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Research article

Semilattice strongly regular relations on ordered n-ary semihypergroups

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Abstract: In this paper, we introduce the concept of j-hyperfilters, for all positive integers $1 \leq j \leq n$ and $n \geq 2$, on (ordered) n-ary semihypergroups and establish the relationships between j-hyperfilters and completely prime j-hyperideals of (ordered) n-ary semihypergroups. Moreover, we investigate the properties of the relation $\mathcal{N}$, which is generated by the same principal hyperfilters, on (ordered) n-ary semihypergroups. As we have known from [21] that the relation $\mathcal{N}$ is the least semilattice congruence on semihypergroups, we illustrate by counterexample that the similar result is not necessarily true on n-ary semihypergroups where $n \geq 3$. However, we provide a sufficient condition that makes the previous conclusion true on n-ary semihypergroups and ordered n-ary semihypergroups where $n \geq 3$. Finally, we study the decomposition of prime hyperideals and completely prime hyperideals by means of their $\mathcal{N}$-classes. As an application of the results, a related problem posed by Tang and Davvaz in [31] is solved.

Keywords: ordered semihypergroup; n-ary semihypergroup; hyperideal; hyperfilter
Mathematics Subject Classification: 20N20, 20N15, 06F05

1. Introduction

The investigation of partially ordered algebraic systems was initiated by Conrad [16, 17]. Particularly, an ordered semigroup (also called a partially ordered semigroup or a po-semigroup) is a well-known example of the partially ordered algebraic systems, which is a semigroup together with a compatible partial order. The notion of ideals and filters on ordered semigroups play a significant role in studying the structure of ordered semigroups, especially the decompositions of ordered semigroups see [2, 23, 34], and there are many applications in various branches of mathematics and computer science. Many mathematicians have studied different aspects of ideals and filters on ordered semigroups. For examples, see [2, 22–25, 34].

The theory of algebraic hyperstructures have been originated by Marty [28] in 1934 when he studied hypergroups, which are considered as a generalization of groups, by using hyperoperations. Later on, several kinds of hyperstructures were studied by many mathematicians, for instance, see [4–7, 19, 20, 26]. In 2011, Davvaz and Heidari [18] applied the notion of ordered semigroups to hyperstructures and presented a new class of hyperstructures called ordered semihypergroups. Such hyperstructure can be considered as an extension and a generalization of ordered semigroups. The relationships between ordered semihypergroups and ordered semigroups were described by Davvaz et al. [11]. Some properties of hyperideals on ordered semihypergroups were investigated by Changphas and Davvaz [3]. Next, Tang et al. [32] introduced the concept of a (right, left) hyperfilter on ordered semihypergroups and discussed their related properties. Omidi and Davvaz [29] generalized the concept of the relation $\mathcal{N}$ on ordered semigroups into ordered semihypergroups and proved some remarkable results. Note that, in an ordered semihypergroup S, the relation $\mathcal{N}$ on S is defined by $\mathcal{N} := \{(x, y) \in S \times S : N(x) = N(y)\}$ where $N(x)$ is the hyperfilter generated by the element x of S. Kehayopulu [21] proved that the relation $\mathcal{N}$ is the least semilattice congruence on semihypergroups. Meanwhile, Kehayopulu illustrate by counterexample that $\mathcal{N}$ is not the least semilattice congruence on ordered semihypergroups in general. The author introduced the concept of complete semilattice congruence on ordered semihypergroups and proved that $\mathcal{N}$ is the smallest complete semilattice congruence. Recently, Tang and Davvaz [31] studied the relation $\mathcal{N}$ and Green's relations, especially the relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}$, on ordered semihypergroups. They showed that every (respectively, right, left) hyperideal can be decomposable into its ($\mathcal{R}$-, $\mathcal{L}$-)$\mathcal{J}$-classes and posed a problem: " Can every prime hyperideal be decomposable into its $\mathcal{N}$-classes?"

On the other hand, the investigation of algebraic hyperstructures have been widely expanded. In 2006, Davvaz and Vougiouklis [14] applied the notion of hypergroups to n-ary algebras and presented the notion of n-ary hypergroups for $n \geq 2$. Leoreanu-Fotea and Davvaz [27] studied some binary relations on n-ary hypergroups. Anvariyeh and Momeni [1] investigated the interesting properties of n-ary relations on n-ary hypergroups. In 2009, Davvaz et al. [13] introduced a new class of hyperstructure called n-ary semihypergroups. This new class can be considered as a natural extension of semigroups, semihypergroups, ternary semigroups, ternary semihypergroups and n-ary semigroups. The interesting results of n-ary semigroups and n-ary semihypergroups, the readers can be referred to [8, 12, 13, 15, 30]. Recently, Daengsaen and Leeratanavalee [9] applied the theory of n-ary algebras to ordered semihypergroups and introduced the notion of ordered n-ary semihypergroups for $n \geq 2$. Such new notion is a natural generalization of ordered semigroups, ordered semihypergroups, ordered ternary semigroups and ordered ternary semihypergroups. Also, they characterized several kinds of regularities of ordered n-ary semihypergroups. In this paper, we attempt to study the notion of hyperfilters on ordered n-ary semihypergroups. In particular, we introduce the concept of j-hyperfilters, for all positive integers $1 \leq j \leq n$ and $n \geq 2$, of ordered n-ary semihypergroups and use such concept to construct the relation $\mathcal{N}$ for studying the properties of semilattice congruence on ordered n-ary semihypergroups where $n \geq 2$.

