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Research article

Application of a tripled fixed point theorem to investigate a nonlinear system of fractional order hybrid sequential integro-differential equations

Muhammed Jamil[1], Rahmat Ali Khan[1], Kamal Shah[1,2,*], Bahaaeldin Abdalla[2] and Thabet Abdeljawad[2,3,*]

[1] Department of Mathematics, University of Malakand, Chakdara Dir (Lower), Khyber Pakhtunkhawa, Pakistan

[2] Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia

[3] Department of Medical Research, China Medical University, Taichung 40402, Taiwan

* **Correspondence:** Email: kamalshah408@gmail.com; tabdeljawad@psu.edu.sa.

Abstract: The goal of this manuscript is to study the existence theory of solution for a nonlinear boundary value problem of tripled system of fractional order hybrid sequential integro-differential equations. The analysis depends on some results from fractional calculus and fixed point theory. As a result, we generalized Darbo's fixed point theorem to form an updated version of tripled fixed point theorem to investigate the proposed system. Also, Hyres-Ulam and generalized Hyres-Ulam stabilities results are established for the considered system. For the illustration of our main results, we provide an example.

Keywords: measure of non-compactness; tripled system; Darbo's fixed point theorem; Hyers-Ulam stability
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1. Introduction

Measure of non-compactness (MNC, in short) was initially introduced as a foremost tool to prove generalization of the cantor intersection theorem by Kuratowski [1] in 1930 . In functional analysis, MNC is a function which associates a number to a non-empty and bounded subset of metric spaces in such a way that a compact set gets measure zero and all non-compact sets have measure greater than zero. The MNC depends on the sets that to what extent they are apart from compactness. Sets which are far-away from compactness will have a greater non-zero value of MNC (see [2]). Darbo in 1955 continued the use of Kuratowski MNC for further analysis (see details in [1]). He presented a fixed point theorem (FPT) based on MNC which now in literature is a prominent result known by Darbo's fixed point theorem (DFPT). DFPT is very useful tool in fixed point theory which is obtained by generalizing Schauder's FPT. Also from Banach FPT, DFPT appends on the existence part, that is contraction of condensing operators. An operator which for any set produces such images which are themselves more compact than the considered set is known as condensing or densifying operator. In more broad sense, the properties of condensing operators are similar to the compact operators. MNC has wide range of applications in theory of fixed points and is helpful in investigation of integral, integro-differential, differential and other operator equations. Also the mentioned concept has been used very well to study integral, integro-differential and differential equations of fractional orders in Banach spaces (see [3]).

In last few decades, many researchers obtained various existence results for above mentioned equations by using MNC and other methods (we refer few papers as [4–13]). The authors [4] have examined boundary value problem (BVP) of hybrid sequential fractional integro-differential equation for existence of solution by using Krasnoselskii's FPT. Deep et al. [14] extended Darbo's FPT by using MNC in a Banach space. They used the extended result for the existence of solution to a tripled system of nonlinear equations containing triple integrals. Karakaya et al. [15] investigated the existence result by using MNC for tripled fixed point problem of a class of densifying operators. For the purpose of application they applied the existence result to a tripled system of differential equations.

Fractional order differential equations (FDEs) can be utilized as a powerful tool to model nonlinear problems of real world with more detailed analysis. Coupled systems of FDEs are concerned with interactions of two quantities, which provide interpretation of real world problems of coupled phenomena, such models which describe chaotic behavior, ecological effects, anomalous diffusion and biological phenomena. The idea of coupled systems can be transformed to more generalized the form of tripled systems of FDEs, where interaction occurs between three quantities. The coupled system of railway track is modeled and investigated which can be extended to a model of tripled system if some external influence interact with existing quantities. This idea of railway track tripled system can be generalized by using FDEs (we refer [16]). In the field of bio-mathematics, we can see some more applications of tripled systems of fractional order epidemic models, such as (susceptible-infected-susceptible) SIS and (susceptible-infected-recovered) SIR models with Caputo fractional order derivative (see [17]). On the other hand quadratic perturbations of nonlinear FDEs have attracted much attention in last two decades. The concerned problems are called hybrid FDEs. The fundamental theory on nonlinear hybrid ordinary differential equations was given by Dhage and Lakshmikantham [20]. Since the theory devoted to differential inequalities for hybrid differential equations play important role in the qualitative study of nonlinear differential equations. The tremendous work in this regards has been done by Lakshmikantham and Leela [21]. Inspired from the aforementioned studies, researchers have also considered sequential type hybrid FDEs and their systems for investigating the qualitative theory of existence of solutions under initial and boundary conditions. Significant results have been produced for the mentioned problems by using hybrid fixed point theory. Here we refer some published work for readers on sequential type hybrid FDEs [22–26].

Motivated from the aforementioned work, we investigate the tripled system of hybrid fractional sequential integro-differential equations (HFSIDEs) with non-linear boundary conditions given by

$$
\begin{cases}
{}^{c}_{0}\mathrm{D}^{\theta}_{y}\Big[\dfrac{{}^{c}_{0}\mathrm{D}^{\sigma}_{y}\mu(y)-\sum_{j=1}^{k}I^{\alpha_j}f_{1j}(y,\mu(y))}{g_1(y,\mu(y))}\Big]=h_1(y,\upsilon(y),I^{\gamma}\upsilon(y)),\\
{}^{c}_{0}\mathrm{D}^{\theta}_{y}\Big[\dfrac{{}^{c}_{0}\mathrm{D}^{\sigma}_{y}\upsilon(y)-\sum_{j=1}^{k}I^{\alpha_j}f_{2j}(y,\upsilon(y))}{g_2(y,\upsilon(y))}\Big]=h_2(y,\omega(y),I^{\gamma}\omega(y)),\\
{}^{c}_{0}\mathrm{D}^{\theta}_{y}\Big[\dfrac{{}^{c}_{0}\mathrm{D}^{\sigma}_{y}\omega(y)-\sum_{j=1}^{k}I^{\alpha_j}f_{3j}(y,\omega(y))}{g_3(y,\omega(y))}\Big]=h_3(y,\mu(y),I^{\gamma}\mu(y)),\\
\mu(0)=\Phi_1(\mu(\eta)),\ {}^{c}_{0}\mathrm{D}^{\sigma}_{y}\mu(0)=0,\ \mu(1)=\Phi_2(\mu(\eta)),\\
\upsilon(0)=\varphi_1(\upsilon(\eta)),\ {}^{c}_{0}\mathrm{D}^{\sigma}_{y}\upsilon(0)=0,\ \upsilon(1)=\varphi_2(\upsilon(\eta)),\\
\omega(0)=\psi_1(\omega(\eta)),\ {}^{c}_{0}\mathrm{D}^{\sigma}_{y}\omega(0)=0,\ \omega(1)=\psi_2\omega(\eta),
\end{cases}
\tag{1.1}
$$

where ${}^{c}_{0}\mathrm{D}^{\theta}_{y}$ and ${}^{c}_{0}\mathrm{D}^{\sigma}_{y}$ denote the Caputo derivatives of non-integer orders θ and σ respectively. Further $0 < \theta \leq 1$ and $1 < \sigma \leq 2$. I^{γ} is fractional order Riemann-Liouville integral operator with an order $\gamma > 0$ and I^{α_j} is the Riemann-Liouville fractional integral operator of order $\alpha_j > 0$. Moreover $g_i,\ h_i,\ f_{ij},\ (i = 1, 2, 3),\ (j = 1, 2, ..., k)$, are continuous functions with each $g_i \neq 0$, for all $y \in \mathbb{J} = [0, 1]$. We develop sufficient conditions for the existence of solution to the proposed problem (1.1) by using Darbo's FPT. The concerned results are based on MNC criteria as we have discussed earlier. For some applications of MNC, we refer [27, 28].

On the other hand stability theory is an important aspect of the qualitative analysis. It is important from numerical and optimization point of view. In the existence literature, various concepts for stability theory has been used. Here we recall some important concept like Laypunove, exponential, Mittag-Leffler type stabilities which have been studied for various problems in FDEs. An important concept of stability which was introduced by Ulam in 1940 and explained by Hyers in 1941 for functional equations known as Hyers-Ulam (HU) stability (see [29]). The mentioned concept has been extended very well for various problems in fractional calculus. Sufficient conditions have been established for HU type stability by using the tools of nonlinear functional analysis. Here we refer few remarkable work as [30–33]. Further, the aforementioned concepts of stability has been extended to coupled hybrid FDEs. We refer some useful work as [34–38]. Inspired from the above discussion, we establish necessary and sufficient conditions for the existence HU and generalized (GHU) stability results for the proposed system. For this need, we utilize, some fundamental concept from nonlinear functional analysis.

2. Elementary definitions and results

In this portion, we present some fundamental definitions, results and properties for Caputo fractional derivative, Riemann-Liouville fractional integrals and results of MNC [1, 2, 9, 14, 15, 18, 19, 27, 28], which build up background knowledge for bringing forth the main results.

