering works have produced many novel and interesting results on the existence of solutions [21, 25–28]. In 2008, Pankov and Rothos [25] considered Eq (1.1) with $a_n = -1$ and $b_n = 2$ when the nonlinearity $f_n(t) = f(t)$ is saturable at ∞. Pankov obtained the following result.

Theorem 1.1 [25]. *Assume that the nonlinearity $f(t)$ satisfies the following assumptions:*

$(h1)$ *$f(t) = o(t)$ as $t \to 0$;*
$(h2)$ $\lim_{|t|\to\infty} \frac{f(t)}{t} = l < \infty$;
$(h3)$ *$f(t) \in C(\mathbb{R})$ and $f(t)t < f'(t)t^2$ for $t \neq 0$;*
$(h4)$ *$g(t) = f(t) - lt$ is bounded.*

Assume $\omega < 0$ and $l + \omega > 0$. Then there exists a non-trivial ground-state solution $u \in l^2$ of Eq (1.1).

In 2010, Pankov [26] considered the existence of gap solitons for Eq (1.1) with saturable nonlinearities when ω belongs to a spectral gap (α,β) of the linear part and $0 \notin \sigma(L)$. The main method of the proof involves applying critical point theory in combination with periodic approximations of solutions.

Theorem 1.2 [26]. *Assume the conditions $(i) - (iv)$ hold:*

(i) *$\forall n \in \mathbb{Z}$, the function $f_n : \mathbb{R} \to \mathbb{R}$ is continuous, $f_n(t) = f_{n+T}(t)$, $f_n(0) = 0$ and $f_n(t) = o(t)$ as $t \to 0$;*
(ii) *$\lim_{|t|\to\infty} \frac{f_n(t)}{t} = l_n$ exists and is finite, and the function $g_n(t) = f_n(t) - l_n t$ is bounded;*
(iii) *For every $r_0 > 0$, there exists $\delta_0 = \delta_0(r_0) > 0$ such that*

$$\frac{1}{2}tf_n(t) - F_n(t) \geq \delta_0,$$

for $|t| \geq r_0$ and $F_n(t) \geq 0$ for $t \in \mathbb{R}$.
(iv) *$\lambda = \min_{n\in\mathbb{Z}} l_n > \hat{\theta}$, where $\hat{\theta}$ is the bottom of the positive part of $\sigma(L)$.*

Then there exists a nonzero solution $u \in l^2$ of Eq (1.1).

Under the condition that the nonlinearity is saturable, Zhou and Yu obtained a new sufficient condition for the existence of homoclinic solutions of the system by using the mountain pass lemma in combination with periodic approximations in 2010 [27]. They proved that it is also necessary in some special cases.

Theorem 1.3 [27]. *Assume that $\omega \in (-\infty,\beta)$, f_n is continuous in t, $f_n(t) = f_{n+T}(t)$ for any $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, $f_n(t) = o(t)$ as $t \to 0$. And the following conditions hold.*

$(H1)$ *$\frac{f_n(t)}{t}$ is strictly increasing in $(0, +\infty)$ and strictly decreasing in $(-\infty, 0)$. Moreover, $\lim_{|t|\to\infty} \frac{f_n(t)}{t} = d_n < \infty$;*
$(H2)$ *$tf_n(t) - 2F_n(t) \to \infty$ as $|t| \to \infty$, and $\limsup_{t\to 0} \frac{f_n^2(t)}{tf_n(t)-2F_n(t)} = p_n < \infty$.*

If $d_n > \beta - \omega$ for $n \in \mathbb{Z}$, then Eq (1.1) has at least a nontrivial solution $u \in l^2$.

Other related gap solitons results can be found in the literature [17–19, 28–31]. The existence of solutions for Eq (1.1) has been widely considered under the condition of various assumptions on the saturable nonlinearity $f_n(t)$. To prove the existence of solutions for Eq (1.1), the main required assumptions are $(h1)$, $(h3)$, (iii), $(H1)$ and $(H2)$ which are given in most papers [19, 30, 32]. The conditions $(h1)$ and $(h3)$ imply the condition $(H1)$, that the function $t \to \frac{f_n(t)}{|t|}$ is strictly increasing in $(-\infty, 0)$ and $(0, +\infty)$. According to $(H1)$, it follows that $tf_n(t) \geq 2F_n(t) \geq 0$ for all $t \in \mathbb{R}$, that is, $tf_n(t)$ has an unchanging sign. We must point out that one essential assumption, $(H1)$ must be used to prove the critical functional satisfying the Palais-Smale (P.S.) condition in many studies. Moreover, the assumption $(H2)$, i.e., $tf_n(t) - 2F_n(t) \to +\infty$ need also be used during the proof in some studies, such as in Theorem 1.3 from [27]. In these situations, most of the authors established the existence of solutions for Eq (1.1); see [27, 31] for examples. Regarding the case that $tf_n(t)$ is allowed to be sign changing, there seems to be only a few papers that apply this for the DNLS equations. For the case in which $tf(t)$ is sign changing, we can find some results [19,33–35]. However, most of these papers deal with the nonlinearity in the sup-cubic case rather than the saturable nonlinearity at infinity.

