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Research article

Existence and concentration of nontrivial solutions for an elastic beam equation with local nonlinearity

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Abstract: In this paper, by using the mountain pass lemma and the skill of truncation function, we investigate the existence and concentration phenomenon of nontrivial weak solutions for a class of elastic beam differential equation with two parameters λ and μ when the nonlinear term satisfies some growth conditions only near the origin. In particular, we obtain a concrete lower bound of the parameter λ, and analyze the relationship between λ and μ. In the end, we investigate the concentration phenomenon of solutions when $\mu \to 0$, and obtain a specific lower bound of the parameter λ which is independent of μ.

Keywords: weak solution; elastic beam equation; local nonlinearity; mountain pass theorem; concentration
Mathematics Subject Classification: 34B15, 49J35

1. Introduction and main results

In this paper, we focus on the following equation

$$\begin{cases} u^{(4)}(x) + 2h(x)u'''(x) + (h^2(x) + h'(x))u''(x) = \lambda f(x, u(x)), & x \in [0, 1], \\ u(0) = u'(0) = u''(1) = u'''(1) - \mu g(u(1)) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$, $\mu \in \mathbb{R}$, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, and $h \in C^1[0, 1]$ is nonnegative. The problem (1.1) with $h = 0$ describes the static equilibrium of an elastic beam which is fixed at the left end of $x = 0$ and is attached to a bearing device at the right end of $x = 1$, where the corresponding force of the bearing device is given by function g, the nonlinear term f is a continuous load which is attached to elastic beam. Moreover, there are many fourth order differential equations which is similar to problem (1.1) in engineering, material mechanics and so on. In recent years, with the development of science and technology, more and more scholars are devoted to the study on the existence and multiplicity of solutions for these fourth order ordinary differential equations. It is well known that different boundary conditions will lead to different physical meanings of these equations. For example, the boundary conditions that $u(0) = 0, u'(0) = 0$, $u(1) = 0$ and $u'(1) = 0$ describe the static equilibrium of an elastic beam fixed at both ends, see [1] and references therein. In [2], the boundary conditions that $u(0) = \alpha u'(0) - \beta u''(0) = \gamma u(1) + \delta u''(1) = u'''(1) = 0$ describe that one end of the elastic beam is fixed and the other is sliding when $\alpha = \delta = 1$ and $\beta = \gamma = 0$. In [3], the boundary conditions that $u(0) = 0, u''(0) = g(u'(0)), u(1) = 0$, and $u''(1) = h(u'(1))$ describe that both ends of the elastic beam are attached to fixed torsional represented by two nonlinear functions.

For the boundary value problems like problem (1.1), the existence and multiplicity of solutions have been investigated extensively by some methods, for example, fixed point theory and variational method. In particular, in [4], Cabada et al. investigated the problem

$$\begin{cases} u^{(4)}(x) = f(x, u(x)), \quad x \in [0, 1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{1.2}$$

where the function $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(x, \mathbb{R}_+) \subset \mathbb{R}_+$ for all $x \in [0, 1]$. They studied the existence, localization and multiplicity of positive solutions by using the critical point theorems in conical shells, Krasnosel'skiĭ's compression-expansion theorem, and unilateral Harnack type inequalities. In [5], Bonanno and Tersian investigated the following problem

$$\begin{cases} u^{(4)}(x) = \lambda f(x, u(x)), \quad x \in [0, 1], \\ u(0) = u'(0) = u''(1) = u'''(1) - \mu g(u(1)) = 0, \end{cases} \tag{1.3}$$

where $\lambda > 0$, $\mu > 0$, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. By using the variational methods, they established the existence result of solutions for problem (1.3). We refer readers to [6–10] for more related results of problem (1.3) with $\mu > 0$ or $\mu < 0$, respectively. The problem with perturbed nonlinear term is also an interesting topic. In [11], Heidarkhani and Gharehgazlouei investigated the problem

$$\begin{cases} u^{(4)}(x) = \lambda f(x, u(x)) + k(u(x)), \quad x \in [0, 1], \\ u(0) = u'(0) = u''(1) = u'''(1) - \mu g(u(1)) = 0, \end{cases} \tag{1.4}$$

where $\lambda > 0$, $\mu \geq 0$, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function, and $k : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with a Lipschitz constant $L > 0$, i.e.,

$$|k(\xi_1) - k(\xi_2)| \leq L|\xi_1 - \xi_2|, \quad \forall \xi_1, \xi_2 \in \mathbb{R},$$

and $k(0) = 0$. They investigated the existence of solution for problem (1.4) by using variational methods. More results about fourth-order boundary value problems with perturbations can be seen in [12–16] and references therein. Moreover, the multi-point boundary value and integral boundary value problems of fourth order ordinary differential equations have also been studied extensively. We refer readers to [17–21] and references therein.

In this paper, our work is mainly motivated by [22, 23]. In [23], Costa and Wang considered a class of elliptic problems with a parameter

$$\begin{cases} -\Delta u = \lambda f(u), & \text{in} \quad \Omega, \\ u(x) = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

where $\lambda > 0$, Ω is a bounded smooth domain in $\mathbb{R}^N (N \geq 3)$ and $f \in C^1(\mathbb{R}, \mathbb{R})$ has superlinear growth only in neighborhood of $u = 0$. They investigated the existence of both signed and sign-changing solutions for problem (1.5) by using truncation function and minimax method and obtained the existence result when λ is large enough. Subsequently, the method was used widely (for example, see [24–28]). However, the concrete values of lower bound of λ were not given in these references. Recently, in [22], by using the idea in [23], the three authors in this paper, Kang, Liu and Zhang, considered a fractional order Kirchhoff-type system with a parameter

$$\begin{cases} A(u(t))[{}_tD_T^{\alpha}\phi_p({}_0D_t^{\alpha}u(t)) + V(t)\phi_p(u(t))] = \lambda\nabla F(t, u(t)), & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.6}$$

where

$$A(u(t)) = \left[a + b\int_0^T (|{}_0D_t^{\alpha}u(t)|^p + V(t)|u(t)|^p)dt\right]^{p-1},$$

$a, b, \lambda > 0, p > 1$ and $1/p < \alpha \leq 1$, $u(t) = (u_1(t), \cdots, u_N(t))^{\tau} \in \mathbb{R}^N$ for a.e. $t \in [0, T]$ and N is a given positive integer, $(\cdot)^{\tau}$ denote the transpose of a vector,$V(t) \in C([0, T], \mathbb{R})$ with $\min_{t\in[0,T]} V(t) > 0$, ${}_0D_T^{\alpha}$ and ${}_tD_T^{\alpha}$ are the left and right Riemann-Liouville fractional derivatives, respectively, $\phi_p(s) := |s|^{p-2}s$, $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ and $\nabla F(t, x)$ is the gradient of F with respect to $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, that is, $\nabla F(t, x) = \left(\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_N}\right)$. They investigated the existence of solutions for problem (1.6) by using mountain pass theorem when the nonlinear term satisfied the superquadric condition only near the origin. They obtained problem (1.6) has at least one nontrivial solution if $\lambda > \lambda_0$, where λ_0 is given in detail.

Based on the idea in [22, 23], in this paper, we investigate the existence of nontrivial weak solutions of problem (1.1) when the nonlinear term $f(x, u)$ with respect to u satisfies the super-quadratic growth condition only near the origin. We obtain a specific lower bound of the parameter λ when $\mu > 0$ and $\mu < 0$ respectively, and analyze the relationships between λ and μ. Moreover, we also investigate the concentration phenomenon of solutions when $\mu \to 0$. Although the idea origins from [22, 23], there are still three differences: (1) the model (1.1) is obviously different from (1.5) and (1.6). In particular, (1.5) and (1.6) have only one parameter λ, while problem (1.1) has two parameters λ and μ. Hence, it is necessary to discuss the relationship between these two parameters; (2) we study the concentration phenomenon of solutions when the parameter $\mu \to 0$; (3) the boundary value condition of (1.1) is not Dirichlet boundary condition. In particular, when $\mu \neq 0$, $u'''(1) = \mu g(u(1)) \neq 0$. In the following theorems, we assume that g satisfies the locally subquadratic condition when $\mu > 0$ and locally superquadric condition when $\mu < 0$ with respect to u near the origin. Finally, comparing with those results for the fourth order differential equations like (1.2) and (1.3), we only suppose f and g satisfy the local conditions near the origin, and consider more general model (1.1) since it is easy to see that (1.1) reduces to (1.2) and (1.3) if $h = 0$. Here, we refer to some related references for the case

$h \neq 0$, (see [29–33]) which focus on the existence and multiplicity of solutions for the second order Hamiltonian system with damped term by variational methods. Moreover, as a comparison, we also refer to a recent reference [34] in which the Klein-Gordon-Maxwell systems was investigated and the nonlinear term was assumed to satisfy super-quadratic conditions near ∞.

Next, we make some assumptions for F, and then state our main results.

(H_1) *there exists a constant* $\delta > 0$ *such that* $F(x,u)$ *is continuously differentiable in* $u \in \mathbb{R}$ *with* $|u| \leq \delta$ *for a.e.* $x \in [0,1]$, *measurable in* x *for every* $u \in \mathbb{R}$ *with* $|u| \leq \delta$, *and there are* $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ *and* $b \in L^1([0,1]; \mathbb{R}^+)$ *such that*

$$|F(x,u)|, |f(x,u)| \leq a(|u|)b(x),$$

for all $u \in \mathbb{R}$ *with* $|u| \leq \delta$ *and a.e.* $x \in [0,1]$, *where* $F(x,u) = \int_0^u f(x,s)ds$;

(H_2) *there exist constants* $q_1 > 2$, $q_2 \in (2, q_1)$, $M_1 > 0$ *and* $M_2 > 0$ *such that*

$$M_1|u|^{q_1} \leq F(x,u) \leq M_2|u|^{q_2},$$

for all $u \in \mathbb{R}$ *with* $|u| \leq \delta$ *and a.e.* $x \in [0,1]$;

(H_3) *there exists a constant* $\beta > 2$ *such that*

$$0 \leq \beta F(x,u) \leq f(x,u)u,$$

for all $u \in \mathbb{R}$ *with* $|u| \leq \delta$ *and a.e.* $x \in [0,1]$.

Theorem 1.1. *Suppose that* $(H_1) - (H_3)$ *hold and* G *satisfies*

(H_4) *there exist constants* $1 < p_2 < p_1 < 2$ *and* $N_1, N_2 > 0$ *such that*

$$N_1|u|^{p_1} \leq G(u) \leq N_2|u|^{p_2},$$

for all $u \in \mathbb{R}$ *with* $|u| \leq \delta$ *and a.e.* $x \in [0,1]$, *where* $G(u) = \int_0^u g(s)ds$;

(H_5) *there exists a constant* $\gamma \in (0,2)$ *such that*

$$0 < g(u)u \leq \gamma G(u),$$

for all $u \in \mathbb{R}$ *with* $|u| \leq \delta$ *and a.e.* $x \in [0,1]$.