Definition 2.1. *The Riemann-Liouville integral of fractional order $\delta > 0$ of a function $\mu \in L^1([a, b], \mathbb{R}_+)$ is defined by*

$$I^{\delta}\mu(y) = \frac{1}{\Gamma(\delta)}\int_0^y (y-s)^{\delta-1}\mu(s)ds,$$

provided that the integral on right side converges.

Definition 2.2. *The Caputo fractional derivative with order $\delta > 0$ of a function $\mu \in C[a, b]$ is defined by*

$$\begin{cases} {}^{c}\mathrm{D}^{\delta}\mu(y) = \dfrac{1}{\Gamma(\mathrm{n}-\delta)} \displaystyle\int_0^y \dfrac{\mu^{(\mathrm{n})}(s)}{(y-s)^{\delta-\mathrm{n}+1}} ds, \ \mathrm{n}-1 < \delta < \mathrm{n}, \\ \dfrac{d^{\mathrm{n}}}{dy^{\mathrm{n}}}[\mu(y)], \ \delta = \mathrm{n}. \end{cases}$$

Lemma 2.3. *[19] For any continuous function $\mu \in C(a, b) \cap L[a, b]$, then the solution of FDE*

$${}^{c}\mathrm{D}^{\delta}\mu(y) = h(y), \ \mathrm{n}-1 < \delta \leq \mathrm{n}$$

is given by

$$\mu(y) = I^{\delta}h(y) + k_0 + k_1 y + \cdots + k_{\mathrm{n}-1}y^{\mathrm{n}-1},$$

where k_i for $i = 0, 1, 2, \ldots, \mathrm{n}-1$ are real constants.

Property 2.4. *[9] Let $\delta, \sigma > 0$ and μ be a continuous function, then if I^{θ} and I^{σ} are Riemann-Liouville fractional integral operators of orders δ and σ respectively, then we have*

$$I^{\theta}[I^{\sigma}\mu(y)] = I^{\theta+\sigma}\mu(y),$$

which is known as semi-group property.

Throughout this paper, we assume B to be a Banach space, β_B be the family of bounded subsets of B, $\bar{B}$ be closure of B and $conv(B)$ be closed convex hull of B, then we proceed to the following results.

Definition 2.5. *[1] For the family β_B a mapping*

$$\varrho : \beta_B \to \mathbb{R}_+$$

is known as MNC defined by B, if it hold

(1) $\varrho(M) = 0$ iff M is a precompact set.
(2) $M \subset N \implies \varrho(M) \leq \varrho(N)$.
(3) $\varrho(M) = \varrho(\bar{M})$, for all $M \in \beta_B$.
(4) $\varrho(convM) = \varrho(M)$.
(5) $\varrho(cM + (1-c)N) \leq c\varrho(M) + (1-c)\varrho(N), \ c \in [0, 1]$.
(6) Let $\{M_k\}$ be a sequence of closed subsets of β_B such that $M_{k+1} \subseteq M_k, (k \geq 1)$ and $\lim_{k\to\infty} \varrho(M_k) = 0$, then $M_\infty = \bigcap_{k=1}^{\infty} M_k$ is a non-empty intersection set and M_∞ is pre-compact.

Definition 2.6. *Let B be a Banach space and $\mathcal{T} : B \to B$ be an operator. If for $y \in B$*

$$\mathcal{T}(y) = y,$$

then y is called fixed point of $\mathcal{T}$.

Theorem 2.7. *Schauder FPT [18]. If M is a nonempty, closed, bounded and convex subset of B. Then every continuous compact mapping $\mathcal{T} : M \to M$ has at least one fixed point solution.*

Theorem 2.8. *Darbo's FPT [27]. If M is a nonempty, closed, bounded and convex subset of B. If $\mathcal{T} : M \to M$ is a continuous mapping and if*

$$\varrho(\mathcal{T}N) \leq c\varrho(N),\ c \in [0,1),\ N \subseteq M,$$

then $\mathcal{T}$ has at least one fixed point solution.

The following theorem is an important generalization of Darbo's fixed point theorem 2.8 proved by Aghajani et al. [6].

Theorem 2.9. *If M is a nonempty, closed, bounded and convex subset of B. If $\mathcal{T} : M \to M$ is a continuous mapping and if*

$$\varrho(\mathcal{T}N) \leq \Psi(\varrho(N)),\ N \subseteq M,$$

where $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ is an upper semi-continuous, non-decreasing function and $\forall y \in \mathbb{R}_+,\ \Psi(y) < y$. Then $\mathcal{T}$ has at least one fixed point solution.

Definition 2.10. *[15] A tripled point (μ, υ, ω) is known as a tripled fixed point of a map $\mathcal{T} : M^3 \to M$ if*

$$\mathcal{T}(\mu, \upsilon, \omega) = \mu,\ \mathcal{T}(\upsilon, \mu, \omega) = \upsilon,\ \mathcal{T}(\omega, \upsilon, \mu) = \omega.$$

Theorem 2.11. *[28] Let $\varrho_1, \varrho_2, \ldots, \varrho_n$ be MNCs in the Banach Spaces $B_1, B_2, \ldots, B_n$ respectively. Assume that $\Psi : \mathbb{R}_+^n \to \mathbb{R}_+$ be a convex mapping such that $\Psi(\mu_1, \mu_2, \ldots, \mu_n) = 0$ iff $\mu_1 = \mu_2 = \cdots = \mu_n = 0$, then*

$$\varrho(B) = \Psi(\varrho_1(M_1), \varrho_2(M_2), \ldots, \varrho_n(M_n)),$$

defines a MNC in $B_1 \times B_2 \times \cdots \times B_n$, where $M_1, M_2, \ldots, M_n$ are the natural projections of B on $B_1, B_2, \ldots, B_n$ respectively.

Corollary 2.12. *Let $\varrho_1, \varrho_2, \varrho_3$ be MNCs in Banach Spaces B_1, B_2, B_3 respectively. Assume that $\Psi : \mathbb{R}_+^3 \to \mathbb{R}_+$ be a convex mapping such that $\Psi(\mu_1, \mu_2, \mu_3) = 0$ iff $\mu_1 = \mu_2 = \mu_3 = 0$. Then*

$$\varrho(B) = \Psi(\varrho_1(M_1), \varrho_2(M_2), \varrho_3(M_3)),$$

defines a MNC in $B_1 \times B_2 \times B_3$, where M_1, M_2, M_3 are the natural projections of B on B_1, B_2, B_3 respectively.

Example 2.13. *Let ϱ be a MNC in a Banach Space B and $\Psi(\mu, \upsilon, \omega) = \max\{\mu, \upsilon, \omega\}$, for $(\mu, \upsilon, \omega) \in \mathbb{R}_+^3$. Clearly Ψ is convex and $\max\{\mu, \upsilon, \omega\} = 0$ iff $\mu = \upsilon = \omega = 0$. We see that the conditions of Corollary 2.12 are satisfied. Hence $\varrho(B) = \max(\varrho_1(M_1), \varrho_2(M_2), \varrho_3(M_3))$ is a MNC in $B_1 \times B_2 \times B_3$, where M_1, M_2, M_3 are the natural projections of B on B_1, B_2, B_3 respectively.*

Example 2.14. *Let ϱ be a MNC in a Banach Space B and $\Psi(\mu, \upsilon, \omega) = \mu + \upsilon + \omega$, for $(\mu, \upsilon, \omega) \in \mathbb{R}_+^3$. Clearly Ψ is convex and $\mu + \upsilon + \omega = 0$ iff $\mu = \upsilon = \omega = 0$. We see that the conditions of Corollary 2.12 are satisfied. Hence $\varrho(B) = \varrho_1(M_1) + \varrho_2(M_2) + \varrho_3(M_3)$ is a MNC in $B_1 \times B_2 \times B_3$, where M_1, M_2, M_3 are the natural projections of B on B_1, B_2, B_3 respectively.*

Theorem 2.15. *[14] If M is a nonempty, closed, bounded and convex subset of B. If $\mathcal{T} : M^3 \to M$ is a continuous mapping and if*

$$\varrho(\mathcal{T}(M_1 \times M_2 \times M_3)) \leq \Psi(\varrho(M_1), \varrho(M_2), \varrho(M_2)),\ \forall\ M_1, M_2, M_3 \subseteq M, \tag{2.1}$$

where $\Psi : \mathbb{R}_+^3 \to \mathbb{R}_+$ is a non-decreasing, upper semi-continuous function. Then $\mathcal{T}$ has at least one tripled fixed point solution.

3. Main results

In this section, first we derive extension of the required theorem for existence theory of solution to the proposed problem. We prove our main result by using Theorem 2.15. Secondly, we evaluate an equivalent integral form of the system of HFSIDEs (1.1) with boundary conditions. The last aim of the section is to investigate existence of solution of system of HFSIDEs (1.1) by using our newly obtained fixed point theorem.

