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Research article

Least squares type estimations for discretely observed nonergodic Gaussian Ornstein-Uhlenbeck processes of the second kind

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Abstract: We consider the nonergodic Gaussian Ornstein-Uhlenbeck processes of the second kind defined by $dX_t = \theta X_t dt + dY_t^{(1)}, t \geq 0, X_0 = 0$ with an unknown parameter $\theta > 0$, where $dY_t^{(1)} = e^{-t} dG_{a_t}$ and $\{G_t, t \geq 0\}$ is a mean zero Gaussian process with the self-similar index $\gamma \in (\frac{1}{2}, 1)$ and $a_t = \gamma e^{\frac{t}{\gamma}}$. Based on the discrete observations $\{X_{t_i} : t_i = i\Delta_n, i = 0, 1, \cdots, n\}$, two least squares type estimators $\hat{\theta}_n$ and $\tilde{\theta}_n$ of θ are constructed and proved to be strongly consistent and rate consistent. We apply our results to the cases such as fractional Brownian motion, sub-fractional Brownian motion, bifractional Brownian motion and sub-bifractional Brownian motion. Moreover, the numerical simulations confirm the theoretical results.

Keywords: nonergodic Gaussian Ornstein-Uhlenbeck processes of the second kind; least squares type estimation; discrete observations; rate consistent
Mathematics Subject Classification: 62F12, 60G22

1. Introduction and main results

Recently, the study of fractional Ornstein-Uhlenbeck processes of the second kind (FOUSK) has attracted interest. For example, Azmoodeh and Morlanes (2013) studied the drift parameter estimation of FOUSK based on continuous observations in the ergodic case. Azmoodeh and Viitasaari (2015) considered the drift parameter estimation of FOUSK based on discrete observations in the ergodic case. For the nonergodic case, the drift parameter estimations of FOUSK based on continuous and discrete observations were studied in EI Onsy et al. (2017) and EI Onsy et al. (2018), respectively. Balde et al. (2018) investigated the infinite-dimensional version of FOUSK. Yu et al. (2017) studied the problem of parameter estimation for Ornstein-Uhlenbeck processes of the second kind driven by α-stable Lévy motions, based on continuous and discrete observations, respectively. Es-Sebaiy et al.

(2019) considered least squares type estimations for discretely observed nonergodic Gaussian Ornstein-Uhlenbeck processes.

Motivated by all these studies, in this paper, we will consider the nonergodic Gaussian Ornstein-Uhlenbeck processes of the second kind (GOUSK) defined by

$$dX_t = \theta X_t dt + dY_t^{(1)}, t \geq 0, X_0 = 0, \tag{1.1}$$

with an unknown parameter $\theta > 0$, where $dY_t^{(1)} = e^{-t} dG_{a_t}$ and $G = \{G_t, t \geq 0\}$ is a mean zero Gaussian process with the self-similar index $\gamma \in (\frac{1}{2}, 1)$ and $a_t = \gamma e^{\frac{t}{\gamma}}$. Based on the discrete observations $\{X_{t_i} : t_i = i\Delta_n, i = 0, 1, \cdots, n\}$, we construct two least squares type estimators $\hat{\theta}_n$ and $\tilde{\theta}_n$ of θ:

$$\hat{\theta}_n = \frac{\sum_{i=1}^n X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})}{\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2} \tag{1.2}$$

and

$$\tilde{\theta}_n = \frac{X_{T_n}^2}{2\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2}, \tag{1.3}$$

where $T_n = n\Delta_n$ denotes the length of the 'observation window'.

Let the covariance function of Gaussian process G be $R(t, s) = \mathbf{E}(G_t G_s), t \geq 0, s \geq 0$. Denote

$$\phi(t, s) = \frac{\partial^2 R(t, s)}{\partial t \partial s}. \tag{1.4}$$

Assume that

$$\phi(t, s) \leq c(\gamma)|t - s|^{2\gamma - 2}, \tag{1.5}$$

where $c(\gamma)$ is a positive constant depending on γ, and $\frac{1}{2} < \gamma < 1$ is self-similar index of G.

For the assumption (1.5), many self-similar Gaussian processes satisfy the condition, such as fractional Brownian motion, sub-fractional Brownian motion, bifractional Brownian motion and sub-bifractional Brownian motion. Therefore our main results hold for the Gaussian processes mentioned above.

Let $\{Z_n\}$ be a sequence of random variables. We say $\{Z_n\}$ is tight (or bounded in probability) if for every $\epsilon > 0$, there exists $M_\epsilon > 0$, such that

$$\mathbf{P}(|Z_n| > M_\epsilon) < \epsilon \quad \text{for all } \ n.$$

Now we state our main results as follows.

Theorem 1.1. Assume that (1.5) holds and $\frac{1}{2} < \gamma < 1$, and $\theta > 0, \Delta_n \to 0$ and $n\Delta_n^{1+\beta} \to \infty$ for some $\beta > 0$ as $n \to \infty$. Then, we have, as $n \to \infty$,

$$\hat{\theta}_n \xrightarrow{\text{a.s.}} \theta, \tag{1.6}$$

and

$$\tilde{\theta}_n \xrightarrow{\text{a.s.}} \theta. \tag{1.7}$$

Theorem 1.2. Assume that (1.5) holds and $\frac{1}{2} < \gamma < 1$, and $\theta > 0, \Delta_n \to 0$ and $n\Delta_n^{1+\beta} \to \infty$ for some $\beta > 0$ as $n \to \infty$. Then, we have

(1) for any $q \geq 0$,

$$\Delta_n^q e^{\theta T_n}(\hat{\theta}_n - \theta) \text{ is not tight,} \tag{1.8}$$

(2) if $n\Delta_n^3 \to 0$ as $n \to \infty$, then

$$\sqrt{T_n}(\hat{\theta}_n - \theta) \text{ is tight.} \tag{1.9}$$

Theorem 1.3. Assume that (1.5) holds and $\frac{1}{2} < \gamma < 1$, and $\theta > 0, \Delta_n \to 0$ and $n\Delta_n^{1+\beta} \to \infty$ for some $\beta > 0$ as $n \to \infty$. Then, we have

(1) for any $q \geq 0$,

$$\Delta_n^q e^{\theta T_n}(\tilde{\theta}_n - \theta) \text{ is not tight,} \tag{1.10}$$

(2) if $n\Delta_n^3 \to 0$ as $n \to \infty$, then

$$\sqrt{T_n}(\tilde{\theta}_n - \theta) \text{ is tight.} \tag{1.11}$$

Remark. We study Gaussian Ornstein-Uhlenbeck processes of the second kind in this paper. We know they are a subset of a much larger class Barndorff-Nielsen and Shephard type Ornstein-Uhlenbeck processes (see Barndorff-Nielsen (2001) and Barndorff-Nielsen and Shephard (2001)). In the future, we will extend our results to more general Ornstein-Uhlenbeck type models (see Salmon and SenGupta (2021), Issaka and SenGupta (2017) and Roberts and SenGupta (2020)).

We have organized our paper as follows: In Sect.2 we present some preliminaries for the Gaussian process G and main lemmas. Sect.3 is devoted to the proofs of Theorems 1.1–1.3. In Sect.4 we apply our results to the cases such as fractional Brownian motion, sub-fractional Brownian motion, bifractional Brownian motion and sub-bifractional Brownian motion, while Sect.5 contains numerical simulations for four fractional Gaussian processes.