The present paper is organized as follows. In Section 2, we recall some basic notions and elementary results of algebraic hyperstructures. In Section 3, we introduce the concept of j-hyperfilters, which is a generalization of left and right hyperfilters of ordered semihypergroups and discuss the connection between j-hyperfilters and completely prime j-hyperideals on ordered n-ary semihypergroups where

$n \geq 2$. In Section 4, we investigate some properties of the relation $\mathcal{N}$ on n-ary semihypergroups. We illustrate by counterexample that $\mathcal{N}$ does not necessarily to be the least semilattice congruence on n-ary semihypergroup, where $n \geq 3$, in general. At the meantime, we provide a sufficient condition that makes the previous result true. In Section 5, we introduce the concept of a complete $*$-semilattice (strongly) regular relation on ordered n-ary semihypergroups, where $n \geq 3$, and prove that $\mathcal{N}$ is the smallest complete $*$-semilattice strongly regular relation. In the last section, we give an example to show that a prime hyperideal of ordered n-ary semihypergroups cannot be decomposable into its $\mathcal{N}$-classes in general. However, we provide some special condition for prime hyperideals that lead to be decomposable into its $\mathcal{N}$-classes. As an application of our results, the related problem posed by Tang and Davvaz is solved.

2. Preliminaries

In this section, we recall some elementary results of algebraic hyperstructures. For more detail, we refer to [8, 9, 12–14].

Let S be a nonempty set and let $\mathcal{P}^*(S)$ denote the set of all nonempty subsets of S. A mapping $f : \underbrace{S \times \cdots \times S}_{S \text{ appears } n \geq 2 \text{ times}} \to \mathcal{P}^*(S)$ is called an *n-ary hyperoperation*. A structure (S, f) is called an *n-ary hypergroupoid* [14]. Throughout this paper, we use the symbol a_j^k to denote a sequence of elements $a_j, a_{j+1}, ..., a_k$ of S. For the case $k < j$, a_j^k is the empty symbol. Then $f(a_1, a_2, ..., a_n) = f(a_1^n)$ and

$$f(a_1, ..., a_j, b_{j+1}, ..., b_k, c_{k+1}, ..., c_n) = f(a_1^j, b_{j+1}^k, c_{k+1}^n).$$

In case $a_1 = ... = a_j = a$ and $c_{k+1} = ... = c_n = c$, we write the second expression in the form $f(a^j, b_{j+1}^k, c^{n-k})$. For the symbols of a sequence of subsets of S, we denote analogously. For $X_1, ..., X_n \in \mathcal{P}^*(S)$, we define

$$f(X_1^n) = f(X_1, ..., X_n) := \bigcup \left\{ f(a_1^n) : a_j \in X_j, j = 1, ..., n \right\}.$$

If $X_1 = \{a_1\}$, then we write $f(\{a_1\}, X_2^n)$ as $f(a_1, X_2^n)$ and analogously in other cases. In case $X_1 = ... = X_j = Y$ and $X_{j+1} = ... = X_n = Z$, we write $f(X_1^n)$ as $f(Y^j, Z^{n-j})$. The hyperoperation f is associative [13] if

$$f(f(a_1^n), a_{n+1}^{2n-1}) = f(a_1^{j-1}, f(a_j^{n+j-1}), a_{n+j}^{2n-1})$$

hold for all $a_1^{2n-1} \in S$ and for all $1 < j \leq n$. In this case, the n-ary hypergroupoid (S, f) is called an *n-ary semihypergroup* (also called an *n-ary hypersemigroup*) [13].

An ordered n-ary semihypergroup $(S, f, \leq)$ (also called a *partially ordered n-ary semihypergroup* or a po-n-ary semihypergroup) [9] is an n-ary semihypergroup (S, f) and a partially ordered set $(S, \leq)$ such that a partial order $\leq$ is compatible with f. Indeed, for any $a, b \in S$,

$$a \leq b \text{ implies } f(c_1^{j-1}, a, c_{j+1}^n) \leq f(c_1^{j-1}, b, c_{j+1}^n)$$

for all $c_1^n \in S$ and for all $1 \leq j \leq n$. Note that, for any $X, Y \in \mathcal{P}^*(S)$, $X \leq Y$ means for every $x \in X$ there exists $y \in Y$ such that $x \leq y$. If H is an n-ary subsemihypergroup of an ordered n-ary semihypergroup $(S, f, \leq)$, i.e $f(H^n) \subseteq H$, then $(H, f, \leq)$ is an ordered n-ary semihypergroup.

Definition 2.1. [9] *Let $(S, f, \leq)$ be an ordered n-ary semihypergroup. For any positive integer $1 \leq j \leq n$ and $n \geq 2$, a nonempty subset I of S is called a j-hyperideal of S if it satisfies the following two conditions:*

(I1) *I is a j-hyperideal of an n-ary semihypergroup (S, f), i.e., $f(a_1^{j-1}, b, a_{j+1}^n) \subseteq I$ for all $b \in I$ and $a_1^{j-1}, a_{j+1}^n \in S$.*

(I2) *If $b \in I$ and $c \in S$ such that $c \leq b$ then $c \in I$.*

If I is a j-hyperideal, for all $1 \leq j \leq n$, then I is said to be a hyperideal.

Here, 1-hyperideal (n-hyperideal) is called a *right hyperideal (left hyperideal)*, respectively. Throughout this paper, we let S stand for an ordered n-ary semihypergroup $(S, f, \leq)$. By applying Lemma 4 in [8], the following lemma is obtained.

Lemma 2.2. [8] *For any positive integer $1 \leq j \leq n$, let $\{I_k : k \in \mathcal{J}\}$ be a collection of j-hyperideals (hyperideals) of S. Then $\bigcup_{k\in\mathcal{J}} I_k$ is a j-hyperideal (hyperideal) of S and $\bigcap_{k\in\mathcal{J}} I_k$ is also a j-hyperideal (hyperideal) of S if $\bigcap_{k\in\mathcal{J}} I_k \neq \emptyset$.*

A nonempty subset P of S is called *prime* [31] if, for every $a_1^n \in S$, $f(a_1^n) \subseteq P$ implies $a_k \in P$ for some $1 \leq k \leq n$. Equivalently, if $a_1^n \notin P$ then $f(a_1^n) \nsubseteq P$.