If $\lambda > \lambda_+^* := \max\{\Lambda_1^+, \Lambda_2^+, \Lambda_3^+\}$ *and* $0 < \mu < \mu_*$, *then problem (1.1) has at least a nontrivial solution* $u_{\lambda,\mu}$. *Moreover,*

$$\begin{aligned}
&\|u_{\lambda,\mu}\|_H^2 \leq \frac{2\theta}{\theta-2}(C_*\lambda^{-\frac{1}{q_1-2}} + C_{**}\mu),\\
&\lim_{\substack{\mu\to0^+\\ \lambda\to+\infty}} \|u_{\lambda,\mu}\|_H = 0 = \lim_{\substack{\mu\to0^+\\ \lambda\to+\infty}} \|u_{\lambda,\mu}\|_\infty,\\
&\sqrt{e^{H_0}} \lim_{\substack{\mu\to\mu_*^-\\ \lambda\to+\infty}} \|u_{\lambda,\mu}\|_\infty \leq \lim_{\substack{\mu\to\mu_*^-\\ \lambda\to+\infty}} \|u_{\lambda,\mu}\|_H \leq \frac{\delta}{2}\sqrt{e^{H_0}},\\
&\sqrt{e^{H_0}} \lim_{\substack{\mu\to0^+\\ \lambda\to\lambda_*}} \|u_{\lambda,\mu}\|_\infty \leq \lim_{\substack{\mu\to0^+\\ \lambda\to\lambda_*}} \|u_{\lambda,\mu}\|_H \leq \sqrt{\frac{2\theta C_*}{\theta-2}\lambda_*^{-\frac{1}{2(q_1-2)}}},
\end{aligned}$$

where

$$\|u\|_H = \left(\int_0^1 e^{H(x)}|u''(x)|^2dx\right)^{\frac{1}{2}}, \quad H(x) = \int_0^x h(s)ds,$$

$$
\begin{aligned}
\Lambda_1^+ &= \max\left\{\frac{2e^{\frac{q_2H_0}{2}}}{M_2e^{H_1}\|\delta\kappa\|_H^{q_2-2}}, \frac{\|\delta\kappa\|_H^2 + 2\mu N(|\delta\kappa(1)|^{p_1} + |\delta\kappa(1)|^{p_2})}{2e^{H_0}M_1\|\delta\kappa\|_2^{q_1}}\right\},\\
\Lambda_2^+ &= 1,\\
\Lambda_3^+ &= \left[\frac{8\theta C_*}{(\theta-2)e^{H_0}\delta^2 - 8\theta C_{**}\mu}\right]^{q_1-2},\\
C_* &= \left(\frac{1}{2(M_1e^{H_0}q_1)^{\frac{2}{q_1-2}}} - \frac{M_1e^{H_0}}{(e^{H_0}M_1q_1)^{\frac{q_1}{q_1-2}}}\right)\left(\frac{\|\kappa_1\|_H}{\|\kappa_1\|_2}\right)^{\frac{2q_1}{q_1-2}}, && (1.7)\\
C_{**} &= e^{H_1}N\left(e^{-\frac{H_0p_1}{2}}\|\kappa_1\|_H^{p_1} + e^{-\frac{H_0p_2}{2}}\|\kappa_1\|_H^{p_2}\right), && (1.8)\\
N &= \max\{N_1, N_2\},\\
\lambda_* &= \max\left\{\max\left\{\frac{2e^{\frac{q_2H_0}{2}}}{M_2e^{H_1}\|\delta\kappa\|_H^{q_2-2}}, \frac{\|\delta\kappa\|_H^2}{2e^{H_0}M_1\|\delta\kappa\|_2^{q_1}}\right\}, 1, \left[\frac{8\theta C_*}{(\theta-2)e^{H_0}\delta^2}\right]^{q_1-2}\right\}, && (1.9)\\
\mu_* &= \frac{(\theta-2)e^{H_0}\delta^2}{8\theta C_{**}}, && (1.10)\\
\theta &= \min\{q_2, \beta\},
\end{aligned}
$$

κ is any given element in E and satisfies $\|\kappa\|_\infty \le 1, \kappa_1 = \delta\kappa$,

$H_0 = \min_{x\in[0,1]} H(x), H_1 = \max_{x\in[0,1]} H(x).$

Remark 1.1. It is easy to see that $\Lambda_3^+ \to +\infty$ if $\mu \to \mu_*$, and then $\lambda \to +\infty$. Hence $\lim_{\substack{\mu\to\mu_*\\ \lambda\to+\infty}} \|u_{\lambda,\mu}\|_\infty$ can be simply written as $\lim_{\mu\to\mu_*} \|u_{\lambda,\mu}\|_\infty$.

Theorem 1.2. *Suppose that $(H_1)-(H_3)$ hold and G satisfies*
(H_6) there exist constants $\alpha_1 > q_2$, $\alpha_2 \in (q_2, \alpha_1)$, $\eta_1 > 0$ and $\eta_2 > 0$ such that

$$\eta_1|u|^{\alpha_1} \le G(u) \le \eta_2|u|^{\alpha_2},$$

for all $u \in \mathbb{R}$ with $|u| \le \delta$ and a.e. $x \in [0, 1]$;
(H_7) there exists a constant $\xi > 2$ such that

$$0 \le \xi G(u) \le g(u)u,$$

for all $u \in \mathbb{R}$ with $|u| \le \delta$ and a.e.$x \in [0, 1]$.

If $\lambda > \lambda_-^* := \max\{\Lambda_1^-, \Lambda_2^-, \Lambda_3^-, \Lambda_4^-\}$ and $\mu < 0$, then system (1.1) has at least a nontrivial solution $u_{\lambda,\mu}$. Moreover,

$$\|u_{\lambda,\mu}\|_H^2 \le \frac{2\rho}{\rho-2}D_*\lambda^{-\frac{1}{q_1-2}}, \tag{1.11}$$

$$\lim_{\lambda\to+\infty} \|u_{\lambda,\mu}\|_H = 0 = \lim_{\lambda\to+\infty} \|u_{\lambda,\mu}\|_\infty, \tag{1.12}$$

$$\sqrt{e^{H_0}} \lim_{\substack{\mu\to0^-\\ \lambda\to\Lambda_*}} \|u_{\lambda,\mu}\|_\infty \le \lim_{\substack{\mu\to0^-\\ \lambda\to\Lambda_*}} \|u_{\lambda,\mu}\|_H \le \sqrt{\frac{2\rho D_*}{\rho-2}\Lambda_*^{-\frac{1}{2(q_1-2)}}}, \tag{1.13}$$

where

$$
\begin{aligned}
\Lambda_1^- &= \max\left\{\frac{2e^{\frac{q_2H_0}{2}}}{M_2e^{H_1}\|\delta\phi\|_H^{q_2-2}}+\frac{8\mu\eta_2e^{H(1)}}{M_2e^{H_1}},\frac{\|\delta\phi\|_H^2}{2M_1e^{H_0}\|\delta\phi\|_2^{q_1}}\right\},\\
\Lambda_2^- &= 1,\\
\Lambda_3^- &= \left[\frac{8\rho D_*}{(\rho-2)\delta^2e^{H_0}}\right]^{q_1-2},\\
\Lambda_4^- &= \frac{2e^{\frac{q_2H_0}{2}}}{M_2e^{H_1}},\\
D_* &= \left(\frac{1}{2(M_1e^{H_0}q_1)^{\frac{2}{q_1-2}}}-\frac{M_1e^{H_0}}{(e^{H_0}M_1q_1)^{\frac{q_1}{q_1-2}}}\right)\left(\frac{\|\phi_1\|_H}{\|\phi_1\|_2}\right)^{\frac{2q_1}{q_1-2}}, \qquad (1.14)\\
\Lambda_* &= \max\left\{\frac{2e^{\frac{q_2H_0}{2}}}{M_2e^{H_1}\|\delta\phi\|_H^{q_2-2}},\frac{\|\delta\phi\|_H^2}{2e^{H_0}M_1\|\delta\phi\|_2^{q_1}},\frac{2e^{\frac{q_2H_0}{2}}}{M_2e^{H_1}},1,\left[\frac{8\theta C_*}{(\theta-2)e^{H_0}\delta^2}\right]^{q_1-2}\right\}, \qquad (1.15)
\end{aligned}
$$

ϕ is any given element in E and satisfies $\|\phi\|_\infty \le 1, \phi_1 = \delta\phi$,

$\rho = \min\{\theta, \alpha_2, \xi\}$.

In Theorem 1.1 and Theorem 1.2, we investigate problem (1.1) with $\mu > 0$ and $\mu < 0$, respectively. It is natural to ask what happens when $\mu \to 0$. Next, in Theorem 1.3, we show the concentration phenomenon of $\{u_{\lambda,\mu}\}$ as $\mu \to 0$, which means that $u_{\lambda,\mu} \to u_\lambda$ as $\mu \to 0$ for some $u_\lambda \in E$ (E defined by (2.1)) and u_λ is a nontrivial solution of the following equation

$$
\begin{cases}
u^{(4)}(x)+2h(x)u'''(x)+(h^2(x)+h'(x))u''(x)=\lambda f(x,u(x)), & x\in[0,1],\\
u(0)=u'(0)=u''(1)=u'''(1)=0.
\end{cases} \qquad (1.16)
$$

Theorem 1.3. *Suppose that* $(H_1)-(H_3)$ *hold and assume that* $(H_4)-(H_5)$ *hold if* $\mu > 0$ *and* $(H_6)-(H_7)$ *hold if* $\mu < 0$. *If* $\{u_{\lambda,\mu}\}$ *is a family of nontrivial solutions of problem (1.1), which are given in Theorem 1.1 and Theorem 1.2, then problem (1.16) has at least a nontrivial solution* u_λ *for all* $\lambda > \lambda^* := \max\{\Lambda_{M_0}, \Lambda_{21}, \Lambda_{22}, \Lambda_3, \Lambda_4^-\}$, *and for any given* $\lambda > \lambda^*$, $u_{\lambda,\mu} \to u_\lambda$, *as* $\mu \to 0$. *Moreover,*

$$
\begin{aligned}
&\|u_\lambda\|_H^2 \le K_*\lambda^{-\frac{1}{q_1-2}},\quad \|u_\lambda\|_\infty^2 \le \frac{1}{e^{H_0}}\cdot K_*\lambda^{-\frac{1}{q_1-2}},\\
&\lim_{\lambda\to\infty}\|u_\lambda\|_H = 0 = \lim_{\lambda\to\infty}\|u_\lambda\|_\infty,
\end{aligned}
$$

where M_0 *is any given positive constant,*

$$
\begin{aligned}
\Lambda_{M_0} &= \max\left\{\frac{2e^{\frac{q_2H_0}{2}}}{\|\delta\kappa\|_H^{q_2-2}e^{H_1}M_2},\frac{\|\delta\kappa\|_H^2+2M_0N(|\delta\kappa|^{p_1}+|\delta\kappa|^{p_2})}{2M_1\|\delta\kappa\|_2^{q_1}e^{H_0}}\right\}, \qquad (1.17)\\
\Lambda_{22} &= \max\left\{\frac{2e^{\frac{q_2H_0}{2}}}{e^{H_1}\|\delta\phi\|_H^{q_2-2}M_2},\frac{\|\delta\phi\|^2}{2M_1\|\delta\phi\|_2^{q_1}}\right\}, \qquad (1.18)\\
\Lambda_{21} &= 1, \qquad (1.19)
\end{aligned}
$$

$$\Lambda_3 = \max\left\{\left(\frac{8\theta C_*}{e^{H_0}(\theta-2)\delta^2}\right)^{q_1-2}, \left(\frac{8\rho D_*}{e^{H_0}(\rho-2)\delta^2}\right)^{q_1-2}\right\}, \tag{1.20}$$

$$K_* = \max\left\{\frac{2\theta}{\theta-2}C_*, \frac{2\theta}{\theta-2}D_*\right\}. \tag{1.21}$$

We organize this paper as follows. In section 2, we present the working space, some conclusions for the working space and a variant of mountain pass theorem. In section 3, we complete the proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3.

