Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(1): 784–803.
DOI: 10.3934/math.2022047
Received: 13 July 2021
Accepted: 27 September 2021
Published: 18 October 2021

Research article

Fractional version of the Jensen-Mercer and Hermite-Jensen-Mercer type inequalities for strongly h-convex function

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Abstract: In this paper we find further versions of generalized Hadamard type fractional integral inequality for k-fractional integrals. For this purpose we utilize the definition of h-convex function. The presented results hold simultaneously for variant types of convexities and fractional integrals.

Keywords: convex function; Hermite-Hadamard inequality; Jensen-Mercer inequality; fractional integrals
Mathematics Subject Classification: 35A23, 26E70, 34N05

1. Introduction

Convex functions and mathematical inequalities play vital role in the advancement of several fields of pure and applied sciences. Various inequalities for convex and related functions have been studied in recent decades, also their variants in fractional calculus are analyzed, see [1–5].

The employment of fractional integral operators for establishing the generalized versions of classical inequalities become a fashion in the study of mathematical inequalities. In this regard the most celebrated inequality called Hadamard inequality has been studied extensively for fractional integral operators. The presence of Mittag-Leffler function in fractional integral operators produces interesting fractional integral inequalities.

The purpose of this paper is to present new Hadamard type inequalities for k-fractional integrals of strongly h-convex functions.

Let $f \in [\phi, \varphi]$ then the Riemann-Liouville fractional integrals $J^{\varepsilon}_{\phi+}f$ and $J^{\varepsilon}_{\varphi-}f$ of order ε with $\varepsilon > 0$, $0 \leq \phi < \varphi$ are defined by [6]:

$$\left(J^{\varepsilon}_{\phi+}\right) f(v) = \frac{1}{\Gamma(\varepsilon)} \int_{\phi}^{v} (v-t)^{\varepsilon-1} f(t)\, dt \qquad (v > \phi), \tag{1.1}$$

and

$$\left(J_{\varphi-}^{\varepsilon}\right) f(v)=\frac{1}{\Gamma(\varepsilon)} \int_{v}^{\varphi}(t-v)^{\varepsilon-1} f(t)\, dt \qquad (v<\varphi). \tag{1.2}$$

respectively, where $\Gamma(\varepsilon)=\int_0^{\infty} e^{-t} t^{\phi-1} dt$ is the Gamma function and $\left(J_{\phi+}^{0}\right) f(v)=\left(J_{\varphi-}^{0}\right) f(v)=f(v)$.

In [7], Jarad et al. defined the new fractional integral operator as follows:

$${}_{\phi}^{\delta}J^{\varepsilon}t(v)=\frac{1}{\Gamma(\delta)} \int_{\phi}^{v}\left(\frac{(v-\phi)^{\varepsilon}-(t-\phi)^{\varepsilon}}{\varepsilon}\right)^{\delta-1} \frac{f(t)}{(t-\phi)^{1-\varepsilon}} dt, \tag{1.3}$$

and

$${}^{\delta}J_{\varphi}^{\varepsilon}t(v)=\frac{1}{\Gamma(\delta)} \int_{v}^{\varphi}\left(\frac{(\varphi-v)^{\gamma}-(\varphi-t)^{\varepsilon}}{\varepsilon}\right)^{\delta-1} \frac{f(t)}{(\varphi-t)^{1-\varepsilon}} dt. \tag{1.4}$$

Note that, by taking $\phi=0$ and $\varepsilon \to 0$, the new conformable fractional integral operator coincides with the generalized fractional integrals [8], by taking $\phi=0$ and $\varepsilon=1$, the (1.3) becomes Riemann-Liouville fractional operator (1.1). Furthermore, by taking $\varphi=0$ and $\varepsilon=1$, the (1.4) reduces to the Riemann-Liouville.

The generalized k-fractional conformable integrals [9] are defined as follows:

$${}_{k}^{\delta}J_{\phi^{+}}^{\varepsilon}t(v)=\frac{1}{k\Gamma_{k}(\delta)} \int_{\phi}^{v}\left(\frac{(v-\phi)^{\varepsilon}-(t-\phi)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}-1} \frac{f(t)}{(t-\phi)^{1-\varepsilon}} dt, \tag{1.5}$$

and

$${}_{k}^{\delta}J_{\varphi^{-}}^{\varepsilon}t(v)=\frac{1}{k\Gamma_{k}(\delta)} \int_{v}^{\varphi}\left(\frac{(\varphi-v)^{\varepsilon}-(\varphi-t)^{\gamma}}{\varepsilon}\right)^{\frac{\delta}{k}-1} \frac{f(t)}{(\varphi-t)^{1-\varepsilon}} dt. \tag{1.6}$$

If k > 0, then the k-Gamma function Γ_k is defined as [10]:

$$\Gamma_{k}(\varepsilon)=\lim_{m \to \infty} \frac{m!k^{m}(mk)^{\frac{\varepsilon}{k}-1}}{(\varepsilon)_{m,k}}. \tag{1.7}$$

If Re ($\varepsilon > 0$), then k-Gamma function in integral form is defined as

$$\Gamma_{k}(\varepsilon)=\int_{0}^{\infty} \exp^{-\frac{\phi^{k}}{k}} \phi^{\varepsilon-1} d\phi$$

with $\varepsilon\Gamma_k(\varepsilon)=\Gamma_k(\varepsilon+k)$.

Let $f: I^{\circ} \subseteq \mathrm{R} \to \mathrm{R}$ be a convex function. Then

$$f\left(\frac{\phi+\varphi}{2}\right) \leq \frac{1}{\varphi-\phi} \int_{\phi}^{\varphi} f(t) dt \leq \frac{f(\phi)+f(\varphi)}{2}.$$

holds $\forall\ \phi, \varphi \in I^{\circ}$ with $\phi \neq \varphi$ and is known as Hermite-Hadamard inequality. The above inequality is reversed, if f is a concave function on I°, see [11–15].

In [16], the following generalization of Hermite-Hadamard inequality for strongly h-convex function is established:

Let $f : I^{\circ} \subseteq R \to R$ is Lebesgue integrable and strongly h-convex function with modulus c > 0. If $h : (0,1) \to (0,\infty)$ be a given function. Then

$$\frac{1}{2h\left(\frac{1}{2}\right)}\left[f\left(\frac{u+v}{2}\right)+\frac{c}{12}(v-u)^2\right] \leq \frac{1}{v-u}\int_u^v f(x)dx \leq (f(u)+f(v))\int_0^1 h(t)dt - \frac{c}{6}(v-u)^2.$$

holds ∀ u,v ∈ I°, u < v.

Some further interesting extensions and refinements of the Hermite-Hadamard inequalities for strongly h-convex mappings have been widely studied in the literature, see [17]. The other well known inequality is Jensen inequality and different variants of Jensen-Mercer inequalities are present in literature, see [18–21].

In this paper we find further versions of generalized Hadamard type fractional integral inequality for k-fractional integrals. For this purpose we utilize the definition of h-convex function.

2. Definitions

Next, we recall some basic definitions which will be used in the present article:
The gamma function is denoted by $\Gamma(.)$ and is defined as follows:

$$\Gamma(\tau) = \int_0^{\infty} e^{-t}t^{\tau-1}dt, \quad \tau > 0,$$

The beta function is denoted by β and is defined as follows:

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

where x, y, q are positive real numbers.

The $L_1[a,b]$ is a class of all real-valued functions integrable in $[a,b]$.

Further, we give definitions of convex and related functions which are useful to understand the inequalities established in this work.

Definition 2.1. *[23] A function $f : [a,b] \to \mathbb{R}$ is said to be convex if*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y),\ \forall x,y \in [a,b], t \in [0,1]. \tag{2.1}$$

holds. If the inequality in (2.1) is reversed, then f is called concave function.

Definition 2.2. *[24, 25] Let $I \subset (0,\infty)$ be an interval and $p \in \mathbb{R}\setminus\{0\}$. Then a function $f : I \to \mathbb{R}$ is said to be p-convex, if*

$$f\left((ta^p+(1-t)b^p)^{\frac{1}{p}}\right) \leq tf(a)+(1-t)f(b). \tag{2.2}$$

holds for $a, b \in I$ and $t \in [0,1]$. If the inequality in (2.2) is reversed, then f is called p-concave function.

Definition 2.3. *[23] Let I, J be intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and $h : J \to \mathbb{R}$ be a non-negative function and $h \neq 0$. A function $f : I \to \mathbb{R}$ is said to be an h-convex, if*

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y). \tag{2.3}$$

holds for $x, y \in I$ and $t \in (0,1)$. If the inequality in (2.3) is reversed, then f is called h-concave function.

Definition 2.4. *[26] Let $h : J \supseteq (0,1) \to \mathbb{R}$ be a positive function. Let $I \subset (0,\infty)$ be an interval and $p \in \mathbb{R} \setminus \{0\}$. A function $f : I \to \mathbb{R}$ is said to be (p,h)-convex, if*

$$f\left((ta^p + (1-t)b^p)^{\frac{1}{p}}\right) \leq h(t)f(a) + h(1-t)f(b). \tag{2.4}$$

holds for $a, b \in I$ and $t \in [0,1]$. If the inequality in (2.4) is reversed, then f is called (p,h)-concave function.