P is called *completely prime* [32] if, for every $a_1^n \in S$, $f(a_1^n) \cap P \neq \emptyset$ implies $a_k \in P$ for some $1 \leq k \leq n$. Equivalently, if $a_1^n \notin P$ then $f(a_1^n) \cap P = \emptyset$.

A (j-)hyperideal I of S is called a *prime (j-)hyperideal (completely prime (j-)hyperideal)* of S if I is a prime (completely prime) subset of S. Clearly, every completely prime (j-)hyperideal of S is a prime (j-)hyperideal of S.

Example 2.3. Let $\mathbb{Z}$ be the set of all integers. Define $f : \mathbb{Z}^n \to \mathcal{P}^*(\mathbb{Z})$ by

$$f(x_1^n) = \{y \in \mathbb{Z} \ : \ y \leq min\{x_1, ..., x_n\}\}$$

for all $x_1^n \in \mathbb{Z}$. Then $(\mathbb{Z}, f, \leq)$ is an ordered n-ary semihypergroup where $\leq$ is a usual partial order on $\mathbb{Z}$. Let $\mathbb{N}$ be the set of all natural numbers. Then $(\mathbb{N}, f, \leq)$ is an ordered n-ary subsemihypergroup of $\mathbb{Z}$.

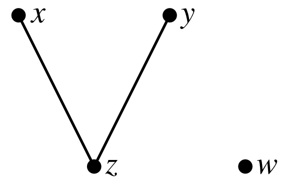
Example 2.4. Let $S = \{w, x, y, z\}$. Define $f : S^n \to \mathcal{P}^*(S)$ by $f(x_1^n) = ((...(x_1 \circ x_2) \circ ...) \circ x_n)$, for all $x_1^n \in S$, where $\circ$ is defined by the following table.

$\circ$	w	x	y	z
w	$\{x, z\}$	$\{x, z\}$	$\{z\}$	$\{z\}$
x	$\{x, z\}$	$\{x\}$	$\{z\}$	$\{z\}$
y	$\{z\}$	$\{z\}$	$\{y\}$	$\{z\}$
z	$\{z\}$	$\{z\}$	$\{z\}$	$\{z\}$

By applying Example 2 in [31], (S, f) is an n-ary semihypergroup. Define a partial order $\leq$ on S as follows:

$$\leq := \{(w, w), (x, x), (y, y), (z, x), (z, y), (z, z)\}.$$

Such relation can be presented by the following diagram.



Then $(S, f, \leq)$ is an ordered n-ary semihypergroup. It is easy to verify that the sets $A_1 = \{x, z\}$, $A_2 = \{y, z\}$, $A_3 = \{w, x, z\}$ and $A_4 = \{x, y, z\}$ are all proper 2-hyperideals of S. Then we obtain the results as follows:

(1) A_1 and A_4 are not prime since there exists $w \in S \setminus A_k$ such that $f(w^n) \subseteq A_k$ for all $k = 1, 4$. It follows that A_1, A_4 are also not completely prime.

(2) A_2 is prime since $f(x_1^n) \nsubseteq A_2$ for all $x_1^n \in S \setminus A_2$. But, it is not completely prime since there exists $w \in S \setminus A_2$ such that $f(w^n) \cap A_2 \neq \emptyset$.

(3) A_3 is completely prime since $f(x_1^n) \cap A_3 = \emptyset$ for all $x_1^n \in S \setminus A_3$. It follows that A_3 is prime.

3. Hyperfilters on ordered n-ary semihypergroups

In this section, we introduce the concept of j-hyperfilters on ordered n-ary semihypergroups, which generalizes the notion of right and left hyperfilters on ordered semihypergroups [32], and investigate their related properties.

Definition 3.1. *Let $(S, f, \leq)$ be an ordered n-ary semihypergroup. For any positive integer $1 \leq j \leq n$ and $n \geq 2$, an n-ary subsemihypergroup K of S is said to be a j-hyperfilter of S if it satisfies the following two conditions:*

(F1) *K is a j-hyperfilter of an n-ary semihypergroup (S, f), i.e., for all $a_1^n \in S$, $f(a_1^n) \cap K \neq \emptyset$ implies $a_j \in K$.*

(F2) *If for any $a \in K$ and $b \in S$ such that $a \leq b$ then $b \in K$.*

If K is a j-hyperfilter, for all $1 \leq j \leq n$, then K is called a hyperfilter.

Here, 1-hyperfilter (n-hyperfilter) is said to be a right hyperfilter (left hyperfilter, respectively). For any $a \in S$ and $1 \leq j \leq n$, we denote by $N^j(a)(N(a))$ the j-hyperfilter (hyperfilter) of S generated by a and called the *principal j-hyperfilter (hyperfilter) generated by a*. A j-hyperfilter (hyperfilter) K of S is said to be *proper* if $K \neq S$. Clearly, Definition 3.1 coincides with Definition 3.1 given in [32] if $n = 2$.

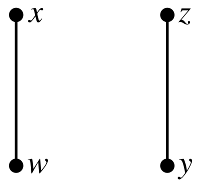
Example 3.2. Let $S = \{w, x, y, z\}$. Define $f : S^n \to \mathcal{P}^*(S)$ by $f(x_1^n) = ((...(x_1 \circ x_2) \circ ...) \circ x_n)$, for all $x_1^n \in S$, where $\circ$ is defined by the following table

$\circ$	w	x	y	z
w	$\{w\}$	$\{w, x\}$	$\{w, y\}$	S
x	$\{x\}$	$\{x\}$	$\{x, z\}$	$\{x, z\}$
y	$\{y\}$	$\{y, z\}$	$\{y\}$	$\{y, z\}$
z	$\{z\}$	$\{z\}$	$\{z\}$	$\{z\}$

and a partial order $\leq$ is defined as follows:

$$\leq := \{(w,w),(w,x),(x,x),(y,y),(y,z),(z,z)\}.$$

Such relation can be presented by the following diagram.