2. Preliminaries

Consider the space

$$E = \{u \in H^2(0,1) | u(0) = u'(0) = 0\}, \tag{2.1}$$

where

$$H^2([0,1]) = \{u : [0,1] \to \mathbb{R} | u \text{ and } u' \text{ are absolutely continuous}, u'' \in L^2([0,1])\}$$

is the Sobolev space. E is a Hilbert space with inner product

$$\langle u, v\rangle := \int_0^1 e^{H(x)} u''(x) v''(x) dx,$$

and norm

$$\|u\|_H := \left(\int_0^1 e^{H(x)} |u''(x)|^2 dx\right)^{\frac{1}{2}},$$

which is equivalent to the usual norm

$$\|u\| := \left(\int_0^1 |u''(x)|^2 dx\right)^{\frac{1}{2}},$$

and

$$\|u\|_0 := \left(\int_0^1 (|u(x)|^2 + |u'(x)|^2 + |u''(x)|^2) dx\right)^{\frac{1}{2}}.$$

Lemma 2.1. ([11]) *The embedding*

$$E \hookrightarrow C^1([0,1]),$$

is compact and

$$\|u\|_\infty \le \|u\|, \quad \|u'\|_\infty \le \|u\|,$$

for all $u \in E$.

Remark 2.1. It is easy to obtain from Lemma 2.1 that

$$\|u\|_\infty \le \|u\| \le \frac{1}{\sqrt{e^{H_0}}}\|u\|_H, \quad \|u'\|_\infty \le \|u\| \le \frac{1}{\sqrt{e^{H_0}}}\|u\|_H,$$

where $H_0 = \min_{s \in [0,1]} H(s)$.

Lemma 2.2. *The embedding*

$$E \hookrightarrow L^2(0,1),$$

is compact and

$$\|u\|_2 \le \frac{1}{2\sqrt{2}}\|u\| \le \frac{1}{2\sqrt{2e^{H_0}}}\|u\|_H, \quad \|u'\|_2 \le \frac{1}{\sqrt{2}}\|u\| \le \frac{1}{\sqrt{2e^{H_0}}}\|u\|_H,$$

for all $u \in E$, where $\|u\|_2 = \left(\int_0^1 |u(x)|^2 dx\right)^{\frac{1}{2}}$.

Proof. The compactness of the embedding is easily proved by Lemma 2.1. Next, we prove the embedding inequalities. By Hölder's inequality, we have

$$\begin{aligned}
\|u\|_2^2 &= \int_0^1 |u(x)|^2 dx \\
&= \int_0^1 \left| \int_0^x u'(s)ds \right|^2 dx \\
&\le \int_0^1 \left(\int_0^x |u'(s)|ds \right)^2 dx \\
&\le \int_0^1 x \left(\int_0^x |u'(s)|^2 ds \right) dx \\
&= \int_0^1 \int_0^x x|u'(s)|^2 ds dx \\
&= \int_0^1 ds \int_s^1 x|u'(s)|^2 dx \\
&= \frac{1}{2} \int_0^1 (1-s^2)|u'(s)|^2 ds \\
&= \frac{1}{2} \int_0^1 (1-s^2) \left| \int_0^s u''(\tau)d\tau \right|^2 ds \\
&\le \frac{1}{2} \int_0^1 (s-s^3) \int_0^s |u''(\tau)|^2 d\tau ds \\
&\le \frac{1}{2} \int_0^1 (s-s^3) ds \left(\int_0^1 |u''(\tau)|^2 d\tau \right) \\
&= \frac{1}{8}\|u\|^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|u'\|_2^2 &= \int_0^1 |u'(x)|^2 dx \\
&= \int_0^1 \left| \int_0^x u''(s)ds \right|^2 dx \\
&\le \int_0^1 \left(\int_0^x |u''(s)|ds \right)^2 dx
\end{aligned}$$

$$\begin{aligned} &\leq \int_0^1 x\left(\int_0^x |u''(s)|^2 ds\right) dx \\ &\leq \int_0^1 x dx\left(\int_0^1 |u''(s)|^2 ds\right) \\ &= \frac{1}{2}\|u\|^2. \end{aligned}$$

□

By Lemma 2.2, it is easy to obtain the following inequalities which is of independent interest.

Lemma 2.3. *For all $u \in E$,*

$$\frac{2\sqrt{2}}{\sqrt{13}}\|u\|_0 \leq \|u\| \leq \|u\|_0.$$

Let X be a Banach space. $\chi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. A sequence $\{u_n\} \subset X$ is called $(\mathrm{PS})_c$ sequence (named after Palais and Smale) if the sequence $\{u_n\}$ satisfies

$$\chi(u_n) \to c, \ \ \chi'(u_n) \to 0.$$

Lemma 2.4. (Mountain Pass Theorem [35]) Let X be a Banach space, $\chi \in C^1(X, \mathbb{R})$, $\omega \in X$ and $r > 0$ be such that $\|\omega\| > r$ and

$$b := \inf_{\|u\|=r} \chi(u) > \chi(0) \geq \chi(w).$$

Then there exists a $(\mathrm{PS})_c$ sequence with

$$c := \inf_{g\in\Gamma} \max_{t\in[0,1]} \chi(g(t)),$$

and

$$\Gamma := \{g \in C([0,1], X) : g(0) = 0, g(1) = \omega\}.$$

3. Proofs

3.1. *Proof of Theorem 1.1*

For each $\lambda > 0, \mu \in \mathbb{R}$, we define the functional $J_{\lambda,\mu} : E \to \mathbb{R}$ as

$$J_{\lambda,\mu}(u) = \frac{1}{2}\|u\|_H^2 - \lambda \int_0^1 e^{H(x)} F(x,u)dx + \mu e^{H(1)} G(u(1)).$$

It is easy to see that the assumption $(H_1) - (H_5)$ can not ensure that $J_{\lambda,\mu}$ is well defined on E. So we follow the method in [23]. Define $m(s) \in C^1(\mathbb{R}, [0,1])$ as an even cut-off function satisfying $sm'(s) \leq 0$ and

$$m(s) = \begin{cases} 1, & \text{if } \ |s| \leqslant \delta/2, \\ 0, & \text{if } \ |s| \geqslant \delta. \end{cases} \tag{3.1}$$

Define $\bar{F}:[0,1]\times\mathbb{R}\to\mathbb{R}$, $\bar{G}:\mathbb{R}\to\mathbb{R}$ as

$$\bar{F}(x,u)=m(|u|)F(x,u)+(1-m(|u|))M_2|u|^{q_2},$$
$$\bar{G}(u)=m(|u|)G(u)+(1-m(|u|))N_1|u|^{p_1}.$$

By (H_1) and the definition of $\bar{F}$, it is easy to obtain that $\bar{F}$ satisfies $(H_1)'$ *$\bar{F}(x,u)$ is continuously differentiable in $u\in\mathbb{R}$ for a.e. $x\in[0,1]$, measurable in x for every $u\in\mathbb{R}$, and there exist $a\in C(\mathbb{R}^+,\mathbb{R}^+)$ and $b\in L^1([0,1];\mathbb{R}^+)$ such that*

$$|\bar{F}(x,u)|\le a_0b(x)+M_2|u|^{q_2},$$

$$|\bar{f}(x,u)|\le(1+m_0)a_0b(x)+M_2q_2|u|^{q_2-1}+m_0M_2|u|^{q_2},$$

for all $u\in\mathbb{R}$ and a.e. $x\in[0,1]$, $a_0=\max_{s\in[0,\delta]}a(s)$ and $m_0=\max_{s\in[\frac{\delta}{2},\delta]}|m'(s)|$, where $\bar{F}(x,u)=\int_0^u\bar{f}(x,s)ds$ (see [22]).

Lemma 3.1. ([22]) *Assume that (H_2) and (H_3) hold. Then*

$(H_2)'$

$$0\le\bar{F}(x,u)\le M_2|u|^{q_2},\ \ \textit{for all } u\in\mathbb{R};$$

$(H_3)'$

$$0<\theta\bar{F}(x,u)\le\bar{f}(x,u)u,\ \ \textit{for all } u\in\mathbb{R}/\{0\},$$

where $\theta=\min\{q_2,\beta\}$.

Lemma 3.2. ([36]) *Assume that (H_4) and (H_5) hold. Then*

$(H_4)'$

$$N_1|u|^{p_1}\le\bar{G}(u)\le\max\{N_1,N_2\}(|u|^{p_1}+|u|^{p_2}),\ \ \textit{for all } u\in\mathbb{R};$$

$(H_5)'$

$$\bar{g}(u)u\le\zeta\bar{G}(u),\ \textit{for all } u\in\mathbb{R}/\{0\},$$

where $\zeta=\max\{p_1,\gamma\}$ and $\bar{g}(u)=\bar{G}'(u)$.

Next, we define the variational functional corresponding to $\bar{F}$ and $\bar{G}$ as

$$\bar{J}_{\lambda,\mu}(u)=\frac{1}{2}\|u\|_H^2-\lambda\int_0^1 e^{H(x)}\bar{F}(x,u)dx+\mu e^{H(1)}\bar{G}(u(1)),\tag{3.2}$$

for all $u\in E$. It follows from Lemma 3.1 and Lemma 3.2 that $\bar{J}_{\lambda,\mu}\in C^1(E,\mathbb{R})$ and

$$\langle\bar{J}'_{\lambda,\mu}(u),v\rangle=\int_0^1 e^{H(x)}u''(x)v''(x)dx-\lambda\int_0^1 e^{H(x)}\bar{f}(x,u)vdx+\mu e^{H(1)}\bar{g}(u(1))v(1),\tag{3.3}$$

for all $u,v\in E$. Hence, for all $u\in E$, we can get

$$\langle\bar{J}'_{\lambda,\mu}(u),u\rangle=\|u\|_H^2-\lambda\int_0^1 e^{H(x)}\bar{f}(x,u)udx+\mu e^{H(1)}\bar{g}(u(1))u(1).$$

Lemma 3.3. *If u is a critical point of $\bar{J}_{\lambda,\mu}$, then u is a weak solution of the following equation*

$$\begin{cases} u^{(4)}(x) + 2h(x)u'''(x) + (h^2(x) + h'(x))u''(x) = \lambda \bar{f}(x, u(x)), \quad x \in [0, 1], \\ u(0) = u'(0) = u''(1) = u'''(1) - \mu\bar{g}(u(1)) = 0. \end{cases} \tag{3.4}$$

Proof. If u is a critical point of $\bar{J}_{\lambda,\mu}$, we have

$$\int_0^1 e^{H(x)}u''(x)v''(x)dx - \lambda \int_0^1 e^{H(x)}\bar{f}(x, u(x))v(x)dx + \mu e^{H(1)}\bar{g}(u(1))v(1) = 0, \tag{3.5}$$

for all $v \in H^2([0, 1])$. An integration by parts gives

$$\begin{aligned} &\int_0^1 e^{H(x)}u''(x)v''(x)dx \\ &= \int_0^1 e^{H(x)}u''(x)dv'(x) \\ &= e^{H(x)}u''(x)v'(x)\,|_0^1 - \int_0^1 \left(e^{H(x)}u'''(x) + e^{H(x)}u''(x)h(x)\right)dv(x) \\ &= e^{H(x)}u''(x)v'(x)\,|_0^1 - \left(e^{H(x)}u'''(x) + e^{H(x)}u''(x)h(x)\right)v(x)\,|_0^1 \\ &+ \int_0^1 \left(e^{H(x)}u^{(4)}(x) + 2e^{H(x)}u'''(x)h(x) + e^{H(x)}u''(x)h^2(x) + e^{H(x)}u''(x)h'(x)\right)v(x)dx. \end{aligned} \tag{3.6}$$

Define

$$H_0^2([0, 1]) = \{u : [0, 1] \to \mathbb{R} | u(0) = u(1) = u'(0) = u'(1) = 0\}.$$

Then for any $v \in H^2[0, 1] \cap H_0^2[0, 1]$, (3.5) and (3.6) implies that

$$\int_0^1 e^{H(x)}\left[u^{(4)}(x) + 2u'''(x)h(x) + u''(x)h^2(x) + u''(x)h'(x) - \lambda\bar{f}(x, u(x))\right]v(x)dx = 0,$$

and then by the arbitrary of v, we have

$$u^{(4)}(x) + 2u'''(x)h(x) + u''(x)h^2(x) + u''(x)h'(x) = \lambda\bar{f}(x, u(x)). \tag{3.7}$$

Next, we prove that u satisfies the boundary condition of (3.4). For any $v \in E$, integrating (3.7) by parts, and by (3.5) and (3.6), we can obtain

$$\begin{aligned} &\int_0^1 e^{H(x)}\left(u^{(4)}(x) + 2u'''(x)h(x) + u''(x)h^2(x) + u''(x)h'(x) - \lambda\bar{f}(x, u(x))\right)v(x)dx \\ &+u''(1)e^{H(1)}\left[v'(1) - h(1)v(1)\right] + \left[\mu\bar{g}(u(1)) - u'''(1)\right]e^{H(1)}v(1) = 0. \end{aligned}$$

Then (3.7) and the arbitrary of v imply that the boundary conditions $u''(1) = 0$ and $u'''(1) - \mu\bar{g}(u(1)) = 0$. Therefore, u is a weak solution for problem (3.4).