Theorem 3.1. *If M is a nonempty subset of a Banach space B such that M is closed, bounded and convex and $\mathcal{T}_i : M \times M \times M \to M,\ i = 1, 2, 3$ are continuous operators, if*

$$\|\mathcal{T}_i(\mu, \upsilon, \omega) - \mathcal{T}_i(\bar{\mu}, \bar{\upsilon}, \bar{\omega})\|_\infty \leq \Psi_i(\|\mu - \bar{\mu}\|_\infty, \|\upsilon - \bar{\upsilon}\|_\infty, \|\omega - \bar{\omega}\|_\infty), \tag{3.1}$$

for all $\mu, \upsilon, \omega, \bar{\mu}, \bar{\upsilon}, \bar{\omega} \in M$, where each $\Psi_i : \mathbb{R}_+^3 \to \mathbb{R}_+$ is an upper semi-continuous, non-decreasing function. Then $\mathcal{T}_i$ has atleast one tripled fixed point.

Proof. To show that the operator $\mathcal{T}_i$ has a fixed point, we prove that $\mathcal{T}_i : M \times M \times M \to M$ satisfy the condition (2.1) of Theorem 2.15. For this purpose, we define MNC in a Banach space B for a fixed positive y on β_B as

$$\varrho(M) = W_0(M) + \limsup_{y\to\infty} diam M(y), \tag{3.2}$$

where $M(y) = \{\mu(y) : \mu \in M\},\ diam M(y) = \sup\{|\mu(y) - \bar{\mu}(y)| : \mu, \bar{\mu} \in M\}$ and

$$W_0(M) = \lim_{r\to\infty} W_0^r(M),$$

$$W_0^r(M) = \lim_{\epsilon\to 0} W^r(M, \epsilon),$$

$$W^r(M, \epsilon) = \sup\{W^r(\mu, \epsilon) : \mu \in M\},$$

$$W^r(\mu, \epsilon) = \sup\{|\mu(y) - \mu(s)| : y, s \in [0, r], |y - s| \leq \epsilon\},\ r > 0,$$

where for $\mu \in M$ and $\epsilon > 0$, $W^r(\mu, \epsilon)$ is the modulus of continuity of μ on the closed compact interval $[0, r]$. Now by Eq (3.1), we have

$$\begin{aligned} &\|\mathcal{T}_i(\mu(y), \upsilon(y), \omega(y)) - \mathcal{T}_i(\mu(s), \upsilon(s), \omega(s))\|_\infty \\ \leq\ &\Psi_j(\|\mu(y) - \mu(s)\|_\infty, \|\upsilon(y) - \upsilon(s)\|_\infty, \|\omega(y) - \omega(s)\|_\infty), \end{aligned}$$

for which if we take the supremum and use the fact Ψ_i is non-decreasing, we get

$$W^r(\mathcal{T}_i(\mu, \upsilon, \omega), \epsilon) \leq \Psi_i(W^r(\mu, \epsilon), W^r(\upsilon, \epsilon), W^r(\omega, \epsilon)),$$

which gives

$$W_0(\mathcal{T}_i(M_1 \times M_2 \times M_3)) \leq \Psi_i(W_0(M_1), W_0(M_2), W_0(M3)). \tag{3.3}$$

Since μ, υ and ω are arbitrary and Ψ_i is non-decreasing, so

$$diam\mathcal{T}_i(M_1 \times M_2 \times M_3)(y) \leq \Psi_i(diam M_1(y), diam M_2(y), diam M_3(y)).$$

Moreover, M_1, M_2 and M_3 are subspaces of M, therefore

$$\begin{aligned}\limsup_{y\to\infty} diam\mathcal{T}_i(M_1\times M_2\times M_3)(y) &\le \limsup_{y\to\infty}\Psi_i(diamM_1(y), diamM_2(y), diamM_3(y))\\ &\le \Psi_i(\limsup_{y\to\infty} diamM_1(y), \limsup_{y\to\infty} diamM_2(y), \limsup_{y\to\infty} diamM_3(y)).\end{aligned}$$

Now using the last inequality with (3.3), then by Eq (3.2) we have

$$\varrho(\mathcal{T}_i(M_1\times M_2\times M_3)) \le \Psi_i(\varrho(M_1), \varrho(M_2), \varrho(M_2)),\ \forall\ M_1, M_2, M_3 \subseteq M.$$

Hence Theorem 2.15 is satisfied, consequently $\mathcal{T}_i$ has a tripled fixed point solution. □

Theorem 3.2. *Let $0<\theta\le 1, 1<\sigma\le 2, \gamma>0$, and the function $h_1 : \mathbb{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is θ-time integrable. Then for the BVP of HFSIDE*

$$\begin{cases} {}_0^c\mathrm{D}_y^\theta\Big[\dfrac{{}_0^c\mathrm{D}_y^\sigma\mu(y)-\sum_{j=1}^r I^{\alpha_j}f_{1j}(y,\mu(y))}{g_1(y,\mu(y))}\Big] = h_1(y,\upsilon(y), I^\gamma\upsilon(y)),\\ \mu(0)=\Phi_1(\mu(\eta)),\ {}_0^c\mathrm{D}_y^\sigma\mu(0)=0,\ \mu(1)=\Phi_2(\mu(\eta)). \end{cases} \tag{3.4}$$

An equivalent integral form is

$$\begin{aligned}\mu(y) = &\int_0^1 \mathcal{G}(y,s)g_1(s,\mu(s))\int_0^s (s-r)^{\theta-1}h_1(r,\upsilon(r),I^\gamma\upsilon(r))drds + \sum_{j=1}^k\int_0^1 \mathcal{G}_j(y,s)f_{1j}(s,\mu(s))ds\\ &+(1-y)\Phi_1(\mu(\eta)) + y\Phi_2(\mu(\eta)),\end{aligned} \tag{3.5}$$

where $\mathcal{G}(y,s)$ and $\mathcal{G}_j(y,s)$ are the Green's functions given by

$$\mathcal{G}(y,s) = \frac{-1}{\Gamma(\sigma)\Gamma(\theta)}\begin{cases} y(1-s)^{\sigma-1}, & y\le s,\\ y(1-s)^{\sigma-1}-(y-s)^{\sigma-1}, & y>s,\end{cases} \tag{3.6}$$

and

$$\mathcal{G}_j(y,s) = \frac{-1}{\Gamma(\alpha_j+\sigma)}\begin{cases} y(1-s)^{\alpha_j+\sigma-1}, & y\le s,\\ y(1-s)^{\alpha_j+\sigma-1}-(y-s)^{\alpha_j+\sigma-1}, & y>s.\end{cases} \tag{3.7}$$

Proof. Employing the Riemann-Liouville integral operator of fractional order θ to hybrid sequential fractional integro-differential equation (3.4) and operating Lemma 2.3, we get

$$\frac{{}_0^c\mathrm{D}_y^\sigma\mu(y)-\sum_{j=1}^k I^{\alpha_j}f_{1j}(y,\mu(y))}{g_1(y,\mu(y))} = I^\theta h_1(y,\upsilon(y),I^\gamma\upsilon(y)) + a_0. \tag{3.8}$$

In view of the initial condition ${}_0^c\mathrm{D}_y^\sigma\mu(0)=0$, we have $a_0=0$ and then Eq (3.8) becomes

$${}_0^c\mathrm{D}_y^\sigma\mu(y) = g_1(y,\mu(y))\int_0^y \frac{(y-s)^{\theta-1}}{\Gamma(\theta)}h_1(s,\upsilon(s),I^\gamma\upsilon(s))ds + \sum_{j=1}^k I^{\alpha_j}f_{1j}(y,\mu(y)). \tag{3.9}$$

Now the employment of the Riemann-Liouville integral operator of fractional order σ to Eq (3.9) with using Lemma 2.3 and semi-group property (2.4), we get

$$\mu(y) = \int_0^y \frac{(y-s)^{\sigma-1}}{\Gamma(\sigma)} g_1(s,\mu(s)) \int_0^s \frac{(s-r)^{\theta-1}}{\Gamma(\theta)} h_1(r,\upsilon(r), I^\gamma \upsilon(r)) dr ds \\ + \sum_{j=1}^{k} I^{\alpha_j+\sigma} f_{1j}(y,\mu(y)) + a_1 + a_2 y. \tag{3.10}$$

The initial condition $\mu(0) = \Phi_1(\mu(\eta))$ gives $a_1 = \Phi_1(\mu(\eta))$ and boundary condition $\mu(1) = \Phi_2(\mu(\eta))$ together with $a_1 = \Phi_1(\mu(\eta))$ gives

$$a_2 = -\int_0^1 \frac{(y-s)^{\sigma-1}}{\Gamma(\sigma)} g_1(s,\mu(s)) \int_0^s \frac{(s-r)^{\theta-1}}{\Gamma(\theta)} h_1(r,\upsilon(r), I^\gamma \upsilon(r)) dr ds \\ - \sum_{j=1}^{k} I^{\alpha_j+\sigma} f_{1j}(1,\mu(1)) - \Phi_1(\mu(\eta)) + \Phi_2(\mu(\eta)).$$