By applying Example 3.3 in [33], $(S, f, \leq)$ is an ordered n-ary semihypergroup. Clearly, the sets $\{w, x\}$ and S are all 1-hyperfilters of S.

Proposition 3.3. *Let $(S, f, \leq)$ be an ordered n-ary semihypergroup with $n \geq 2$. For any $1 \leq j \leq n$, let K_1, K_2 be two j-hyperfilters of S. Then $K_1 \cap K_2$ is a j-hyperfilter of S if $K_1 \cap K_2 \neq \emptyset$.*

Proof. Let $a_1^n \in K_1 \cap K_2$. Then $a_1^n \in K_i$ for all $i = 1, 2$. Since K_i is an n-ary subsemihypergroup, we obtain $f(a_1^n) \subseteq K_1 \cap K_2$. Thus $K_1 \cap K_2$ is an n-ary semihypergroup of S. Next, let $b_1^n \in S$ and $c \in f(b_1^n) \cap (K_1 \cap K_2)$. Then $c \in f(b_1^n) \cap K_i$ for all $i = 1, 2$. Since K_i is a j-hyperfilter of S, we have $b_j \in K_i$ for all $i = 1, 2$. So $b_j \in K_1 \cap K_2$. Finally, let $a \in K_1 \cap K_2$ such that $a \leq b \in S$. Since K_i is a j-hyperfilter, we obtain $b \in K_i$ for all $i = 1, 2$. So $b \in K_1 \cap K_2$. Therefore $K_1 \cap K_2$ is a j-hyperfilter of S. □

By Proposition 3.3, we obtain the following result.

Corollary 3.4. *Let $(S, f, \leq)$ be an ordered n-ary semihypergroup with $n \geq 2$ and K_1, K_2 be two hyperfilters of S. $K_1 \cap K_2$ is a hyperfilter of S if $K_1 \cap K_2 \neq \emptyset$.*

Proposition 3.3 and Corollary 3.4 show that the intersection of finitely many j-hyperfilters(hyperfilters) of S is always a j-hyperfilter(hyperfilter) if their intersection is nonempty. In 2015, Tang et al. [32] showed that the union of two hyperfilters of an ordered semihypergroup H (for a special case $n = 2$) is not necessarily a hyperfilter of H (see Example 3.5 in [32]). Using the idea of Lemma 3.6 in [32], we obtain the results as follows.

Proposition 3.5. *Let $(S, f, \leq)$ be an ordered n-ary semihypergroup with $n \geq 3$. For any $1 < j < n$, let K_1, K_2 be two j-hyperfilters of S. Then $K_1 \cup K_2$ is a j-hyperfilter of S if and only if $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$.*

Proof. Now, let j be a fixed positive integer with $1 < j < n$ and $n \geq 3$. Let $K_1 \cup K_2$ be a j-hyperfilter of S. Assume that $K_1 \nsubseteq K_2$ and $K_2 \nsubseteq K_1$. There are $a, b \in K_1 \cup K_2$ such that $a \in K_1, a \notin K_2, b \notin K_1$ and $b \in K_2$. Since $K_1 \cup K_2$ is an n-ary subsemihypergroup, we have $f(a^{j-1}, b^{n-j+1}) \subseteq K_1 \cup K_2$. If $f(a^{j-1}, b^{n-j+1}) \cap K_1 \neq \emptyset$ then, since K_1 is a j-hyperfilter of S, we obtain $b \in K_1$. It is impossible. So $f(a^{j-1}, b^{n-j+1}) \cap K_1 = \emptyset$. This implies that $f(a^{j-1}, b^{n-j+1}) \subseteq K_2$. Since K_2 is an n-ary subsemihypergroup, we obtain $f(b^{j-1}, f(a^{j-1}, b^{n-j+1}), b^{n-j}) \subseteq K_2$. Then

$$f(b^{j-1}, f(a^{j-1}, b^{n-j+1}), b^{n-j}) = \begin{cases} f(b^{j-1}, a^{j-1}, b^{n-2j+1}, f(b^n)) & \text{if } 1 < j \leq \dfrac{n}{2}, \\ f(b^{j-1}, a^{n-j}, f(a^{2j-n-1}, b^{2n-2j+1})) & \text{if } \dfrac{n}{2} < j < n. \end{cases}$$

Since K_2 is a j-hyperfilter, we obtain $a \in K_2$. It is impossible. Thus $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$. Conversely, if $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$ then $K_1 \cup K_2$ is a j-hyperfilter of S. □

Corollary 3.6. *Let $(S, f, \leq)$ be an ordered n-ary semihypergroup with $n \geq 2$ and K_1, K_2 be two hyperfilters of S. Then $K_1 \cup K_2$ is a hyperfilter of S if and only if $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$.*

Theorem 3.7. *Let $(S, f, \leq)$ be an ordered n-ary semihypergroup with $n \geq 2$ and $K \in \mathcal{P}^*(S)$. For any $1 \leq j \leq n$, the following assertions are equivalent.*

(i) *K is a j-hyperfilter of S.*

(ii) *$S \setminus K = \emptyset$ or $S \setminus K$ is a completely prime j-hyperideal of S.*