Motivated by the above works, in this paper, we further consider the gap soliton problems of Eq (1.1) with saturable nonlinearity $f_n(t)$ at infinity. Let $\delta = \beta - \omega > 0$; we assume the following assumptions are satisfied:

$(F1)$ $\lim_{t\to 0} \frac{f_n(t)}{t} = q$ uniformly for all $n \in \mathbb{Z}$, where $|q| < \frac{\delta}{8}$;

$(F2)$ There exists a constant $d > 0$ such that $\lim_{|t|\to\infty} \frac{f_n(t)}{t} = d < +\infty$ uniformly for all $n \in \mathbb{Z}$ and the function $g_n(t) = f_n(t) - dt$ is bounded for each $n \in \mathbb{Z}$;

$(F3)$ There exists a constant $\gamma \in (0, \delta)$ such that $tf_n(t) - 4F_n(t) \geq -\gamma t^2$, $\forall n \in \mathbb{Z}$.

The term $tf_n(t)$ is allowed to be sign-changing based on the assumption $(F3)$ and the assumption $(F1)$ implies that the nonlinear term represents the mixed nonlinearities that can be superlinear or asymptotically linear at the origin. By using the mountain pass lemma in combination with periodic approximations, we establish the existence of gap solitons for Eq (1.1) in l^2.

Remark 1.1. *Before proceeding further, we will first give a function that satisfies the conditions* $(F1)-(F3)$*, but not* $(h1)$*:* $f_n(t) = o(t)$ *as* $t \to 0$*. For all* $n \in \mathbb{Z}$*, let* $f_n(t) = \frac{t^3-2t}{1+t^2}$ *for* $t \in \mathbb{R}$*.* $(F1) - (F2)$ *are obvious, but the aforementioned function does not satisfy* $(h1)$*. Since* $F_n(t) = \frac{t^2}{2} - \frac{3}{2}\ln(1+t^2)$*, we can find a constant* $\gamma = 5 > 0$ *such that* $tf_n(t) - 4F_n(t) + 5t^2 = \frac{4t^4+t^2+6(1+t^2)\ln(1+t^2)}{(1+t^2)} \geq 0$ *and* $tf_n(t) = \frac{t^4-2t^2}{1+t^2}$ *is sign-changing.*

2. Preliminaries

In order to establish the variational framework associated with Eq (1.1) and apply the critical point theory, we will give some basic notations and lemmas that will be used to prove our main results. In the Hilbert space $E = l^2$, we consider the functional

$$J(u) = \frac{1}{2}(Lu - \omega u, u) - \sum_{n=-\infty}^{+\infty} F_n(u_n),$$

where $(\cdot,\cdot)$ is the inner product in l^2, and $F_n(t) = \int_0^t f_n(s)ds$, $n \in \mathbb{Z}$. The corresponding norm in E is denoted by $\|\cdot\|$. Then $J \in C^1(E, \mathbb{R})$ and we can compute the Frećhet derivative as

$$\langle J'(u), v\rangle = (Lu - \omega u, v) - \sum_{n=-\infty}^{+\infty} f_n(u_n)v_n$$

for all $u,\ v \in E$.

Equation (1.1) is the corresponding Euler-Lagrange equation for J. Thus, the nonzero critical points of J are nontrivial solutions of Eq (1.1).

Let S be the following set of all two-sided sequences:

$$S = \{u = \{u_n\} | u_n \in \mathbb{R}, n \in \mathbb{Z}\}.$$

Then, S is a vector space with $au + bv = \{au_n + bv_n\}$ for $u, v \in S$ and $a, b \in \mathbb{R}$.

For any fixed positive integer k, we define the subspace E_k of S that consists of all kT-periodic sequences:

$$E_k = \{u = \{u_n\} \subset S \,|u_{n+2kT} = u_n \text{ for } n \in \mathbb{Z}\}.$$

Obviously, E_k is the $2kT$-dimensional Hilbert space. E_k can be equipped with the inner product $(\cdot, \cdot)_k$ and norm $\|\cdot\|_k$ as follows:

$$(u, v)_k = \sum_{n=-kT}^{kT-1} u_n \cdot v_n \text{ for } u, v \in E_k$$

and

$$\|u\|_k = \left(\sum_{n=-kT}^{kT-1} |u_n|^2 \right)^{\frac{1}{2}} \text{ for } u \in E_k.$$

We also define a norm $\|\cdot\|_{k\infty}$ in E_k by

$$\|u\|_{k\infty} = \max\{|u_n| : n \in \mathbb{Z}\} \text{ for } u \in E_k.$$

Consider the following functional J_k in E_k:

$$J_k(u) = \frac{1}{2}(L_k u - \omega u, u)_k - \sum_{n=-kT}^{kT-1} F_n(u_n); \tag{2.1}$$

then,

$$\langle J_k'(u), v\rangle = (L_k u - \omega u, v)_k - \sum_{n=-kT}^{kT-1} f_n(u_n) v_n \tag{2.2}$$

for all $u,\ v \in E_k$, where L_k is the operator L acting in E_k. This is a C^1-functional in a finite dimensional space and its critical points are exactly kT-periodic solutions of Eq (1.1).