Lemma 3.4. *If $(H_1) - (H_5)$ hold. For all $\lambda > \Lambda_1^+$ and $\mu > 0$, $\bar{J}_{\lambda,\mu}$ satisfies the following conditions:*
(i) there exist two positive constants d_λ and ν_λ such that $\bar{J}_{\lambda,\mu}|_{\partial B_{\nu_\lambda}} \geq d_\lambda$, where B_r denote a ball with

center 0 and radius r;
(ii) there is $\omega \in E/\bar{B}_{\nu_\lambda}$ such that $\bar{J}_{\lambda,\mu}(\omega) < 0$.
Proof. By Lemma 2.1 and 2.2, Lemma 3.1 and 3.2, we have

$$\begin{aligned}
\bar{J}_{\lambda,\mu}(u) &= \frac{1}{2}\|u\|_H^2 - \lambda \int_0^1 e^{H(x)}\bar{F}(x,u)dx + \mu e^{H(1)}\bar{G}(u(1)) \\
&\geq \frac{1}{2}\|u\|_H^2 - \lambda \int_0^1 e^{H(x)}\bar{F}(x,u)dx \\
&\geq \frac{1}{2}\|u\|_H^2 - \lambda e^{H_1} M_2 \int_0^1 |u|^{q_2} dx \\
&\geq \frac{1}{2}\|u\|_H^2 - \lambda e^{H_1} M_2 \|u\|_\infty^{q_2-2} \int_0^1 |u|^2 dx \\
&= \frac{1}{2}\|u\|_H^2 - \lambda e^{H_1} M_2 \|u\|_\infty^{q_2-2}\|u\|_2^2 \\
&\geq \frac{1}{2}\|u\|_H^2 - \frac{\lambda M_2 e^{H_1}}{8e^{\frac{q_2 H_0}{2}}}\|u\|_H^{q_2},
\end{aligned}$$

where $H_1 = \max_{x\in[0,1]} H(x)$. For any given $\lambda > 0$, we choose $\nu_\lambda = \left(\frac{2e^{\frac{q_2 H_0}{2}}}{\lambda M_2 e^{H_1}}\right)^{\frac{1}{q_2-2}}$. Then for all $\|u\|_H = \nu_\lambda$, we have

$$\bar{J}_{\lambda,\mu}(u) \geq d_\lambda := \frac{1}{2}\nu_\lambda^2 - \frac{\lambda M_2 e^{H_1}}{8e^{\frac{q_2 H_0}{2}}}\nu_\lambda^{q_2} > 0. \tag{3.8}$$

Since $\kappa \in E$ with

$$\|\kappa\|_\infty \leq 1, \tag{3.9}$$

and

$$\Lambda_1^+ = \max\left\{\frac{2e^{\frac{q_2 H_0}{2}}}{M_2 e^{H_1}\|\delta\kappa\|_H^{q_2-2}}, \frac{\|\delta\kappa\|_H^2 + 2\mu N(|\delta\kappa(1)|^{p_1} + |\delta\kappa(1)|^{p_2})}{2e^{H_0} M_1 \|\delta\kappa\|_2^{q_1}}\right\}.$$

Then for all $\lambda > \Lambda_1^+$, we have

$$\|\delta\kappa\|_H \geq \nu_\lambda.$$

By (3.9), we have $\|\delta\kappa\|_\infty \leq \delta$. By $(H_2), (H_4)$ and the definitions of $\bar{F}$ and $\bar{G}$, for all $|u| \leq \frac{\delta}{2}$, we have

$$\bar{F}(x,u) = F(x,u) \geq M_1|u|^{q_1}, \quad \bar{G}(u) = G(u) \leq N_2|u|^{p_2} \leq \max\{N_1, N_2\}(|u|^{p_1} + |u|^{p_2}). \tag{3.10}$$

We also have

$$\begin{aligned}
\bar{F}(x,u) &= m(|u|)F(x,u) + (1 - m(|u|))M_2|u|^{q_2} \\
&\geq m(|u|)M_1|u|^{q_1} + (1 - m(|u|))M_1|u|^{q_1} \\
&= M_1|u|^{q_1}, \qquad (3.11)
\end{aligned}$$

and

$$\begin{aligned}\bar{G}(u) &= m(|u|)G(u) + (1 - m(|u|))N_1|u|^{p_1} \\ &\leq G(u) + N_1|u|^{p_1} \\ &\leq N_2|u|^{p_2} + N_1|u|^{p_1} \\ &\leq \max\{N_1, N_2\}(|u|^{p_1} + |u|^{p_2}),\end{aligned} \tag{3.12}$$

for all $\frac{\delta}{2} < |u| \leq \delta$. Hence by Hölder inequality, for all $\lambda > \Lambda_1^+$, we can obtain

$$\begin{aligned}\bar{J}_{\lambda,\mu}(\delta\kappa) &= \frac{1}{2}\|\delta\kappa\|_H^2 - \lambda\int_0^1 e^{H(x)}\bar{F}(x,\delta\kappa)dx + \mu e^{H(1)}\bar{G}(\delta\kappa(1)) \\ &\leq \frac{1}{2}\|\delta\kappa\|_H^2 - \lambda e^{H_0}M_1\int_0^1 |\delta\kappa|^{q_1}dx + \mu e^{H(1)}N(|\delta\kappa(1)|^{p_1} + |\delta\kappa(1)|^{p_2}) \\ &\leq \frac{1}{2}\|\delta\kappa\|_H^2 - \lambda e^{H_0}M_1\|\delta\kappa\|_2^{q_1} + \mu e^{H(1)}N(|\delta\kappa(1)|^{p_1} + |\delta\kappa(1)|^{p_2}) \\ &< 0,\end{aligned}$$

where $N = \max\{N_1, N_2\}$. Let $\omega = \delta\kappa$. Then the proof is completed. □

Let $\chi = \bar{J}_{\lambda,\mu}$. Then for any given $\lambda > \Lambda_1^+$ and $\mu > 0$, Lemma 2.4 and Lemma 3.4 imply that $\bar{J}_{\lambda,\mu}$ has a $(\mathrm{PS})_{c_{\lambda,\mu}}$ sequence $\{u_n\} := \{u_{n,\lambda,\mu}\}$, that is, there exists a sequence $\{u_n\}$ satisfying

$$\bar{J}_{\lambda,\mu}(u_n) \to c_{\lambda,\mu}, \quad \bar{J}'_{\lambda,\mu}(u_n) \to 0, \ \text{ as } n \to \infty, \tag{3.13}$$

where

$$c_{\lambda,\mu} := \inf_{g\in\Gamma}\max_{t\in[0,1]} \bar{J}_{\lambda,\mu}(g(t)), \qquad \Gamma := \{g \in C([0,1], X) : g(0) = 0, g(1) = \omega\}. \tag{3.14}$$

Lemma 3.5. *Suppose that (H_1)−(H_5) hold. Then for any given $\lambda > \Lambda_1^+$ and $\mu > 0$, the $(PS)_{c_{\lambda,\mu}}$ sequence $\{u_n\}$ has a convergent subsequence in E, that is, there exists a $u_{\lambda,\mu} \in E$ such that $\|u_n - u_{\lambda,\mu}\|_H \to 0$.*

Proof. The proof is similar to the argument in [37]. Note that $\zeta < \theta$, by Lemma 3.1, Lemma 3.2 and (3.13), there exists a positive constant $M > 0$ such that

$$\begin{aligned}M + \|u_n\|_H &\geq \bar{J}_{\lambda,\mu}(u_n) - \frac{1}{\theta}\langle \bar{J}'_{\lambda,\mu}(u_n), u_n\rangle \\ &= \frac{1}{2}\|u_n\|_H^2 - \lambda\int_0^1 e^{H(x)}\bar{F}(x,u_n)dx + \mu e^{H(1)}\bar{G}(u_n(1)) - \frac{1}{\theta}\|u_n\|_H^2 \\ &\quad + \lambda\int_0^1 \frac{1}{\theta}e^{H(x)}\bar{f}(x,u_n)dx - \frac{\mu}{\theta}e^{H(1)}\bar{g}(u_n(1))u_n(1) \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_H^2 - \lambda\int_0^1 e^{H(x)}\left[\bar{F}(x,u_n) - \frac{1}{\theta}\bar{f}(x,u_n)u_n\right]dx \\ &\quad - \mu e^{H(1)}\left[\frac{1}{\theta}\bar{g}(u_n(1))u_n(1) - \bar{G}(u_n(1))\right]\end{aligned}$$

$$\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|_H^2 - \mu e^{H(1)}\left[\frac{\zeta}{\theta}\bar{G}(u_n(1)) - \bar{G}(u_n(1))\right]$$
$$\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|_H^2. \tag{3.15}$$

So the $(PS)_{c_{\lambda,\mu}}$ sequence $\{u_n\}$ is bounded in E. Then by Lemma 2.1, there exists a subsequence, denoted by $\{u_n\}$, for some $u := u_{\lambda,\mu} \in E$, such that

$$u_n \rightharpoonup u \quad \text{in} \quad E, \qquad\qquad u_n \to u \quad \text{in} \quad C^1([0,1]). \tag{3.16}$$

By (3.3), we have

$$\begin{aligned}\langle \bar{J}'_{\lambda,\mu}(u_n), u_n - u\rangle &= \int_0^1 e^{H(x)}u_n''(u_n''-u'')dx - \lambda\int_0^1 e^{H(x)}\bar{f}(x,u_n)(u_n-u)dx \\ &\quad +\mu e^{H(1)}\bar{g}(u_n(1))(u_n(1)-u(1)).\end{aligned} \tag{3.17}$$

So we get

$$\begin{aligned}&\langle \bar{J}'_{\lambda,\mu}(u_n) - \bar{J}'_{\lambda,\mu}(u), u_n - u\rangle \\ =\ & \int_0^1 e^{H(x)}u_n''(u_n''-u'')dx - \lambda\int_0^1 e^{H(x)}\bar{f}(x,u_n)(u_n-u)dx + \mu e^{H(1)}\bar{g}(u_n(1))(u_n(1)-u(1)) \\ & -\left[\int_0^1 e^{H(x)}u''(u_n''-u'')dx - \lambda\int_0^1 e^{H(x)}\bar{f}(x,u)(u_n-u)dx + \mu e^{H(1)}\bar{g}(u(1))(u_n(1)-u(1))\right] \\ =\ & \int_0^1 e^{H(x)}(u_n''-u'')^2dx - \lambda\int_0^1 e^{H(x)}(\bar{f}(x,u_n)-\bar{f}(x,u))(u_n-u)dx \\ & +\mu e^{H(1)}(\bar{g}(u_n(1))-\bar{g}(u(1)))(u_n(1)-u(1)) \\ \geq\ & \|u_n-u\|_H^2 - \lambda\int_0^1 e^{H(x)}|\bar{f}(x,u_n)-\bar{f}(x,u)||u_n-u|dx \\ & -\mu e^{H(1)}\left|\bar{g}(u_n(1))-\bar{g}(u(1))\right|\left|u_n(1)-u(1)\right|.\end{aligned} \tag{3.18}$$