For values of a_1 and a_2, Eq (3.10) takes the form

$$\mu(y) = \int_0^y \frac{(y-s)^{\sigma-1}}{\Gamma(\sigma)} g_1(s,\mu(s)) \int_0^s \frac{(s-r)^{\theta-1}}{\Gamma(\theta)} h_1(r,\upsilon(r), I^\gamma \upsilon(r)) dr ds + \sum_{j=1}^{k} I^{\alpha_j+\sigma} f_{1j}(y,\mu(y)) \\ - y\int_0^1 \frac{(1-s)^{\sigma-1}}{\Gamma(\sigma)} g_1(s,\mu(s)) \int_0^s \frac{(s-r)^{\theta-1}}{\Gamma(\theta)} h_1(r,\upsilon(r), I^\gamma \upsilon(r)) dr ds - y\sum_{j=1}^{k} I^{\alpha_j+\sigma} f_{1j}(1,\mu(1)) \\ + (1-y)\Phi_1(\mu(\eta)) + y\Phi_2(\mu(\eta)).$$

Thus,

$$\mu(y) = \int_0^1 \mathcal{G}(y,s) g_1(s,\mu(s)) \int_0^s (s-r)^{\theta-1} h_1(r,\upsilon(r), I^\gamma \upsilon(r)) dr ds \\ + \sum_{j=1}^{k} \int_0^1 \mathcal{G}_j(y,s) f_{1j}(s,\mu(s)) ds + (1-y)\Phi_1(\mu(\eta)) + y\Phi_2(\mu(\eta)),$$

where $\mathcal{G}(y,s)$ and $\mathcal{G}_j(y,s)$ are Green's functions given as in (3.6) and (3.7). Hence we obtained the desired integral form of the problem (3.4). □

In view of Theorem 3.2, the triple (μ,υ,ω) is the solution of the tripled system (1.1) if and only if (μ,υ,ω) satisfies the following tripled system of integral equations

$$
\begin{cases}
\mu(y) = \displaystyle\int_0^1 \mathcal{G}(y,s) g_1(s,\mu(s)) \int_0^s (s-r)^{\theta-1} h_1(r, \upsilon(r), I^\gamma \upsilon(r)) dr ds \\
\qquad + \displaystyle\sum_{j=1}^{k} \int_0^1 \mathcal{G}_j(y,s) f_{1j}(s,\mu(s)) ds + (1-y)\Phi_1(\mu(\eta)) + y\Phi_2(\mu(\eta)), \\
\upsilon(y) = \displaystyle\int_0^1 \mathcal{G}(y,s) g_2(s,\upsilon(s)) \int_0^s (s-r)^{\theta-1} h_2(r, \omega(r), I^\gamma \omega(r)) dr ds \\
\qquad + \displaystyle\sum_{j=1}^{k} \int_0^1 \mathcal{G}_j(y,s) f_{2j}(s,\upsilon(s)) ds + (1-y)\varphi_1(\upsilon(\eta)) + y\varphi_2(\upsilon(\eta)), \\
\omega(y) = \displaystyle\int_0^1 \mathcal{G}(y,s) g_3(s,\omega(s)) \int_0^s (s-r)^{\theta-1} h_3(r, \mu(r), I^\gamma \mu(r)) dr ds \\
\qquad + \displaystyle\sum_{j=1}^{k} \int_0^1 \mathcal{G}_j(y,s) f_3 j(s,\omega(s)) ds + (1-y)\psi_1(\omega(\eta)) + y\psi_2(\omega(\eta)).
\end{cases}
\tag{3.11}
$$

For developing result concerning existence of solution, we put forth the following hypothesis.

(H_1) The functions $f_{ij} \in C(\mathbb{J}\times B, B)$, $g_i \in C(\mathbb{J}\times B, B\backslash\{0\})$ and $h_i \in C(\mathbb{J}\times B^2, B)$ are continuous and there exist positive functions $F_{ij}(y)$, $\Theta_i(y)$, and $\lambda_i(y)$ with bounds $\|F_{ij}\|$, $\|\Theta_i\|$, and $\|\lambda_i\|$ respectively, for $i = 1, 2, 3$, such that

$$
\begin{aligned}
|f_{1j}(y,\mu(y)) - f_{1j}(y,\bar{\mu}(y)| &\le F_{1j}(y)|\mu(y) - \bar{\mu}(y)|, \\
|f_{2j}(y,\upsilon(y)) - f_{2j}(y,\bar{\upsilon}(y)| &\le F_{2j}(y)|\upsilon(y) - \bar{\upsilon}(y)|, \\
|f_{3j}(y,\omega(y)) - f_{3j}(y,\bar{\omega}(y)| &\le F_{3j}(y)|\omega(y) - \bar{\omega}(y)|, \\
|g_1(y,\mu(y)) - g_1(y,\bar{\mu}(y)| &\le \Theta_1(y)|\mu(y) - \bar{\mu}(y)|, \\
|g_2(y,\upsilon(y)) - g_2(y,\bar{\upsilon}(y)| &\le \Theta_2(y)|\upsilon(y) - \bar{\upsilon}(y)|, \\
|g_3(y,\omega(y)) - g_3(y,\bar{\omega}(y)| &\le \Theta_3(y)|\omega(y) - \bar{\omega}(y)|, \\
|h_1(y,\upsilon(y),\bar{\upsilon}(y)) - h_1(y,\upsilon^*(y),\bar{\upsilon^*}(y)| &\le \lambda_1(y)(|\upsilon(y) - \upsilon^*(y)| + |\bar{\upsilon}(y) - \bar{\upsilon^*}(y)|) \\
|h_2(y,\omega(y),\bar{\omega}(y)) - h_2(y,\omega^*(y),\bar{\omega^*}(y)| &\le \lambda_2(y)(|\omega(y) - \omega^*(y)| + |\bar{\omega}(y) - \bar{\omega^*}(y)|), \\
|h_3(y,\mu(y),\bar{\mu}(y)) - h_3(y,\mu^*(y),\bar{\mu^*}(y)| &\le \lambda_3(y)(|\mu(y) - \mu^*(y)| + |\bar{\mu}(y) - \bar{\mu^*}(y)|),
\end{aligned}
$$

for $y \in \mathbb{J}$ and $\mu, \upsilon, \omega,\ \bar{\mu}, \bar{\upsilon}, \bar{\omega} \in B$.

(H_2) For $j = 1, 2, \ldots, k$, $|f_{1j}(y,\mu(y))| \le \phi_{1j}(y)$, $|f_{2j}(y,\upsilon(y))| \le \phi_{2j}(y)$, $|f_{3j}(y,\omega(y))| \le \phi_{3j}(y)$, where $\phi_{ij}(y) \in C(\mathbb{J}, \mathbb{R}_+)$ for $i = 1, 2, 3$.

$$|g_1(y,\mu(y))| \le \chi_1(y), |g_2(y,\upsilon(y))| \le \chi_2(y),\ |g_3(y,\omega(y))| \le \chi_3(y),\ |h_1(y,\upsilon(y),\bar{\upsilon}(y)| \le \Omega_1(y),$$

$$|h_2(y,\omega(y),\bar{\omega}(y)| \le \Omega_2(y), |h_3(y,\mu(y),\bar{\mu}(y)| \le \Omega_3(y),\ |\Phi_1(\mu(y))| \le \beta_1(y), |\Phi_2(\mu(y))| \le \rho_1(y),$$

$$|\varphi_1(\upsilon(y))| \le \beta_2(y), |\varphi_2(\upsilon(y))| \le \rho_2(y), |\psi_1(\omega(y))| \le \beta_3(y),\ |\psi_2(\mu(y))| \le \rho_3(y),$$

where $\Omega_i(y), \chi_i(y), \beta_i(y), \rho_i(y) \in C(\mathbb{J}, \mathbb{R}_+)$ for $i = 1, 2, 3$.

(H_3) For $i = 1,2,3$, there exists $r_i > 0$, such that

$$\frac{\|\chi_i\|\|\Omega_i\|}{\Gamma(\sigma+1)\Gamma(\theta+1)} + \sum_{j=1}^{k} \frac{\|\phi_{ij}\|}{\Gamma(\alpha_j+\sigma+1)} + \|\beta_i\| + \|\rho_i\| \leq r_i.$$

(H_4) For $\mu, \upsilon, \omega \in [0,1]$, there exists $L_{1i}, L_{2i}, L_{3i} \in [0,1)$, such that

$$\Phi_i(y) \leq L_{1i} y,$$
$$\varphi_i(y) \leq L_{2i} y,$$
$$\psi_i(y) \leq L_{3i} y,$$

where $i = 1,2$.

Now as an application we use our developed existence result of fixed point solution in Theorem 3.1 for the tripled system BVP of HFSIDEs (1.1).