Definition 2.5. *A function $h : I^\circ \subseteq R \to R$ is called superadditive function if*

$$h(u + v) \geq h(u) + h(v). \tag{2.5}$$

$\forall$ *$u,v \in I^\circ$.*

Definition 2.6. *[27] Let $h : J \supseteq (0,1) \to \mathbb{R}$ be a positive function and $I \subset (0,\infty)$ be an interval. A function $f : I^\circ \subseteq R \to R$ is said to be strongly convex function with modulus c, $u,v \in I^\circ$ and $t \in [0,1]$ if the following inequality holds*

$$f(tu + (1-t)v) \leq tf(u) + 1 - tf(v) - ct(1-t)(u-v)^2. \tag{2.6}$$

Definition 2.7. *[16] Let $h : J \supseteq (0,1) \to \mathbb{R}$ be a positive function and $I \subset (0,\infty)$ be an interval. A function $f : I^\circ \subseteq R \to R$ is said to be strongly convex function with modulus c, $u,v \in I^\circ$ and $t \in [0,1]$ if the following inequality holds*

$$f(tu + (1-t)v) \leq h(t) f(u) + h(1-t) f(v) - ct(1-t)(u-v)^2. \tag{2.7}$$

3. Main results

3.1. Jensen-Mercer type inequality

Our main purpose of this section is to prove the Jensen-Mercer inequality for strongly h-convex function.To present Jensen-Mercer inequality first we will prove the following lemma:

Let $0 < u_1 \leq u_2 \leq ... \leq u_n$ be real numbers and let $t_j (1 \leq j \leq n)$ be positive weights associated with these u_j and $\sum_{J=1}^{n} u_j = 1$. let $h : I^\circ \to$ R be a non negative function defined over an interval $I^\circ \subset R$ such that $(0,1) \in I^\circ$.

Lemma 3.1. *Let $f : I^\circ \to R$ be a strongly h-convex function with modulus c > 0, then*

$$f\left(u_1 + u_n - u_j\right) \leq M' \left(f(u_1) + f(u_n)\right) - f(u_j) - 2ct(1-t)(u_1 - u_n)^2. \tag{3.1}$$

for any $u_j \in I^\circ, (1 \leq j \leq n)$ and for all $t \in (0,1)$, where $M' = \sup\{h(t) : t \in (0,1)\}$.

Proof. Write $v_j = u_1 + u_n - u_j$. Then $u_1 + u_n = u_j + v_j$ so that the pairs u_1, u_n and u_j, v_j posses the same mid point. Since that is the case, there exist a t such that

$$u_j = tu_1 + (1-t)u_n \quad \text{and} \quad v_j = (1-t)u_1 + tu_n.$$

where $0 \le t \le 1$ and $1 \le j \le n$. Then, applying strongly h-convexity of f

$$\begin{aligned} f(v_j) =& f((1-t)u_1 + tu_n) \\ \le& h(1-t)f(u_1) + h(t)f(u_n) - ct(1-t)(u_1-u_n)^2 \\ =& h(1-t)f(u_1) + h(t)f(u_n) + h(t)f(u_1) + h(1-t)f(u_n) \\ & - (h(t)f(u_1) + h(1-t)f(u_n) - ct(1-t)(u_1-u_n)^2) - 2ct(1-t)(u_1-u_n)^2 \\ =& (h(t)+h(1-t))(f(u_1)+f(u_n)) - (h(t)f(u_1) + h(1-t)f(u_n) \\ & - ct(1-t)(u_1-u_n)^2) - 2ct(1-t)(u_1-u_n)^2 \\ \le& (h(t)+h(1-t))(f(u_1)+f(u_n)) - f(tu_1 + (1-t)u_n) - 2ct(1-t)(u_1-u_n)^2 \\ \le& 2M(f(u_1)+f(u_n)) - f(u_j) - 2ct(1-t)(u_1-u_n)^2 \\ \le& M'(f(u_1)+f(u_n)) - f(u_j) - 2ct(1-t)(u_1-u_n)^2. \end{aligned}$$

where $M' = \sup\{h(t) : t \in (0,1)\}$ and since $v_j = u_1 + u_n - u_j$, then

$$f\left(u_1 + u_n - u_j\right) \le M'(f(u_1)+f(u_n)) - f(u_j) - 2ct(1-t)(u_1-u_n)^2.$$

This completes the proof. □

Remark 3.2. *1. Taking $M' = \sup\{h(t) : t \in (0,1)\} = 1$ and $c = 0$ in Lemma 3.1, we get Lemma 1.3 of [20].*
2. Taking $c = 0$ in Lemma 3.1, we get Lemma 3.1 of [18].
3. Taking $M' = \sup\{h(t) : t \in (0,1)\} = 1$ in Lemma 3.1, we get Lemma 2.2 of [19].

Theorem 3.3. *Let $f : I^\circ \subset R \to R$ be a strongly h-convex function with modulus $c > 0$ and $0 < u_1 \le u_2 \le \ldots \le u_n$ be real numbers in I°. If $t_j (1 \le j \le n)$ be positive numbers such that $\sum_{j=1}^n t_j = 1$ and $\bar{u} = \sum_{j=1}^n t_j u_j$. Then*

$$\begin{aligned} & f\left(u_1 + u_n - \sum_{j=1}^n t_j u_j\right) \\ & \quad \le M'(f(u_1)+f(u_n)) - \sum_{j=1}^n h(t_j)f(u_j) - c\left(2\sum_{j=1}^n h(t_j)t(1-t)(u_1-u_n)^2 + \sum_{j=1}^n t_j(u_j-\bar{u})^2\right), \end{aligned} \tag{3.2}$$

where $M' = \sup\{h(t) : t \in (0,1)\}$.

Proof. Since $\sum_{j=1}^n t_j = 1$ we have

$$f\left(u_1 + u_n - \sum_{j=1}^n t_j u_j\right) = f\left(\sum_{j=1}^n t_j\left(u_1 + u_n - u_j\right)\right) \le \sum_{j=1}^n h(t_j)f(u_1+u_n-u_j) - c\sum_{j=1}^n t_j(u_j-\bar{u})^2.$$

By using Lemma 3.1, we have

$$
\begin{aligned}
f\left(u_1+u_n-\sum_{j=1}^{n} t_j u_j\right) &\le \sum_{j=1}^{n} h(t_j) f(u_1+u_n-u_j) - c\sum_{j=1}^{n} t_j (u_j-\bar{u})^2 \\
&\le \sum_{j=1}^{n} h(t_j)[M'(f(u_1)+f(u_n)) - f(u_j) - 2ct(1-t)(u_1-u_n)^2] - c\sum_{j=1}^{n} t_j(u_j-\bar{u})^2 \\
&\le M'(f(u_1)+f(u_n)) - \sum_{j=1}^{n} h(t_j) f(u_j) - c\left(2\sum_{j=1}^{n} h(t_j) t(1-t)(u_1-u_n)^2 + \sum_{j=1}^{n} t_j (u_j-\bar{u})^2\right).
\end{aligned}
$$

This completes the proof. □

Remark 3.4. *1. Taking $c = 0$ in Theorem 3.3,we get Theorem 3.2 of [18].*

2. *Taking $h(t) = t$ and $M' = \sup\{h(t) : t \in (0,1)\} = 1$ in Theorem 3.3, we get Theorem 2.2 of [19].*
3. *Taking $c = 0$, $h(t) = t$ and $M' = \sup\{h(t) : t \in (0,1)\} = 1$ in Theorem 3.3, we get Theorem 1.2 of [20]).*

3.2. Hermite-Jensen-Mercer type inequalities

Theorem 3.5. *Assume $f : I^\circ \subseteq R \to R$ be a strongly h-convex function with modulus $c > 0$ and $\phi, \varphi \in I^\circ$ with $\phi < \varphi$. If h is a super-additive function and take $\varepsilon, \delta > 0$. Then*

$$
\begin{aligned}
f\left(\phi+\varphi-\frac{u+v}{2}\right) \le & h\left(\frac{1}{2}\right)\frac{2^{\varepsilon\frac{\delta}{k}}\varepsilon^{\frac{\delta}{k}}\Gamma_k(\delta+k)}{(v-u)^{\varepsilon\frac{\delta}{k}}}\left\{{}^{\delta}_{k}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^+} f(\phi+\varphi-u) + {}^{\delta}_{k}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^-} f(\phi+\varphi-v)\right\} \\
& - c\left(\frac{1}{4}\right)\left(\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(u-v)^2 \\
\le & h\left(\frac{1}{2}\right)2M'[f(\phi)+f(\varphi)] - h\left(\frac{1}{2}\right)h(1)[f(u)+f(v)] \\
& - c\left(\frac{1}{2}h\left(\frac{1}{2}\right)\left(1-\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(u-v)^2 + \frac{1}{4}\left(\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(u-v)^2\right). \quad (3.3)
\end{aligned}
$$

holds $\forall$ u,v $\in [\phi, \varphi]$, where $M' = \sup\{h(t) : t \in (0,1)\}$.

Proof. Since f is strongly h-convex function then we write

$$
\begin{aligned}
f\left(\phi+\varphi-\frac{u_1+v_1}{2}\right) &= f\left(\frac{\phi+\varphi-u_1+\phi+\varphi-v_1}{2}\right) \\
&\le h\left(\frac{1}{2}\right)(f(\phi+\varphi-u_1)+f(\phi+\varphi-v_1)) - c(\frac{1}{4})\left((\phi+\varphi-u_1)-(\phi+\varphi-v_1)\right)^2.
\end{aligned}
$$

$\forall\, u_1, v_1 \in [\phi, \varphi]$.