We notice that $\sigma(L_k)$ is finite, $\sigma(L_k) \subset \sigma(L)$ and $\|L_k\| \leq \|L\|$ for all $k \in \mathbb{Z}$, as described in [23]. Furthermore, $\cup_{k\in\mathbb{Z}}\sigma(L_k)$ is a dense subset of $\sigma(L)$.
Then, we have that

$$(Lu - \omega u, u) \geq \delta\|u\|^2 \text{ for } u \in E. \tag{2.3}$$

$$(L_k u - \omega u, u)_k \geq \delta\|u\|_k^2 \text{ for } u \in E_k. \tag{2.4}$$

Let X be a real Banach space and $J \in C^1(X, \mathbb{R})$. A sequence $\{u_n\} \subset X$ is called a P.S. sequence for J if $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$. We say that J satisfies the P.S. condition if any P.S. sequence for J possesses a convergent subsequence in X.

Let B_r be the open ball centered at 0 of radius r in H, and ∂B_r denotes its boundary. The following lemma will play an important role in the proof of our main results.

Lemma 2.1 (Mountain Pass Lemma [36]). *Let H be a real Hilbert space, and assume that $J \in C^1(H, \mathbb{R})$ satisfies the P.S. condition if $J(0) = 0$ and the following conditions hold.*
(G_1) There exist constants $\rho > 0$ and $\alpha > 0$ such that $J(x) \geq \alpha$ for all $x \in \partial B_\rho$, where $B_\rho = \{x \in H : \|x\| < \rho\}$.

(G_2) *There exists* $e \in H\backslash\bar{B}_\rho$ *such that* $J(e) < 0$.
Then $c = \inf_{h\in\Gamma} \sup_{s\in[0,1]} J(h(s)) \geq \alpha$ *is a positive critical value of* J*, where*

$$\Gamma = \{h \in C([0,1], H) | h(0) = 0, h(1) = e\}.$$

3. Main results

If some of these conditions, i.e., $(F1) - (F3)$ are violated or changed, Eq (1.1) has no nontrivial solutions in l^2. Thus, we have the following proposition.

Proposition 3.1. *Assume that the conditions* $(F1)$ *and* $(F2)$ *hold and* $\delta \to +\infty$*. Then, Eq* (1.1) *has no nontrivial solutions in* l^2*.*

Proof. Arguing by contradiction, we assume that Eq (1.1) has a nontrivial solutions for $u \in l^2$, then, u is a critical point of J that satisfies

$$(Lu - \omega u, u) = \sum_{n=-\infty}^{+\infty} f_n(u_n)u_n.$$

Since $(F1)$ and $(F2)$ are satisfied, they imply that there exists a finite constant $a_1 > 0$ such that $|tf_n(t)| \leq a_1|t|^2$ uniformly for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. By Eq (2.3), we have that

$$\delta\|u\|^2 \leq (Lu - \omega u, u) = \sum_{n=-\infty}^{+\infty} f_n(u_n)u_n < a_1\|u\|^2.$$

This is impossible from $\delta \to +\infty$.

Remark 3.1. *If we consider that the conditions* $(F1)$*,* $(F2)$ *and* $a_1 < \beta - \omega$ *are satisfied, Eq* (1.1) *still has no nontrivial solutions in* l^2*(see [27]).*

We define the linear operator as

$$\tilde{L}_k u = L_k u - du, \ u \in E_k.$$

Let $G_n(t) = \int_0^t g_n(s)ds$ be the primitive functions of g_n for all $n \in \mathbb{Z}$. Then we can represent the functional J_k and its derivative in the following forms:

$$J_k(u) = \frac{1}{2}(\tilde{L}_k u - \omega u, u)_k - \sum_{n=-kT}^{kT-1} G_n(u_n) \tag{3.1}$$

and

$$\langle J_k'(u), v\rangle = (\tilde{L}_k u - \omega u, v)_k - \sum_{n=-kT}^{kT-1} g_n(u_n)v_n \tag{3.2}$$

for all $u, \ v \in E_k$.

Lemma 3.1. *Assume that the condition* $(F2)$ *holds and* $\omega \notin \sigma(\tilde{L}_k)$*; then, the functional* J_k *satisfies the*

P.S. condition.

Proof. Let $\{u^{(j)}\} \subset E_k$ be a sequence such that $J_k(u^{(j)})$ is bounded and $J'_k(u^{(j)}) \to 0$ as $j \to \infty$. Since E_k is finite dimensional, it is enough to show that the sequence $\{u^{(j)}\}$ is bounded.

Let E_k^+ and E_k^- be the positive spectral subspace and the negative spectral subspace of the operator $\tilde{L}_k - \omega$ in E_k, respectively; then, $E_k = E_k^+ \bigoplus E_k^-$.