By $(H_1)'$, the boundedness of $\{u_n\}$ in E, and $h(x) \in C^1([0,1])$, we have $e^{H(x)}|\bar{f}(x,u_n) - \bar{f}(x,u)|$ is bounded in $[0,1]$. Moreover, by $u_n \to u$ in $C^1([0,1])$ and the boundedness of $e^{H(x)}|\bar{g}(u_n(1)) - \bar{g}(u(1))|$ on $[0,1]$, we have

$$\lambda\int_0^1 e^{H(x)}|\bar{f}(x,u_n)-\bar{f}(x,u)||u_n-u|dx \to 0, \qquad \mu e^{H(1)}\left|\bar{g}(u_n(1))-\bar{g}(u(1))\right|\left|u_n(1)-u(1)\right| \to 0, \tag{3.19}$$

and it is easy to see from (3.13) and (3.16) that

$$\langle \bar{J}'_{\lambda,\mu}(u_n) - \bar{J}'_{\lambda,\mu}(u), u_n - u\rangle \to 0, \qquad \text{as} \quad n\to\infty. \tag{3.20}$$

Therefore, by (3.18)–(3.20), we get

$$\|u_n-u\|_H^2 \to 0, \qquad as \quad n\to\infty.$$

By the continuity of $\bar{J}_{\lambda,\mu}$, we obtain that $\bar{J}_{\lambda,\mu}(u_{\lambda,\mu}) = c_{\lambda,\mu}$, where $c_{\lambda,\mu}$ is defined by (3.14). Then (3.8) implies that $c_{\lambda,\mu} \geq d_\lambda > 0$. Hence, $u_{\lambda,\mu}$ is a nontrivial critical point of $\bar{J}_{\lambda,\mu}$ in E for any given $\lambda > \Lambda_1^+$.

Next, we will show that $u_{\lambda,\mu}$ precisely is the nontrivial weak solution of problem (1.1) for any given $\lambda > \lambda_+^*$. In order to get this, we need to make an estimate for the critical level $c_{\lambda,\mu}$. We introduce the functional $\widetilde{I}_{\lambda,\mu} : E \to \mathbb{R}$ as follows

$$\widetilde{I}_{\lambda,\mu}(u) = \frac{1}{2}\|u\|_H^2 - \lambda e^{H_0} M_1 \int_0^1 |u|^{q_1} dx + \mu e^{H(1)} N(|u|^{p_1} + |u|^{p_2}).$$

Lemma 3.6. *Suppose* $(H_1) - (H_5)$ *hold. Then for all* $\lambda \geq \max\{\Lambda_1^+, \Lambda_2^+\}$ *and* $\mu > 0$, *we have*

$$c_{\lambda,\mu} \leq C_* \lambda^{-\frac{1}{q_1-2}} + C_{**}\mu,$$

where C_* *and* C_{**} *are defined by (1.7) and (1.8), respectively.*
Proof. Define $f_i : [0,\infty) \to \mathbb{R}$, $i = 1,2,3$, by

$$\begin{aligned} f_1(s) &= \frac{s^2}{2}\|\kappa_1\|_H^2 - \lambda^{\frac{1}{q_1}} \frac{s^2}{2}\|\kappa_1\|_H^2, \\ f_2(s) &= \mu e^{H(1)} N(s^{p_1}|\kappa_1|^{p_1} + s^{p_2}|\kappa_1|^{p_2}), \\ f_3(s) &= -\lambda e^{H_0} M_1 s^{q_1} \int_0^1 |\kappa_1|^{q_1} dx + \lambda^{\frac{1}{q_1}} \frac{s^2}{2}\|\kappa_1\|_H^2, \end{aligned}$$

where $\kappa_1 = \delta\kappa$ and κ is defined in (3.9). Then $f_1(s) + f_2(s) + f_3(s) = \widetilde{I}_{\lambda,\mu}(s\kappa_1)$. Let

$$f_3'(s) = -\lambda e^{H_0} M_1 q_1 \|\kappa_1\|_{q_1}^{q_1} s^{q_1-1} + \lambda^{\frac{1}{q_1}} \|\kappa_1\|_H^2 s = 0.$$

Thus for each given $\lambda > 0$, we have $s = \left(\dfrac{\lambda^{\frac{1}{q_1}}\|\kappa_1\|_H^2}{\lambda e^{H_0} M_1 q_1 \|\kappa_1\|_{L^{q_1}}^{q_1}}\right)^{\frac{1}{q_1-2}}$. Then

$$\max_{s\geq 0} f_3(s) = \left(\frac{1}{2(M_1 e^{H_0} q_1)^{\frac{2}{q_1-2}}} - \frac{e^{H_0} M_1}{(e^{H_0} M_1 q_1)^{\frac{q_1}{q_1-2}}}\right)\left(\frac{\|\kappa_1\|_H}{\|\kappa_1\|_{q_1}}\right)^{\frac{2q_1}{q_1-2}} \lambda^{-\frac{1}{q_1-2}}.$$

Obviously, $f_1(0) = 0$ and

$$f_1'(s) \;\; = \;\; \|\kappa_1\|_H^2 s - \lambda^{\frac{1}{q_1}} \|\kappa_1\|_H^2 s.$$

So if $\lambda > \Lambda_2^+ = 1$, $f_1(s)$ is decreasing on $s \in [0,1]$ and then $f_1(s) < 0$ for all $s \in [0,1]$. Moreover, for all $\mu > 0$, we can get

$$\max_{s\in[0,1]} f_2(s) \leq \mu e^{H_1} N(\|\kappa_1\|_\infty^{p_1} + \|\kappa_2\|_\infty^{p_2}).$$

By (3.9), we have

$$\|s\kappa_1\|_\infty \leq \|\delta\kappa\|_\infty \leq \delta, \tag{3.21}$$

for all $s \in [0,1]$. So for all $\lambda > \Lambda_2^+$, by (3.10)–(3.12), we have

$$c_{\lambda,\mu} \leq \max_{s\in[0,1]} \bar{J}_{\lambda,\mu}(s\kappa_1) \leq \max_{s\in[0,1]} \widetilde{I}_\lambda(s\kappa_1) \leq \max_{s\in[0,1]} f_1(s) + \max_{s\in[0,1]} f_2(s) + \max_{s\geq 0} f_3(s)$$

$$
\begin{aligned}
&\le \max_{s\in[0,1]} f_2(s) + \max_{s\ge 0} f_3(s) = \left(\frac{1}{2(M_1e^{H_0}q_1)^{\frac{2}{q_1-2}}} - \frac{M_1e^{H_0}}{(e^{H_0}M_1q_1)^{\frac{q_1}{q_1-2}}}\right)\left(\frac{\|\kappa_1\|_H}{\|\kappa_1\|_{q_1}}\right)^{\frac{2q_1}{q_1-2}} \lambda^{-\frac{1}{q_1-2}}\\
&+\mu e^{H_1}N(\|\kappa_1\|_\infty^{p_1} + \|\kappa_2\|_\infty^{p_2})\\
&\le \left(\frac{1}{2(M_1e^{H_0}q_1)^{\frac{2}{q_1-2}}} - \frac{M_1e^{H_0}}{(e^{H_0}M_1q_1)^{\frac{q_1}{q_1-2}}}\right)\left(\frac{\|\kappa_1\|_H}{\|\kappa_1\|_2}\right)^{\frac{2q_1}{q_1-2}} \lambda^{-\frac{1}{q_1-2}} + \mu e^{H_1}N\left(e^{-\frac{H_0p_1}{2}}\|\kappa_1\|_H^{p_1} + e^{-\frac{H_0p_2}{2}}\|\kappa_2\|_H^{p_2}\right)\\
&= C_*\lambda^{-\frac{1}{q_1-2}} + C_{**}\mu.
\end{aligned}
$$

Proof of Theorem 1.1. Note that $u_{\lambda,\mu}$ is a critical point of $\bar{J}_{\lambda,\mu}$ with critical value $c_{\lambda,\mu}$. Since $\langle \bar{J}'(u_{\lambda,\mu}), u_{\lambda,\mu}\rangle = 0$, similar to the argument in (3.15) and by Lemma 3.6, we have

$$\|u_{\lambda,\mu}\|_H^2 \le \frac{2\theta}{\theta-2}\bar{J}_{\lambda,\mu}(u_{\lambda,\mu}) = \frac{2\theta}{\theta-2}c_{\lambda,\mu} \le \frac{2\theta}{\theta-2}(C_*\lambda^{-\frac{1}{q_1-2}} + C_{**}\mu). \tag{3.22}$$

Note that $\mu \in (0,\mu_*)$, where μ_* is defined by (1.10). If

$$\lambda > \Lambda_3^+ := \left[\frac{8\theta C_*}{(\theta-2)e^{H_0}\delta^2 - 8\theta C_{**}\mu}\right]^{q_1-2},$$

then by Remark 2.1 and (3.22), we have

$$\|u_{\lambda,\mu}\|_\infty \le \frac{1}{\sqrt{e^{H_0}}}\|u_{\lambda,\mu}\|_H \le \frac{\delta}{2}. \tag{3.23}$$

So for all $\lambda > \Lambda_3^+$ and all $x \in [0,1]$,

$$|u_{\lambda,\mu}(x)| \le \|u_{\lambda,\mu}\|_\infty \le \frac{\delta}{2},$$

and then

$$\bar{F}(x,u(x)) = F(x,u(x)),\ \ \bar{G}(u(1)) = G(u(1)).$$

Furthermore, for all $v \in E$, we have

$$\bar{J}_{\lambda,\mu}(u_{\lambda,\mu}) = J_{\lambda,\mu}(u_{\lambda,\mu}) = c_{\lambda,\mu} > 0, \quad \langle \bar{J}'_{\lambda,\mu}(u_{\lambda,\mu}), v\rangle = \langle J'_{\lambda,\mu}(u_{\lambda,\mu}), v\rangle = 0.$$

Thus, $u_{\lambda,\mu}$ is precisely the nontrivial weak solution of problem (1.1) when $\lambda > \lambda_+^* := \max\{\Lambda_1^+, \Lambda_2^+, \Lambda_3^+\}$.

Next, we discuss the connection between μ, λ and $\Lambda_i^+ (i = 1,2,3)$.

(a_1) If $\mu \to 0^+$, we have

$$
\begin{aligned}
&\Lambda_1^+ \longrightarrow \Lambda_{11} = \max\left\{\frac{2e^{\frac{q_2H_0}{2}}}{M_2e^{H_1}\|\delta\kappa\|_H^{q_2-2}}, \frac{\|\delta\kappa\|_H^2}{2e^{H_0}M_1\|\delta\kappa\|_2^{q_1}}\right\},\\
&\Lambda_2^+ \to \Lambda_{21} = 1,\\
&\Lambda_3^+ \to \Lambda_{31} = \left[\frac{8\theta C_*}{(\theta-2)e^{H_0}\delta^2}\right]^{q_1-2},
\end{aligned}
$$

which shows that if $\mu \to 0^+$, then $\lambda_+^* \to \lambda_*$, where λ_* is defined by (1.9). Hence, if μ is small enough, the range of λ can be extended to (λ_*, ∞).

(a_2) If $\mu \to \mu_*^-$, we have

$$\Lambda_1^+ \longrightarrow \max\left\{\frac{2e^{\frac{q_2 H_0}{2}}}{M_2 e^{H_1}\|\delta\kappa\|_H^{q_2-2}}, \frac{4\theta C_{**}\|\delta\kappa\|_H^2 + (\theta-2)\delta^2 e^{H_0} N(|\delta\kappa(1)|^{p_1} + |\delta\kappa|^{p_2})}{8\theta C_{**} e^{H_0} M_1 \|\delta\kappa\|_2^{q_1}}\right\},$$
$$\Lambda_2^+ \longrightarrow 1,$$
$$\Lambda_3^+ \longrightarrow +\infty,$$

and then $\lambda \to +\infty$, which means that if μ is close to μ_*, problem (1.1) has a nontrivial weak solution when λ is sufficiently large.