If we take $u_1 = \frac{t}{2}u + \frac{2-t}{2}v$ and $v_1 = \frac{2-t}{2}u + \frac{t}{2}v$. Then for u,v $\in [\phi, \varphi]$ and $t \in [0, 1]$, we have

$$
f\left(\phi+\varphi-\frac{u_1+v_1}{2}\right)
$$

$$\leq h\left(\frac{1}{2}\right)\left(f\left(\phi+\varphi-\left(\frac{t}{2}u+\frac{2-t}{2}v\right)\right)+f\left(\phi+\varphi-\left(\frac{2-t}{2}u+\frac{t}{2}v\right)\right)\right)-c\left(\frac{1}{4}\right)(1-t)^2(u-v)^2. \quad (3.4)$$

Multiplying (3.4) by $\left(\frac{1-(1-t)^\varepsilon}{\gamma}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}$ and integrating the resulting inequality with respect to t over [0,1] and then combining the obtained inequality with the definition of integral operator gives

$$\begin{aligned}
f\left(\phi+\varphi-\frac{u+v}{2}\right)&\int_0^1\left(\frac{1-(1-t)^\varepsilon}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}dt\\
&\leq\int_0^1\left(\frac{1-(1-t)^\varepsilon}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}\Bigg[h\left(\frac{1}{2}\right)\Bigg\{f\left(\phi+\varphi-\left(\frac{t}{2}u+\frac{2-t}{2}v\right)\right)\\
&\quad+f\left(\phi+\varphi-\left(\frac{2-t}{2}u+\frac{t}{2}v\right)\right)\Bigg\}-c\left(\frac{1}{4}\right)(1-t)^2(u-v)^2\Bigg]dt\\
&=h\left(\frac{1}{2}\right)\Bigg\{\int_{\phi+\varphi-v}^{\phi+\varphi-\frac{u+v}{2}}\left(\frac{1-\left(\frac{(\phi+\varphi-\frac{u+v}{2})-t_1}{\frac{v-u}{2}}\right)^\varepsilon}{\varepsilon}\right)^{\frac{\delta}{k}-1}\left(\frac{(\phi+\varphi-\frac{u+v}{2})-t_1}{\frac{v-u}{2}}\right)^{\varepsilon-1}f(t_1)\\
&\quad\times\left(\frac{2}{v-u}dt_1\right)+\int_{\phi+\varphi-\frac{u+v}{2}}^{\phi+\varphi-u}\left(\frac{1-\left(\frac{t_2-(\phi+\varphi-\frac{u+v}{2})}{\frac{v-u}{2}}\right)^\varepsilon}{\varepsilon}\right)^{\frac{\delta}{k}-1}\left(\frac{t_2-(\phi+\varphi-\frac{u+v}{2})}{\frac{v-u}{2}}\right)^{\varepsilon-1}\\
&\quad\times\left(f(t_2)\frac{2}{v-u}dt_2\right)\Bigg\}-c\left(\frac{1}{4}\right)(u-v)^2\int_0^1\left(\frac{1-(1-t)^\varepsilon}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon+1}dt\\
&=h\left(\frac{1}{2}\right)\left(\frac{2}{v-u}\right)^{\varepsilon\frac{\delta}{k}}\Bigg\{k\Gamma_k(\delta)\,{}_k^\delta J^\gamma_{(\phi+\varphi-\frac{u+v}{2})^+}f(\phi+\varphi-u)\\
&\quad+k\Gamma_k(\delta)\,{}_k^\delta J^\gamma_{(\phi+\varphi-\frac{u+v}{2})^-}f(\phi+\varphi-v)\Bigg\}-c\left(\frac{1}{4}\right)\left(\frac{1}{\varepsilon^{\frac{\delta}{k}}}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(u-v)^2.
\end{aligned}$$

Note that

$$\int_0^1\left(\frac{1-(1-t)^\varepsilon}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}dt=\frac{1}{\frac{\delta}{k}\varepsilon^{\frac{\delta}{k}}}. \quad (3.5)$$

By simple calculations, we have

$$\begin{aligned}
f\left(\phi+\varphi-\frac{u+v}{2}\right)\leq&h\left(\frac{1}{2}\right)\frac{2^{\varepsilon\frac{\delta}{k}}\varepsilon^{\frac{\delta}{k}}\Gamma_k(\delta+k)}{(v-u)^{\gamma\frac{\delta}{k}}}\left\{{}_k^\delta J^\varepsilon_{(\phi+\varphi-\frac{u+v}{2})^+}f(\phi+\varphi-u)+{}_k^\delta J^\varepsilon_{(\phi+\varphi-\frac{u+v}{2})^-}f(\phi+\varphi-v)\right\}\\
&-c\left(\frac{1}{4}\right)\left(\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(u-v)^2. \quad (3.6)
\end{aligned}$$

which completes the proof of first inequality of (3.3).

For second part of inequality (3.3) take the definition of strongly h-convex function and by similar discussion, we get

$$f\left(\phi+\varphi-\left(\frac{t}{2}u+\frac{2-t}{2}v\right)\right)\leq M'[f(\phi)+f(\varphi)]-\left(h\left(\frac{t}{2}\right)f(u)+h\left(\frac{2-t}{2}\right)f(v)\right)-c\left(\frac{t}{2}\left(\frac{2-t}{2}\right)\right)(u-v)^2$$

and

$$f\left(\phi+\varphi-\left(\frac{2-t}{2}u+\frac{t}{2}v\right)\right)\leq M'[f(\phi)+f(\varphi)]-\left(h\left(\frac{2-t}{2}\right)f(u)+h\left(\frac{t}{2}\right)f(v)\right)-c\left(\left(\frac{2-t}{2}\right)\frac{t}{2}\right)(u-v)^2.$$

Adding the above two inequalities give

$$f\left(\phi+\varphi-\left(\frac{t}{2}u+\frac{2-t}{2}v\right)\right)+f\left(\phi+\varphi-\left(\frac{2-t}{2}u+\frac{t}{2}v\right)\right)$$
$$\leq 2M'\left[f(\phi)+f(\varphi)\right]-\left[\left(h\left(\frac{t}{2}\right)+h\left(\frac{2-t}{2}\right)\right)f(u)+\left(h\left(\frac{2-t}{2}\right)+h\left(\frac{t}{2}\right)\right)f(v)\right]-2c\left(\left(\frac{2-t}{2}\right)\frac{t}{2}\right)(u-v)^2.$$

By using super-additivity of function, we have

$$f\left(\phi+\varphi-\left(\frac{t}{2}u+\frac{2-t}{2}v\right)\right)+f\left(\phi+\varphi-\left(\frac{2-t}{2}u+\frac{t}{2}v\right)\right)$$
$$\leq 2M'[f(\phi)+f(\varphi)]-h(1)[f(u)+f(v)]-2c\left(\frac{2t-t^2}{4}\right)(u-v)^2. \tag{3.7}$$

Multiplying (3.7) by $\left(\frac{1-(1-t)^\gamma}{\gamma}\right)^{\frac{\delta}{k}-1}(1-t)^{\gamma-1}$ and then integrating the resulting inequality with respect to t over [0,1], we obtain

$$\int_0^1\left(\frac{1-(1-t)^\varepsilon}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}\left\{f\left(\phi+\varphi-\left(\frac{t}{2}u+\frac{2-t}{2}v\right)\right)+f\left(\phi+\varphi-\left(\frac{2-t}{2}u+\frac{t}{2}v\right)\right)\right\}dt$$
$$\leq\left\{2M'\left[f(\phi)+f(\varphi)\right]-h(1)\left[f(u)+f(v)\right]-2c\left(\frac{2t-t^2}{4}\right)(u-v)^2\right\}\left(\int_0^1\left(\frac{1-(1-t)^\varepsilon}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}dt\right).$$

By computing the above integrals, we have

$$\left(\frac{2}{v-u}\right)^{\varepsilon\frac{\delta}{k}}\left\{k\Gamma_k(\delta)^{\delta}_{k}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^+}f(\phi+\varphi-u)+k\Gamma_k(\delta)^{\delta}_{k}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^-}f(\phi+\varphi-v)\right\}$$
$$\leq\frac{1}{\frac{\delta}{k}\varepsilon^{\frac{\delta}{k}}}\left\{2M'[f(\phi)+f(\varphi)]-h(1)[f(u)+f(v)]\right\}-\frac{1}{2}c\left(\frac{1}{\frac{\delta}{k}\varepsilon^{\frac{\delta}{k}}}-\frac{1}{\varepsilon^{\frac{\delta}{k}}}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(u-v)^2.$$

that is,

$$\frac{2^{\varepsilon\frac{\delta}{k}}\varepsilon^{\frac{\delta}{k}}\Gamma_k(\delta+k)}{(v-u)^{\gamma\frac{\delta}{k}}}\left\{^{\delta}_{k}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^+}f(\phi+\varphi-u)+^{\delta}_{k}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^-}f(\phi+\varphi-v)\right\}$$
$$\leq 2M'[f(\phi)+f(\varphi)]-h(1)[f(u)+f(v)]-\frac{1}{2}c\left(1-\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(u-v)^2. \tag{3.8}$$

Multiplying both sides of (3.8) by $h\left(\frac{1}{2}\right)$ and then subtract $c\left(\frac{1}{4}\right)\left(\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(u-v)^2$ leads to the conclusion that

$$h\left(\frac{1}{2}\right)\frac{2^{\varepsilon\frac{\delta}{k}}\varepsilon^{\frac{\delta}{k}}\Gamma_k(\delta+k)}{(v-u)^{\gamma\frac{\delta}{k}}}\left\{^{\delta}_{k}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^+}f(\phi+\varphi-u)+^{\delta}_{k}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^-}f(\phi+\varphi-v)\right\}-c\left(\frac{1}{4}\right)\left(\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(u-v)^2$$

$$\leq h\left(\frac{1}{2}\right)2M^{'}[f(\phi)+f(\varphi)]-h\left(\frac{1}{2}\right)h(1)[f(u)+f(v)]$$
$$-c\left(\frac{1}{2}h\left(\frac{1}{2}\right)\left(1-\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(u-v)^2+\frac{1}{4}\left(\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(u-v)^2\right). \tag{3.9}$$

Combining (3.6) and (3.9) lead to (3.3). □

Remark 3.6. *1. Taking $h(t)=t$, $M^{'}=sup\,\{h(t):t\in(0,1)\}=1$ and $c=0$ in Theorem 3.5, we get Theorem 2.1 of [28].*

2. Taking $h(t)=t$, $M^{'}=sup\,\{h(t):t\in(0,1)\}=1$, $c=0$, $k=1$, $u=\phi$, and $v=\varphi$ in Theorem 3.5, we get Theorem 2.1 of [29].