2. Preliminaries and main lemmas

In this section, we firstly recall some elements of the Malliavin calculus. We refer to Nualart (2006) for detailed account these notions(see Moshrefi-Torbati and Hammond (1998), Meerschaert et al. (2017), Ei-Nabulsi (2012,2015,2017), and Ei-Nabulsi and Golmankhaneh (2021)). Let $\mathcal{H}$ be a real separable Hilbert space associated with the Gaussian process G, which is defined by the closure of the linear space ε generated by the indicator functions $\{\mathbf{I}_{[0,t]}, t \in [0, T]\}$ with respect to the scalar product

$$\langle \mathbf{I}_{[0,t]}, \mathbf{I}_{[0,s]} \rangle_{\mathcal{H}} = R(t, s).$$

We know that the covariance of G can be written as

$$R(t, s) = \int_0^t \int_0^s \phi(u, v) du dv, \tag{2.1}$$

where $\phi(u, v)$ is defined by (1.4).

We can find a linear space of functions contained in $\mathcal{H}$ in the following way. Let $|\mathcal{H}|$ be the linear space of measurable functions φ on $[0, T]$ such that

$$||\varphi||^2_{|\mathcal{H}|} = \int_0^T \int_0^T |\varphi(u)||\varphi(v)|\phi(u, v) du dv < \infty.$$

It is not difficult to show that $|\mathcal{H}|$ is a Banach space with the norm $|| \bullet ||_{|\mathcal{H}|}$ and ε is dense in $|\mathcal{H}|$.

Moreover, for all $\varphi, \psi \in |\mathcal{H}|$, it can be proved that

$$\mathbf{E}\left(\int_0^T \varphi(u)dG_u \int_0^T \psi(v)dG_v\right) = \int_0^T \int_0^T \varphi(u)\psi(v)\phi(u,v)dudv. \tag{2.2}$$

For every $q \geq 1$, let $\mathcal{H}_q$ be the qth Wiener chaos of G, namely, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(G(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where H_q is the qth Hermite polynomial defined as $H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}\left(e^{-\frac{x^2}{2}}\right)$. The mapping $I_q(h^{\otimes q}) = H_q(G(h))$ provides a linear isometry between the symmetric tensor product $\mathcal{H}^{\odot q}$ (equipped with the modified norm $\|\cdot\|_{\mathcal{H}^{\odot q}} = \sqrt{q!}\|\cdot\|_{\mathcal{H}^{\otimes q}}$) and $\mathcal{H}_q$. Specifically, for all $f, g \in \mathcal{H}^{\odot q}$ and $q \geq 1$, one has

$$\mathbf{E}[I_q(f)I_q(g)] = q!\langle f, g\rangle_{\mathcal{H}^{\otimes q}}. \tag{2.3}$$

For the multiple stochastic integral $I_q(f)$, it has the following property: for any $p \geq 2$,

$$\left(\mathbf{E}[|I_q(f)|^p]\right)^{1/p} \leq c(p,q)\left(\mathbf{E}[|I_q(f)|^2]\right)^{1/2}, \tag{2.4}$$

where $c(p,q)$ is a positive constant only depending on p and q.

It is easy to obtain the solution of (1.1):

$$X_t = e^{\theta t}\int_0^t e^{-\theta s}dY_s^{(1)}, \quad t \geq 0, \tag{2.5}$$

where the integral with respect to $Y_s^{(1)}$ is a Young integral (see Young (1936)).

Denote

$$\eta_t = \int_0^t e^{-\theta s}dY_s^{(1)}, \quad t \geq 0, \tag{2.6}$$

then,

$$X_t = e^{\theta t}\eta_t. \tag{2.7}$$

By (2.6) and $dY_s^{(1)} = e^{-s}dG_{a_s}$, $a_s = \gamma e^{\frac{s}{\gamma}}$ and let $a_s = u$, we get

$$\begin{aligned}\eta_t &= \int_0^t e^{-(\theta+1)s}dG_{a_s} \\ &= \gamma^{(\theta+1)\gamma}\int_{a_0}^{a_t} u^{-(\theta+1)\gamma}dG_u.\end{aligned} \tag{2.8}$$

By (2.2) and $\frac{1}{2} < \gamma < 1$, we have, for $0 \leq s < t$,

$$\mathbf{E}[(\eta_t - \eta_s)^2] = \gamma^{2(\theta+1)\gamma}\int_{a_s}^{a_t}\int_{a_s}^{a_t}(uv)^{-(\theta+1)\gamma}\phi(u,v)dudv. \tag{2.9}$$

In order to prove Theorems 1.1-1.3, we need the following some lemmas.

Lemma 2.1. Let $\eta = \{\eta_t, t \geq 0\}$ be given by (2.6). Assume that (1.5) holds and $\frac{1}{2} < \gamma < 1$. Then,

(1) For all $\epsilon \in (0, \gamma)$, the process η has a modification with $(\gamma - \epsilon)$-Hölder continuous paths, still denoted η in the sequel.

(2) As $t \to \infty$,

$$\eta_t \to \eta_\infty := \gamma^{(\theta+1)\gamma} \int_{a_0}^{\infty} u^{-(\theta+1)\gamma} dG_u, \tag{2.10}$$

almost surely and in $L^2(\Omega)$.

Proof. Since the proof is similar to that of Lemma 2.2 in EI Onsy et al. (2017), we omit the details.

Lemma 2.2. Let $l > 0$ and $\{Z_n\}_{n\in\mathbf{N}}$ be a sequence of random variables. Suppose that, for every $p \geq 1$, there exists a constant $c_p > 0$ such that, for all $n \in \mathbf{N}$,

$$(\mathbf{E}[|Z_n|^p])^{1/p} \leq c_p \cdot n^{-l}.$$

Then, for all $\epsilon > 0$, there exists a random variable ξ_ϵ such that, for any $n \in \mathbf{N}$,

$$|Z_n| \leq \xi_\epsilon \cdot n^{-l+\epsilon} \quad \text{a.s.},$$

moreover, $\mathbf{E}[|\xi_\epsilon|^p] < \infty$ for all $p \geq 1$.

Proof. See the proof in Kloeden and Neuenkirch (2007).

Lemma 2.3. Let

$$R_n = \sum_{i=1}^{n-1} e^{-2\theta(n-i)\Delta_n} (\eta_{t_i}^2 - \eta_{t_{i-1}}^2), \tag{2.11}$$

and

$$S_n = \Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2. \tag{2.12}$$

Then, we have

$$e^{-2\theta T_n} S_n = \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \left(\eta_{t_{n-1}}^2 - R_n\right). \tag{2.13}$$

Assume that (1.5) holds and $\frac{1}{2} < \gamma < 1$, and $\theta > 0, \Delta_n \to 0$ and $n\Delta_n^{1+\beta} \to \infty$ for some $\beta > 0$ as $n \to \infty$, then,

$$R_n \xrightarrow{\text{a.s.}} 0. \tag{2.14}$$

In particular, as $n \to \infty$, we have

$$e^{-2\theta T_n} S_n \xrightarrow{\text{a.s.}} \frac{\eta_\infty^2}{2\theta}, \tag{2.15}$$

where η_∞ is defined by (2.10).

Proof. By (2.7), we obtain

$$\begin{aligned}
e^{-2\theta T_n} S_n &= e^{-2\theta T_n} \Delta_n \sum_{i=1}^{n} e^{2\theta(i-1)\Delta_n} \eta_{t_{i-1}}^2 \\
&= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \sum_{i=1}^{n} e^{-2\theta(n-i+1)\Delta_n} \left(e^{2\theta\Delta_n} - 1\right) \eta_{t_{i-1}}^2 \\
&= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \sum_{i=1}^{n} \left(e^{-2\theta(n-i)\Delta_n} - e^{-2\theta(n-i+1)\Delta_n}\right) \eta_{t_{i-1}}^2
\end{aligned}$$

$$= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1}\left[\eta^2_{t_{n-1}} - \sum_{i=1}^{n-1} e^{-2\theta(n-i)\Delta_n}\left(\eta^2_{t_i} - \eta^2_{t_{i-1}}\right)\right].$$

$$= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1}(\eta^2_{t_{n-1}} - R_n),$$

which shows that (2.13) holds.