Finally, by $(a_1), (a_2)$, (3.22) and (3.23), it is easy to obtain that

$$\lim_{\substack{\mu\to 0^+\\ \lambda\to+\infty}} \|u_{\lambda,\mu}\|_H = 0 = \lim_{\substack{\mu\to 0^+\\ \lambda\to+\infty}} \|u_{\lambda,\mu}\|_\infty,$$

$$\sqrt{e^{H_0}} \lim_{\substack{\mu\to 0^+\\ \lambda\to\lambda_*}} \|u_{\lambda,\mu}\|_\infty \le \lim_{\substack{\mu\to 0^+\\ \lambda\to\lambda_*}} \|u_{\lambda,\mu}\|_H \le \sqrt{\frac{2\theta C_*}{\theta-2}\lambda_*^{-\frac{1}{2(q_1-2)}}},$$

$$\sqrt{e^{H_0}} \lim_{\substack{\mu\to \mu_*^-\\ \lambda\to+\infty}} \|u_{\lambda,\mu}\|_\infty \le \lim_{\substack{\mu\to \mu_*^-\\ \lambda\to+\infty}} \|u_{\lambda,\mu}\|_H \le \frac{\delta}{2}\sqrt{e^{H_0}}.$$

3.2. Proof of Theorem 1.2

By (3.1), we define $\bar{\bar{G}} : \mathbb{R} \to \mathbb{R}$ as

$$\bar{\bar{G}}(u) = m(|u|)G(u) + (1 - m(|u|))\eta_2|u|^{\alpha_2}.$$

Similar to Lemma 3.1, we have the following conclusion.
Lemma 3.7. *Assume* $(H_6) - (H_7)$ *hold. Then*
$(H_6)'$

$$0 \le \bar{\bar{G}}(u) \le \eta_2|u|^{\alpha_2}, \quad \textit{for all } u \in \mathbb{R};$$

$(H_7)'$

$$0 \le \varrho\bar{\bar{G}}(u) \le \bar{\bar{g}}(u)u, \quad \textit{for all } u \in \mathbb{R}/\{0\},$$

where $\varrho = \min\{\alpha_2, \xi\}, \quad \bar{\bar{g}}(u) = \bar{\bar{G}}'(u)$.

Next, we define the variational functional corresponding to $\bar{F}$ and $\bar{\bar{G}}$ as

$$\bar{\bar{J}}_{\lambda,\mu}(u) = \frac{1}{2}\|u\|_H^2 - \lambda\int_0^1 e^{H(x)}\bar{F}(x,u)dx + \mu e^{H(1)}\bar{\bar{G}}(u(1)),$$

for all $u \in E$. By Lemma 3.1 and Lemma 3.7, we have $\bar{\bar{J}}_{\lambda,\mu} \in C^1(E,\mathbb{R})$ and

$$\langle \bar{\bar{J}}'_{\lambda,\mu}(u), v\rangle = \int_0^1 e^{H(x)}u''(x)v''(x)dx - \lambda\int_0^1 e^{H(x)}\bar{f}(x,u)v dx + \mu e^{H(1)}\bar{\bar{g}}(u(1))v(1),$$

for all $u, v \in E$. Hence

$$\langle \bar{\bar{J}}'_{\lambda,\mu}(u), u\rangle = \|u\|_H^2 - \lambda \int_0^1 e^{H(x)} \bar{f}(x,u)u dx + \mu e^{H(1)} \bar{\bar{g}}(u(1))u(1),$$

for all $u \in E$.

Lemma 3.8. *If $(H_1)-(H_3)$, (H_6) and (H_7) hold. If $\lambda > \max\{\Lambda_1^-, \Lambda_4^-\}$ and $\mu < 0$, then $\bar{\bar{J}}_{\lambda,\mu}$ satisfies the following conditions:*
(i) there exist two positive constants $d_{\lambda,\mu}$ and $\rho_{\lambda,\mu}$ such that $\bar{\bar{J}}_{\lambda,\mu}|_{\partial B_{\rho_{\lambda,\mu}}} \geq d_{\lambda,\mu}$;
(ii) there is $\omega \in E/\bar{B}_{\rho_{\lambda,\mu}}$ such that $\bar{\bar{J}}_{\lambda,\mu}(\omega) < 0$.
Proof. By Lemma 2.1 and 2.2, Remark 2.1 and Lemma 3.7, we have

$$\begin{aligned}
\bar{\bar{J}}_{\lambda,\mu}(u) &= \frac{1}{2}\|u\|_H^2 - \lambda \int_0^1 e^{H(x)} \bar{F}(x,u)dx + \mu e^{H(1)} \bar{\bar{G}}(u(1)) \\
&\geq \frac{1}{2}\|u\|_H^2 - \lambda e^{H_1} M_2 \int_0^1 |u|^{q_2} dx + \mu e^{H(1)} \eta_2 |u(1)|^{\alpha_2} \\
&\geq \frac{1}{2}\|u\|_H^2 - \lambda e^{H_1} M_2 \|u\|_\infty^{q_2-2} \int_0^1 |u|^2 dx + \mu e^{H(1)} \eta_2 \|u\|_\infty^{\alpha_2} \\
&\geq \frac{1}{2}\|u\|_H^2 - \frac{\lambda M_2 e^{H_1}}{8e^{\frac{q_2 H_0}{2}}}\|u\|_H^{q_2} + \frac{\mu e^{H(1)}\eta_2}{e^{\frac{\alpha_2 H_0}{2}}}\|u\|_H^{\alpha_2}.
\end{aligned}$$

Note that $\lambda > \Lambda_4^- = \frac{2e^{\frac{q_2H_0}{2}}}{M_2 e^{H_1}}$. If we take $\|u\|_H = \rho_{\lambda,\mu} := \left(\frac{2e^{\frac{q_2H_0}{2}}}{\lambda e^{H_1} M_2 - 8\mu e^{H(1)}\eta_2}\right)^{\frac{1}{q_2-2}}$, then $\|u\|_H < 1$. Thus for all $u \in \partial B_{\rho_{\lambda,\mu}}$, we have

$$\begin{aligned}
\bar{\bar{J}}_{\lambda,\mu}(u) &\geq \frac{1}{2}\|u\|_H^2 - \frac{\lambda M_2 e^{H_1}}{8e^{\frac{q_2 H_0}{2}}}\|u\|_H^{q_2} + \frac{\mu e^{H(1)}\eta_2}{e^{\frac{\alpha_2 H_0}{2}}}\|u\|_H^{q_2} \\
&= d_{\lambda,\mu} := \frac{1}{2}\rho_{\lambda,\mu}^2 - \frac{\lambda e^{H_1} M_2 - 8\mu e^{H(1)}\eta_2}{8e^{\frac{q_2 H_0}{2}}}\rho_{\lambda,\mu}^{q_2} > 0. \qquad (3.24)
\end{aligned}$$

We choose $\phi \in E$ such that

$$\|\phi\|_\infty \leq 1. \qquad (3.25)$$

Note that

$$\Lambda_1^- = \max\left\{\frac{2e^{\frac{q_2H_0}{2}}}{M_2 e^{H_1}\|\delta\phi\|_H^{q_2-2}} + \frac{8\mu\eta_2 e^{H(1)}}{M_2 e^{H_1}}, \frac{\|\delta\phi\|_H^2}{2M_1 e^{H_0}\|\delta\phi\|_2^{q_1}}\right\}.$$

For all $\lambda > \Lambda_1^-$, we have

$$\|\delta\phi\|_H \geq \rho_{\lambda,\mu}.$$

By (3.25), we have $\|\delta\phi\|_\infty \leq \delta$. By (H_2), (H_4) and the definition of $\bar{F}$ and $\bar{\bar{G}}$, for all $|u| \leq \frac{\delta}{2}$, we have

$$\bar{F}(x,u) = F(x,u) \geq M_1|u|^{q_1}, \quad \bar{\bar{G}}(u) = G(u) \geq \eta_1|u|^{\alpha_1}. \qquad (3.26)$$

We also have

$$\begin{aligned}\bar{F}(x,u) &= m(|u|)F(x,u) + (1-m(|u|))M_2|u|^{q_2}\\ &\geq m(|u|)M_1|u|^{q_1} + (1-m(|u|))M_1|u|^{q_1}\\ &= M_1|u|^{q_1},\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}\bar{\bar{G}}(u) &= m(|u|)G(u) + (1-m(|u|))\eta_2|u|^{\alpha_2}\\ &\geq m(|u|)\eta_1|u|^{\alpha_1} + (1-m(|u|))\eta_1|u|^{\alpha_1}\\ &= \eta_1|u|^{\alpha_1},\end{aligned} \tag{3.28}$$

for all $\frac{\delta}{2} < |u| < \delta$. Hence, by Hölder inequality, for any $\lambda > \Lambda_1^-$, we have

$$\begin{aligned}\bar{\bar{J}}_{\lambda,\mu}(\delta\phi) &= \frac{1}{2}\|\delta\phi\|_H^2 - \lambda\int_0^1 e^{H(x)}\bar{F}(x,\delta\phi)dx + \mu e^{H(1)}\bar{\bar{G}}(\delta\phi(1))\\ &\leq \frac{1}{2}\|\delta\phi\|_H^2 - \lambda M_1 e^{H_0}\int_0^1 |\delta\phi|^{q_1}dx\\ &\leq \frac{1}{2}\|\delta\phi\|_H^2 - \lambda M_1 e^{H_0}\|\delta\phi\|_2^{q_1}\\ &< 0.\end{aligned}$$

Let $\omega = \delta\phi$. Then the proof is completed.

Let $\chi = \bar{J}_{\lambda,\mu}$. Then for any given $\lambda > \max\{\Lambda_1^-, \Lambda_4^-\}$, Lemma 2.4 and Lemma 3.8 imply that $\bar{\bar{J}}_{\lambda,\mu}$ has a $(PS)_{c_{\lambda,\mu}}$ sequence $\{u_n\} := \{u_{n,\lambda,\mu}\}$, that is, there exists a sequence $\{u_n\}$ satisfying

$$\bar{\bar{J}}_{\lambda,\mu}(u_n) \to c_{\lambda,\mu}, \quad \bar{\bar{J}}'_{\lambda,\mu}(u_n) \to 0, \ \text{ as } n \to \infty, \tag{3.29}$$

where

$$c_{\lambda,\mu} := \inf_{g\in\Gamma}\max_{t\in[0,1]} \bar{\bar{J}}_{\lambda,\mu}(g(t)), \qquad \Gamma := \{g \in C([0,1],X) : g(0) = 0, g(1) = \omega\}. \tag{3.30}$$

Lemma 3.9. *Suppose that $(H_1)-(H_3)$, (H_6) and (H_7) hold. Then for any given $\lambda > \max\{\Lambda_1^-, \Lambda_4^-\}$ and $\mu < 0$, the $(PS)_{c_{\lambda,\mu}}$ sequence $\{u_n\}$ has a convergent subsequence in E, that is, there exists a $u_{\lambda,\mu} \in E$ such that $\|u_n - u_{\lambda,\mu}\|_H \to 0$.*