Hence, we can find a positive constant $\eta > 0$ such that

$$\pm (\tilde{L}_k u - \omega u, u)_k \geq \eta \|u\|_k^2 \ \text{ for } u \in E_k^\pm. \tag{3.3}$$

For each $j \in \mathbb{Z}$, we write $u^{(j)} = u^{(j)+} + u^{(j)-}$; then,

$$\begin{aligned} \eta \|u^{(j)+}\|_k^2 &\leq (\tilde{L}_k u^{(j)+} - \omega u^{(j)+}, u^{(j)+})_k \\ &= (\tilde{L}_k u^{(j)} - \omega u^{(j)}, u^{(j)+})_k \\ &= \sum_{n=-kT}^{kT-1} g_n(u_n^{(j)}) u_n^{(j)+} \\ &\leq \|u^{(j)+}\|_k + \sum_{n=-kT}^{kT-1} |g_n(u_n^{(j)})||u_n^{(j)+}|. \end{aligned} \tag{3.4}$$

In view of $(F2)$, we can take $\varepsilon = \frac{\eta}{2} > 0$ and a sufficiently large $M > 0$ such that

$$0 < \left| \frac{g_n(t)}{t} \right| < \frac{\eta}{2} \ \text{ for } |t| \geq M \text{ and } n \in \mathbb{Z}. \tag{3.5}$$

We have that

$$\begin{aligned} Q_k^{(j)} &= \{n : |u_n^{(j)}| < M, n \in \mathbb{Z}(-kT, kT-1)\}, \\ R_k^{(j)} &= \{n : |u_n^{(j)}| \geq M, n \in \mathbb{Z}(-kT, kT-1)\}. \end{aligned}$$

Let $M_0 = \max\limits_{n \in Q_k^{(j)}} \{|g_n(u_n^{(j)})|\}$. Then

$$\begin{aligned} \sum_{n=-kT}^{kT-1} |g_n(u_n^{(j)})|^2 &= \sum_{n \in Q_k^{(j)}} |g_n(u_n^{(j)})|^2 + \sum_{n \in R_k^{(j)}} |g_n(u_n^{(j)})|^2 \\ &\leq 2kT M_0^2 + \sum_{n \in R_k^{(j)}} \frac{\eta^2}{4} |u_n^{(j)}|^2 \\ &\leq 2kT M_0^2 + \frac{\eta^2}{4} \|u^{(j)}\|_k^2. \end{aligned}$$

The above inequality implies that

$$\left(\sum_{n=-kT}^{kT-1} |g_n(u_n^{(j)})|^2 \right)^{\frac{1}{2}} \leq \sqrt{2kT} M_0 + \frac{\eta}{2} \|u^{(j)}\|_k. \tag{3.6}$$

Combining Eq (3.4), the Cauchy-Schwartz inequality and Eq (3.6), we obtain the following inequality

$$\eta \|u^{(j)+}\|_k^2 \leq (1 + \sqrt{2kT} M_0) \|u^{(j)+}\|_k + \tfrac{\eta}{2} \|u^{(j)}\|_k \|u^{(j)+}\|_k. \tag{3.7}$$

Similarly, we get

$$\eta \|u^{(j)-}\|_k^2 \leq (1 + \sqrt{2kT} M_0) \|u^{(j)-}\|_k + \tfrac{\eta}{2} \|u^{(j)}\|_k \|u^{(j)-}\|_k. \tag{3.8}$$

Since $\|u^{(j)}\|_k^2 = \|u^{(j)+}\|_k^2 + \|u^{(j)-}\|_k^2$ and $\|u^{(j)+}\|_k + \|u^{(j)-}\|_k \leq \sqrt{2}\|u^{(j)}\|_k$, by Eqs (3.7) and (3.8), we have that

$$\eta\|u^{(j)}\|_k^2 \leq \sqrt{2}(1 + \sqrt{2kT}M_0)\|u^{(j)}\|_k + \tfrac{\sqrt{2}\eta}{2}\|u^{(j)}\|_k^2.$$

Hence, the sequence $\{u^{(j)}\}$ is bounded.

In the following theorem, we will use Lemma 2.1 and Lemma 3.1 to prove the existence of the nonzero critical point $u^{(k)}$ of J_k in E_k.

Theorem 3.1. *Assume that the conditions* $(F1)$-$(F2)$ *hold and* $\omega \notin \sigma(\tilde{L}_k)$. *If* $d > \beta - \omega$, *then* J_k *has at least one nonzero critical point* $u^{(k)}$ *in* E_k. *Moreover,* $\|u^{(k)}\|_k$ *is bounded and there exist positive constants* ξ *and* μ *such that*

$$\xi \leq \|u^{(k)}\|_{k\infty} \leq \mu. \tag{3.9}$$

Proof. The functional J_k satisfies the P.S. condition according to Lemma 3.1. Next, we need only to verify that J_k satisfies the conditions G_1 and G_2 in Lemma 2.1. In fact, $J_k(0) = 0$; let $\varepsilon = \frac{\delta}{4} - \frac{q}{2} > 0$; then, there exists some positive constant $\rho > 0$ such that

$$F_n(t) \leq \frac{\delta}{4}t^2 \ \text{ for } \forall n \in \mathbb{Z} \text{ and } |t| \leq \rho. \tag{3.10}$$

Since $\|u\|_{k\infty} \leq \|u\|_k$, we have that $\sum\limits_{n=-kT}^{kT-1} F_n(u_n) \leq \frac{\delta}{4}\|u\|_k^2$ for all $u \in E_k$ with $\|u\|_k \leq \rho$; then,

$$\begin{aligned} J_k(u) &= \tfrac{1}{2}(Lu - \omega u, u)_k - \sum_{n=-kT}^{kT-1} F_n(u_n) \\ &\geq \tfrac{\delta}{2}\|u\|_k^2 - \tfrac{\delta}{4}\|u\|_k^2 \\ &= \tfrac{\delta}{4}\|u\|_k^2. \end{aligned} \tag{3.11}$$

Taking $\alpha = \frac{\delta}{4}\rho^2$, we get $J_k(u) \geq \alpha$ for all $u \in \partial B_\rho$; hence, J_k satisfies the condition G_1 of Lemma 2.1.