Using (2.9) and (1.5) and making the change of variables $r = \frac{u}{a_{t_{i-1}}}$, $s = \frac{v}{a_{t_{i-1}}}$, we get

$$\begin{aligned}
\mathbf{E}[(\eta_{t_i} - \eta_{t_{i-1}})^2] &= \gamma^{2(\theta+1)\gamma}\int_{a_{t_{i-1}}}^{a_{t_i}}\int_{a_{t_{i-1}}}^{a_{t_i}} (uv)^{-(\theta+1)\gamma}\phi(u,v)dudv \\
&\leq c(\gamma)\gamma^{2(\theta+1)\gamma}\int_{a_{t_{i-1}}}^{a_{t_i}}\int_{a_{t_{i-1}}}^{a_{t_i}} (uv)^{-(\theta+1)\gamma}|u-v|^{2\gamma-2}dudv \\
&= c(\gamma)\gamma^{2\gamma}e^{-2\theta(i-1)\Delta_n}\int_1^{\frac{\Delta_n}{\gamma}}\int_1^{\frac{\Delta_n}{\gamma}} (rs)^{-(\theta+1)\gamma}|r-s|^{2\gamma-2}drds \\
&\leq c(\gamma)\gamma^{2\gamma}e^{-2\theta(i-1)\Delta_n}\int_1^{\frac{\Delta_n}{\gamma}}\int_1^{\frac{\Delta_n}{\gamma}} |r-s|^{2\gamma-2}drds \\
&= \frac{c(\gamma)\gamma^{2\gamma}e^{-2\theta(i-1)\Delta_n}}{\gamma(2\gamma-1)}\left(e^{\frac{\Delta_n}{\gamma}} - 1\right)^{2\gamma}. \qquad (2.16)
\end{aligned}$$

By (2.10) and η is Gaussian, we also obtain, for every $p \geq 1$,

$$\left(\mathbf{E}\left[\left|\eta^2_{t_i} - \eta^2_{t_{i-1}}\right|^p\right]\right)^{1/p} \leq c(p)\left(\mathbf{E}\left[(\eta_{t_i} - \eta_{t_{i-1}})^2\right]\right)^{1/2}, \qquad (2.17)$$

where $c(p)$ is a constant depending on p. In fact, we have, by Cauchy-Schwarz inequality,

$$\begin{aligned}
\left(\mathbf{E}\left[\left|\eta^2_{t_i} - \eta^2_{t_{i-1}}\right|^p\right]\right)^{1/p} &= \left(\mathbf{E}\left[\left|\eta_{t_i} + \eta_{t_{i-1}}\right|^p\left|\eta_{t_i} - \eta_{t_{i-1}}\right|^p\right]\right)^{1/p} \\
&\leq \left\{\left[\mathbf{E}\left|\eta_{t_i} + \eta_{t_{i-1}}\right|^{2p}\right]^{1/2}\left[\mathbf{E}\left|\eta_{t_i} - \eta_{t_{i-1}}\right|^{2p}\right]^{1/2}\right\}^{1/p} \\
&\leq \left[\mathbf{E}\left|2\eta_\infty\right|^{2p}\right]^{1/(2p)}\left[\mathbf{E}\left|\eta_{t_i} - \eta_{t_{i-1}}\right|^{2p}\right]^{1/(2p)} \\
&\leq c(p)\left(\mathbf{E}\left[(\eta_{t_i} - \eta_{t_{i-1}})^2\right]\right)^{1/2},
\end{aligned}$$

the last inequality comes from the fact: if ξ is Gaussian and $p \geq 2$, then

$$\left[\mathbf{E}|\xi|^p\right]^{1/p} \leq c(p)\left[\mathbf{E}|\xi|^2\right]^{1/2}.$$

By (2.17) and (2.16) and the Minkowski inequality, we deduce that

$$\left(\mathbf{E}\left[|R_n|^p\right]\right)^{1/p} \leq \sum_{i=1}^{n-1} e^{-2\theta(n-i)\Delta_n}\left(\mathbf{E}\left[\left|\eta^2_{t_i} - \eta^2_{t_{i-1}}\right|^p\right]\right)^{1/p}$$

$$\leq \frac{c(p)\sqrt{c(\gamma)}\gamma^{\gamma}}{\sqrt{\gamma(2\gamma-1)}} e^{-\theta n\Delta_n}\left(e^{\frac{\Delta_n}{\gamma}}-1\right)^{\gamma}\sum_{i=1}^{n-1} e^{-\theta(n-i-1)\Delta_n}$$
$$= \frac{c(p)\sqrt{c(\gamma)}\gamma^{\gamma}}{\sqrt{\gamma(2\gamma-1)}} e^{-\theta n\Delta_n}\left(e^{\frac{\Delta_n}{\gamma}}-1\right)^{\gamma}\frac{1-e^{-\theta(n-1)\Delta_n}}{1-e^{-\theta\Delta_n}}$$
$$\leq \frac{c(p)\sqrt{c(\gamma)}\gamma^{\gamma}}{\sqrt{\gamma(2\gamma-1)}} e^{-\theta n\Delta_n}\left(e^{\frac{\Delta_n}{\gamma}}-1\right)^{\gamma}\frac{1}{1-e^{-\theta\Delta_n}}$$
$$= \frac{c(p)\sqrt{c(\gamma)}}{\theta\sqrt{\gamma(2\gamma-1)}}(\Delta_n)^{\gamma-1}e^{-\theta n\Delta_n}\left(\frac{e^{\frac{\Delta_n}{\gamma}}-1}{\frac{\Delta_n}{\gamma}}\right)^{\gamma}\cdot\frac{e^{\theta\Delta_n}}{\frac{e^{\theta\Delta_n}-1}{\theta\Delta_n}}$$
$$\leq \frac{c(p)\sqrt{c(\gamma)}c}{\theta\sqrt{\gamma(2\gamma-1)}}(\Delta_n)^{\gamma-1}e^{-\theta n\Delta_n}, \tag{2.18}$$

where c is a constant, and the last inequality comes from the fact: as $x \to 0$,

$$\frac{e^x-1}{x} \to 1.$$

Note that for any $\alpha > 0$, as $n \to \infty$,

$$(\Delta_n)^{\gamma-1}e^{-\theta n\Delta_n} = o(n^{-\alpha}). \tag{2.19}$$

In fact,

$$n^{\alpha}(\Delta_n)^{\gamma-1}e^{-\theta T_n} = \frac{T_n^{\alpha+\frac{\alpha+1-\gamma}{\beta}}e^{-\theta T_n}}{\left(n\Delta_n^{1+\beta}\right)^{\frac{\alpha+1-\gamma}{\beta}}}. \tag{2.20}$$

Since $n\Delta_n^{1+\beta} \to \infty$ for some $\beta > 0$ as $n \to \infty$, we have

$$\left(n\Delta_n^{1+\beta}\right)^{\frac{\alpha+1-\gamma}{\beta}} \to \infty. \tag{2.21}$$

In addition,

$$\lim_{n\to\infty} T_n^{\alpha+\frac{\alpha+1-\gamma}{\beta}}e^{-\theta T_n} = 0. \tag{2.22}$$

Thus (2.19) is obtained from (2.20)–(2.22).

By (2.19), we have, for any $\delta > 0$,

$$(\Delta_n)^{\gamma-1}e^{-\theta T_n} = n^{-\alpha-\delta}.$$

Hence (2.18) becomes

$$\left(\mathbf{E}\left[|R_n|^p\right]\right)^{1/p} \leq \frac{c(p)\sqrt{c(\gamma)}c}{\theta\sqrt{\gamma(2\gamma-1)}}n^{-\alpha-\delta}.$$

By Lemma 2.2, there exists a random variable ξ_α such that

$$|R_n| \leq \xi_\alpha n^{-\alpha}, \quad \text{a.s.}$$

for all $n \in \mathbf{N}$. Moreover, $\mathbf{E}|\xi_\alpha|^p < \infty$ for all $p \geq 1$. Therefore (2.14) holds.