Theorem 3.3. *Let the hypothesis (H_1)–(H_4) holds, Then there exists at least one triple point solution for the BVP (1.1), in the space $\mathbb{E} = B \times B \times B$.*

Proof. First we set $\sup_{y\in\mathbb{J}} |F_{ij}(y)| = \|F_{ij}\|$, $\sup_{y\in\mathbb{J}} |\phi_j(y)| = \|\phi_j\|$, $j = 1,2,\ldots,k$, $\sup_{y\in\mathbb{J}} |\Theta_i(y)| = \|\Theta_i\|$, $\sup_{y\in\mathbb{J}} |\lambda_i(t)| = \|\lambda_i\|$, $\sup_{y\in\mathbb{J}} |\Omega_i(y)| = \|\Omega_i\|$ and $\sup_{y\in\mathbb{J}} |\chi_i(y)| = \|\chi_i\|$. where for all the supremum norms $i = 1,2,3$.

Now we consider a product space $\mathbb{E} = B \times B \times B$ and define a continuous operator $\mathcal{T}_i : \mathbb{E} \longrightarrow B$, i = 1,2,3, such that

$$\begin{aligned}
\mathcal{T}_1(\mu(y), \upsilon(y), \omega(y)) &= \int_0^1 \mathcal{G}(y,s) g_1(s, \mu(s)) \int_0^s (s-r)^{\theta-1} h_1(r, \upsilon(r), I^\gamma \upsilon(r)) dr ds \\
&+ \sum_{j=1}^{k} \int_0^1 \mathcal{G}_j(y,s) f_{1j}(s, \mu(s)) ds + (1-y)\Phi_1(\mu(\eta)) + y\Phi_2(\mu(\eta)),
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_2(\mu(y), \upsilon(y), \omega(y)) &= \int_0^1 \mathcal{G}(y,s) g_2(s, \upsilon(s)) \int_0^s (s-r)^{\theta-1} h_2(r, \omega(r), I^\gamma \omega(r)) dr ds \\
&+ \sum_{j=1}^{k} \int_0^1 \mathcal{G}_j(y,s) f_{2j}(s, \upsilon(s)) ds + (1-y)\varphi_1(\upsilon(\eta)) + y\varphi_2(\upsilon(\eta)),
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_3(\mu(y), \upsilon(y), \omega(y)) &= \int_0^1 \mathcal{G}(y,s) g_3(s, \omega(s)) \int_0^s (s-r)^{\theta-1} h_3(r, \mu(r), I^\gamma \mu(r)) dr ds \\
&+ \sum_{j=1}^{k} \int_0^1 \mathcal{G}_j(y,s) f_{3j}(s, \omega(s)) ds + (1-y)\psi_1(\omega(\eta)) + y\psi_2(\omega(\eta)).
\end{aligned}$$

Now solution of the problem (1.1) is the fixed point of the operator $\mathcal{T}_i$. Since g_i, h_i and f_{ij}, for $i = 1,2,3$ and $j = 1,2,\ldots,k$, are continuous functions, so the operators $\mathcal{T}_i$ are continuous. To ensure that fixed point of $\mathcal{T}_i$ exist, we verify conditions of Theorem 3.1.

Accordingly, for $\mu, \upsilon, \omega \in B$,

$$\begin{aligned}
\|\mathcal{T}_1(\mu(y),\upsilon(y),\omega(y))\| &= \Big\| \int_0^1 \mathcal{G}(y,s)g_1(s,\mu(s)) \int_0^s (s-r)^{\theta-1} h_1(r,\upsilon(r),I^\gamma \upsilon(r))drds \\
&+ \sum_{j=1}^k \int_0^1 \mathcal{G}_j(y,s) f_{1j}(s,\mu(s))ds + (1-y)\Phi_1(\mu(\eta)) + y\Phi_2(\mu(\eta))\Big\| \\
&\leq \int_0^1 \mathcal{G}(y,s)\|g_1(s,\mu(s))\| \int_0^s (s-r)^{\theta-1}\|h_1(r,\upsilon(r),I^\gamma \upsilon(r))\|drds \\
&+ \sum_{j=1}^k \int_0^1 \mathcal{G}_j(y,s)\|f_{1j}(s,\mu(s))\|ds + \|\Phi_1(\mu(\eta))\| + \|\Phi_2(\mu(\eta))\| \\
&\leq \frac{\|\chi_1\|\|\Omega_1\|}{\Gamma(\sigma+1)\Gamma(\theta+1)} + \sum_{j=1}^k \frac{\|\phi_{1j}\|}{\Gamma(\alpha_j+\sigma+1)} + \|\beta_1\| + \|\rho_1\|.
\end{aligned}$$

Similarly, we can show that

$$\|\mathcal{T}_2(\mu(y),\upsilon(y),\omega(y))\| \leq \frac{\|\chi_2\|\|\Omega_2\|}{\Gamma(\sigma+1)\Gamma(\theta+1)} + \sum_{j=1}^k \frac{\|\phi_{2j}\|}{\Gamma(\alpha_j+\sigma+1)} + \|\beta_2\| + \|\rho_2\|$$

and

$$\|\mathcal{T}_3(\mu(y),\upsilon(y),\omega(y))\| \leq \frac{\|\chi_3\|\|\Omega_3\|}{\Gamma(\sigma+1)\Gamma(\theta+1)} + \sum_{j=1}^k \frac{\|\phi_{3j}\|}{\Gamma(\alpha_j+\sigma+1)} + \|\beta_3\| + \|\rho_3\|.$$

Thus by hypothesis (H_3), for $i = 1,2,3$, we have

$$\|\mathcal{T}_i(\mu(y),\upsilon(y),\omega(y))\| \leq r_i,$$

so we get $\mathcal{T}_i(B,B,B) \subset B$. Furthermore

$$\begin{aligned}
&\|\mathcal{T}_1(\mu(y),\upsilon(y),\omega(y)) - \mathcal{T}_1(\bar{\mu}(y),\bar{\upsilon}(y),\bar{\omega}(y))\| \\
=& \Big\| \int_0^1 \mathcal{G}(y,s)g_1(s,\mu(s)) \int_0^s (s-r)^{\theta-1} h_1(r,\upsilon(r),I^\gamma \upsilon(r))drds + \sum_{j=1}^k \int_0^1 \mathcal{G}_j(y,s) f_{1j}(s,\mu(s))ds \\
+& (1-y)\Phi_1(\mu(\eta)) + y\Phi_2(\mu(\eta)) - \int_0^1 \mathcal{G}(y,s)g_1(s,\bar{\mu}(s)) \int_0^s (s-r)^{\theta-1} h_1(r,\bar{\upsilon}(r),I^\gamma \bar{\upsilon}(r))drds \\
-& \sum_{j=1}^k \int_0^1 \mathcal{G}_j(y,s) f_{1j}(s,\bar{\mu}(s))ds - (1-y)\Phi_1(\bar{\mu}(\eta)) - y\Phi_2(\bar{\mu}(\eta))\Big\| \\
=& \Big\| \int_0^1 \mathcal{G}(y,s)g_1(s,\mu(s)) \int_0^s (s-r)^{\theta-1} h_1(r,\upsilon(r),I^\gamma \upsilon(r))drds + \sum_{j=1}^k \int_0^1 \mathcal{G}_j(y,s) f_{1j}(s,\mu(s))ds \\
+& (1-y)\Phi_1(\mu(\eta)) + y\Phi_2(\mu(\eta)) - \int_0^1 \mathcal{G}(y,s)g_1(s,\bar{\mu}(s)) \int_0^s (s-r)^{\theta-1} h_1(r,\upsilon(r),I^\gamma \upsilon(r))drds \\
+& \int_0^1 \mathcal{G}(y,s)g_1(s,\bar{\mu}(s)) \int_0^s (s-r)^{\theta-1} h_1(r,\upsilon(r),I^\gamma \upsilon(r))drds
\end{aligned}$$

$$
\begin{aligned}
- & \int_0^1 \mathcal{G}(y,s)g_1(s,\bar{\mu}(s))\int_0^s (s-r)^{\theta-1}h_1(r,\bar{\upsilon}(r),I^{\gamma}\bar{\upsilon}(r))drds \\
- & \sum_{j=1}^{k}\int_0^1 \mathcal{G}_j(y,s)f_{1j}(s,\bar{\mu}(s))ds-(1-y)\Phi_1(\bar{\mu}(\eta))-y\Phi_2(\bar{\mu}(\eta))\Big\| \\
\leq & \int_0^1 \|\mathcal{G}(y,s)\|\|g_1(s,\mu(s))-g_1(s,\bar{\mu(s)})\|\int_0^s (s-r)^{\theta-1}\|h_1(r,\upsilon(r),I^{\gamma}\upsilon(r))\|drds \\
+ & \int_0^1 \|\mathcal{G}(y,s)\|\|g_1(s,\mu(s))\|\int_0^s (s-r)^{\theta-1}\|h_1(r,\upsilon(r),I^{\gamma}\upsilon(r))-h_1(r,\bar{\upsilon}(r),I^{\gamma}\bar{\upsilon}(r))\|drds \\
+ & \sum_{j=1}^{k}\int_0^1 \|\mathcal{G}_j(y,s)\|\|f_{1j}(s,\mu(s))-f_{1j}(s,\bar{\mu}(s))\|ds+\|\Phi_1(\mu(\eta))-\Phi_1(\bar{\mu}(\eta))\| \\
+ & \|\Phi_2(\mu(\eta))-\Phi_2(\bar{\mu}(\eta))\|,
\end{aligned}
$$