3. Taking $(t)=t$, $M^{'}=sup\,\{h(t):t\in(0,1)\}=1$, $c=0$, $\varepsilon=k=1$, $u=\phi$ and $v=\varphi$ in Theorem 3.5, we get Theorem 2 of [30].

Theorem 3.7. *Let $f:I^{\circ}\subseteq R\rightarrow R$ be a strongly h-convex function with modulus $c>0$ and $\phi,\varphi\in I^{\circ}$ with $f<\varphi$. If h is a super-additive function and take $\varepsilon,\delta>0$. Then*

$$\begin{aligned}
f\left(\phi+\varphi-\frac{u+v}{2}\right)\leq & h\left(\frac{1}{2}\right)2M^{'}\left[f(\phi)+f(\varphi)\right]-h\left(\frac{1}{2}\right)h(1)\frac{\varepsilon^{\frac{\delta}{k}}\Gamma_k(\delta+k)}{(v-u)^{\varepsilon\frac{\delta}{k}}}\left\{{}^{\delta}_{k}J^{\varepsilon}_{u^+}f(v)+^{\delta}_{k}J^{\varepsilon}_{v^-}f(u)\right\}\\
&-c\left[\frac{1}{2}h\left(\frac{1}{2}\right)\left(1-\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^2\right.\\
&\left.+\frac{1}{4}\left(\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^2\right]\\
\leq & h\left(\frac{1}{2}\right)2M^{'}\left[f(\phi)+f(\varphi)\right]-h(1)f\left(\frac{u+v}{2}\right)-c\left[\frac{1}{4}h(1)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right.\right.\right.\\
&\left.\left.-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^2+\frac{1}{2}h\left(\frac{1}{2}\right)\left(1-\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)\\
&\times\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^2\\
&\left.+\frac{1}{4}\left(\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^2\right].
\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}
f\left(\phi+\varphi-\frac{u+v}{2}\right)&\leq h\left(\frac{1}{2}\right)\frac{\varepsilon^{\frac{\delta}{k}}\Gamma_k(\delta+k)}{(v-u)^{\varepsilon\frac{\delta}{k}}}\left\{{}^{\delta}_{k}J^{\varepsilon}_{(\phi+\varphi-v)^+}f(\phi+\varphi-u)+^{\delta}_{k}J^{\varepsilon}_{(\phi+\varphi-u)^-}f(\phi+\varphi-v)\right\}\\
&\quad-c\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(u-v)^2\\
&\leq h(1)\left(h\left(\frac{1}{2}\right)2M^{'}[f(\phi)+f(\varphi)]-h(1)h\left(\frac{1}{2}\right)[f(u)+f(v)]\right)\\
&\quad-c\left(2h(1)h\left(\frac{1}{2}\right)\frac{\delta}{k}\left(\beta\left(\frac{\beta}{k},\frac{1}{\varepsilon}+1\right)-\beta\left(\frac{\beta}{k},\frac{2}{\varepsilon}+1\right)\right)(u-v)^2\right.
\end{aligned}$$

$$+2h\left(\frac{1}{2}\right)\frac{\delta}{k}\left(\beta\left(\frac{\beta}{k},\frac{1}{\varepsilon}+1\right)-\beta\left(\frac{\beta}{k},\frac{2}{\varepsilon}+1\right)\right)(v-u)^2$$
$$+\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(u-v)^2\Bigg). \tag{3.11}$$

holds $\forall\ u,v \in [\phi,\varphi]$, *where* $M^{'} = sup\ \{h(t) : t \in (0,1)\}$.

Proof. Employing Jensen-Mercer inequality (3.3), we have

$$f\left(\phi+\varphi-\frac{u_1+v_1}{2}\right)\le h\left(\frac{1}{2}\right)2M^{'}[f(\phi)+f(\varphi)]-h\left(\frac{1}{2}\right)h(1)[f(u_1)+f(v_1)]$$
$$-c\left(\frac{1}{2}h\left(\frac{1}{2}\right)\left(1-\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(u_1-v_1)^2+\frac{1}{4}\left(\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(u_1-v_1)^2\right). \tag{3.12}$$

$\forall\ u_1, v_1 \in [\phi,]$.

By changing variables $u_1 = tu + (1-t)v$ and $v_1 = (1-t)u + tv$ for all u,v $\in [\phi,\varphi]$ and $t \in [0,1]$ in (3.12), we get

$$f\left(\phi+\varphi-\frac{u+v}{2}\right)\le h\left(\frac{1}{2}\right)2M^{'}\left[f(c)+f(\varphi)\right]-h\left(\frac{1}{2}\right)h(1)\left[f(tu+(1-t)v)+f((1-t)u+tv)\right]$$
$$-c\left(\frac{1}{2}h\left(\frac{1}{2}\right)\left(1-\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(1-2t)^2(v-u)^2+\frac{1}{4}\left(\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(1-2t)^2(v-u)^2\right). \tag{3.13}$$

Multiplying (3.13) by $\left(\frac{1-(1-t)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}$ and integrating the obtained inequality with respect to t over [0,1] gives

$$f\left(\phi+\varphi-\frac{u+v}{2}\right)\int_0^1\left(\frac{1-(1-t)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}dt$$
$$\le\int_0^1\left(\frac{1-(1-t)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}\times\left[\left(h\left(\frac{1}{2}\right)2M^{'}\left[f(\phi)+f(\varphi)\right]\right.\right.$$
$$\left.-h\left(\frac{1}{2}\right)h(1)\left[f(tu+(1-t)v)+f((1-t)u+tv)\right]\right)$$
$$\left.-c\left(\frac{1}{2}h\left(\frac{1}{2}\right)\left(1-\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(1-2t)^2(v-u)^2+\frac{1}{4}\left(\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)(1-2t)^2(v-u)^2\right)\right]dt.$$

that is,

$$f\left(\phi+\varphi-\frac{u+v}{2}\right)\le h\left(\frac{1}{2}\right)2M^{'}\left[f(\phi)+f(\varphi)\right]-h\left(\frac{1}{2}\right)h(1)\frac{\varepsilon^{\frac{\delta}{k}}\Gamma_k(\delta+k)}{(v-u)^{\varepsilon\frac{\delta}{k}}}\left\{{}_k^{\delta}J_{u^+}^{\varepsilon}f(v)+{}_k^{\delta}J_{v^-}^{\varepsilon}f(u)\right\}$$
$$-c\left[\frac{1}{2}h\left(\frac{1}{2}\right)\left(1-\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^2\right.$$
$$\left.+\frac{1}{4}\left(\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^2\right], \tag{3.14}$$

this completes the proof of first inequality of (3.10).

Now for second part of inequality (3.10) take the definition of strongly h-convex function, we have

$$f\left(\frac{u+v}{2}\right)=f\left(\frac{tu+(1-t)v+(1-t)u+tv}{2}\right)$$
$$\leq h\left(\frac{1}{2}\right)\{f(tu+(1-t)v)+f((1-t)u+tv)\}-c\left(\frac{1}{4}\right)((tu+(1-t)v)-((1-t)u+tv))^{2}. \qquad (3.15)$$

Multiplying (3.15) by $\left(\frac{1-(1-t)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}$ and then integrating the resulting inequality with respect to t over [0,1], we obtain

$$f\left(\frac{u+v}{2}\right)\int_0^1\left(\frac{1-(1-t)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}dt$$
$$\leq\int_0^1\left(\frac{1-(1-t)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}\left(h\left(\frac{1}{2}\right)\{f(tu+(1-t)v)+f((1-t)u+tv)\}-c\left(\frac{1}{4}\right)(1-2t)^{2}(v-u)^{2}\right)dt.$$

By some calculations, we get

$$f\left(\frac{u+v}{2}\right)\leq h\left(\frac{1}{2}\right)\frac{\varepsilon^{\frac{\delta}{k}}\Gamma_k(\delta+k)}{(v-u)^{\varepsilon\frac{\delta}{k}}}\left\{{}_k^{\delta}J_{u^+}^{\varepsilon}f(v)+_k^{\delta}J_{v^-}^{\varepsilon}f(u)\right\},$$
$$-c\left(\frac{1}{4}\right)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^{2}$$
$$-f\left(\frac{u+v}{2}\right)\geq -h\left(\frac{1}{2}\right)\frac{\varepsilon^{\frac{\delta}{k}}\Gamma_k(\delta+k)}{(v-u)^{\varepsilon\frac{\delta}{k}}}\left\{{}_k^{\delta}J_{u^+}^{\varepsilon}f(v)+_k^{\delta}J_{v^-}^{\varepsilon}f(u)\right\}$$
$$+c\left(\frac{1}{4}\right)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^{2}. \qquad (3.16)$$