Finally, we can get (2.15) using (2.13), (2.14), (2.10) and $\lim_{n\to\infty}\frac{\Delta_n}{e^{2\theta\Delta_n}-1} = \frac{1}{2\theta}$. The proof of Lemma 2.3 is finished.

3. Proofs of Theorems

In this section, we will give proofs of Theorems 1.1-1.3.

By (2.7), we can rewrite $\hat{\theta}_n$ and $\tilde{\theta}_n$ as follows, respectively,

$$\begin{aligned}\hat{\theta}_n &= \frac{\sum_{i=1}^n e^{\theta t_i}\eta_{t_i}X_{t_{i-1}} - \sum_{i=1}^n X_{t_{i-1}}^2}{S_n} \\ &= \frac{\sum_{i=1}^n e^{\theta t_i}(\eta_{t_i}-\eta_{t_{i-1}})X_{t_{i-1}} + (\sum_{i=1}^n e^{\theta t_i}\eta_{t_{i-1}}X_{t_{i-1}} - \sum_{i=1}^n X_{t_{i-1}}^2)}{S_n} \\ &= \frac{\sum_{i=1}^n e^{\theta t_i}(\eta_{t_i}-\eta_{t_{i-1}})X_{t_{i-1}} + (e^{\theta\Delta_n}-1)\sum_{i=1}^n X_{t_{i-1}}^2)}{S_n} \\ &= \frac{V_n}{S_n} + \frac{e^{\theta\Delta_n}-1}{\Delta_n}, \end{aligned} \tag{3.1}$$

where

$$V_n = \sum_{i=1}^n e^{\theta t_i}(\eta_{t_i}-\eta_{t_{i-1}})X_{t_{i-1}},$$

and

$$\tilde{\theta}_n = \frac{\eta_{T_n}^2}{2e^{-2\theta T_n}S_n}. \tag{3.2}$$

Proof of Theorem 1.1. For (1.6), according to (2.15) and (3.1), it suffices to prove that, as $n \to \infty$,

$$e^{-2\theta T_n}V_n \xrightarrow{a.s.} 0. \tag{3.3}$$

By the Minkowski inequality and (2.16), we have

$$\begin{aligned}\left[\mathbf{E}\left|e^{-2\theta T_n}V_n\right|^2\right]^{1/2} &= e^{-2\theta T_n}\left\{\mathbf{E}\left[\sum_{i=1}^n e^{\theta t_i}(\eta_{t_i}-\eta_{t_{i-1}})X_{t_{i-1}}\right]^2\right\}^{1/2} \\ &\le e^{-2\theta T_n}\sum_{i=1}^n e^{\theta i\Delta_n}(\mathbf{E}X_{t_{i-1}}^2)^{1/2}\left[\mathbf{E}(\eta_{t_i}-\eta_{t_{i-1}})^2\right]^{1/2} \\ &\le e^{-2\theta T_n}\frac{\sqrt{c(\gamma)}\gamma^\gamma}{\sqrt{\gamma(2\gamma-1)}}e^{\theta\Delta_n}\left(e^{\frac{\Delta_n}{\gamma}}-1\right)^\gamma\sum_{i=1}^n(\mathbf{E}X_{t_{i-1}}^2)^{1/2} \\ &\le e^{-2\theta T_n}\frac{\sqrt{c(\gamma)}\gamma^\gamma}{\sqrt{\gamma(2\gamma-1)}}e^{\theta\Delta_n}\left(e^{\frac{\Delta_n}{\gamma}}-1\right)^\gamma(\mathbf{E}\eta_\infty^2)^{1/2}\sum_{i=1}^n e^{\theta(i-1)\Delta_n} \\ &= \frac{\sqrt{c(\gamma)}\gamma^\gamma}{\sqrt{\gamma(2\gamma-1)}}e^{\theta\Delta_n}(\mathbf{E}\eta_\infty^2)^{1/2}\left(e^{\frac{\Delta_n}{\gamma}}-1\right)^\gamma e^{-2\theta T_n}\frac{1-e^{\theta T_n}}{1-e^{\theta\Delta_n}} \\ &= \frac{\sqrt{c(\gamma)}\gamma^\gamma}{\sqrt{\gamma(2\gamma-1)}}e^{\theta\Delta_n}(\mathbf{E}\eta_\infty^2)^{1/2}\left(e^{\frac{\Delta_n}{\gamma}}-1\right)^\gamma e^{-\theta T_n}\frac{1-e^{-\theta T_n}}{e^{\theta\Delta_n}-1}\end{aligned}$$

$$\leq \frac{\sqrt{c(\gamma)}\gamma^\gamma}{\sqrt{\gamma(2\gamma-1)}} e^{\theta\Delta_n} (\mathbf{E}\eta_\infty^2)^{1/2} \left(e^{\frac{\Delta_n}{\gamma}} - 1\right)^\gamma e^{-\theta T_n} \frac{1}{e^{\theta\Delta_n} - 1}$$

$$= \frac{\sqrt{c(\gamma)}}{\theta\sqrt{\gamma(2\gamma-1)}} (\mathbf{E}\eta_\infty^2)^{1/2} (\Delta_n)^{\gamma-1} \left(\frac{e^{\frac{\Delta_n}{\gamma}} - 1}{\frac{\Delta_n}{\gamma}}\right)^\gamma e^{-\theta T_n} \cdot \frac{e^{\theta\Delta_n}}{\frac{e^{\theta\Delta_n}-1}{\theta\Delta_n}}$$

$$\leq \frac{\sqrt{c(\gamma)}c}{\theta\sqrt{\gamma(2\gamma-1)}} (\mathbf{E}\eta_\infty^2)^{1/2} (\Delta_n)^{\gamma-1} e^{-\theta T_n}, \tag{3.4}$$

where c is a generic constant, and the last inequality also comes from the fact: as $x \to 0$,

$$\frac{e^x - 1}{x} \to 1.$$

Using similar arguments as the proof of (2.14), we can obtain (3.3). Thus (1.6) holds.

We can easily obtain (1.7) from (3.2), (2.15) and $\lim_{t\to\infty} \eta_t = \eta_\infty$ a.s., and this completes the proof of Theorem 1.1.

Proof of Theorem 1.2. (1) Firstly, we consider the case $q \geq \frac{1}{2}$. By (3.1), we get

$$\Delta_n^q e^{\theta T_n} (\hat{\theta}_n - \theta) = \Delta_n^q e^{\theta T_n} \left(\frac{V_n}{S_n} + \frac{e^{\theta\Delta_n} - 1}{\Delta_n} - \theta\right)$$

$$= \frac{\Delta_n^q e^{-\theta T_n} V_n}{e^{-2\theta T_n} S_n} + \Delta_n^{q-1} e^{\theta T_n} \left(e^{\theta\Delta_n} - 1 - \theta\Delta_n\right). \tag{3.5}$$

Since $e^{\theta\Delta_n} - 1 - \theta\Delta_n \sim \frac{\theta^2}{2}\Delta_n^2$ as $\Delta_n \to 0$, we obtain

$$\lim_{n\to\infty} \Delta_n^{q-1} e^{\theta T_n} \left(e^{\theta\Delta_n} - 1 - \theta\Delta_n\right) = \lim_{n\to\infty} \frac{\theta^2}{2} \Delta_n^{q+1} e^{\theta T_n}$$

$$= \lim_{n\to\infty} \frac{\theta^2}{2} \left(n\Delta_n^{1+\beta}\right)^{\frac{q+1}{\beta}} \frac{e^{\theta T_n}}{T_n^{\frac{q+1}{\beta}}}.$$

Because $n\Delta_n^{1+\beta} \to \infty$ for some $\beta > 0$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{e^{\theta T_n}}{T_n^{\frac{q+1}{\beta}}} = \infty.$$

Then, we have

$$\lim_{n\to\infty} \Delta_n^{q-1} e^{\theta T_n} \left(e^{\theta\Delta_n} - 1 - \theta\Delta_n\right) = \infty. \tag{3.6}$$

On the other hand, by (3.4), we get, as $n \to \infty$,

$$\mathbf{E}\left|\Delta_n^q e^{-\theta T_n} V_n\right| \leq \Delta_n^q e^{-\theta T_n} \left(\mathbf{E} V_n^2\right)^{1/2}$$

$$\leq \frac{\sqrt{c(\gamma)}c}{\theta\sqrt{\gamma(2\gamma-1)}} (\mathbf{E}\eta_\infty^2)^{1/2} \Delta_n^{q+\gamma-1}$$

$$\to 0, \tag{3.7}$$

since $\gamma > \frac{1}{2}$. Consequently, by (3.5)–(3.7) and (2.15), for any $q \geq \frac{1}{2}$, $\Delta_n^q e^{\theta T_n}(\hat{\theta}_n - \theta)$ is not tight.