Proof. (i) $\Longrightarrow$ (ii) Let K be a j-hyperfilter of S. Suppose that $S \setminus K \neq \emptyset$. We show that $S \setminus K$ is a completely prime j-hyperideal of S. First of all, we show that K is a j-hyperideal of S. Let $a \in S \setminus K$ and $b_1^{j-1}, b_{j+1}^n \in S$. If $f(b_1^{j-1}, a, b_{j+1}^n) \cap K \neq \emptyset$ then, since K is a j-hyperfilter, we get $a \in K$. It is impossible. So $f(b_1^{j-1}, a, b_{j+1}^n) \cap K = \emptyset$. It implies that $f(b_1^{j-1}, a, b_{j+1}^n) \subseteq S \setminus K$. Next, let $x \in S$ and $y \in S \setminus K$ such that $x \leq y$. If $x \notin S \setminus K$ then $x \in K$. By Definition 3.1 (F2), we have $y \in K$. It is impossible. Hence $x \in S \setminus K$. Thus $S \setminus K$ is a j-hyperideal of S. Next, we show that $S \setminus K$ is completely prime. Let $a_1^n \in S$ such that $f(a_1^n) \cap (S \setminus K) \neq \emptyset$. Assume that $a_i \notin S \setminus K$ for all $i = 1, ..., n$. Then $a_i \in K$ for all $i = 1, ..., n$. Since K is an n-ary subsemihypergroup, we obtain $f(a_1^n) \subseteq K$. It implies that $f(a_1^n) \cap (S \setminus K) = \emptyset$, which is impossible. Hence $a_i \in S \setminus K$ for some $i = 1, ..., n$. Thus $S \setminus K$ is a completely prime j-hyperideal of S.

(ii) $\Longrightarrow$ (i) If $S \setminus K = \emptyset$ then $K = S$ is a j-hyperfilter of S. Now, suppose that $S \setminus K$ is a completely prime j-hyperideal of S. We will show that K is a j-hyperfilter of S. First of all, let $a_1^n \in K$. If $f(a_1^n) \nsubseteq K$ then $f(a_1^n) \cap (S \setminus K) \neq \emptyset$. Since $S \setminus K$ is completely prime, we obtain $a_i \in S \setminus K$ for some $i = 1, ..., n$. It is impossible. Thus $f(a_1^n) \subseteq K$. Next, let $b_1^n \in S$ and $f(b_1^n) \cap K \neq \emptyset$. If $b_j \notin K$ then $b_j \in S \setminus K$. Since $S \setminus K$ is a j-hyperideal, we obtain $f(b_1^n) \subseteq f(S^{j-1}, b_j, S^{n-j}) \subseteq f(S^{j-1}, (S \setminus K), S^{n-j}) \subseteq S \setminus K$. This is impossible. So $b_j \in K$. Finally, let $x \in K$ and $y \in S$ such that $x \leq y$. If $y \notin K$ then $y \in S \setminus K$. Since $S \setminus K$ is a j-hyperideal, by Definition 2.1 (I2), we get $x \in S \setminus K$. It is impossible. So $y \in K$. Therefore K is a j-hyperfilter of S. □

Using the similar proof of the previous theorem, we obtain the result as follows.

Corollary 3.8. *Let $(S, f, \leq)$ be an ordered n-ary semihypergroup with $n \geq 2$ and $K \in \mathcal{P}^*(S)$. Then the following assertions are equivalent.*

(i) *K is a hyperfilter of S.*

(ii) *$S \setminus K = \emptyset$ or $S \setminus K$ is a completely prime hyperideal of S.*

Example 3.9. From the ordered n-ary semihypergroup $(S, f, \leq)$ given in Example 3.2, we know that the set $\{w, x\}$ is a proper 1-hyperfilter of S. By Theorem 3.7, we have $\{y, z\} = S \setminus \{w, x\}$ is a completely prime 1-hyperideal of S. In fact, one can verify that a proper completely prime 1-hyperideal of S is $\{y, z\}$ since $f(x_1^n) \cap \{y, z\} = \emptyset$ for all $x_1^n \in S \setminus \{y, z\}$.

Lemma 3.10. *Let $(S, f, \leq)$ be an ordered n-ary semihypergroup with $n \geq 2$ and $a, b \in S$. For any $1 \leq j \leq n$, the following assertions hold:*

(i) *If* $b \in N^j(a)$ *then* $N^j(b) \subseteq N^j(a)$.

(ii) *If* $a \leq b$ *then* $N^j(b) \subseteq N^j(a)$.

(iii) *If* $a \leq b$ *then* $N^j(a) \setminus N^j(b)$ *is a completely prime j-hyperideal of* $N^j(a)$ *where* $N^j(a) \setminus N^j(b) \neq \emptyset$.

Proof. (i) Let $b \in N^j(a)$. Since $N^j(b)$ is the smallest j-hyperfilter of S containing b, we have $N^j(b) \subseteq N^j(a)$.

(ii) Let $a \leq b$. Since $N^j(a)$ is a j-hyperfilter of S and $a \in N^j(a)$, by Definition 3.1 (F2), we obtain $b \in N^j(a)$. By (i), we have $N^j(b) \subseteq N^j(a)$.

(iii) Let $a \leq b$. Then $N^j(b) \subseteq N^j(a)$. By Theorem 3.7, we have $N^j(a) \setminus N^j(b) = \emptyset$ or $N^j(a) \setminus N^j(b)$ is a completely prime j-hyperideal of $N^j(a)$. Thus $N^j(a) \setminus N^j(b)$ is a completely prime j-hyperideal of $N^j(a)$ where $N^j(a) \setminus N^j(b) \neq \emptyset$. □

Remark 3.11. Lemma 3.10 is still true if the principal j-hyperfilters $N^j(a)$ and $N^j(b)$ are replaced by the principal hyperfilters $N(a)$ and $N(b)$, respectively. Moreover, the results on this section also hold on n-ary semihypergroups (S, f).

4. Semilattice strongly regular relations on n-ary semihypergroups

In this section, we introduce the concept of semilattice (strongly) regular relations on n-ary semihypergroups and investigate their related properties.