Multiplying (3.16) by $h(1)$, we have

$$-h(1)f\left(\frac{u+v}{2}\right)\geq -h\left(\frac{1}{2}\right)h(1)\frac{\varepsilon^{\frac{\delta}{k}}\Gamma_k(\delta+k)}{(v-u)^{\varepsilon\frac{\delta}{k}}}\left\{{}_k^{\delta}J_{u^+}^{\varepsilon}f(v)+_k^{\delta}J_{v^-}^{\varepsilon}f(u)\right\}$$
$$+c\left(\frac{1}{4}\right)h(1)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^{2}. \qquad (3.17)$$

Adding $h\left(\frac{1}{2}\right)2M'\left[f(\phi)+f(\varphi)\right]$ to both sides of (3.17), we get

$$h\left(\frac{1}{2}\right)2M'\left[f(\phi)+f(\varphi)\right]-h(1)f\left(\frac{u+v}{2}\right)$$
$$\geq h\left(\frac{1}{2}\right)2M'\left[f(\phi)+f(\varphi)\right]-h\left(\frac{1}{2}\right)h(1)\frac{\varepsilon^{\frac{\delta}{k}}\Gamma_k(\delta+k)}{(n-m)^{\varepsilon\frac{\delta}{k}}}\left\{{}_k^{\delta}J_{u^+}^{\varepsilon}f(v)+_k^{\delta}J_{v^-}^{\varepsilon}f(u)\right\}$$
$$+c\left(\frac{1}{4}\right)h(1)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^{2}.$$

Adding $-c\left[\frac{1}{2}h\left(\frac{1}{2}\right)\left(1-\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^2+\frac{1}{4}\left(\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^2\right]$ to both sides of above inequality, we have

$$\begin{aligned}
&h\left(\frac{1}{2}\right)2M'\left[f(\phi)+f(\varphi)\right]-h\left(\frac{1}{2}\right)h(1)\frac{\varepsilon^{\frac{\delta}{k}}\Gamma_k(\delta+k)}{(v-u)^{\varepsilon\frac{\delta}{k}}}\left\{{}^{\delta}_{k}J^{\varepsilon}_{u^+}f(v)+^{\delta}_{k}J^{\varepsilon}_{v^-}f(u)\right\}\\
&\quad-c\left[\frac{1}{2}h\left(\frac{1}{2}\right)\left(1-\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^2\right.\\
&\quad\left.+\frac{1}{4}\left(\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^2\right]\\
&\leq h\left(\frac{1}{2}\right)2M'\left[f(\phi)+f(\varphi)\right]-h(1)f\left(\frac{u+v}{2}\right)-c\left[\frac{1}{4}h(1)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^2\right.\\
&\quad+\frac{1}{2}h\left(\frac{1}{2}\right)\left(1-\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^2\\
&\quad\left.+\frac{1}{4}\left(\frac{\delta}{k}\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)\right)\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(v-u)^2\right]. \qquad (3.18)
\end{aligned}$$

Combining (3.14) and (3.18) yields (3.10).

Next, we prove inequality (3.11) by using strong h-convexity of f, we have

$$\begin{aligned}
f\left(\phi+\varphi-\frac{u_1+v_1}{2}\right)&=f\left(\frac{\phi+\varphi-u_1+\phi+\varphi-v_1}{2}\right)\\
&\leq h\left(\frac{1}{2}\right)\{f(\phi+\varphi-u_1)+f(\phi+\varphi-v_1)\}-c\left(\frac{1}{4}\right)((\phi+\varphi-u_1)+(\phi+\varphi-v_1))^2. \qquad (3.19)
\end{aligned}$$

for all $u_1, v_1 \in [\phi, \varphi]$.

For $u_1 = tu + (1-t)v$ and $v_1 = (1-t)u + tv$. Then (3.19) leads to the conclusion that

$$f\left(\phi+v-\frac{u+v}{2}\right)\leq h\left(\frac{1}{2}\right)\{f(\phi+v-(tu+(1-t)v))+f(c+v-((1-t)u+tv))\}-c\left(\frac{1}{4}\right)(1-2t)^2(u-v)^2. \qquad (3.20)$$

Multiplying (3.20) by $\left(\frac{1-(1-t)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}$ and then integrating the resulting inequality with respect to t over [0,1], we have

$$\begin{aligned}
&f\left(\phi+\varphi-\frac{u+v}{2}\right)\int_0^1\left(\frac{1-(1-t)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}dt\leq\int_0^1\left(\frac{1-(1-t)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}\\
&\qquad\times\left(h\left(\frac{1}{2}\right)\{f(\phi+\varphi-(tu+(1-t)v))+f(\phi+\varphi-((1-t)u+tv))\}-c\left(\frac{1}{4}\right)(1-2t)^2(u-v)^2\right)dt,
\end{aligned}$$

By computing the above integrals, we get

$$f\left(\phi+\varphi-\frac{u+v}{2}\right)\leq h\left(\frac{1}{2}\right)\frac{\varepsilon^{\frac{\delta}{k}}\Gamma_k(\delta+k)}{(v-u)^{\varepsilon\frac{\delta}{k}}}\left\{{}^{\delta}_{k}J^{\gamma}_{(\phi+\varphi-v)^+}f(\phi+\varphi-u)+^{\delta}_{k}J^{\varepsilon}_{(\phi+\varphi-u)^-}f(\phi+\varphi-v)\right\}.$$

$$-c\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(u-v)^2. \tag{3.21}$$

Employing strong h-convexity of f gives

$$\begin{aligned}
&f(t(\phi+\varphi-u)+(1-t)(\phi+\varphi-v))\\
&\leq h(t)f(\phi+\varphi-u)+h(1-t)f(\phi+\varphi-v)-ct(1-t)\left((\phi+\varphi-u)-(\phi+\varphi-v)\right)^2,
\end{aligned}$$

and

$$\begin{aligned}
&f((1-t)(\phi+\varphi-u)+t(\phi+\varphi-v))\\
&\leq h(1-t)f(\phi+\varphi-u)+h(t)f(\phi+\varphi-v)-c(1-t)t\left((\phi+\varphi-u)-(\phi+\varphi-v)\right)^2.
\end{aligned}$$

Adding above two inequalities and employing supper-additivity of function, we have

$$\begin{aligned}
&f(t(\phi+\varphi-u)+(1-t)(\phi+\varphi-v))+f((1-t)(\phi+\varphi-u)+t(\phi+\varphi-v))\\
&\quad\leq f(\phi+\varphi-u)(h(t)+h(1-t))+f(\phi+\varphi-v)(h(1-t)+h(t))-2ct(1-t)(v-u)^2\\
&\quad\leq h(1)\left(f(\phi+\varphi-u)+f(c+\varphi-v)\right)-2ct(1-t)(v-u)^2.
\end{aligned}$$

Now employing the Jensen-Mercer inequality, we get

$$\begin{aligned}
&f(t(\phi+\varphi-u)+(1-t)(\phi+\varphi-v))+f((1-t)(\phi+\varphi-u)+t(\phi+\varphi-v))\\
&\leq h(1)\left(2M'[f(\phi)+f(\varphi)]-h(1)[f(u)+f(v)]-2ct(1-t)(u-v)^2\right)-2ct(1-t)(v-u)^2.
\end{aligned} \tag{3.22}$$

Multiplying (3.22) by $\left(\frac{1-(1-t)^\varepsilon}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}$ and integrate the resulting inequality with respect to t over [0,1], we have

$$\begin{aligned}
&\int_0^1\left(\frac{1-(1-t)^\varepsilon}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}\{f(t(\phi+\varphi-u)+(1-t)(\phi+\varphi-v))\\
&\qquad\qquad +f((1-t)(\phi+\varphi-u)+t(\phi+\varphi-v))\}\,dt\\
&\quad\leq\Big(h(1)\left(2M'[f(\phi)+f(\varphi)]-h(1)[f(u)+f(v)]-2ct(1-t)(u-v)^2\right)\\
&\qquad\qquad -2ct(1-t)(v-u)^2\Big)\int_0^1\left(\frac{1-(1-t)^\varepsilon}{\varepsilon}\right)^{\frac{\delta}{k}-1}(1-t)^{\varepsilon-1}dt.
\end{aligned}$$

Adding $-c\left(1+4\frac{\delta}{k}\left(\beta\left(\frac{\delta}{k},\frac{2}{\varepsilon}+1\right)-\beta\left(\frac{\delta}{k},\frac{1}{\varepsilon}+1\right)\right)\right)(u-v)^2$ to both sides of above inequality and combining the result with (3.21) yields (3.11). □

Remark 3.8. *1. Taking $h(t)=t$, $c=0$ and $M'=\sup\{h(t):t\in(0,1)\}=1$ in Theorem 3.7 , we get Theorem 2.3 of [28] .*

2. Taking $\varepsilon=\delta=k=1$, $c=0$, $h(t)=t$ and $M'=\sup\{h(t):t\in(0,1)\}=1$ in Theorem 3.7, we get Theorem 2.1 of [31].