For the case $0 \leq q < \frac{1}{2}$, note that

$$\Delta_n^q e^{\theta T_n}(\hat{\theta}_n - \theta) = \Delta_n^{q-\frac{1}{2}} \left(\Delta_n^{\frac{1}{2}} e^{\theta T_n}(\hat{\theta}_n - \theta)\right),$$

and $\lim_{n\to\infty} \Delta_n^{q-\frac{1}{2}} = \infty$ and the previous case $q = \frac{1}{2}$. The proof of (1.8) is finished.

(2) By (3.1), it is obvious that

$$\sqrt{T_n}(\hat{\theta}_n - \theta) = \sqrt{n\Delta_n^3}\frac{e^{\theta\Delta_n} - 1 - \theta\Delta_n}{\Delta_n^2} + \frac{\sqrt{T_n}e^{-2\theta T_n}V_n}{e^{-2\theta T_n}S_n}. \tag{3.8}$$

Since $n\Delta_n^3 \to 0$ and $\frac{e^{\theta\Delta_n}-1-\theta\Delta_n}{\Delta_n^2} \to \frac{\theta^2}{2}$, we have, as $n \to \infty$,

$$\sqrt{n\Delta_n^3}\frac{e^{\theta\Delta_n} - 1 - \theta\Delta_n}{\Delta_n^2} \to 0. \tag{3.9}$$

Furthermore, by (3.4), we get, as $n \to \infty$,

$$\begin{aligned}
\mathbf{E}\left|\sqrt{T_n}e^{-2\theta T_n}V_n\right| &\leq \sqrt{T_n}\left[\mathbf{E}\left|e^{-2\theta T_n}V_n\right|^2\right]^{1/2} \\
&\leq \frac{\sqrt{c(\gamma)}c}{\theta\sqrt{\gamma(2\gamma-1)}}(\mathbf{E}\eta_\infty^2)^{1/2}\Delta_n^{\gamma-1}\sqrt{T_n}e^{-\theta T_n} \\
&= \frac{\sqrt{c(\gamma)}c}{\theta\sqrt{\gamma(2\gamma-1)}}(\mathbf{E}\eta_\infty^2)^{1/2}\frac{T_n^{\frac{1}{2}+\frac{1-\gamma}{\beta}}e^{-\theta T_n}}{\left(n\Delta_n^{1+\beta}\right)^{\frac{1-\gamma}{\beta}}} \\
&\to 0.
\end{aligned} \tag{3.10}$$

By (3.8)–(3.10) and (2.15), we deduce (1.9). Hence, we complete the proof of Theorem 1.2.

Proof of Theorem 1.3. (1) Fix $q \geq \frac{1}{2}$, by (3.2) and (2.13), we obtain

$$\begin{aligned}
&\Delta_n^q e^{\theta T_n}(\tilde{\theta}_n - \theta) \\
&= \Delta_n^q e^{\theta T_n}\left(\frac{\eta_{T_n}^2}{2e^{-2\theta T_n}S_n} - \theta\right) \\
&= \frac{\Delta_n^q e^{\theta T_n}}{2e^{-2\theta T_n}S_n}\left[(\eta_{t_n}^2 - \eta_{t_{n-1}}^2) + \left(1 - \frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right)\eta_{t_{n-1}}^2 - 2\theta\left(e^{-2\theta T_n}S_n - \frac{\Delta_n}{e^{2\theta\Delta_n}-1}\eta_{t_{n-1}}^2\right)\right] \\
&= \frac{\Delta_n^q e^{\theta T_n}}{2e^{-2\theta T_n}S_n}\left[(\eta_{t_n}^2 - \eta_{t_{n-1}}^2) + \left(1 - \frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right)\eta_{t_{n-1}}^2 + \left(\frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right)R_n\right].
\end{aligned} \tag{3.11}$$

By (2.16), we have, as $n \to \infty$,

$$\left[\mathbf{E}\left(\Delta_n^q e^{\theta T_n}(\eta_{t_n}^2 - \eta_{t_{n-1}}^2)\right)^2\right]^{1/2} \leq \Delta_n^q\left[\mathbf{E}\left(\eta_{t_n} + \eta_{t_{n-1}}\right)^4\right]^{1/4}\left[\mathbf{E}\left(e^{\theta T_n}(\eta_{t_n} - \eta_{t_{n-1}})\right)^4\right]^{1/4}$$

$$
\begin{aligned}
&\leq 2\sqrt{3}\left[\mathbf{E}\eta_\infty^2\right]^{1/2}\Delta_n^q\left[\mathbf{E}\left(e^{\theta T_n}(\eta_{t_n}-\eta_{t_{n-1}})\right)^2\right]^{1/2}\\
&\leq \frac{2\sqrt{3}\sqrt{c(\gamma)}}{\sqrt{\gamma(2\gamma-1)}}(\mathbf{E}\eta_\infty^2)^{1/2}e^{\theta\Delta_n}\Delta_n^{q+\gamma}\left(\frac{e^{\frac{\Delta_n}{\gamma}}-1}{\frac{\Delta_n}{\gamma}}\right)^\gamma\\
&\to 0. \qquad (3.12)
\end{aligned}
$$

We also get

$$
\begin{aligned}
\Delta_n^q e^{\theta T_n}\left(1-\frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right) &= \Delta_n^{q+1}e^{\theta T_n}\frac{e^{2\theta\Delta_n}-1-2\theta\Delta_n}{\Delta_n^2}\frac{\Delta_n}{e^{2\theta\Delta_n}-1}\\
&= \left(n\Delta_n^{1+\beta}\right)^{\frac{q+1}{\beta}}\frac{e^{\theta T_n}}{T_n^{\frac{q+1}{\beta}}}\frac{e^{2\theta\Delta_n}-1-2\theta\Delta_n}{\Delta_n^2}\frac{\Delta_n}{e^{2\theta\Delta_n}-1}\\
&\to\infty,\ \text{ as } n\to\infty, \qquad (3.13)
\end{aligned}
$$

since $\lim_{n\to\infty} n\Delta_n^{1+\beta}=\infty$ for some $\beta>0$, $\lim_{n\to\infty}\frac{e^{\theta T_n}}{T_n^{\frac{q+1}{\beta}}}=\infty$, $\lim_{n\to\infty}\frac{e^{2\theta\Delta_n}-1-2\theta\Delta_n}{\Delta_n^2}=2\theta^2$ and $\lim_{n\to\infty}\frac{\Delta_n}{e^{2\theta\Delta_n}-1}=\frac{1}{2\theta}$.