Let (S, f) be an n-ary semihypergroup with $n \geq 2$ and ρ be a relation on S. For any two elements $A, B \in \mathcal{P}^*(S)$, we write

$A\bar{\rho}B$ if and only if for any $a \in A$ there exists $b \in B$ such that $a\rho b$ and for all $b' \in B$ there exists $a' \in A$ such that $a'\rho b'$,

$A\bar{\bar{\rho}}B$ if and only if $a\rho b$ for all $a \in A$ and $b \in B$, see [10].

For any $1 \leq j \leq n$, an equivalence relation ρ is called *j-regular relation (strongly j-regular relation)* if, for every $a, b, x_1^n \in S$,

$$a\rho b \Longrightarrow f(x_1^{j-1}, a, x_{j+1}^n)\bar{\rho}f(x_1^{j-1}, b, x_{j+1}^n)$$

$$(a\rho b \Longrightarrow f(x_1^{j-1}, a, x_{j+1}^n)\bar{\bar{\rho}}f(x_1^{j-1}, b, x_{j+1}^n)).$$

Here, ρ is said to be a *regular relation (strongly regular relation)* if it is a j-regular relation (strongly j-regular relation) for all $1 \leq j \leq n$.

Lemma 4.1. *Let* (S, f) *be an n-ary semihypergroup with* $n \geq 2$. *For any* $A_1^n, B_1^n \in \mathcal{P}^*(S)$, *the following statements hold:*

(i) *If* ρ *is regular and* $A_k\bar{\rho}B_k$, *for all* $k = 1, ..., n$, *then* $f(A_1^n)\bar{\rho}f(A_1^n)$.

(ii) *If* ρ *is strongly regular and* $A_k\bar{\bar{\rho}}B_k$, *for all* $k = 1, ..., n$, *then* $f(A_1^n)\bar{\bar{\rho}}f(A_1^n)$.

Proof. The proof is straightforward. □

Definition 4.2. *Let* (S, f) *be an n-ary semihypergroup with* $n \geq 2$. *A regular relation (strongly regular relation)* ρ *on* S *is said to be a semilattice regular relation (semilattice strongly regular relation) if*

$$x\bar{\rho}f(x^n) \text{ and } f(x_1^n)\bar{\rho}f(x_{\alpha(1)}^{\alpha(n)})$$

$$(x\bar{\bar{\rho}}f(x^n) \text{ and } f(x_1^n)\bar{\bar{\rho}}f(x_{\alpha(1)}^{\alpha(n)}))$$

for all $x, x_1^n \in S$ *and for all* $\alpha \in S_n$ *where* S_n *is the set of all permutations on* $\{1, 2, ..., n\}$.

Note that any semilattice strongly regular relation, semilattice equivalence relation [29] and semilattice congruence [21] on semihypergroups (for case $n = 2$) are the same.

Example 4.3. Let $S = \{1, 2, 3, 4, 5\}$. Define $f : S \times S \times S \to \mathcal{P}^*(S)$ by $f(x_1, x_2, x_3) = (x_1 \circ x_2) \circ x_3$, for all $x_1, x_2, x_3 \in S$, where $\circ$ is defined by the following table.

$\circ$	1	2	3	4	5
1	$\{1,2\}$	$\{1,2\}$	$\{3\}$	$\{3\}$	$\{3\}$
2	$\{1,2\}$	$\{1,2\}$	$\{3\}$	$\{3\}$	$\{3\}$
3	$\{1,2\}$	$\{1,2\}$	$\{3\}$	$\{3\}$	$\{3\}$
4	$\{1,2\}$	$\{1,2\}$	$\{3\}$	$\{4,5\}$	$\{4\}$
5	$\{1,2\}$	$\{1,2\}$	$\{3\}$	$\{4\}$	$\{5\}$

By applying Example 3.2 in [32], (S, f) is a ternary semihypergroup (an 3-ary semihypergroup). Let

$$\begin{aligned}
\sigma_1 &:= \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\},\\
\sigma_2 &:= \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,2),(1,3),(2,1),(2,3),(3,1),(3,2),\\
&\quad\ (4,5),(5,4)\}.
\end{aligned}$$

It is easy to show that σ_1, σ_2 are equivalence relations on S. Furthermore, one can verify that σ_2 is a semilattice strongly regular relation on S. On the other hand, σ_1 is not a semilattice strongly regular relation on S because we have $f(4,4,4) = \{4,5\}$ but $(4,5) \notin \sigma_1$, so σ_1 does not satisfy with the condition $4\,\overline{\overline{\sigma_1}}f(4,4,4)$ in Definition 4.2.

For any $K \in \mathcal{P}^*(S)$, we write $\delta_K := \{(a,b) \in S \times S : a, b \in K \text{ or } a, b \notin K\}$. Clearly, δ_K is an equivalence relation on S.

Theorem 4.4. *Let* (S, f) *be an n-ary semihypergroup with* $n \geq 2$ *and* K *be a completely prime hyperideal of* S. *Then* δ_K *is a semilattice strongly regular relation on* S.

Proof. Let K be a completely prime hyperideal of S.

(1) We show that δ_K is a strongly regular relation. Firstly, let j be a fixed positive integer with $1 \leq j \leq n$. Let $a, b, x_1^{j-1}, x_{j+1}^n \in S$ such that $a\delta_K b$. To show that $f(x_1^{j-1}, a, x_{j+1}^n)\overline{\overline{\delta_K}}f(x_1^{j-1}, b, x_{j+1}^n)$, let $u \in f(x_1^{j-1}, a, x_{j+1}^n)$ and $v \in f(x_1^{j-1}, b, x_{j+1}^n)$. Since $a\delta_K b$, we have $a, b \in K$ or $a, b \notin K$.

<u>Case 1.1</u>: $a, b \in K$. Since K is a j-hyperideal, we get $u \in f(x_1^{j-1}, a, x_{j+1}^n) \subseteq K$ and $v \in f(x_1^{j-1}, b, x_{j+1}^n) \subseteq K$. So $u\delta_K v$.