Proof. Note that $\rho = \min\{\theta, \varrho\}$. By Lemma 3.1, Lemma 3.7 and (3.29), there exists a positive constant $M > 0$ such that

$$\begin{aligned}M + \|u_n\|_H &\geq \bar{\bar{J}}_{\lambda,\mu}(u_n) - \frac{1}{\rho}\langle \bar{\bar{J}}'_{\lambda,\mu}(u_n), u_n\rangle\\ &= \frac{1}{2}\|u_n\|_H^2 - \lambda\int_0^1 e^{H(x)}\bar{F}(x,u_n)dx + \mu e^{H(1)}\bar{\bar{G}}(u_n(1)) - \frac{1}{\rho}\|u_n\|_H^2\\ &\quad +\lambda\int_0^1 e^{H(x)}\frac{1}{\rho}\bar{f}(x,u_n)u_ndx - \frac{\mu}{\rho}e^{H(1)}\bar{\bar{g}}(u_n(1))u_n(1)\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{2}-\frac{1}{\rho}\right)\|u_n\|_H^2 - \lambda\int_0^1 e^{H(x)}\left[\bar{F}(x,u_n)-\frac{1}{\rho}\bar{f}(x,u_n)u_n\right]dx \\
&\quad -\mu e^{H(1)}\left[\frac{1}{\rho}\bar{\bar{g}}(u_n(1))u(1)-\bar{\bar{G}}(u_n(1))\right] \\
&\geq \left(\frac{1}{2}-\frac{1}{\rho}\right)\|u_n\|_H^2 - \lambda\int_0^1 e^{H(x)}\left[\bar{F}(x,u_n)-\frac{\theta}{\rho}\bar{F}(x,u_n)\right]dx \\
&\quad -\mu e^{H(1)}\left[\frac{\varrho}{\rho}\bar{\bar{G}}(u_n(1))-\bar{\bar{G}}(u_n(1))\right] \\
&\geq \left(\frac{1}{2}-\frac{1}{\rho}\right)\|u_n\|_H^2. \qquad (3.31)
\end{aligned}
$$

So the $(PS)_{c_{\lambda,\mu}}$ sequence $\{u_n\}$ is bounded in E, when $n\to\infty$. The rest proof is similar to the argument in Lemma 3.5.

By the continuity of $\bar{\bar{J}}_{\lambda,\mu}$, we obtain that $\bar{\bar{J}}_{\lambda,\mu}(u_{\lambda,\mu}) = c_{\lambda,\mu}$, where $c_{\lambda,\mu}$ is defined by (3.30). Then (3.24) implies that $c_{\lambda,\mu} \geq d_{\lambda,\mu} > 0$. Hence $u_{\lambda,\mu}$ is a nontrivial critical point of $\bar{\bar{J}}_{\lambda,\mu}$ in E for any given $\lambda > \max\{\Lambda_1^-, \Lambda_4^-\}$.

Next, we will show that $u_{\lambda,\mu}$ precisely is the nontrivial weak solution of problem (1.1) for any given $\lambda > \lambda_-^*$. In order to get this, we need to make an estimate for the critical level $c_{\lambda,\mu}$. We introduce the functional $\bar{I}_{\lambda,\mu}: E\to\mathbb{R}$ as follows

$$\bar{I}_{\lambda,\mu}(u) = \frac{1}{2}\|u\|_H^2 - \lambda M_1 e^{H_0}\int_0^1 |u|^{q_1}dx + \mu e^{H(1)}\bar{\bar{G}}(u(1)).$$

Lemma 3.10. *Suppose* $(H_1)-(H_3)$*,* (H_6) *and* (H_7) *hold. Then for all* $\lambda > \max\{\Lambda_1^-, \Lambda_2^-, \Lambda_4^-\}$ *and* $\mu < 0$*, we have*

$$c_{\lambda,\mu} \leq D_*\lambda^{-\frac{1}{q_1-2}},$$

where D_* *is defined by (1.14).*

Proof. Define $k_i: [0,\infty)\to\mathbb{R}$, $i=1,2,3$, by

$$
\begin{aligned}
k_1(s) &= \|\phi_1\|_H^2\frac{s^2}{2} - \lambda^{\frac{1}{q_1}}\|\phi_1\|_H^2\frac{s^2}{2}, \\
k_2(s) &= \mu e^{H(1)}\bar{\bar{G}}(s\phi_1(1)), \\
k_3(s) &= -\lambda M_1 e^{H_0}s^{q_1}\int_0^1 |\phi_1|^{q_1}dx + \lambda^{\frac{1}{q_1}}\|\phi_1\|_H^2\frac{s^2}{2},
\end{aligned}
$$

where $\phi_1 = \delta\phi$ and ϕ is defined in (3.25). Then $k_1(s)+k_2(s)+k_3(s) = \bar{I}_{\lambda,\mu}(s\phi_1)$. Let

$$k_3'(s) = -\lambda M_1 e^{H_0} q_1\|\phi_1\|_{q_1}^{q_1}s^{q_1-1} + \lambda^{\frac{1}{q_1}}\|\phi_1\|_H^2 s = 0.$$

Thus for each given $\lambda > 0$, we have $s = \left(\frac{\lambda^{\frac{1}{q_1}}\|\phi_1\|_H^2}{\lambda e^{H_0}M_1q_1\|\phi_1\|_{q_1}^{q_1}}\right)^{\frac{1}{q_1-2}}$. Then

$$\max_{s\geq 0} k_3(s) = \left(\frac{1}{2(M_1e^{H_0}q_1)^{\frac{2}{q_1-2}}} - \frac{M_1e^{H_0}}{(e^{H_0}M_1q_1)^{\frac{q_1}{q_1-2}}}\right)\left(\frac{\|\phi_1\|_H}{\|\phi_1\|_{q_1}}\right)^{\frac{2q_1}{q_1-2}}\lambda^{-\frac{1}{q_1-2}}.$$

Obviously, $f_1(0) = 0$ and

$$k_1'(s) \;=\; \|\phi_1\|_H^2 s - \lambda^{\frac{1}{q_1}}\|\phi_1\|_H^2 s.$$

So if $\lambda > \Lambda_2^- = 1$, $k_1(s)$ is decreasing on $s \in [0,1]$ and then $k_1(s) \le 0$. Moreover, obviously,

$$\max_{s\in[0,1]} k_2(s) \le 0.$$

By (3.25), we have

$$\|s\phi_1\|_\infty \le \|\delta\phi\|_\infty \le \delta \tag{3.32}$$

for all $s \in [0,1]$. Then for all $\lambda > \max\{\Lambda_1^-, \Lambda_2^-, \Lambda_4^-\}$, by (3.26)–(3.27), we have

$$\begin{aligned}
c_{\lambda,\mu} &\le \max_{s\in[0,1]} \bar{\bar{J}}_{\lambda,\mu}(s\phi_1) \le \max_{s\in[0,1]} \bar{I}_\lambda(s\phi_1) \le \max_{s\in[0,1]} k_1(s) + \max_{s\in[0,1]} k_2(s) + \max_{s\ge 0} k_3(s)\\
&\le \max_{s\ge 0} k_3(s) = \left(\frac{1}{2(M_1e^{H_0}q_1)^{\frac{2}{q_1-2}}} - \frac{M_1e^{H_0}}{(e^{H_0}M_1q_1)^{\frac{q_1}{q_1-2}}}\right)\left(\frac{\|\phi_1\|_H}{\|\phi_1\|_{q_1}}\right)^{\frac{2q_1}{q_1-2}} \lambda^{-\frac{1}{q_1-2}}\\
&\le \left(\frac{1}{2(M_1e^{H_0}q_1)^{\frac{2}{q_1-2}}} - \frac{M_1e^{H_0}}{(e^{H_0}M_1q_1)^{\frac{q_1}{q_1-2}}}\right)\left(\frac{\|\phi_1\|_H}{\|\phi_1\|_2}\right)^{\frac{2q_1}{q_1-2}} \lambda^{-\frac{1}{q_1-2}}\\
&= D_*\lambda^{-\frac{1}{q_1-2}}.
\end{aligned}$$

Proof of Theorem 1.2. Note that $u_{\lambda,\mu}$ is a critical point of $\bar{\bar{J}}_{\lambda,\mu}$ with critical value $c_{\lambda,\mu}$. Since $\langle \bar{\bar{J}}'(u_{\lambda,\mu}), u_{\lambda,\mu}\rangle = 0$, similar to the argument in (3.31) and by Lemma 3.10, we have

$$\|u_{\lambda,\mu}\|_H^2 \le \frac{2\rho}{\rho-2}\bar{\bar{J}}_{\lambda,\mu}(u_{\lambda,\mu}) = \frac{2\rho}{\rho-2}c_{\lambda,\mu} \le \frac{2\rho}{\rho-2}D_*\lambda^{-\frac{1}{q_1-2}}. \tag{3.33}$$

Since

$$\lambda > \Lambda_3^- := \left[\frac{8\rho D_*}{(\rho-2)\delta^2 e^{H_0}}\right]^{q_1-2},$$

by Remark 2.1 and (3.33), we have

$$\|u_{\lambda,\mu}\|_\infty \le \frac{1}{\sqrt{e^{H_0}}}\|u_{\lambda,\mu}\|_H \le \frac{\delta}{2}. \tag{3.34}$$

So for all $\lambda > \Lambda_3^-$, we have

$$|u_{\lambda,\mu}(x)| \le \|u_{\lambda,\mu}\|_\infty \le \frac{\delta}{2}, \quad \text{for all } x \in [0,1],$$

and then

$$\bar{F}(x, u(x)) = F(x, u(x)), \quad \bar{\bar{G}}(u(1)) = G(u(1)).$$

Furthermore, for all $v \in E$, we have

$$\bar{\bar{J}}_{\lambda,\mu}(u_{\lambda,\mu}) = J_{\lambda,\mu}(u_{\lambda,\mu}) = c_{\lambda,\mu} > 0, \quad \langle \bar{\bar{J}}'_{\lambda,\mu}(u_{\lambda,\mu}), v\rangle = \langle J'_{\lambda,\mu}(u_{\lambda,\mu}), v\rangle = 0.$$

Thus, $u_{\lambda,\mu}$ is precisely the nontrivial weak solution of problem (1.1) when $\lambda > \lambda_-^* := \max\{\Lambda_1^-, \Lambda_2^-, \Lambda_3^-, \Lambda_4^-\}$.

Next, we discuss the connection between μ, λ and $\Lambda_i^-(i = 1, 2, 3, 4)$. If $\mu \to 0^-$, we can get

$$\begin{aligned}
\Lambda_1^- &\to \Lambda_{22} := \max\left\{\frac{2e^{\frac{q_2 H_0}{2}}}{M_2 e^{H_1}\|\delta\phi\|^{q_2-2}}, \frac{\|\delta\phi\|_H^2}{2M_1 e^{H_0}\|\delta\phi\|_2^{q_1}}\right\},\\
\Lambda_2^- &= \Lambda_{21} := 1,\\
\Lambda_3^- &= \Lambda_{23} := \left[\frac{8\rho D_*}{(\rho-2)\delta^2 e^{H_0}}\right]^{q_1-2},\\
\Lambda_4^- &= \Lambda_4 := \frac{2e^{\frac{q_2 H_0}{2}}}{M_2 e^{H_1}},
\end{aligned}$$

which shows that if $\mu \to 0^-$, then $\lambda_-^* \to \Lambda_*$. Hence, if $|\mu|$ is small enough, the range of λ can be extended to (Λ_*, ∞), where Λ_* is defined by (1.15).

Finally, by (3.33) and (3.34), it is easy to see that (1.11)–(1.13) hold.