Moreover, by (2.18), we deduce, as $n\to\infty$,

$$
\begin{aligned}
\left[\mathbf{E}|\Delta_n^q e^{\theta T_n}R_n|^2\right]^{1/2} &= \Delta_n^q e^{\theta T_n}\left[\mathbf{E}|R_n|^2\right]^{1/2}\\
&\leq \frac{c(2)\sqrt{c(\gamma)}c}{\theta\sqrt{\gamma(2\gamma-1)}}\Delta_n^{q+\gamma-1}\\
&\to 0. \qquad (3.14)
\end{aligned}
$$

where $c(2)$ denotes $c(p)=c(2)$ in (2.18). Combining (3.11)–(3.14) and (2.15), we conclude that for every $q\geq\frac{1}{2}$, $\Delta_n^q e^{\theta T_n}(\tilde{\theta}_n-\theta)$ is not tight.

For the case $0\leq q<\frac{1}{2}$, we obtain it similarly to the proof of (1) in Theorem 1.2. Hence (1.10) holds.

(2) By (3.2) and (2.13), note that $T_n=t_n=n\Delta_n$, we can write

$$
\begin{aligned}
&\sqrt{T_n}(\tilde{\theta}_n-\theta)\\
&= \sqrt{T_n}\left(\frac{\eta_{T_n}^2}{2e^{-2\theta T_n}S_n}-\theta\right)\\
&= \frac{\sqrt{T_n}}{2e^{-2\theta T_n}S_n}\left[(\eta_{t_n}^2-\eta_{t_{n-1}}^2)+\left(1-\frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right)\eta_{t_{n-1}}^2+\left(\frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right)R_n\right]. \qquad (3.15)
\end{aligned}
$$

Similarly to (3.12)–(3.14), we obtain, as $n\to\infty$,

$$
\left[\mathbf{E}\left(\sqrt{T_n}(\eta_{t_n}^2-\eta_{t_{n-1}}^2)\right)^2\right]^{1/2}\leq\frac{2\sqrt{3}\sqrt{c(\gamma)}}{\sqrt{\gamma(2\gamma-1)}}\left(\mathbf{E}\eta_\infty^2\right)^{1/2}\sqrt{T_n}e^{-\theta T_n}e^{\theta\Delta_n}\Delta_n^\gamma\to 0, \qquad (3.16)
$$

$$
\sqrt{T_n}\left(1-\frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right)=\sqrt{n\Delta_n^3}\frac{e^{2\theta\Delta_n}-1-2\theta\Delta_n}{\Delta_n^2}\frac{\Delta_n}{e^{2\theta\Delta_n}-1}\to 0, \qquad (3.17)
$$

and

$$
\begin{aligned}
\left[\mathbf{E}|\sqrt{T_n}R_n|^2\right]^{1/2} &\leq \frac{c(2)\sqrt{c(\gamma)}c}{\theta\sqrt{\gamma(2\gamma-1)}}\sqrt{T_n}e^{-\theta T_n}\Delta_n^{\gamma-1} \\
&= \frac{c(2)\sqrt{c(\gamma)}c}{\theta\sqrt{\gamma(2\gamma-1)}}\frac{T_n^{\frac{1}{2}+\frac{1-\gamma}{\beta}}e^{-\theta T_n}}{\left(n\Delta_n^{1+\beta}\right)^{\frac{1-\gamma}{\beta}}} \\
&\to 0. \qquad (3.18)
\end{aligned}
$$

By (3.15)–(3.18), we can easily get (1.11).

Thus, we finish the proof of Theorem 1.3.

4. Applications to fractional Gaussian processes

This section is devoted to some examples of the Gaussian process G. For example fractional Brownian motion, sub-fractional Brownian motion, bifractional Brownian motion and sub-bifractional Brownian motion.

4.1. Fractional Brownian motion

The fractional Brownian motion (fBm) $B^H = \left\{B_t^H, t \geq 0\right\}$ with Hurst parameter $H \in (0,1)$ is defined as a centered Gaussian process starting from zero with covariance

$$
R(t,s) = \mathbf{E}\left(B_t^H B_s^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \qquad (4.1)
$$

Note that, when $H = \frac{1}{2}$, $B^{\frac{1}{2}}$ is a standard Brownian motion. By (4.1), we get

$$
\phi(t,s) = \frac{\partial^2 R(t,s)}{\partial t \partial s} = H(2H-1)|t-s|^{2H-2}. \qquad (4.2)
$$

It is well-known that the self-similar index of fBm is H. Thus when $\frac{1}{2} < H < 1$, fBm satisfies the assumption (1.5). Hence Theorems 1.1–1.3 hold for the fBm $B^H(\frac{1}{2} < H < 1)$, namely, corresponding to Theorems 4, 6, and 7 in EI Onsy et al. (2019).

4.2. Sub-fractional Brownian motion

The sub-fractional Brownian motion (sfBm) $S^H = \left\{S_t^H, t \geq 0\right\}$ with Hurst parameter $H \in (0,1)$ is defined as a centered Gaussian process starting from zero with covariance

$$
R(t,s) = \mathbf{E}\left(S_t^H S_s^H\right) = t^{2H} + s^{2H} - \frac{1}{2}\left((t+s)^{2H} + |t-s|^{2H}\right). \qquad (4.3)
$$

Note that, when $H = \frac{1}{2}$, $S^{\frac{1}{2}}$ is a standard Brownian motion. The self-similar index for sfBm is also H. For more on sub-fractional Brownian motion, we can see Kuang and Xie (2015,2017), Kuang and Liu (2015,2018) and so on.

By (4.3), we have

$$\phi(t,s)=\frac{\partial^2 R(t,s)}{\partial t\partial s}=H(2H-1)\left[|t-s|^{2H-2}-(t+s)^{2H-2}\right]. \tag{4.4}$$

When $\frac{1}{2}<H<1$, by (4.4), we get

$$\phi(t,s)\leq H(2H-1)|t-s|^{2H-2}, \tag{4.5}$$

which shows that the assumption (1.5) holds for sfBm. Hence Theorems 1.1–1.3 hold for the sfBm $S^H(\frac{1}{2}<H<1)$.

4.3. *Bifractional Brownian motion*

The bifractional Brownian motion (bfBm) $B^{H,K}=\left\{B_t^{H,K},t\geq 0\right\}$ with parameters $H\in(0,1)$ and $K\in(0,1]$ is defined as a centered Gaussian process starting from zero with covariance

$$R(t,s)=\mathbf{E}\left(B_t^{H,K}B_s^{H,K}\right)=\frac{1}{2^K}\left(\left(t^{2H}+s^{2H}\right)^K-|t-s|^{2HK}\right). \tag{4.6}$$

Note that, the case $K=1$ corresponds to the fBm with Hurst parameter H. The self-similar index of bfBm is HK. By (4.6), we obtain

$$\phi(t,s)=\frac{\partial^2 R(t,s)}{\partial t\partial s}=2^{2-K}H^2K(K-1)\left(t^{2H}+s^{2H}\right)^{K-2}(ts)^{2H-1}$$
$$+2^{1-K}HK(2HK-1)|t-s|^{2HK-2}. \tag{4.7}$$

Since $K\leq 1$, when $1<2HK<2$, we have

$$\phi(t,s)\leq 2^{1-K}HK(2HK-1)|t-s|^{2HK-2}. \tag{4.8}$$

The assumption (1.5) holds for the bfBm $B^{H,K}(1<2HK<2)$. Thus we also obtain Theorems 1.1-1.3 for the bfBm $B^{H,K}(1<2HK<2)$.