which by using hypothesis (H_1)–(H_4) yields

$$
\begin{aligned}
& \|\mathcal{T}_1(\mu(y),\upsilon(y),\omega(y))-\mathcal{T}_1(\bar{\mu}(y),\bar{\upsilon}(y),\bar{\omega}(y))\|_\infty \\
\leq & \left(\frac{\|\Omega_1\|\|\Theta_1\|}{\Gamma(\sigma+1)\Gamma(\theta+1)}+\sum_{j=1}^{k}\frac{\|F_{1j}\|}{\Gamma(\alpha_j+\sigma+1)}+L_{11}+L_{12}\right)\|\mu(y)-\bar{\mu}(y)\|_\infty \\
+ & \frac{\|\chi_1\|\|\lambda_1\|(1+\Gamma(\gamma+1))}{\Gamma(\sigma+1)\Gamma(\theta+1)\Gamma(\gamma+1)}\|\upsilon(y)-\bar{\upsilon}(y)\|_\infty \\
\leq & \Psi_1(\|\mu(y)-\bar{\mu}(y)\|_\infty,\|\upsilon(y)-\bar{\upsilon}(y)\|_\infty,\|\omega(y)-\bar{\omega}(y)\|_\infty).
\end{aligned}
$$

Similarly we can deduce the second inequality as

$$
\begin{aligned}
& \|\mathcal{T}_2(\mu(y),\upsilon(y),\omega(y))-\mathcal{T}_2(\bar{\mu}(y),\bar{\upsilon}(y),\bar{\omega}(y))\| \\
= & \Big\|\int_0^1 \mathcal{G}(y,s)g_21(s,\mu(s))\int_0^s (s-r)^{\theta-1}h_2(r,\upsilon(r),I^{\gamma}\upsilon(r))drds+\sum_{j=1}^{k}\int_0^1 \mathcal{G}_j(y,s)f_{2j}(s,\mu(s))ds \\
+ & (1-y)\varphi_1(\mu(\eta))+y\varphi_2(\mu(\eta))-\int_0^1 \mathcal{G}(y,s)g_2(s,\bar{\mu}(s))\int_0^s (s-r)^{\theta-1}h_2(r,\bar{\upsilon}(r),I^{\gamma}\bar{\upsilon}(r))drds \\
- & \sum_{j=1}^{k}\int_0^1 \mathcal{G}_j(y,s)f_{2j}(s,\bar{\mu}(s))ds-(1-y)\varphi_2(\bar{\mu}(\eta))-y\varphi_2(\bar{\mu}(\eta))\Big\| \\
= & \Big\|\int_0^1 \mathcal{G}(y,s)g_2(s,\mu(s))\int_0^s (s-r)^{\theta-1}h_2(r,\upsilon(r),I^{\gamma}\upsilon(r))drds+\sum_{j=1}^{k}\int_0^1 \mathcal{G}_j(y,s)f_{2j}(s,\mu(s))ds \\
+ & (1-y)\varphi_1(\mu(\eta))+y\varphi_2(\mu(\eta))-\int_0^1 \mathcal{G}(y,s)g_2(s,\bar{\mu}(s))\int_0^s (s-r)^{\theta-1}h_2(r,\upsilon(r),I^{\gamma}\upsilon(r))drds \\
+ & \int_0^1 \mathcal{G}(y,s)g_2(s,\bar{\mu}(s))\int_0^s (s-r)^{\theta-1}h_2(r,\upsilon(r),I^{\gamma}\upsilon(r))drds \\
- & \int_0^1 \mathcal{G}(y,s)g_2(s,\bar{\mu}(s))\int_0^s (s-r)^{\theta-1}h_2(r,\bar{\upsilon}(r),I^{\gamma}\bar{\upsilon}(r))drds
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{j=1}^{k} \int_0^1 \mathcal{G}_j(y,s) f_{2j}(s,\bar{\mu}(s))ds - (1-y)\varphi_1(\bar{\mu}(\eta)) - y\varphi_2(\bar{\mu}(\eta))\Big\| \\
&\leq \int_0^1 \|\mathcal{G}(y,s)\| \|g_2(s,\mu(s)) - g_2(s,\bar{\mu(s)})\| \int_0^s (s-r)^{\theta-1} \|h_2(r,\upsilon(r), I^{\gamma}\upsilon(r))\| drds \\
&+ \int_0^1 \|\mathcal{G}(y,s)\| \|g_2(s,\mu(s))\| \int_0^s (s-r)^{\theta-1} \|h_2(r,\upsilon(r), I^{\gamma}\upsilon(r)) - h_2(r,\bar{\upsilon}(r), I^{\gamma}\bar{\upsilon}(r))\| drds \\
&+ \sum_{j=1}^{k} \int_0^1 \|\mathcal{G}_j(y,s)\| \|f_{2j}(s,\mu(s)) - f_{2j}(s,\bar{\mu}(s))\| ds + \|\varphi_1(\mu(\eta)) - \varphi_1(\bar{\mu}(\eta))\| \\
&+ \|\varphi_2(\mu(\eta)) - \varphi_2(\bar{\mu}(\eta))\|,
\end{aligned}
$$

on using hypothesis (H_1)–(H_4), we have

$$
\begin{aligned}
& \|\mathcal{T}_2(\mu(y),\upsilon(y),\omega(y)) - \mathcal{T}_2(\bar{\mu}(y),\bar{\upsilon}(y),\bar{\omega}(y))\|_\infty \\
\leq & \left(\frac{\|\Omega_2\|\|\Theta_2\|}{\Gamma(\sigma+1)\Gamma(\theta+1)} + \sum_{j=1}^{k} \frac{\|F_{2j}\|}{\Gamma(\alpha_j+\sigma+1)} + L_{21} + L_{22} \right) \|\upsilon(y) - \bar{\upsilon}(y)\|_\infty \\
+ & \frac{\|\chi_2\|\|\lambda_2\|(1+\Gamma(\gamma+1))}{\Gamma(\sigma+1)\Gamma(\theta+1)\Gamma(\gamma+1)} \|\omega(y) - \bar{\omega}(y)\|_\infty \\
\leq & \Psi_2(\|\mu(y) - \bar{\mu}(y)\|_\infty, \|\upsilon(y) - \bar{\upsilon}(y)\|_\infty, \|\omega(y) - \bar{\omega}(y)\|_\infty).
\end{aligned}
$$

On same fashion, we can deduce the third inequality as given by

$$
\begin{aligned}
& \|\mathcal{T}_3(\mu(y),\upsilon(y),\omega(y)) - \mathcal{T}_3(\bar{\mu}(y),\bar{\upsilon}(y),\bar{\omega}(y))\|_\infty \\
\leq & \left(\frac{\|\Omega_3\|\|\Theta_3\|}{\Gamma(\sigma+1)\Gamma(\theta+1)} + \sum_{j=1}^{k} \frac{\|F_{3j}\|}{\Gamma(\alpha_j+\sigma+1)} + L_{31} + L_{32} \right) \|\omega(y) - \bar{\omega}(y)\|_\infty \\
+ & \frac{\|\chi_3\|\|\lambda_3\|(1+\Gamma(\gamma+1))}{\Gamma(\sigma+1)\Gamma(\theta+1)\Gamma(\gamma+1)} \|\mu(y) - \bar{\mu}(y)\|_\infty \\
\leq & \Psi_3(\|\mu(y) - \bar{\mu}(y)\|_\infty, \|\upsilon(y) - \bar{\upsilon}(y)\|_\infty, \|\omega(y) - \bar{\omega}(y)\|_\infty).
\end{aligned}
$$

Hence

$$
\|\mathcal{T}_i(\mu,\upsilon,\omega) - \mathcal{T}_i(\bar{\mu},\bar{\upsilon},\bar{\omega})\|_\infty \leq \Psi_i(\|\mu - \bar{\mu}\|_\infty, \|\upsilon - \bar{\upsilon}\|_\infty, \|\omega - \bar{\omega}\|_\infty).
$$

Thus all the conditions of Theorem 3.1 are satisfied, consequently the fixed point problem has one tripled fixed point in E. Hence inview of this result, our proposed problem (1.1) has atleast one solution. □

4. HU and GHU stability

This section is dedicated to the study of HU stability and GHU stability for our proposed tripled system (1.1) of HFSIDEs. We take benefit from the definitions given in [39] to give definitions of HU stability and GHU stability for the desired investigation of stability analysis.