Next, we will verify the condition G_2.

Let β_k be the lowest point in $\sigma(L_k)$. According to the spectral theory of periodic difference operators, endpoints of spectral gaps are either T-periodic or T-antiperiodic, eigenvalues of the difference operator L. Since β is one of endpoints, we see that either $\beta_k = \beta$ for all integers $k \geq 1$, or that $\beta_{2k} = \beta$ for all integers $k \geq 1$. Since $\omega \notin \sigma(\tilde{L}_k)$ and $d > \beta - \omega$, it is easy to verify that $E_k^- \neq \emptyset$ for $k \geq 1$.

We denote λ by the lowest positive point in $\sigma(\tilde{L}_k - \omega)$ in E_k. Let $z^k \in E_k^+$ be a unit eigenvector of $\tilde{L}_k - \omega$ with the eigenvalue λ. We can find that $y \neq 0$ in E_k^-. Let $u = z^k + \tau\frac{y}{\|y\|_k}$, where $\tau \in \mathbb{R}$. We have that

$$\begin{aligned} J_k(u) &= J_k\left(z^k + \tau\tfrac{y}{\|y\|_k}\right) \\ &= \tfrac{1}{2}\left((\tilde{L}_k - \omega)z^k, z^k\right)_k + \tfrac{1}{2}\left((\tilde{L}_k - \omega)\tfrac{\tau y}{\|y\|_k}, \tfrac{\tau y}{\|y\|_k}\right)_k - \sum_{n=-kT}^{kT-1} G_n(u_n) \\ &\leq \tfrac{\lambda}{2} - \tfrac{\eta}{2}\tau^2 - \sum_{n=-kT}^{kT-1} G_n(u_n). \end{aligned} \tag{3.12}$$

By $(F2)$, $|G_n(t)|$ has, at most, linear growth for all $n \in \mathbb{Z}$; we have that

$$\begin{aligned} J_k(u) &\leq \tfrac{\lambda}{2} - \tfrac{\eta}{2}\tau^2 + C\|u\|_k \\ &= -\tfrac{\eta}{2}\|u\|_k^2 + C\|u\|_k + \tfrac{\lambda+\eta}{2} \to -\infty \ \text{ as } \|u\|_k \to \infty. \end{aligned} \tag{3.13}$$

Thus, we can choose $\tau_0 \in \mathbb{R}$ such that $\|u_0\|_k = \sqrt{1+\tau_0^2} > \rho$ and $J_k(u_0) = J_k\left(z^k + \tau_0 \frac{y}{\|y\|_k}\right) < 0$.

All of the assumptions of Lemma 2.1 have been verified; then, J_k possesses a critical value $c_k \geq a$ with

$$c_k = \inf_{h\in\Gamma_k} \sup_{s\in[0,1]} J_k(h(s)),$$

where

$$\Gamma_k = \{h \in C([0,1], E_k) | h(0) = 0, h(1) = u_0 = z^k + \tau_0 \frac{y}{\|y\|_k} \in E_k \backslash B_\rho\}.$$

Given that a critical point $u^{(k)}$ of J_k corresponds to c_k in E_k, $u^{(k)}$ is nonzero as $c_k > 0$. We denote $h \in \Gamma_k$ as $h(s) = s\left(z^k + \tau_0 \frac{y}{\|y\|_k}\right)$ for $s \in [0,1]$; then,

$$J_k(u^{(k)}) \leq \sup_{s\in[0,1]} J_k\left(s\left(z^k + \tau_0 \frac{y}{\|y\|_k}\right)\right). \tag{3.14}$$

In fact,

$$\begin{aligned} J_k\left(s\left(z^k + \tau_0 \tfrac{y}{\|y\|_k}\right)\right) &\leq \tfrac{\lambda}{2} s^2 - \tfrac{\eta\tau_0^2}{2} s^2 + C\sqrt{1+\tau_0^2}\, s \\ &\leq \tfrac{\lambda}{2} + C\sqrt{1+\tau_0^2}. \end{aligned} \tag{3.15}$$