Lemma 3.9. *[28] Let $\varepsilon, \delta > 0, \phi < \varphi$ and $f : [\phi, \varphi] \to R$ be a differentiable mapping such that $f' \in L[\phi, \varphi]$. Then the inequality*

$$\frac{2^{\varepsilon^{\frac{\delta}{k}}-1}\Gamma_k(\delta+k)}{(v-u)^{\varepsilon^{\frac{\delta}{k}}}}\left\{{}_k^{\delta}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^+}f(\phi+\varphi-u)+{}_k^{\delta}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^-}f(\phi+\varphi-v)\right\}-f\left(\phi+\varphi-\frac{u+v}{2}\right)$$
$$=\frac{(v-u)\varepsilon^{\frac{\delta}{k}}}{4}\int_0^1\left(\frac{1-(1-t)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}}\left\{f'\left(\phi+\varphi-\left(\frac{2-t}{2}u+\frac{t}{2}v\right)\right)-f'\left(\phi+\varphi-\left(\frac{t}{2}u+\frac{2-t}{2}v\right)\right)\right\}dt. \tag{3.23}$$

holds for all $u, v \in [\phi, \varphi]$.

Theorem 3.10. *Let $f : I^\circ \subseteq R \to R$ ba a differentiable function on I°, $\phi, \varphi \in I^\circ$ with $\phi < \varphi$ and $f' \in L[\phi, \varphi]$. If $|f'|$ is a strongly h-convex function with modulus $c > 0$ on $[\phi, \varphi]$ and take $\varepsilon, \delta > 0$. Then*

$$\left|\frac{2^{\varepsilon^{\frac{\delta}{k}}-1}\varepsilon_k(\delta+k)}{(v-u)^{\varepsilon^{\frac{\delta}{k}}}}\left\{{}_k^{\delta}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^+}f(\phi+\varphi-u)+{}_k^{\delta}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^-}f(\phi+\varphi-v)\right\}-f\left(\phi+\varphi-\frac{u+v}{2}\right)\right|$$
$$\leq\frac{(v-u)\varepsilon^{\frac{\delta}{k}}}{4}\left[\left\{M'\left[\left|f'(\phi)\right|+\left|f'(\varphi)\right|\right]\left(\frac{1}{\varepsilon^{\frac{\delta}{k}+1}}\beta\left(\frac{\delta}{k}+1,\frac{1}{\varepsilon}\right)\right)-\left(\left|f'(u)\right|\beta_1\left(h,\frac{2-t}{2}\right)+\left|f'(v)\right|\beta_2\left(h,\frac{t}{2}\right)\right)\right.\right.$$
$$-c\frac{1}{4\varepsilon^{\frac{\delta}{k}+1}}\left(\beta\left(\frac{\delta}{k}+1,\frac{1}{\varepsilon}\right)-\beta\left(\frac{\delta}{k}+1,\frac{3}{\varepsilon}\right)\right)(u-v)^2\right\}+\left\{M'\left[\left|f'(\phi)\right|+\left|f'(\varphi)\right|\right]\left(\frac{1}{\varepsilon^{\frac{\delta}{k}+1}}\beta\left(\frac{\delta}{k}+1,\frac{1}{\varepsilon}\right)\right)\right.$$
$$\left.\left.-\left(\left|f'(u)\right|\beta_2\left(h,\frac{t}{2}\right)+\left|f'(v)\right|\beta_1\left(h,\frac{2-t}{2}\right)\right)-c\frac{1}{4\varepsilon^{\frac{\delta}{k}+1}}\left(\beta\left(\frac{\delta}{k}+1,\frac{1}{\varepsilon}\right)-\beta\left(\frac{\delta}{k}+1,\frac{3}{\varepsilon}\right)\right)(u-v)^2\right\}\right], \tag{3.24}$$

holds $\forall\ u, v \in [\phi, \varphi]$, where $\beta_1\left(h,\frac{2-t}{2}\right)=\int_0^1\left(\frac{1-(1-t)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}}h\left(\frac{2-t}{2}\right)dt$, $\beta_2\left(h,\frac{t}{2}\right)=\int_0^1\left(\frac{1-(1-t)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}}h\left(\frac{t}{2}\right)dt$ and $M'=\sup\{h(t):t\in(0,1)\}$.

Proof. Employing Lemma 3.9, inequality (3.2) and definition of strong h-convexity of $|f'|$ yield the desired result. □

Remark 3.11.
1. *Taking $h(t) = t$, M' =sup $\{h(t) : t \in (0, 1)\}$ = 1 and c = 0 in Theorem* 3.10*, we get Theorem 2.10 of [28].*
2. *Taking $h(t) = t$, $M' = \sup\{h(t) : t \in (0, 1)\} = 1$, $c = 0$, $k = 1$, $u = \phi$ and $v = \varphi$ in Theorem 3.10, we get Theorem 3.1 of [29].*
3. *Taking $\varepsilon = k = 1$, $h(t) = t$, $M' = \sup\{h(t) : t \in (0, 1)\} = 1$, $c = 0$, $u = \phi$ and $v = \varphi$ in Theorem 3.10, we get Theorem 5 of [32] in the case of $q = 1$.*

Theorem 3.12. *Let $f : I^\circ \subseteq R \to R$ be a differentiable function on I°, $\phi, \varphi \in I^\circ$ with $\phi < \varphi$ and $f' \in L[\phi, \varphi]$. If $|f'|^q$ is a strongly h-convex function with modulus $c > 0$ on $[\phi, \varphi]$ for $q > 1$ and take $\varepsilon, \delta > 0$. Then*

$$\left|\frac{2^{\varepsilon^{\frac{\delta}{k}}-1}\varepsilon_k(\delta+k)}{(v-u)^{\varepsilon^{\frac{\delta}{k}}}}\left\{{}_k^{\delta}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^+}f(\phi+\varphi-u)+{}_k^{\delta}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^-}f(\phi+\varphi-v)\right\}-f\left(\phi+\varphi-\frac{u+v}{2}\right)\right.$$

$$\leq \frac{(v-u)\varepsilon^{\frac{\delta}{k}}}{4}\Bigg[\Bigg\{\left(\frac{1}{\varepsilon^{\frac{\delta}{k}+1}}\beta\left(\frac{\delta}{k}+1,\frac{1}{\varepsilon}\right)\right)^{1-\frac{1}{q}}\Bigg\{M^{'}\left[|f^{'}(\phi)|^{q}+|f^{'}(\varphi)|^{q}\right]\left(\frac{1}{\varepsilon^{\frac{\delta}{k}+1}}\beta\left(\frac{\delta}{k}+1,\frac{1}{\varepsilon}\right)\right)\Bigg|$$

$$-\left(\left(|f^{'}(u)|^{q}\beta_{2}\left(h,\frac{t}{2}\right)+|f^{'}(v)|^{q}\beta_{1}\left(h,\frac{2-t}{2}\right)\right)-c\frac{1}{4\varepsilon^{\frac{\delta}{k}+1}}\left(\beta\left(\frac{\delta}{k}+1,\frac{1}{\varepsilon}\right)-\beta\left(\frac{\delta}{k}+1,\frac{3}{\varepsilon}\right)\right)(u-v)^{2}\right)^{\frac{1}{q}}\Bigg\}\Bigg\}$$

$$+\Bigg\{\left(\frac{1}{\varepsilon^{\frac{\delta}{k}+1}}\beta\left(\frac{\delta}{k}+1,\frac{1}{\varepsilon}\right)\right)^{1-\frac{1}{q}}\Bigg\{M^{'}\left[|f^{'}(\phi)|^{q}+|f^{'}(\varphi)|^{q}\right]\left(\frac{1}{\varepsilon^{\frac{\delta}{k}+1}}\beta\left(\frac{\delta}{k}+1,\frac{1}{\varepsilon}\right)\right)$$

$$-\left(\left(|f^{'}(u)|^{q}\beta_{1}\left(h,\frac{2-t}{2}\right)+|f^{'}(v)|^{q}\beta_{2}\left(h,\frac{t}{2}\right)\right)-c\frac{1}{4\varepsilon^{\frac{\delta}{k}+1}}\left(\beta\left(\frac{\delta}{k}+1,\frac{1}{\varepsilon}\right)-\beta\left(\frac{\delta}{k}+1,\frac{3}{\varepsilon}\right)\right)(u-v)^{2}\right)^{\frac{1}{q}}\Bigg\}\Bigg\}\Bigg], \tag{3.25}$$

holds $\forall\ u,v\in[\phi,\varphi]$, *where* $\beta_{1}\left(h,\frac{2-t}{2}\right)$, $\beta_{2}\left(h,\frac{t}{2}\right)$ *and* $M^{'}$ *are same as in Theorem 3.10.*

Proof. Employing Lemma 3.9, inequality (3.2), power-mean inequality and definition of strongly h-convex function of $|f^{'}|$ yield the desired result. □

Remark 3.13. *Taking* $h(t)=t$, $M^{'}=\sup\{h(t):t\in(0,1)\}=1$ *and* $c=0$ *in Theorem* 3.12*, we get Theorem 2.12 of [28].*