<u>Case 1.2</u>: $a, b \notin K$. Then we have 2 subcases to be considered as follows.

<u>Case 1.2.1</u>: There exists $x_i \in K$ for some $i = 1, ..., j-1, j+1, ..., n$. Since K is an i-hyperideal, we obtain $u \in f(x_1^{j-1}, a, x_{j+1}^n) \subseteq K$ and $v \in f(x_1^{j-1}, b, x_{j+1}^n) \subseteq K$. So $u\delta_K v$.

<u>Case 1.2.2</u>: $x_i \notin K$ for all $i = 1, ..., j-1, j+1, ..., n$. Then $f(x_1^{j-1}, a, x_{j+1}^n) \cap K = \emptyset$ and $f(x_1^{j-1}, b, x_{j+1}^n) \cap K = \emptyset$. In fact, if $f(x_1^{j-1}, a, x_{j+1}^n) \cap K \neq \emptyset$ then, since K is completely prime, we obtain $x_i \in K$ for some $i = 1, ..., n$ (we write $x_i = a$ if $i = j$), which is a contradiction. Similarly, if $f(x_1^{j-1}, b, x_{j+1}^n) \cap K \neq \emptyset$

then we also get a contradiction. Thus $f(x_1^{j-1}, a, x_{j+1}^n) \cap K = \emptyset$ and $f(x_1^{j-1}, b, x_{j+1}^n) \cap K = \emptyset$. Hence $u \in f(x_1^{j-1}, a, x_{j+1}^n) \subseteq S \setminus K$ and $v \in f(x_1^{j-1}, b, x_{j+1}^n) \subseteq S \setminus K$. It follows that $u\delta_K v$. From Case 1.1 and 1.2, we conclude that $f(x_1^{j-1}, a, x_{j+1}^n)\overline{\overline{\delta_K}}f(x_1^{j-1}, b, x_{j+1}^n)$ and so δ_K is a strongly j-regular relation. For arbitrary positive integer j, we obtain δ_K is a strongly regular relation.

(2) We show that δ_K is semilattice. Let $a, a_1^n \in S$ and $\alpha \in S_n$. Firstly, we show that $a\overline{\overline{\delta_K}}f(a^n)$. Let $u \in f(a^n)$. We consider two cases as follows.

Case 2.1: $a \in K$. Since K is a hyperideal of S, we have $u \in f(a^n) \subseteq K$. So $a\delta_K u$.

Case 2.2: $a \notin K$. Then $f(a^n) \cap K = \emptyset$. In fact, if $f(a^n) \cap K \neq \emptyset$ then, since K is completely prime, we have $a \in K$. It is impossible. So $f(a^n) \cap K = \emptyset$. It follows $u \notin K$. So $a\delta_K u$. From Cases 2.1 and 2.2, we conclude that $a\overline{\overline{\delta_K}}f(a^n)$. Next, we show that $f(a_1^n)\overline{\overline{\delta_K}}f(a_{\alpha(1)}^{\alpha(n)})$. Let $u \in f(a_1^n)$ and $v \in f(a_{\alpha(1)}^{\alpha(n)})$. We consider two cases as follows.

Case 1: $f(a_1^n) \cap K \neq \emptyset$. Since K is completely prime, we obtain $a_i \in K$ for some $i = 1, ..., n$. Since α is bijective, there is $i' \in \{1, ..., n\}$ such that $\alpha(i') = i$. Since K is an i'-hyperideal and an i-hyperideal, we obtain $f(a_{\alpha(1)}^{\alpha(n)}) = f(a_{\alpha(1)}, ..., a_{\alpha(i')}, ..., a_{\alpha(n)}) \subseteq K$ and $f(a_1^n) = f(a_1, ..., a_i, ..., a_n) \subseteq K$. Thus $u, v \in K$ and it follows that $u\delta_K v$.

Case 2: $f(a_1^n) \cap K = \emptyset$. Then $f(a_{\alpha(1)}^{\alpha(n)}) \cap K = \emptyset$. Indeed, if $f(a_{\alpha(1)}^{\alpha(n)}) \cap K \neq \emptyset$ then, since K is completely prime, we have $a_{\alpha(i')} \in K$ for some $i' \in \{1, ..., n\}$. We set $i = \alpha(i')$. Since K is an i-hyperideal, we obtain $f(a_1^n) = f(a_1, ..., a_i, ..., a_n) \subseteq K$. It is impossible. Thus $f(a_{\alpha(1)}^{\alpha(n)}) \cap K = \emptyset$. Hence $u \in f(a_1^n) \subseteq S \setminus K$ and $v \in f(a_{\alpha(1)}^{\alpha(n)}) \subseteq S \setminus K$ and it follows that $u\delta_K v$. From Cases 1 and 2, we conclude that $f(a_1^n)\overline{\overline{\delta_K}}f(a_{\alpha(1)}^{\alpha(n)})$ and the proof is complete. □

For any positive integer $1 \leq j \leq n$ and $n \geq 2$, let $\mathcal{N}_j$ ($\mathcal{N}$) denote the relation on an (ordered) n-ary semihypergroup S which is defined by

$$\mathcal{N}_j = \left\{(a, b) \in S \times S : N^j(a) = N^j(b)\right\}$$

$$(\mathcal{N} = \{(a, b) \in S \times S : N(a) = N(b)\}).$$

Clearly, $\mathcal{N}_j$ and $\mathcal{N}$ are equivalence relations on S. For any $a \in S$, we denote by $(a)_{\mathcal{N}_j}$ the $\mathcal{N}_j$-class containing a and the set $S/\mathcal{N}_j := \left\{(a)_{\mathcal{N}_j} : a \in S\right\}$. For the relation $\mathcal{N}$, we define analogously.