Hence,

$$J_k(u^{(k)}) \leq \frac{\lambda}{2} + C\sqrt{1+\tau_0^2} = M_1. \tag{3.16}$$

Let $0 \leq \gamma < \delta$. We see from $(F3)$, Eq (2.1) and (2.2) that

$$\begin{aligned} M_1 &\geq J_k(u^{(k)}) \\ &= \sum_{n=-kT}^{kT-1} \left(\tfrac{1}{2} f_n(u_n^{(k)}) u_n^{(k)} - F_n(u_n^{(k)})\right) \\ &= \sum_{n=-kT}^{kT-1} \left(\tfrac{1}{4} f_n(u_n^{(k)}) u_n^{(k)} + \tfrac{\gamma}{4}(u_n^{(k)})^2 - F_n(u_n^{(k)})\right) + \tfrac{1}{4}\sum_{n=-kT}^{kT-1} \left(f_n(u_n^{(k)}) u_n^{(k)} - \gamma (u_n^{(k)})^2\right) \\ &\geq \sum_{n=-kT}^{kT-1} \left(\tfrac{1}{4} f_n(u_n^{(k)}) u_n^{(k)} - \tfrac{\gamma}{4}(u_n^{(k)})^2\right) \\ &\geq \tfrac{\delta-\gamma}{4} \|u^{(k)}\|_k^2. \end{aligned} \tag{3.17}$$

So the sequence $\{u^{(k)}\}$ is bounded in E_k. In particular, Eq (3.16) implies that $|u_n^{(k)}| \leq 2\sqrt{\frac{M_1}{\delta-\gamma}}$ for $n \in \mathbb{Z}$, that is,

$$\|u^{(k)}\|_{k\infty} \leq 2\sqrt{\frac{M_1}{\delta-\gamma}} = \mu.$$

On the other hand, from Eq (2.4), we have that

$$\begin{aligned} \tfrac{\delta}{2}\|u^{(k)}\|_k^2 &\leq (Lu^{(k)} - \omega u^{(k)}, u^{(k)})_k \\ &= \langle J_k'(u^{(k)}), u^{(k)}\rangle + \sum_{n=-kT}^{kT-1} f_n(u_n^{(k)}) u_n^{(k)} \\ &\leq \sum_{n=-kT}^{kT-1} \left| f_n(u_n^{(k)}) u_n^{(k)} \right|. \end{aligned} \tag{3.18}$$

From $(F1)$, we have that $\varepsilon = \frac{\delta}{4} - |q| > 0$ and find a positive number ξ such that

$$0 \leq |f_n(t)t| < \frac{\delta}{4}t^2 \ \text{ for } n \in \mathbb{Z} \text{ and } |t| \leq \xi. \tag{3.19}$$

Consequently, from Eq (3.18), we get $|u_n^{(k)}| \geq \xi$ for $n \in \mathbb{Z}$, so

$$\|u^{(k)}\|_{k\infty} \geq \xi.$$

Now, we can give the proof of our main result.

Theorem 3.2. *Assume that the conditions* $(F1)$-$(F3)$ *hold. If $d > \beta - \omega$ and $\omega \notin \sigma(L - d)$, then Eq* (1.1) *possesses at least one nontrivial solution u in l^2.*

Proof $u^{(k)} = \{u_n^{(k)}\} \in E_k$ is a critical point obtained via Theorem 3.1, and there exists $n_k \in \mathbb{Z}$ such that

$$\xi \leq |u_{n_k}^{(k)}| \leq \mu. \tag{3.20}$$

Note that

$$a_n u_{n+1}^{(k)} + a_{n-1} u_{n-1}^{(k)} + (b_n - \omega) u_n^{(k)} = f_n(u_n^{(k)}), \ n \in \mathbb{Z}. \tag{3.21}$$

By the periodicity of the coefficients in Eq (3.21), we see that $\{u_{n+T}^{(k)}\}$ is also a solution of Eq (3.21). Making some shifts if necessary, and without loss of generality, we can assume that $0 \leq n_k \leq T - 1$ in Eq (3.20). Moreover, passing to a subsequence of $\{u^{(k)}\}$ if necessary, we can also assume that $n_k = n^*$ for $k \geq 1$ and some integer n^* such that $0 \leq n^* \leq T - 1$. It follows from Eq (3.20) that we can choose a subsequence, still denoted by $\{u^{(k)}\}$, such that

$$u_n^{(k)} \to u_n \ \text{ as } k \to \infty \text{ for } n \in \mathbb{Z}.$$

Obviously, from Eq (3.21), given $k \to \infty$, we can obtain the following:

$$a_n u_{n+1} + a_{n-1} u_{n-1} + (b_n - \omega) u_n = f_n(u_n), \ n \in \mathbb{Z}, \tag{3.22}$$

that is, u is a nonzero solution of Eq (1.1), as Eq(3.20) implies $\xi \leq |u_{n^*}^{(k)}|$. Now, we will verify that $u = u_n \in l^2$. For each $s \in \mathbb{N}$, let $k > s$. Then, it follows from (3.17) that

$$\sum_{n=-s}^{s} (u_n^{(k)})^2 \leq \|u^{(k)}\|_k^2 \leq \frac{4M_1}{\delta - \gamma}.$$

Let $k \to \infty$. We obtain $\sum\limits_{n=-s}^{s} (u_n)^2 \leq \frac{4M_1}{\delta - \gamma}$. By the arbitrariness of s, we know that $u = u_n \in l^2$. The proof is completed.