Theorem 3.14. *Let* $f: I^{\circ}\subseteq R\to R$ *be a differentiable function on* I°, $\phi,\varphi\in I^{\circ}$ *with* $\phi<\varphi$ *and* $f^{'}\in L[\phi,\varphi]$. *If* $|f^{'}|^{q}$ *is a strongly h-convex function with modulus* $c>0$ *on* $[\phi,\varphi]$ *for* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and take* $\varepsilon,\delta>0$*. Then*

$$\left|\frac{2^{\varepsilon\frac{\delta}{k}-1}\varepsilon_{k}(\delta+k)}{(v-u)^{\varepsilon\frac{\delta}{k}}}\left\{{}_{k}^{\delta}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^{+}}f(\phi+\varphi-u)+{}_{k}^{\delta}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^{-}}f(\phi+\varphi-v)\right\}-f\left(\phi+\varphi-\frac{u+v}{2}\right)\right|$$

$$\leq\frac{(v-u)\varepsilon^{\frac{\delta}{k}}}{4}\left(\frac{1}{\varepsilon^{\frac{\delta}{k}p+1}}\beta\left(\frac{\delta}{k}p+1,\frac{1}{\varepsilon}\right)\right)^{\frac{1}{p}}\left\{\left(M^{'}\left[\left|f^{'}(\phi)\right|^{q}+\left|f^{'}(\varphi)\right|^{q}\right]-\left(\left|f^{'}(u)\right|^{q}C_{1}\left(h,\frac{2-t}{2}\right)+\left|f^{'}(v)\right|^{q}\ C_{2}\left(h,\frac{t}{2}\right)\right)\right.\right.$$

$$\left.\left.-c\frac{1}{6}(u-v)^{2}\right)^{\frac{1}{q}}+\left(M^{'}\left[\left|f^{'}(\phi)\right|^{q}+\left|f^{'}(\varphi)\right|^{q}\right]-\left(\left|f^{'}(u)\right|^{q}C_{2}\left(h,\frac{t}{2}\right)+\left|f^{'}(v)\right|^{q}C_{1}\left(h,\frac{2-t}{2}\right)\right)-c\frac{1}{6}(u-v)^{2}\right)^{\frac{1}{q}}\right\}, \tag{3.26}$$

holds $\forall\ u,v\in[\phi,\varphi]$, *where* $C_{1}\left(h,\frac{2-t}{2}\right)=\int_{0}^{1}h\left(\frac{2-t}{2}\right)dt$, $\quad C_{2}\left(h,\frac{t}{2}\right)=\int_{0}^{1}h\left(\frac{t}{2}\right)dt$ *and* $M^{'}=$ $\sup\{h(t):t\in(0,1)\}$.

Proof. From applying Lemma 3.9, inequality 3.2, Hölder integral inequalities and definition of strongly h-convex function simultaneously give the desired result. □

Remark 3.15. *Taking* $h(t)=t$, $M^{'}=\sup\{h(t):t\in(0,1)\}=1$ *and* $c=0$ *in Theorem* 3.14*, we get Theorem 2.14 of [28].*

Theorem 3.16. *Assume* $f: I^{\circ}\subseteq R\to R$ *be a differentiable function on* I°, $\phi,\varphi\in I^{\circ}$ *with* $\phi<\varphi$ *and* $f^{'}\in L[\phi,\varphi]$. *If* $|f^{'}|^{q}$ *is a strongly h-convex function with modulus* $c>0$ *on* $[\phi,\varphi]$ *for* $p,q>1$ *with*

$\frac{1}{p}+\frac{1}{q}=1$ *and take* $\varepsilon,\delta>0$. *Then*

$$\left|\frac{2^{\varepsilon\frac{\delta}{k}-1}\varepsilon_k(\delta+k)}{(v-u)^{\varepsilon\frac{\delta}{k}}}\left\{{}_k^{\delta}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^+}f(\phi+\varphi-u)+{}_k^{\delta}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^-}f(\phi+\varphi-v)\right\}-f\left(\phi+\varphi-\frac{u+v}{2}\right)\right|$$
$$\leq\frac{(v-u)\varepsilon^{\frac{\delta}{k}}}{4}\Bigg\{\Bigg(M'\left(\left|f'(\phi)\right|^q+\left|f'(\varphi)\right|^q\right)\left(\frac{\beta(\frac{\delta}{k}+1,\frac{1}{\varepsilon})}{\varepsilon^{\frac{\delta}{k}+1}}\right)-\left(\beta1\left(h,\frac{2-t}{2}\right)\left|f'(u)\right|^q+\beta_2\left(h,\frac{t}{2}\right)\left|f'(v)\right|^q\right)$$
$$-c\frac{1}{4\varepsilon^{\frac{\delta}{k}+1}}\left(\beta\left(\frac{\delta}{k}+1,\frac{1}{\varepsilon}\right)-\beta\left(\frac{\delta}{k}+1,\frac{3}{\varepsilon}\right)\right)(u-v)^2\Bigg)^{\frac{1}{q}}+\Bigg(M'\left(\left|f'(\phi)\right|^q+\left|f'(\varphi)\right|^q\right)\left(\frac{\beta(\frac{\delta}{k}+1,\frac{1}{\varepsilon})}{\varepsilon^{\frac{\delta}{k}+1}}\right)$$
$$-\left(\beta2\left(h,\frac{t}{2}\right)\left|f'(u)\right|^q+\beta_1\left(h,\frac{2-t}{2}\right)\left|f'(v)\right|^q\right)-c\frac{1}{4\varepsilon^{\frac{\delta}{k}+1}}\left(\beta\left(\frac{\delta}{k}+1,\frac{1}{\varepsilon}\right)-\beta\left(\frac{\delta}{k}+1,\frac{3}{\varepsilon}\right)\right)(u-v)^2\Bigg)^{\frac{1}{q}}\Bigg\},\tag{3.27}$$

holds $\forall\ u,v\in[\phi,\varphi]$, *where* $\beta_1\left(h,\frac{2-t}{2}\right)$, $\beta_2\left(h,\frac{t}{2}\right)$ *and* M' *are same as in Theorem 3.10.*

Proof. Using Lemma 3.9, inequality (3.2), Hölder integral inequalities and definition of strongly h-convex function, we obtain our desired result. □

Remark 3.17. *Taking* $h(t)=t$, $M'=\sup\{h(t):t\in(0,1)\}=1$ *and* $c=0$ *in Theorem* 3.16, *we get Theorem 2.16 of [28].*

3.3. New inequalities in the sense of improved Hölder inequality

Theorem 3.18. *Let* $f:I^{\circ}\subseteq R\to R$ *be a differentiable function on* I°, $\phi,\varphi\in I^{\circ}$ *with* $\phi<\varphi$ *and* $f'\in L[\phi,\varphi]$. *If* $|f'|^q$ *is a strongly h-convex function with modulus* $c>0$ *on* $[\phi,\varphi]$ *for* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and take* $\varepsilon,\delta>0$. *Then*

$$\left|\frac{2^{\varepsilon\frac{\delta}{k}-1}\varepsilon_k(\delta+k)}{(v-u)^{\varepsilon\frac{\delta}{k}}}\left\{{}_k^{\delta}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^+}f(\phi+\varphi-u)+{}_k^{\delta}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^-}f(\phi+\varphi-v)\right\}-f\left(\phi+\varphi-\frac{u+v}{2}\right)\right|$$
$$\leq\frac{(v-u)\varepsilon^{\frac{\delta}{k}}}{4}\Bigg[\Bigg\{\left(\frac{\beta(\frac{2}{\varepsilon},\frac{\delta}{k}+1)}{\varepsilon^{\frac{\delta}{k}+1}}\right)^{\frac{1}{p}}\Bigg(\frac{M'\left(\left|f'(\phi)\right|^q+\left|f'(\varphi)\right|^q\right)}{2}$$
$$-\left(\beta_1\left((1-t)h,\frac{2-t}{2}\right)\left|f'(u)\right|^q+\beta_2\left((1-t)h,\frac{t}{2}\right)\left|f'(v)\right|^q\right)-\frac{c}{16}(u-v)^2\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{\beta(\frac{1}{\varepsilon},\frac{\delta}{k}p+1)-\beta(\frac{2}{\varepsilon},\frac{\delta}{k}p+1)}{\varepsilon^{\frac{\delta}{k}p+1}}\right)^{\frac{1}{p}}\Bigg(\frac{M'\left(\left|f'(\phi)\right|^q+\left|f'(\varphi)\right|^q\right)}{2}-\left(\beta_3\left(th,\frac{2-t}{2}\right)\left|f'(u)\right|^q+\beta_4\left(th,\frac{t}{2}\right)\left|f'(v)\right|^q\right)$$
$$-c\frac{5}{48}(u-v)^2\Bigg)^{\frac{1}{q}}\Bigg\}+\Bigg\{\left(\frac{\beta(\frac{2}{\varepsilon},\frac{\delta}{k}p+1)}{\varepsilon^{\frac{\delta}{k}p+1}}\right)^{\frac{1}{p}}\Bigg(\frac{M'\left(\left|f'(\phi)\right|^q+\left|f'(\varphi)\right|^q\right)}{2}$$
$$-\left(\beta_2\left((1-t)h,\frac{t}{2}\right)\left|f'(u)\right|^q+\beta_1\left((1-t)h,\frac{2-t}{2}\right)\left|f'(v)\right|^q\right)-\frac{c}{16}(u-v)^2\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{\beta(\frac{1}{\varepsilon},\frac{\delta}{k}p+1)-\beta(\frac{2}{\varepsilon},\frac{\delta}{k}p+1)}{\varepsilon^{\frac{\delta}{k}p+1}}\right)^{\frac{1}{p}}\Bigg(\frac{M'\left(\left|f'(\phi)\right|^q+\left|f'(\varphi)\right|^q\right)}{2}$$

$$-\left(\beta_4\left(th,\frac{t}{2}\right)\left|f'(u)\right|^q+\beta_3\left(th,\frac{2-t}{2}\right)\left|f'(v)\right|^q\right)-c\frac{5}{48}(u-v)^2\Bigg)^{\frac{1}{q}}\Bigg\}\Bigg], \tag{3.28}$$

holds for all u,v $\in[\phi,\varphi]$, *where* $\beta_1\left((1-t)h,\frac{2-t}{2}\right)=\int_0^1(1-t)h\left(\frac{2-t}{2}\right)dt$, $\beta_2\left((1-t)h,\frac{t}{2}\right)=\int_0^1(1-t)h\left(\frac{t}{2}\right)dt$, $\beta_3\left(th,\frac{2-t}{2}\right)=\int_0^1 th\left(\frac{2-t}{2}\right)dt$, $\beta_4\left(th,\frac{t}{2}\right)=\int_0^1 th\left(\frac{t}{2}\right)dt$ *and* $M'=\sup\{h(t):t\in(0,1)\}$.