Definition 4.1. *The tripled system* (3.11) *of integral equations is said to be HU stable if there exist positive real numbers* $D_i (i = 1, 2, 3)$, *such that for every* $\epsilon_i > 0$ $(i = 1, 2, 3)$ *and for any arbitrary solution* $(\mu^*, \upsilon^*, \omega^*)$ *of the system of inequities*

$$\begin{cases} \Big|\mu^*(y) - \displaystyle\int_0^1 \mathcal{G}(y,s) g_1(s,\mu^*(s)) \int_0^s (s-r)^{\theta-1} h_1(r,\upsilon^*(r), I^\gamma \upsilon^*(r)) dr ds \\ \quad + \displaystyle\sum_{j=1}^k \int_0^1 \mathcal{G}_j(y,s) f_{1j}(s,\mu^*(s)) ds + (1-y)\Phi_1(\mu^*(\eta)) + y\Phi_2(\mu^*(\eta))\Big| \le \epsilon_1, \\ \Big|\upsilon^*(y) - \displaystyle\int_0^1 \mathcal{G}(y,s) g_2(s,\upsilon^*(s)) \int_0^s (s-r)^{\theta-1} h_2(r,\omega^*(r), I^\gamma \omega^*(r)) dr ds \\ \quad + \displaystyle\sum_{j=1}^k \int_0^1 \mathcal{G}_j(y,s) f_{2j}(s,\upsilon^*(s)) ds + (1-y)\varphi_1(\upsilon^*(\eta)) + y\varphi_2(\upsilon^*(\eta))\Big| \le \epsilon_2, \\ \Big|\omega^*(y) - \displaystyle\int_0^1 \mathcal{G}(y,s) g_3(s,\omega^*(s)) \int_0^s (s-r)^{\theta-1} h_3(r,\mu^*(r), I^\gamma \mu^*(r)) dr ds \\ \quad + \displaystyle\sum_{j=1}^k \int_0^1 \mathcal{G}_j(y,s) f_{3j}(s,\omega^*(s)) ds + (1-y)\psi_1(\omega^*(\eta)) + y\psi_2(\omega^*(\eta))\Big| \le \epsilon_3, \end{cases}$$

there exist a unique solution (μ, υ, ω) *of* (3.11), *such that*

$$\begin{aligned} |\mu(y) - \mu^*(y)| &\le D_1\epsilon, \\ |\upsilon(y) - \upsilon^*(y)| &\le D_2\epsilon, \\ |\omega(y) - \omega^*(y)| &\le D_3\epsilon, \end{aligned}$$

where $\epsilon = \max\{\epsilon_1, \epsilon_2, \epsilon_3\}$.

Definition 4.2. *The tripled system* (3.11) *of integral equations is said to be GHU stable if there exist functions* $\Upsilon_i : B \to [0, \infty)$, $(i = 1, 2, 3)$, *such that for every* $\epsilon_i > 0$ $(i = 1, 2, 3)$ *and for any arbitrary solution* $(\mu^*, \upsilon^*, \omega^*)$ *of the system of inequalities*

$$\begin{cases} \Big|\mu^*(y) - \displaystyle\int_0^1 \mathcal{G}(y,s) g_1(s,\mu^*(s)) \int_0^s (s-r)^{\theta-1} h_1(r,\upsilon^*(r), I^\gamma \upsilon^*(r)) dr ds \\ \quad + \displaystyle\sum_{j=1}^k \int_0^1 \mathcal{G}_j(y,s) f_{1j}(s,\mu^*(s)) ds + (1-y)\Phi_1(\mu^*(\eta)) + y\Phi_2(\mu^*(\eta))\Big| \le \epsilon_1, \\ \Big|\upsilon^*(y) - \displaystyle\int_0^1 \mathcal{G}(y,s) g_2(s,\upsilon^*(s)) \int_0^s (s-r)^{\theta-1} h_2(r,\omega^*(r), I^\gamma \omega^*(r)) dr ds \\ \quad + \displaystyle\sum_{j=1}^k \int_0^1 \mathcal{G}_j(y,s) f_{2j}(s,\upsilon^*(s)) ds + (1-y)\varphi_1(\upsilon^*(\eta)) + y\varphi_2(\upsilon^*(\eta))\Big| \le \epsilon_2, \\ \Big|\omega^*(y) - \displaystyle\int_0^1 \mathcal{G}(y,s) g_3(s,\omega^*(s)) \int_0^s (s-r)^{\theta-1} h_3(r,\mu^*(r), I^\gamma \mu^*(r)) dr ds \\ \quad + \displaystyle\sum_{j=1}^k \int_0^1 \mathcal{G}_j(y,s) f_{3j}(s,\omega^*(s)) ds + (1-y)\psi_1(\omega^*(\eta)) + y\psi_2(\omega^*(\eta))\Big| \le \epsilon_3, \end{cases}$$

there exist a unique solution (μ, υ, ω) *of* (3.11)*, such that*

$$\begin{aligned}|\mu(y) - \mu^*(y)| &\leq \Upsilon_1(\epsilon),\\ |\upsilon(y) - \upsilon^*(y)| &\leq \Upsilon_2(\epsilon),\\ |\omega(y) - \omega^*(y)| &\leq \Upsilon_3(\epsilon),\end{aligned}$$

where $\epsilon = \max\{\epsilon_1, \epsilon_2, \epsilon_3\}$.

Theorem 4.3. *Assume that* (H_1)*–*(H_4) *hold, then the solution of the considered system* (1.1) *of HFSIDEs is HU stable.*

Proof. To show that the tripled system (1.1) is HU stable, we use (3.11) which is an equivalent integral form of (1.1). Assume that $(\mu^*, \upsilon^*, \omega^*)$ is an arbitrary solution and let (μ, υ, ω) be the exact solution of (3.11), then by hypothesis (H_1)–(H_4), we have

$$\begin{aligned}|\mu(y) - \mu^*(y)| &\leq \Big(\frac{\|\Omega_1\|\|\Theta_1\|}{\Gamma(\sigma+1)\Gamma(\theta+1)} + \sum_{j=1}^{k}\frac{\|F_{1j}\|}{\Gamma(\alpha_j+\sigma+1)} + L_{11} + L_{12}\Big)\|\mu(y) - \mu^*(y)\|_\infty\\ &+ \frac{\|\chi_1\|\|\lambda_1\|(1+\Gamma(\gamma+1))}{\Gamma(\sigma+1)\Gamma(\theta+1)\Gamma(\gamma+1)}\|\upsilon(y) - \upsilon^*(y)\|_\infty.\end{aligned}$$

$$\|\mu - \mu^*\| \leq C_1\epsilon_1 + C_2\epsilon_2, \tag{4.1}$$

where

$$C_1 = \Big(\frac{\|\Omega_1\|\|\Theta_1\|}{\Gamma(\sigma+1)\Gamma(\theta+1)} + \sum_{j=1}^{k}\frac{\|F_{1j}\|}{\Gamma(\alpha_j+\sigma+1)} + L_{11} + L_{12}\Big),$$

$$C_2 = \frac{\|\chi_1\|\|\lambda_1\|(1+\Gamma(\gamma+1))}{\Gamma(\sigma+1)\Gamma(\theta+1)\Gamma(\gamma+1)}.$$

Let use $D_1 = C_1 + C_2$, and $\epsilon = \max\{\epsilon_1, \epsilon_2\}$, we get from (4.1)

$$\|\mu - \mu^*\| \leq D_1\epsilon. \tag{4.2}$$

In similar manner we can derive that

$$\|\upsilon - \upsilon^*\| \leq D_2\epsilon \tag{4.3}$$

and

$$\|\omega - \omega^*\| \leq D_3\epsilon. \tag{4.4}$$

Thus by the use of Definition 4.1, the solution of the problem (3.11) is HU stable. Consequently, the solution of the system (1.1) of HFSIDEs is HU stable. □

Theorem 4.4. *Assume that* (H_1)*–*(H_4) *hold, then the solution of the tripled system* (1.1) *of HFSIDEs is GHU stable.*

Proof. Let for $\Upsilon_i (i = 1, 2, 3)$ are functions defined by $\Upsilon_i : B \to [0, \infty)$, with each $\Upsilon_i(0) = 0$. Then from the inequalities (4.2)–(4.4), we choose $D_1\epsilon = \Upsilon_1(\epsilon)$, $D_2\epsilon = \Upsilon_2(\epsilon)$ and $D_3\epsilon = \Upsilon_3(\epsilon)$, which led us to the conclusion that

$$\begin{aligned} \|\mu - \mu^*\| &\leq \Upsilon_1(\epsilon), \\ \|\upsilon - \upsilon^*\| &\leq \Upsilon_2(\epsilon), \\ \|\omega - \omega^*\| &\leq \Upsilon_3(\epsilon). \end{aligned}$$

Hence bythe Definition 4.2, the solution of the problem (3.11) is GHU stable. Consequently, the solution of the tripled system (1.1) of HFSIDEs is GHU stable. □

5. Examples

In this section, we present an example for the the tripled system BVP (1.1) of HFSIDEs to test our respective existence and stability results.