4. Conclusions

In this paper, through the use of variational methods, we consider the existence of gap solitons for the class of difference equations described by Eq (1.1) in one spatial dimension. Under general asymptotically linear conditions on nonlinearity $f_n(\cdot)$ at infinity, we find a gap soliton for Eq (1.1). Most of the pioneering work requires that the function $t \to \frac{f_n(t)}{|t|}$ is strictly increasing in $(-\infty, 0)$ and $(0, +\infty)$, and that $tf_n(t) \geq 2F_n(t) \geq 0$ for all $t \in \mathbb{R}$; see Theorems 1.1, 1.2 and 1.3. Obviously, the assumptions of these conditions are stronger than ours and $tf_n(t)$ has an unchanging sign, but our conditions require that $tf_n(t)$ is sign-changing according to $(F3)$. This is different from the previous assumptions of $(h3)$, (iii) or $(H2)$. Moreover, for Theorems 1.1, 1.2 and 1.3, the corresponding authors have decided to assume $f(t) = o(t)$ at the origin; we give a general assumption $(F1)$ in this paper; it implies that the nonlinear term represents the mixed nonlinearities that can be superlinear or asymptotically linear at the origin. We must point out that our work is different from previous works, and that the results of this paper improve, extend and complement some related results in the literature [25–27]. Our solutions are obtained by applying the variational approach in combination with periodic approximations; additionally, the functional J is required to satisfy the P.S. conditions; hence, some technical methods are used to deal with the process of proof.

The dynamic behavior of solitons has recently attracted much interest. In future work, we will analyze the existence problem for multiple solitons in Schrödinger lattice systems and the stability of gap solitons in lattices, as well as investigate the nonlinear evolution of unstable solitons under perturbations. Moreover, we plan to explore the stability of the solitons by investigating the linearization and to verify our results via direct simulations.

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Conflict of interest

All authors declare that they have no conflict of interest.

References

1. W. G. Kelly, A. C. Peterson, *Difference Equations: An Introduction with Applications,* Academic Press, San Diego, New York Basel, 1991.
2. Y. H. Long, L. Wang, Global dynamics of a delayed two-patch discrete SIR disease model, *Commun. Nonlinear Sci. Numer. Simul.,* **83** (2020), 105117. https://doi.org/10.1016/j.cnsns.2019.105117
3. J. S. Yu, J. Li, Discrete-time models for interactive wild and sterile mosquitoes with general time steps, *Math. Biosci.,* **346** (2022), 108797. https://doi.org/10.1016/j.mbs.2022.108797
4. B. Zheng, J. S. Yu, Existence and uniqueness of periodic orbits in a discrete model on Wolbachia infection frequency, *Adv. Nonlinear Anal.,* **11** (2022), 212–224. https://doi.org/10.1515/anona-2020-0194
5. Y. H. Long, H. Zhang, Three nontrivial solutions for second-order partial difference equation via morse theory, *J. Funct. Spaces,* **2022** (2022), 1564961. https://doi.org/10.1155/2022/1564961

6. Y. H. Long, X. Q. Deng, Existence and multiplicity solutions for discrete Kirchhoff type problems, *Appl. Math. Lett.,* **126** (2022), 107817. https://doi.org/10.1016/j.aml.2021.107817
7. Y. H. Long, Multiple results on nontrivial solutions of discrete Kirchhoff type problems, *J. Appl. Math. Comput.,* **2022** (2022), 1–17. https://doi.org/10.1007/s12190-022-01731-0
8. Y. H. Long, Nontrivial solutions of discrete Kirchhoff type problems via Morse theory, *Adv. Nonlinear Anal.,* **11** (2022), 1352–1364. https://doi.org/10.1515/anona-2022-0251
9. D. N. Christodoulides, F. Lederer, Y. Silberberg, Discretizing light behaviour in linear and nonlinear waveguide lattices, *Nature.,* **424** (2003), 817–823.
10. X. Liu, B. A. Malomed, J. Zeng, Localized modes in nonlinear fractional systems with deep lattices, *Adv. Theory Simul.,* **5** (2022), 2100482. https://doi.org/10.48550/arXiv.2201.01038
11. G. Kopidakis, S. Aubry, G. P. Tsironis, Targeted energy transfer through discrete breathers in nonlinear systems, *Phys. Rev. Lett.,* **87** (2001), 165501. http://arxiv.org/10.1103/ PhysRevLett.87.165501
12. A. Pankov, Periodic nonlinear Schrödinger equation with application to photonic crystals, *Milan J. Math.,* **73** (2005), 259–287. http://arxiv.org/abs/math/0404450
13. S. Aubry, Breathers in nonlinear lattices: existence, linear stability and quantization, *Phys. D,* **103** (1997), 201–250. http://dx.doi.org/10.1016/S0167-2789(96)00261-8
14. D. Henning, G. P. Tsironis, Wave transmission in nonlinear lattices, *Phys. Rep.,* **307** (1999), 333–432. http://dx.doi.org/10.1016/S0370-1573(98)00025-8
15. G. H. Lin, J. S. Yu, Z. Zhou, Homoclinic solutions of discrete nonlinear Schrödinger equations with partially sublinear nonlinearities, *Electron. J. Differ. Equ.,* **2019** (2019), 1–14. https://ejde.math.txstate.edu
16. G. H. Lin, Z. Zhou, J. S. Yu, Ground state solutions of discrete asymptotically linear Schrödinger equations with bounded and non-periodic potentials, *J. Dynam. Differential Equations,* **32** (2020), 527–555. http://dx.doi.org/10.1007/s10884-019-09743-4
17. F. C. Moreira, S. B. Cavalcanti, Gap solitons in one-dimensional (2) hetero-structures induced by the thermo-optic effect, *Opt. Mater.,* **122** (2021), 111666. http://dx.doi.org/10.1016/j.optmat.2021.111666
18. H. Meng, Y. Zhou, X. Li, Gap solitons in Bose CEinstein condensate loaded in a honeycomb optical lattice: Nonlinear dynamical stability, tunneling, and self-trapping, *Phys. A,* **577** (2021), 126087. http://dx.doi.org/10.1016/j.physa.2021.126087
19. G. H. Lin, J. S. Yu, Existence of a ground-state and infinitely many homoclinic solutions for a periodic discrete system with sign-changing mixed nonlinearities, *J. Geom. Anal.,* **32** (2022), 127. http://dx.doi.org/10.1007/s12220-022-00866-7
20. G. H. Lin, J. S. Yu, Homoclinic solutions of periodic discrete Schrödinger equations with local superquadratic conditions, *SIAM J. Math. Anal.,* **54** (2022), 1966–2005. http://dx.doi.org/10.1137/21M1413201
21. S. Gatz, J. Herrmann, Soliton propagation in materials with saturable nonlinearity, *J. Opt. Soc. Amer. B,* **8** (1991), 2296–2302. http://dx.doi.org/10.1364/JOSAB.8.002296
22. S. Gatz, J. Herrmann, Soliton propagation and soliton collision in double-doped fibers with a non-Kerr-like nonlinear refractive-index change, *Opt. Lett.,* **17** (1992), 484–486. http://dx.doi.org/10.1364/OL.17.000484