Proof. Using Lemma 3.9, inequality (3.2), Hölder-Íscan integral inequality given in Theorem 1.4 of [33] and definition of strongly h-convex function of $|f'|^q$ yield the desired result. □

Remark 3.19. *Taking* $h(t)=t$, $c=0$ *and* $M'=\sup\{h(t):t\in(0,1)\}=1$ *in Theorem 3.18, we get Theorem 3.1 of [28].*

Theorem 3.20. *Let* $f: I^\circ\subseteq R\to R$ *be a differentiable function on* I°, $\phi,\varphi\in I^\circ$ *with* $\phi<\varphi$ *and* $f'\in L[\phi,\varphi]$. *If* $|f'|^q$ *is a strongly h-convex function with modulus c on* $[\phi,\varphi]$, *where* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and take* $\varepsilon,\delta>0$. *Then*

$$\left|\frac{2^{\varepsilon\frac{\delta}{k}-1}\varepsilon_k(\delta+k)}{(v-u)^{\varepsilon\frac{\delta}{k}}}\left\{{}_k^{\delta}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^+}f(\phi+\varphi-u)+{}_k^{\delta}J^{\varepsilon}_{(\phi+\varphi-\frac{u+v}{2})^-}f(\phi+\varphi-v)\right\}-f\left(\phi+\varphi-\frac{u+v}{2}\right)\right|$$

$$\le\frac{(v-u)\varepsilon^{\frac{\delta}{k}}}{4}\Bigg[\Bigg\{\left(\frac{\beta(\frac{2}{\varepsilon},\frac{\delta}{k}+1)}{\varepsilon^{\frac{\delta}{k}+1}}\right)^{1-\frac{1}{q}}\Bigg(M'\left(\left|f'(\phi)\right|^q+\left|f'(\varphi)\right|^q\right)\left(\frac{\beta(\frac{2}{\varepsilon},\frac{\delta}{k}+1)}{\varepsilon^{\frac{\delta}{k}+1}}\right)-\Bigg(\beta_1\left((1-t)h,\frac{2-t}{2}\right)$$

$$\left|f'(u)\right|^q+\beta_2\left((1-t)h,\frac{t}{2}\right)\left|f'(v)\right|^q\Bigg)-c\frac{1}{4\varepsilon^{\frac{\delta}{k}+1}}\left(\beta\left(\frac{\delta}{k}+1,\frac{2}{\varepsilon}\right)-\beta\left(\frac{\delta}{k}+1,\frac{4}{\varepsilon}\right)\right)(u-v)^2\Bigg)^{\frac{1}{q}}$$

$$+\left(\frac{\beta(\frac{1}{\varepsilon},\frac{\delta}{k}+1)-\beta(\frac{2}{\varepsilon},\frac{\delta}{k}+1)}{\varepsilon^{\frac{\delta}{k}+1}}\right)^{1-\frac{1}{q}}\Bigg(M'\left(\left|f'(\phi)\right|^q+\left|f'(\varphi)\right|^q\right)\left(\frac{\beta(\frac{1}{\varepsilon},\frac{\delta}{k}+1)-\beta(\frac{2}{\varepsilon},\frac{\delta}{k}+1)}{\varepsilon^{\frac{\delta}{k}+1}}\right)$$

$$-\left(\beta_3\left(th,\frac{2-t}{2}\right)\left|f'(u)\right|^q+\beta_4\left(th,\frac{t}{2}\right)\left|f'(v)\right|^q\right)-c\frac{1}{4\varepsilon^{\frac{\delta}{k}+1}}\Bigg(\beta\left(\frac{\delta}{k}+1,\frac{4}{\varepsilon}\right)-\beta\left(\frac{\delta}{k}+1,\frac{3}{\varepsilon}\right)-\beta\left(\frac{\delta}{k}+1,\frac{2}{\varepsilon}\right)$$

$$+\beta\left(\frac{\delta}{k}+1,\frac{1}{\varepsilon}\right)\Bigg)(u-v)^2\Bigg)^{\frac{1}{q}}\Bigg\}+\Bigg\{\left(\frac{\beta(\frac{2}{\varepsilon},\frac{\delta}{k}+1)}{\varepsilon^{\frac{\delta}{k}+1}}\right)^{1-\frac{1}{q}}\Bigg(M'\left(\left|f'(\phi)\right|^q+\left|f'(\varphi)\right|^q\right)\left(\frac{\beta(\frac{2}{\varepsilon},\frac{\delta}{k}+1)}{\varepsilon^{\frac{\delta}{k}+1}}\right)$$

$$-\left(\beta_2\left((1-t)h,\frac{t}{2}\right)\left|f'(u)\right|^q+\beta_1\left((1-t)h,\frac{2-t}{2}\right)\left|f'(v)\right|^q\right)-c\frac{1}{4\varepsilon^{\frac{\delta}{k}+1}}\left(\beta\left(\frac{\delta}{k}+1,\frac{2}{\varepsilon}\right)-\beta\left(\frac{\delta}{k}+1,\frac{4}{\varepsilon}\right)\right)$$

$$\times(u-v)^2\Bigg)^{\frac{1}{q}}+\left(\frac{\beta(\frac{1}{\varepsilon},\frac{\delta}{k}+1)-\beta(\frac{2}{\varepsilon},\frac{\delta}{k}+1)}{\varepsilon^{\frac{\delta}{k}+1}}\right)^{1-\frac{1}{q}}\Bigg(M'\left(\left|f'(\phi)\right|^q+\left|f'(\varphi)\right|^q\right)\left(\frac{\beta(\frac{1}{\varepsilon},\frac{\delta}{k}+1)-\beta(\frac{2}{\varepsilon},\frac{\delta}{k}+1)}{\varepsilon^{\frac{\delta}{k}+1}}\right)$$

$$-\left(\beta_4\left(th,\frac{t}{2}\right)\left|f'(u)\right|^q+\beta_3\left(th,\frac{2-t}{2}\right)\left|f'(v)\right|^q\right)-c\frac{1}{4\varepsilon^{\frac{\delta}{k}+1}}\Bigg(\beta\left(\frac{\delta}{k}+1,\frac{4}{\varepsilon}\right)$$

$$-\beta\left(\frac{\delta}{k}+1,\frac{3}{\varepsilon}\right)-\beta\left(\frac{\delta}{k}+1,\frac{2}{\varepsilon}\right)+\beta\left(\frac{\delta}{k}+1,\frac{1}{\varepsilon}\right)\Bigg)(u-v)^2\Bigg)^{\frac{1}{q}}\Bigg\}\Bigg], \tag{3.29}$$

holds for all u,v $\in[\phi,\varphi]$, *where* $\beta_1\left((1-t)h,\frac{2-t}{2}\right)=\int_0^1(1-t)\left(\frac{1-(1-t)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}}h\left(\frac{2-t}{2}\right)dt$,

$$\beta_2\left((1-t)h,\frac{t}{2}\right)=\int_0^1(1-t)\left(\frac{1-(1-t)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}}h\left(\frac{t}{2}\right)dt,\quad \beta_3\left(th,\frac{2-t}{2}\right)=\int_0^1 t\left(\frac{1-(1-t)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}}h\left(\frac{2-t}{2}\right)dt,$$

$$\beta_4\left(th,\frac{t}{2}\right)=\int_0^1 t\left(\frac{1-(1-t)^{\varepsilon}}{\varepsilon}\right)^{\frac{\delta}{k}}h\left(\frac{t}{2}\right)dt \text{ and } M'=\sup\{h(t):t\in(0,1)\}.$$

Proof. Employing Lemma 3.9, inequality (3.2), improved power-mean integral inequality given in Theorem 1.5 of [33] and definition of strong h-convexity of $|f'|^q$ yield the required result. □

Remark 3.21. *Taking $h(t)=t$, $c=0$ and $M=\sup\{h(t):t\in(0,1)\}=1$ in Theorem 3.20, we get Theorem 3.2 of [28].*

4. Conclusions

The convex functions play an important role in applied mathematics as well as in optimization theory. In past years, researchers paid a huge attention to establish properties of different variants of convex functions. In this paper, we studied *h*-convex functions in the setting of *k*-fractional integrals. We established various important versions of Hermite-Hadamard type inequalities for *h*-convex. Our results generalize and extend many existing results in literature.

Acknowledgement

The study was supported by Fundamental Research Funds for the Central Universities (Grant No. 2020RC14) and Young teachers undergraduate teaching top talents Project of Shandong University of Science and Technology (No. BJRC20190508).

Conflict of interest

The author declares no conflict of interests.

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