Example 5.1. *Consider the following BVP tripled system of HFSIDEs*

$$\begin{cases} {}_0^c\mathrm{D}_y^{\frac{2}{5}}\left[\dfrac{{}_0^c\mathrm{D}_y^{\frac{5}{3}}\mu(y) - \sum_{j=1}^k I^{\alpha_j} f_{1j}(y, \mu(y))}{y^2 \sec(\frac{\pi y}{6})\Big(\frac{3|\mu(y)|}{3(|\mu(y)|+1)}\Big)}\right] = y + \cos(\dfrac{\pi y}{6})\Big(\dfrac{e^{-y}|\upsilon(y)|}{|\upsilon(y)| + 13} + I^{2.8}\upsilon(y)\Big), \\ {}_0^c\mathrm{D}_y^{\frac{2}{5}}\left[\dfrac{{}_0^c\mathrm{D}_y^{\frac{5}{3}}\upsilon(y) - \sum_{j=1}^k I^{\alpha_j} f_{2j}(y, \upsilon(y))}{y \tan(y)\Big(\frac{|\upsilon(y)|}{|\upsilon(y)|+2}\Big)}\right] = \sin(\dfrac{\pi y}{2})\Big(\dfrac{|\omega(y)|}{|\omega(y)| + 1} + I^{3.5}\omega(y)\Big), \\ {}_0^c\mathrm{D}_y^{\frac{2}{5}}\left[\dfrac{{}_0^c\mathrm{D}_y^{\frac{5}{3}}\omega(y) - \sum_{j=1}^k I^{\alpha_j} f_{3j}(y, \omega(y))}{\cot(y)\Big(\frac{|\upsilon(y)|+3}{|\upsilon(y)|+4}\Big)}\right] = \sqrt{y}cos(y)\Big(\dfrac{|\mu(y)| + 5}{|\mu(y)| + 6} + I^{4.2}\mu(y)\Big), \\ \mu(0) = \mu^2(0.5),\ {}_0^c\mathrm{D}_y^{\frac{5}{3}}\mu(0) = 0,\ \mu(1) = \sin(\mu(0.5)), \\ \upsilon(0) = \cos(\upsilon(0.5)),\ {}_0^c\mathrm{D}_y^{\frac{5}{3}}\upsilon(0) = 0,\ \upsilon(1) = \dfrac{1}{6}\sin(\upsilon(0.5)), \\ \omega(0) = \omega^2(0.5),\ {}_0^c\mathrm{D}_y^{\frac{5}{3}}\omega(0) = 0,\ \omega(1) = 2^{\omega(0.5)}, \end{cases} \tag{5.1}$$

where

$$\begin{aligned} \sum_{j=1}^{3} I^{\alpha_j} f_{1j}(y, \mu(y)) =& I^{\frac{3}{5}}\Big(\sin(\frac{y}{4})|\mu(y)| + e^y \sin y\Big) + I^{\frac{4}{7}}\Big(\sqrt{y}\sin y|\mu(y)| + \frac{5}{2 - y^3}\Big) \\ &+ I^{\frac{3}{8}}\Big(|\mu(y)|\cot(\frac{\sqrt{y}}{3}) + \frac{y^2}{e^y + 1}\Big), \end{aligned} \tag{5.2}$$

$$\begin{aligned} \sum_{j=1}^{3} I^{\alpha_j} f_{2j}(y, \upsilon(y)) =& I^{\frac{3}{5}}\Big(\cos(\frac{\pi y}{7})|\upsilon(y)| + e^{y^2}\Big) + I^{\frac{4}{7}}\Big(\sqrt{y^3}\cos y|\upsilon(y)| + \sqrt{y + 1}\Big) \\ &+ I^{\frac{3}{8}}\Big(|\upsilon(y)|\tan(\frac{\sqrt{y^3}}{5}) + \frac{y}{e^{2y} + 3}\Big), \end{aligned} \tag{5.3}$$

$$\sum_{j=1}^{3} I^{\alpha_j} f_{3j}(y,\omega(y)) = I^{\frac{3}{5}}\Big(\frac{y^2\sqrt{y+1}}{e^y}|\omega(y)| + e^y \sin y\Big) + I^{\frac{4}{7}}\Big(\sqrt{y}\sin y|\omega(y)| + \frac{5}{2-y^3}\Big)$$
$$+ I^{\frac{3}{8}}\Big(|\omega(y)|\tan\big(\frac{\sqrt{y}}{3}\big) + \frac{y^2}{e^y+1}\Big). \tag{5.4}$$

Now from Eqs (5.1)–(5.4)*, we have* $\theta = \frac{2}{5},\ \sigma = \frac{5}{3},\ \alpha_1 = \frac{3}{5},\ \alpha_2 = \frac{4}{7},\ \alpha_3 = \frac{3}{8},\ \gamma = 2.8,\ \eta = 0.5,$

$$f_{11}(y,\mu(y)) = \sin(\frac{y}{4})|\mu(y)| + e^y \sin y,\ \ f_{12}(y,\mu(y)) = \sqrt{y}\sin y|\mu(y)| + \frac{5}{2-y^3},$$

$$f_{13}(y,\mu(y)) = |\mu(y)|\cot\big(\frac{\sqrt{y}}{3}\big) + \frac{y^2}{e^y+1},\ \ f_{21}(y,\upsilon(y)) = \cos\big(\frac{\pi y}{7}\big)|\upsilon(y)| + e^{y^2},$$

$$f_{22}(y,\upsilon(y)) = \sqrt{y^3}\cos y|\upsilon(y)| + \sqrt{y+1},\ \ f_{23}(y,\upsilon(y)) = |\upsilon(y)|\tan\big(\frac{\sqrt{y^3}}{5}\big) + \frac{y}{e^{2y}+3},$$

$$f_{31}(y,\omega(y)) = \frac{y^2\sqrt{y+1}}{e^y}|\omega(y)| + e^y \sin y,\ \ f_{32}(y,\omega(y)) = \sqrt{y}\sin y|\omega(y)| + \frac{5}{2-y^3},$$

$$f_{33}(y,\omega(y)) = |\omega(y)|\tan\big(\frac{\sqrt{y}}{3}\big) + \frac{y^2}{e^y+1},\ \ g_1(y,\mu(y)) = y^2\sec(\frac{\pi y}{6})\Big(\frac{3|\mu(y)|}{3(|\mu(y)|+1)}\Big)$$

$$g_2(y,\upsilon(y)) = y\tan(y)\Big(\frac{|\upsilon(y)|}{|\upsilon(y)|+2}\Big),\ \ g_3(y,\omega(y)) = \cot(y)\Big(\frac{|\upsilon(y)|+3}{|\upsilon(y)|+4}\Big).$$

Moreover,

$$h_1(y,\upsilon(y),I^\gamma\upsilon(y)) = y + \cos(\frac{\pi y}{6})\Big(\frac{e^{-y}|\upsilon(y)|}{|\upsilon(y)|+13} + I^{2.8}\upsilon(y)\Big),$$

$$h_2(y,\omega(y),I^\gamma\omega(y)) = \sin(\frac{\pi y}{2})\Big(\frac{|\omega(y)|}{|\omega(y)|+1} + I^{3.5}\omega(y)\Big)$$

and

$$h_3(y,\mu(y),I^\gamma\mu(y)) = \sqrt{y}cos(y)\Big(\frac{|\mu(y)|+5}{|\mu(y)|+6} + I^{4.2}\mu(y)\Big).$$

Then it is simple to show that hypothesis (H_1)–(H_4) *holds. Hence by Theorem 3.3 BVP tripled system* (5.1) *of HFSIDEs has atleast one solution. Furthermore, by Theorem 4.3 the solution of the system* (5.1) *is HU stable and by Theorem 4.4, the solution of the system* (5.1) *is GHU stable.*

6. Conclusions

Varieties of existence and uniqueness results appears in a range from theoretical aspects in the literature of analysis. FDEs appears in mathematical modeling of different process and phenomenon in various fields like blood flow phenomena, electro-dynamics, visco-elasticity and biophysics. Modeling through systems of differential equations is an important class of bio-mathematics, physics, applied chemistry and many more areas. Also the area has been extended recently to FDEs as well. BVPs have many applications in engineering and physical sciences. Therefore, systems of BVP of FDEs have been investigated very well. In this paper, we have established sufficient conditions for the existence theory and HU and GHU stability analysis for the tripled system of HFSIDEs under boundary conditions. Our proposed system (1.1) can easily reduce to a system of fractional order Volterra integro-differential equations. The said equations have been studied in various articles which have lots of applications. For instance authors [40] have presented the model of physical system of fractional order Volterra integro-differential equations which is characterized by Levy jumps. The authors solved the Levy jumps problem by reduction of fractional order Schrodinger equation to fractional order Volterra integro-differential equations with hyper singular kernel. So the tripled system of HFSIDEs studied in this work has many applications for modelling different phenomena. Moreover in this study, as a result of Darbo's fixed point theorem and literature of MNC, we concluded a new fixed point result given as a Theorem 3.1. For the application purpose, we utilized Theorem 3.1 for the existence of solution to the considered tripled system (1.1) of hybrid fractional integro-differential equations. HU and GHU stabilities are also investigated for the problem (1.1). In last section, we presented an example which justify all our acquired results. In future, we can extend the above results for tripled systems of HFSIDEs under non-singular kernel type derivatives.

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Conflict of interest

The authors declare no conflicts of interest.

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