4.4. *Sub-bifractional Brownian motion*

El-Nouty and Journé (2013) introduced the process $S^{H,K}=\{S_t^{H,K},t\geq 0\}$ with indices $H\in(0,1)$ and $K\in(0,1]$, named the sub-bifractional Brownian motion (sbfBm) and defined as follows:

$$S_t^{H,K}=\frac{1}{2^{(2-K)/2}}\left(B_t^{H,K}+B_{-t}^{H,K}\right),$$

where $\{B_t^{H,K},t\in\mathbf{R}\}$ is a bifractional Brownian motion (bfBm) with indices $H\in(0,1)$ and $K\in(0,1]$. Clearly, the sbfBm is a centered Gaussian process such that$S_0^{H,K}=0$, with probability 1, and $\mathrm{Var}\left(S_t^{H,K}\right)=\left(2^K-2^{2HK-1}\right)t^{2HK}$. Note that since $(2H-1)K-1<K-1\leq 0$, it follows that $2HK-1<K$. We can easily verify that $S^{H,K}$ is self-similar with index HK. When $K=1$, $S^{H,1}$ is the sub-fractional Brownian motion (sfBm). Straightforward computations show that for all $s,t\geq 0$,

$$R(t,s)=\mathbf{E}\left(S_t^{H,K}S_s^{H,K}\right)=(t^{2H}+s^{2H})^K-\frac{1}{2}\left[(t+s)^{2HK}+|t-s|^{2HK}\right], \tag{4.9}$$

and

$$C_1|t-s|^{2HK} \leq \mathbf{E}\left[\left(S_t^{H,K} - S_s^{H,K}\right)^2\right] \leq C_2|t-s|^{2HK}, \tag{4.10}$$

where

$$C_1 = \min\{2^K - 1, 2^K - 2^{2HK-1}\}, \quad C_2 = \max\{1, 2 - 2^{2HK-1}\}.$$

So $S^{H,K}$ has $(HK - \epsilon)$-Hölder continuous paths for any $\epsilon \in (0, HK)$ thanks to Kolmogorov's continuity criterion. Kuang (2019) studied the collision local time of sub-bifractional Brownian Motions. Kuang and Li (2020) obtained Berry-Esséen bounds and proved the almost sure central limit theorem for the quadratic variation of the sub-bifractional Brownian motion.

By (4.9), we have

$$\begin{aligned}\phi(t,s) = \frac{\partial^2 R(t,s)}{\partial t \partial s} &= 4H^2K(K-1)\left(t^{2H} + s^{2H}\right)^{K-2}(ts)^{2H-1} \\ &+HK(2HK-1)\left[|t-s|^{2HK-2} - (t+s)^{2HK-2}\right].\end{aligned} \tag{4.11}$$

Since $K \leq 1$, when $1 < 2HK < 2$, we have

$$\phi(t,s) \leq HK(2HK-1)|t-s|^{2HK-2}, \tag{4.12}$$

which means the assumption (1.5) also holds for the sbfBm $S^{H,K}(1 < 2HK < 2)$. Hence Theorems 1.1–1.3 also hold for the sbfBm $S^{H,K}(1 < 2HK < 2)$.

5. Numerical simulations

In this section, we firstly simulate the sample paths of the process X given by (1.1). By (2.7) and (2.8), we have

$$X_t = e^{\theta t}\gamma^{(\theta+1)\gamma}\int_{a_0}^{a_t} u^{-(\theta+1)\gamma}dG_u.$$

Let $t_i = i\Delta_n, i = 0, 1, \cdots, n, X_{t_0} = 0$, then,

$$\begin{aligned}X_{t_i} &= e^{\theta t_i}\gamma^{(\theta+1)\gamma}\int_{a_0}^{a_{t_i}} u^{-(\theta+1)\gamma}dG_u \\ &= e^{\theta t_i}\gamma^{(\theta+1)\gamma}\left(\int_{a_0}^{a_{t_{i-1}}} u^{-(\theta+1)\gamma}dG_u + \int_{a_{t_{i-1}}}^{a_{t_i}} u^{-(\theta+1)\gamma}dG_u\right) \\ &= e^{\theta\Delta_n}X_{t_{i-1}} + e^{\theta t_i}\gamma^{(\theta+1)\gamma}\int_{a_{t_{i-1}}}^{a_{t_i}} u^{-(\theta+1)\gamma}dG_u \\ &\approx e^{\theta\Delta_n}X_{t_{i-1}} + \gamma^{(\theta+1)\gamma}e^{\theta i\Delta_n}\left(\frac{a_{t_{i-1}} + a_{t_i}}{2}\right)^{-(\theta+1)\gamma}\cdot\left(G_{a_{t_i}} - G_{a_{t_{i-1}}}\right) \\ &= e^{\theta\Delta_n}X_{t_{i-1}} + \gamma^{(\theta+1)\gamma}e^{\theta i\Delta_n}\left[\frac{\gamma}{2}\left(e^{\frac{i\Delta_n}{\gamma}} + e^{\frac{(i-1)\Delta_n}{\gamma}}\right)\right]^{-(\theta+1)\gamma}\cdot\left(G_{a_{t_i}} - G_{a_{t_{i-1}}}\right),\end{aligned} \tag{5.1}$$

for $i = 1, 2, \cdots, n$.

Let

$$G_{a_{t_i}} = \sqrt{\mathrm{Var}(G_{a_{t_i}})} \cdot \xi_i, \quad i = 0, 1, \cdots, n, \tag{5.2}$$

where $\xi_i (i = 0, 1, \cdots, n)$ are standard normal random variables.

We know that

$$\hat{\theta}_n = \frac{\sum_{i=1}^n X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})}{\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2}$$

and

$$\tilde{\theta}_n = \frac{X_{T_n}^2}{2\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2}.$$

Thus, we can obtain the simulations of $\hat{\theta}_n$ and $\tilde{\theta}_n$.

Case 1: if G is fBm $B^H(\frac{1}{2} < H < 1)$, then (5.1) and (5.2) become (5.3) and (5.4), respectively,

$$X_{t_i} = e^{\theta\Delta_n} X_{t_{i-1}} + H^{(\theta+1)H} e^{\theta i\Delta_n} \left[\frac{H}{2}\left(e^{\frac{i\Delta_n}{H}} + e^{\frac{(i-1)\Delta_n}{H}}\right)\right]^{-(\theta+1)H} \cdot \left(B^H_{a_{t_i}} - B^H_{a_{t_{i-1}}}\right), \tag{5.3}$$

and

$$B^H_{a_{t_i}} = \left(He^{\frac{i\Delta_n}{H}}\right)^H \cdot \xi_i, \quad i = 0, 1, \cdots, n. \tag{5.4}$$

Case 2: if G is sfBm $S^H(\frac{1}{2} < H < 1)$, then (5.1) and (5.2) become (5.5) and (5.6), respectively,

$$X_{t_i} = e^{\theta\Delta_n} X_{t_{i-1}} + H^{(\theta+1)H} e^{\theta i\Delta_n} \left[\frac{H}{2}\left(e^{\frac{i\Delta_n}{H}} + e^{\frac{(i-1)\Delta_n}{H}}\right)\right]^{-(\theta+1)H} \cdot \left(S^H_{a_{t_i}} - S^H_{a_{t_{i-1}}}\right), \tag{5.5}$$

and

$$S^H_{a_{t_i}} = \sqrt{2 - 2^{2H-1}}\left(He^{\frac{i\Delta_n}{H}}\right)^H \cdot \xi_i, \quad i = 0, 1, \cdots, n. \tag{5.6}$$

Case 3: if G is bfBm $B^{H,K}(1 < 2HK < 2)$, then (5.1) and (5.2) are replaced by (5.7) and (5.8), respectively,

$$X_{t_i} = e^{\theta\Delta_n} X_{t_{i-1}} + (HK)^{(\theta+1)HK} e^{\theta i\Delta_n} \left[\frac{HK}{2}\left(e^{\frac{i\Delta_n}{HK}} + e^{\frac{(i-1)\Delta_n}{HK}}\right)\right]^{-(\theta+1)HK} \cdot \left(B^{H,K}_{a_{t_i}} - B^{H,K}_{a_{t_{i-1}}}\right), \tag{5.7}$$

and

$$B^{H,K}_{a_{t_i}} = \left(HKe^{\frac{i\Delta_n}{HK}}\right)^{HK} \cdot \xi_i, \quad i = 0, 1, \cdots, n. \tag{5.8}$$