23. G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Mathematical Surveys and Monographs, No. 72. Providence, RI: American Mathematical Society, 2000. http://dx.doi.org/10.1090/surv /072
24. A. Pankov, Gap solitons in periodic discrete nonlinear Schrödinger equations, *Nonlinearity.,* **19** (2006), 27–40. http://dx.doi.org/10.1088/0951-7715/19/1/002
25. A. Pankov, V. Rothos, Periodic and decaying solutions in discrete nonlinear Schrödinger with saturable nonlinearity, *Proc. R. Soc. A-Math. Phys. Eng. Sci.,***464** (2008), 3219–3236. https://doi.org/10.1098/rspa.2008.0255
26. A. Pankov, Gap solitons in periodic discrete nonlinear Schrödinger equations with saturable nonlinearities. *J. Math. Anal. Appl.,* **371** (2010), 254–265. http://dx.doi.org/10.1016/j.jmaa.2010.05.041
27. Z. Zhou, J. S. Yu, On the existence of homoclinic solutions of a class of discrete nonlinear periodic systems, *J. Differential Equations,* **249** (2010), 1199–1212. http://dx.doi.org/10.1016/j.jde.2010.03.010
28. Z. Zhou, J. S. Yu, Y. M. Chen, Homoclinic solutions in periodic difference equations with saturable nonlinearity, *Sci. China Math.,* **54** (2011), 83–93. http://dx.doi.org/10.1007/s11425-010-4101-9
29. H. P. Shi, H. Zhang, Existence of gap solitons in periodic discrete nonlinear Schrödinger equations, *J. Math. Anal. Appl.,* **361** (2010), 411–419. http://dx.doi.org/10.1016/j.jmaa.2009.07.026
30. S. W. Ma, Z. Q. Wang, Multibump solutions for discrete periodic nonlinear Schrödinger equations, *Z. Angew. Math. Phys.,* **64** (2013), 1413–1442. http://dx.doi.org/10.1007/s00033-012-0295-8
31. G. H. Lin, Z. Zhou, Homoclinic solutions in periodic difference equations with mixed nonlinearities, *Math. Methods Appl. Sci.,* **39** (2016), 245–260. http://dx.doi.org/10.1002/mma.3474
32. Z. Zhou, J. S. Yu, Homoclinic solutions in periodic nonlinear difference equations with superlinear nonlinearity, *Acta. Math. Sin.-English Ser.,* **29** (2013), 1809–1822. http://dx.doi.org/10.1007/s10114-013-0736-0
33. J. Zhang, X. H. Tang, W. Zhang, Existence of multiple solutions of Kirchhoff type equation with sign-changing potential, *Appl. Math. Comput.,* **242** (2014), 491–499. http://dx.doi.org/10.1016/j.amc.2014.05.070
34. S. T. Chen, X. H. Tang, Infinitely many solutions for super-quadratic Kirchhoff-type equations with sign-changing potential, *Appl. Math. Lett.,* **67** (2016), 40–45. http://dx.doi.org/10.1016/j.aml.2016.12.003
35. W. R. Sun, L. Liu, L. Wang, Dynamics of fundamental solitons and rogue waves on the mixed backgrounds, *Eur. Phys. J. Plus,* **136** (2021), 1–9. http://dx.doi.org/10.1140/epjp/s13360-021-01379-y
36. A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.,* **14** (1973), 349–381. http://dx.doi.org/10.1016/0022-1236(73)90051-7