3.3. Proof of Theorem 1.3

Let $\{u_n := u_{\lambda,\mu_n}\} \subset E$ be the critical points of $\bar{J}_{\lambda,\mu_n}$ (if $\mu_n > 0$) and $\widetilde{J}_{\lambda,\mu_n}$ (if $\mu_n < 0$) with respect to c_{λ,μ_n}. Then we have

$$\begin{aligned}
\bar{J}_{\lambda,\mu_n}(u_n) &= \frac{1}{2}\|u_n\|_H^2 - \lambda\int_0^1 e^{H(x)}\bar{F}(x,u_n)dx + \mu_n e^{H(1)}\bar{G}(u_n(1)) = c_{\lambda,\mu_n}, \quad \text{if } \mu_n > 0,\\
\widetilde{J}_{\lambda,\mu_n}(u_n) &= \frac{1}{2}\|u_n\|_H^2 - \lambda\int_0^1 e^{H(x)}\bar{F}(x,u_n)dx + \mu_n e^{H(1)}\widetilde{G}(u_n(1)) = c_{\lambda,\mu_n}, \quad \text{if } \mu_n < 0,
\end{aligned}$$

and

$$\langle \bar{J}'_{\lambda,\mu_n}(u_n), v\rangle = 0, \quad \text{if } \mu_n > 0, \tag{3.35}$$
$$\langle \widetilde{J}'_{\lambda,\mu_n}(u_n), v\rangle = 0, \quad \text{if } \mu_n < 0, \tag{3.36}$$

for all $v \in E$. Define the functional $\bar{J}_\lambda : E \to \mathbb{R}$ as

$$\bar{J}_\lambda(u) = \frac{1}{2}\|u\|_H^2 - \lambda\int_0^1 e^{H(x)}\bar{F}(x,u)dx.$$

Obviously, $\bar{J}_{\lambda,\mu_n}(u_n) \to \bar{J}_\lambda(u_n)$ as $\mu_n \to 0^+$ and $\widetilde{J}_{\lambda,\mu_n}(u_n) \to \bar{J}_\lambda(u_n)$ as $\mu_n \to 0^-$. Moreover, it is easy to see that the critical point of $\bar{J}_\lambda$ is a weak solution of the following elastic beam equation

$$\begin{cases} u^{(4)}(x) + 2h(x)u'''(x) + (h^2(x) + h'(x))u''(x) = \lambda\bar{f}(x,u(x)), & x \in [0,1],\\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases} \tag{3.37}$$

Next, we prove that $\{u_n\}$ is a bounded (PS) sequence of $\bar{J}_\lambda$. Since $\mu_n \to 0$, there exists a positive constant N_0 such that $-M_0 \le \mu_n \le M_0$ for all $n \ge N_0$. Note that Λ_1^+ is decreasing to Λ_{11} as $\mu_n \to 0^+$ and $q_1 > 2$. Then Lemma 3.6 implies that

$$c_{\lambda,\mu_n} \le C_*\left(\max\{\Lambda_1^+, \Lambda_2^+\}\right)^{-\frac{1}{q_1-2}} + C_{**}\mu_n \le C_*\left(\max\{\Lambda_{11}, \Lambda_{21}\}\right)^{-\frac{1}{q_1-2}} + C_{**}M_0, \tag{3.38}$$

for all $n \geq N_0$ and all $\lambda > \max\{\Lambda_{M_0}, \Lambda_{21}\} \geq \max\{\Lambda_1^+, \Lambda_2^+\}$, where Λ_{M_0} is defined by (1.17). Moreover, since Λ_1^- is increasing to Λ_{22} as $\mu_n \to 0^-$, we have

$$\Lambda_1^- \geq \Lambda_{-M_0} := \max\left\{\frac{2e^{\frac{q_2 H_0}{2}}}{M_2\|\delta\phi\|_H^{q_2-2}e^{H_1}} + \frac{-8M_0\eta_2 e^{H(1)}}{M_2 e^{H_1}}, \frac{\|\delta\phi\|_H^2}{2M_1\|\delta\phi\|_2^{q_1}e^{H_0}}\right\},$$

for all $n \geq N_0$. Then Lemma 3.10 implies that

$$c_{\lambda,\mu_n} \leq C_*(\max\{\Lambda_1^-, \Lambda_2^-\})^{-\frac{1}{q_1-2}} \leq C_*(\max\{\Lambda_{-M_0}, \Lambda_2^-\})^{-\frac{1}{q_1-2}}, \tag{3.39}$$

for all $n \geq N_0$ and all $\lambda > \max\{\Lambda_{22}, \Lambda_{21}\} \geq \max\{\Lambda_1^-, \Lambda_2^-, \Lambda_4^-\}$. (3.38) and (3.39) imply that c_{λ,μ_n} is bounded for all $n \geq N_0$ and all $\lambda > \max\{\Lambda_{M_0}, \Lambda_{21}, \Lambda_{22}, \Lambda_4^-\}$. Then it follows from (3.22) and (3.33) that there exists a positive constant K_0 such that $\|u_n\|_H \leq K_0$ for all $n \geq N_0$. Thus we have

$$\|u_n\|_H \leq K_1 := \max\{K_0, \|u_1\|_H, \cdots, \|u_{N_0-1}\|_H\}, \tag{3.40}$$

for all n and all $\lambda > \max\{\Lambda_{M_0}, \Lambda_{21}, \Lambda_{22}, \Lambda_4^-\}$. By (3.40), (H2)$'$ and Remark 2.1, it is easy to see that $\bar{J}_\lambda(u_n)$ is bounded.

If $\mu_n > 0$, then by (3.35), for any given $v \in E$, we have

$$\begin{aligned}
\langle \bar{J}_\lambda'(u_n), v\rangle &= \int_0^1 e^{H(x)}u_n''v''dx - \lambda\int_0^1 e^{H(x)}\bar{f}(x,u_n)vdx \\
&= \int_0^1 e^{H(x)}u_n''v''dx - \lambda\int_0^1 e^{H(x)}\bar{f}(x,u_n)vdx + \mu_n e^{H(1)}\bar{g}(u_n(1))v(1) - \mu_n e^{H(1)}\bar{g}(u_n(1))v(1) \\
&= \langle \bar{J}_{\lambda,\mu_n}'(u_n), v\rangle - \mu_n e^{H(1)}\bar{g}(u_n(1))v(1) \\
&= -\mu_n\bar{g}(u_n(1))v(1).
\end{aligned} \tag{3.41}$$

Since $\mu_n \to 0^+$ as $n \to \infty$, the continuity of g, Remark 2.1, (3.40) and (3.41) imply that

$$\bar{J}_\lambda'(u_n) \to 0 \text{ as } n \to \infty. \tag{3.42}$$

Similarly, if $\mu_n < 0$, we also have

$$\widetilde{J}_\lambda'(u_n) \to 0 \text{ as } n \to \infty. \tag{3.43}$$

Hence, $\{u_n\}$ is a bounded (PS) sequence of $\bar{J}_\lambda$. By make a standard procedure for $\bar{J}_\lambda$ (see [37]), we can obtain a convergent subsequence, still denoted by $\{u_n\}$, such that $u_n \to u_\lambda$ for some $u_\lambda \in E$. Consequently, by (3.42), we have $\bar{J}_\lambda'(u_\lambda) = 0$ and $\bar{J}_\lambda(u_n) \to \bar{J}_\lambda(u_\lambda) := c_\lambda$ as $n \to \infty$, which shows that u_λ is a critical point of $\bar{J}_\lambda(u)$. Moreover, it follows from (3.8) and (3.24) that

$$c_\lambda = \bar{J}_\lambda(u_\lambda) = \lim_{n\to\infty}\bar{J}_\lambda(u_n) = \lim_{n\to\infty}\bar{J}_{\lambda,\mu_n}(u_n) = \lim_{n\to\infty}c_{\lambda,\mu_n} \geq d_\lambda > 0, \ \text{ if } \mu_n > 0, \tag{3.44}$$

and

$$c_\lambda = \bar{J}_\lambda(u_\lambda) = \lim_{n\to\infty}\bar{J}_\lambda(u_n) = \lim_{n\to\infty}\widetilde{J}_{\lambda,\mu_n}(u_n) = \lim_{n\to\infty}c_{\lambda,\mu_n} \geq \lim_{n\to\infty}d_{\lambda,\mu_n}$$

$$= \lim_{n\to\infty}\left[\frac{1}{2}\left(\frac{e^{\frac{q_2H_0}{2}}}{e^{H_1}M_2\lambda - 8\mu_n\eta_2 e^{H(1)}}\right)^{\frac{2}{q_2-2}} - \left(\frac{M_2e^{H_1}\lambda}{8e^{\frac{q_2H_0}{2}}} - \frac{\mu_n\eta_2e^{H(1)}}{e^{\frac{q_2H_0}{2}}}\right)\left(\frac{e^{\frac{q_2H_0}{2}}}{M_2e^{H_1}\lambda - 8\mu_n\eta_2e^{H(1)}}\right)^{\frac{q_2}{q_2-2}}\right]$$

$$= \frac{1}{2}\left(\frac{e^{\frac{q_2H_0}{2}}}{M_2e^{H_1}\lambda}\right)^{\frac{2}{q_2-2}} - \left(\frac{M_2e^{H_1}\lambda}{8e^{\frac{q_2H_0}{2}}}\right)\left(\frac{e^{\frac{q_2H_0}{2}}}{M_2e^{H_1}\lambda}\right)^{\frac{q_2}{q_2-2}} > 0, \quad \text{if } \mu_n < 0, \tag{3.45}$$

which implies that u_λ is a nontrivial. It follows from Lemma 3.6 and Lemma 3.10 that

$$c_\lambda = \lim_{\mu_n\to 0^+} c_{\lambda,\mu_n} \le C_*\lambda^{-\frac{1}{q_1-2}}, \quad c_\lambda = \lim_{\mu_n\to 0^-} c_{\lambda,\mu_n} \le D_*\lambda^{-\frac{1}{q_1-2}}. \tag{3.46}$$

Then similar to the argument in (3.22) and (3.33) with $\mu = 0$, we can obtain that

$$\|u_\lambda\|_H^2 \le \frac{2\theta}{\theta-2}c_\lambda \le \frac{2\theta}{\theta-2}\max\left\{C_*\lambda^{-\frac{1}{q_1-2}}, D_*\lambda^{-\frac{1}{q_1-2}}\right\}.$$

Hence, we have

$$\|u_\lambda\|_H^2 \le K_*\lambda^{-\frac{1}{q_1-2}}, \quad \text{as } \mu \to 0. \tag{3.47}$$

Then when $\lambda > \lambda^* = \max\{\Lambda_{M_0}, \Lambda_{21}, \Lambda_{22}, \Lambda_3, \Lambda_4^-\}$, where Λ_3 is defined by (1.20) and K_* is defined by (1.21), Remark 2.1 implies that

$$\|u_\lambda\|_\infty^2 \le \frac{1}{e^{H_0}}\|u_\lambda\|_H^2 \le \frac{1}{e^{H_0}}\cdot K_*\lambda^{-\frac{1}{q_1-2}} \le \frac{\delta^2}{4}. \tag{3.48}$$

Hence, $\bar{F}(x, u_\lambda) = F(x, u_\lambda)$. Thus u_λ is a nontrival solution of Eq (1.16) if

$$\lambda > \lambda^* = \max\{\Lambda_{M_0}, \Lambda_{21}, \Lambda_{22}, \Lambda_3, \Lambda_4^-\},$$

and by (3.47) and (3.48), it is easy to see that

$$\lim_{\lambda\to\infty}\|u_\lambda\|_H = 0 = \lim_{\lambda\to\infty}\|u_\lambda\|_\infty.$$

The proof is completed.

4. Conclusions

Some sufficient conditions about existence of a nontrivial solution for Eq (1.1) are obtained. (1.1) is a generalization of (1.2) and (1.3) which can be used to describe the static equilibrium of an elastic beam. The nonlinear terms F and G are assumed to satisfy (H1)−(H7) which are some growth conditions only near the origin. The concrete lower bounds of the parameter λ are given for the cases $\mu > 0$ and $\mu < 0$, respectively. Finally, in Theorem 1.3, the concentration phenomenon of $\{u_{\lambda,\mu}\}$ is revealed as $\mu \to 0$.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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