Case 4: if G is sbfBm $S^{H,K}(1 < 2HK < 2)$, then (5.1) and (5.2) become (5.9) and (5.10), respectively,

$$X_{t_i} = e^{\theta\Delta_n} X_{t_{i-1}} + (HK)^{(\theta+1)HK} e^{\theta i\Delta_n} \left[\frac{HK}{2}\left(e^{\frac{i\Delta_n}{HK}} + e^{\frac{(i-1)\Delta_n}{HK}}\right)\right]^{-(\theta+1)HK} \cdot \left(S^{H,K}_{a_{t_i}} - S^{H,K}_{a_{t_{i-1}}}\right), \tag{5.9}$$

and

$$S^{H,K}_{a_{t_i}} = \sqrt{2^K - 2^{2HK-1}}\left(HKe^{\frac{i\Delta_n}{HK}}\right)^{HK} \cdot \xi_i, \quad i = 0, 1, \cdots, n. \tag{5.10}$$

Now we take $\Delta_n = 0.0002, n = 2 \times 10^5$, and simulate 500 sample paths of X for different values of H, θ or H, K, θ. The results of simulations are summarized in Tables 1–4 respectively. From these results, it can be seen that the mean of both constructed parameter estimators are close to the true parameter values and the standard deviations of $\hat{\theta}_n$ and $\tilde{\theta}_n$ are very small. Hence the numerical simulations confirm the theoretical research.

Table 1. Mean and standard deviation of two estimators for fBm.

	$H = 0.55$	$H = 0.60$	$H = 0.65$	$H = 0.70$
$\theta = 0.8$				
Mean($\hat{\theta}_n$)	0.8000	0.8000	0.8000	0.8000
SD($\hat{\theta}_n$)	8.9194e-08	2.6459e-07	1.4519e-07	5.6355e-07
Mean($\tilde{\theta}_n$)	0.8001	0.8001	0.8001	0.8001
SD($\tilde{\theta}_n$)	8.9197e-08	2.6461e-07	1.4520e-07	5.6338e-07
$\theta = 1.7$				
Mean($\hat{\theta}_n$)	1.7001	1.7001	1.7001	1.7001
SD($\hat{\theta}_n$)	1.0967e-14	3.1736e-14	1.2694e-14	1.2644e-14
Mean($\tilde{\theta}_n$)	1.7003	1.7003	1.7003	1.7003
SD($\tilde{\theta}_n$)	1.0865e-14	3.1832e-14	1.2580e-14	1.3038e-14
$\theta = 3.7$				
Mean($\hat{\theta}_n$)	3.7007	3.7007	3.7007	3.7007
SD($\hat{\theta}_n$)	1.1281e-14	1.0869e-14	1.1878e-14	1.1458e-14
Mean($\tilde{\theta}_n$)	3.7014	3.7014	3.7014	3.7014
SD($\tilde{\theta}_n$)	1.0194e-14	9.9964e-15	1.1395e-14	1.1431e-14

Table 2. Mean and standard deviation of two estimators for sfBm.

	$H = 0.55$	$H = 0.60$	$H = 0.65$	$H = 0.70$
$\theta = 0.8$				
Mean($\hat{\theta}_n$)	0.7967	0.8000	0.8000	0.8000
SD($\hat{\theta}_n$)	2.6371e-02	3.9579e-07	2.0294e-07	1.2505e-06
Mean($\tilde{\theta}_n$)	0.8000	0.8001	0.8001	0.8001
SD($\tilde{\theta}_n$)	2.0488e-04	3.9564e-07	2.0297e-07	1.2463e-06
$\theta = 1.7$				
Mean($\hat{\theta}_n$)	1.7001	1.7001	1.7001	1.7001
SD($\hat{\theta}_n$)	1.2324e-14	1.6913e-14	1.3377e-14	1.6702e-14
Mean($\tilde{\theta}_n$)	1.7003	1.7003	1.7003	1.7003
SD($\tilde{\theta}_n$)	1.1958e-14	1.6863e-14	1.3260e-14	1.6726e-14
$\theta = 3.7$				
Mean($\hat{\theta}_n$)	3.7007	3.7007	3.7007	3.7007
SD($\hat{\theta}_n$)	1.1491e-14	1.0355e-14	1.0807e-14	1.0539e-14
Mean($\tilde{\theta}_n$)	3.7014	3.7014	3.7014	3.7014
SD($\tilde{\theta}_n$)	1.1411e-14	1.0602e-14	1.0469e-14	1.0630e-14

Table 3. Mean and standard deviation of two estimators for bfBm.

	$H = 0.6$ $K = 0.9$	$H = 0.8$ $K = 0.9$	$H = 0.9$ $K = 0.9$
$\theta = 0.8$			
Mean($\hat{\theta}_n$)	0.8000	0.8000	0.8000
SD($\hat{\theta}_n$)	3.4198e-07	2.7941e-06	3.5679e-07
Mean($\tilde{\theta}_n$)	0.8001	0.8001	0.8001
SD($\tilde{\theta}_n$)	3.4207e-07	2.6092e-06	3.5678e-07
$\theta = 1.7$			
Mean($\hat{\theta}_n$)	1.7001	1.7001	1.7001
SD($\hat{\theta}_n$)	1.0361e-14	4.0730e-12	1.0888e-14
Mean($\tilde{\theta}_n$)	1.7003	1.7003	1.7003
SD($\tilde{\theta}_n$)	9.4681e-15	4.0742e-12	1.0035e-14
$\theta = 3.7$			
Mean($\hat{\theta}_n$)	3.7007	3.7007	3.7007
SD($\hat{\theta}_n$)	1.0713e-14	1.2432e-14	1.1264e-14
Mean($\tilde{\theta}_n$)	3.7014	3.7014	3.7014
SD($\tilde{\theta}_n$)	1.0130e-14	1.1353e-14	1.0026e-14

Table 4. Mean and standard deviation of two estimators for sbfBm.

	$H = 0.6$ $K = 0.9$	$H = 0.8$ $K = 0.9$	$H = 0.9$ $K = 0.9$
$\theta = 0.8$			
Mean($\hat{\theta}_n$)	0.8000	0.8000	0.8000
SD($\hat{\theta}_n$)	5.1295e-07	2.9938e-07	7.7103e-07
Mean($\tilde{\theta}_n$)	0.8001	0.8001	0.8001
SD($\tilde{\theta}_n$)	5.1320e-07	2.9930e-07	7.7119e-07
$\theta = 1.7$			
Mean($\hat{\theta}_n$)	1.7001	1.7001	1.7001
SD($\hat{\theta}_n$)	1.2144e-14	1.1284e-14	1.4808e-14
Mean($\tilde{\theta}_n$)	1.7003	1.7003	1.7003
SD($\tilde{\theta}_n$)	1.1640e-14	1.1001e-14	1.4746e-14
$\theta = 3.7$			
Mean($\hat{\theta}_n$)	3.7007	3.7007	3.7007
SD($\hat{\theta}_n$)	1.1830e-14	1.0794e-14	1.1455e-14
Mean($\tilde{\theta}_n$)	3.7014	3.7014	3.7014
SD($\tilde{\theta}_n$)	1.0983e-14	9.7612e-15	1.1214e-14

6. Conclusions

EI Onsy et al. (2018) considered the parameter estimation for discretely observed nonergodic fractional Ornstein-Uhlenbeck process of the second kind. We extend their results to the case of Gaussian Ornstein-Uhlenbeck process of the second kind. We prove that two least squares type estimators are strongly consistent and rate consistent. Moreover, we give the numerical simulations which confirm the theoretical results. In the future, we will extend our results to more general Ornstein-Uhlenbeck type models such as Barndorff-Nielsen and Shephard type Ornstein-Uhlenbeck processes. On the other hand, we will consider the case of $0 < \gamma < \frac{1}{2}$ for the self-similar index γ of Gaussian process G.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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