Theorem 4.5. *Let (S, f) be an n-ary semihypergroup with $n \geq 2$. Then the following statements hold:*

(i) *$\mathcal{N}$ is a semilattice strongly regular relation.*

(ii) $\mathcal{N} = \bigcap \{\delta_K : K \in \mathcal{CPI}(\mathcal{S})\}$,

where $\mathcal{CPI}(\mathcal{S})$ is the set of all completely prime hyperideals of S.

Proof. (i) First of all, we show that $\mathcal{N}$ is a strongly regular relation on S. Let $a, b, x_1^n \in S$ such that $a\mathcal{N}b$. Then $N(a) = N(b)$. For any fixed positive integer j with $1 \leq j \leq n$, let $u \in f(x_1^{j-1}, a, x_{j+1}^n)$ and $v \in f(x_1^{j-1}, b, x_{j+1}^n)$. Since $f(x_1^{j-1}, a, x_{j+1}^n) \cap N(u) \neq \emptyset$ and $N(u)$ is a hyperfilter, by Definition 3.1 (F1), we obtain $a, x_1^{j-1}, x_{j+1}^n \in N(u)$. By Lemma 3.10, we obtain $N(a) \subseteq N(u)$. Since $b \in N(b) = N(a)$, we get $b \in N(u)$. Since $b, x_1^{j-1}, x_{j+1}^n \in N(u)$ and $N(u)$ is an n-ary subsemihypergroup, we obtain $v \in f(x_1^{j-1}, b, x_{j+1}^n) \subseteq N(u)$. By Lemma 3.10, we have $N(v) \subseteq N(u)$. Using the similar previous process, we also obtain that $N(u) \subseteq N(v)$. So $N(u) = N(v)$ and it follows that $u\mathcal{N}v$. Hence $f(x_1^{j-1}, a, x_{j+1}^n)\overline{\overline{\mathcal{N}}}f(x_1^{j-1}, b, x_{j+1}^n)$. So $\mathcal{N}$ is a strongly regular relation on S. Next, we show that $\mathcal{N}$ is a semilattice strongly regular relation on S. Let $a, a_1^n \in S$ and $\alpha \in S_n$. To show that $a\overline{\overline{\mathcal{N}}}f(a^n)$, let $u \in f(a^n)$. Then $f(a^n) \cap N(u) \neq \emptyset$. By Definition 3.1 (F1), we obtain $a \in N(u)$. By Lemma 3.10, we get $N(a) \subseteq N(u)$. Since $N(a)$ is an n-ary subsemihypergroup and $a \in N(a)$, we obtain $u \in f(a^n) \subseteq N(a)$. By Lemma 3.10, we have $N(u) \subseteq N(a)$. So $N(a) = N(u)$ and it implies that $a\overline{\overline{\mathcal{N}}}f(a^n)$. Finally, we show that $f(a_1^n)\overline{\overline{\mathcal{N}}}f(a_{\alpha(1)}^{\alpha(n)})$. Let $u \in f(a_1^n)$ and $v \in f(a_{\alpha(1)}^{\alpha(n)})$. Since $f(a_1^n) \cap N(u) \neq \emptyset$, by Definition 3.1 (F1), we obtain $a_1^n \in N(u)$. It follows that $v \in f(a_1^n) \subseteq N(u)$ and so $N(v) \subseteq N(u)$. Conversely, since $f(a_{\alpha(1)}^{\alpha(n)}) \cap N(v) \neq \emptyset$, by using the same process, we have $N(u) \subseteq N(v)$. So $N(u) = N(v)$ and it follows that $u\mathcal{N}v$. Hence $f(a_1^n)\overline{\overline{\mathcal{N}}}f(a_{\alpha(1)}^{\alpha(n)})$. Thus $\mathcal{N}$ is a semilattice strongly regular relation on S.

(ii) Firstly, we will show that $\mathcal{N} \subseteq \bigcap\{\delta_K : K \in \mathcal{CPI}(\mathcal{S})\}$. Let $(x, y) \in \mathcal{N}$. Then $N(x) = N(y)$. Assume that $(x, y) \notin \delta_K$ for some $K \in \mathcal{CPI}(\mathcal{S})$. Then there exist two cases:

Case 1.1: $x \in K$ and $y \notin K$. Then $y \in S \setminus K \subseteq S$. Since $S \setminus (S \setminus K) = K$ is completely prime, by Corollary 3.8, we have $S \setminus K$ is a hyperfilter of S. Since $y \in S \setminus K$, by Lemma 3.10, we obtain $x \in N(x) = N(y) \subseteq S \setminus K$. It is impossible.

Case 1.2: $x \notin K$ and $y \in K$. Using the similar proof as in Case 1.1, we also get a contradiction. From Cases 1.1 and 1.2, we conclude that $(x, y) \in \delta_K$ for all $K \in \mathcal{CPI}(\mathcal{S})$. Thus $\mathcal{N} \subseteq \bigcap\{\delta_K : K \in \mathcal{CPI}(\mathcal{S})\}$. Conversely, let $(x, y) \in \delta_K$ for all $K \in \mathcal{CPI}(\mathcal{S})$. Assume that $(x, y) \notin \mathcal{N}$. Then there exist 2 cases, i.e. $x \notin N(y)$ or $y \notin N(x)$. Indeed, if $x \in N(y)$ and $y \in N(x)$ then, by Lemma 3.10, we get $N(x) \subseteq N(y)$ and $N(y) \subseteq N(x)$. Consequently, $N(x) = N(y)$ and so $(x, y) \in \mathcal{N}$. This is a contradiction. Thus $x \notin N(y)$ or $y \notin N(x)$.

Case 2.1: $x \notin N(y)$. Then $x \in S \setminus N(y)$. Since $N(y)$ is a hyperfilter and $S \setminus N(y) \neq \emptyset$, by Corollary 3.8, we have $S \setminus N(y)$ is a completely prime hyperideal. Since $x \in S \setminus N(y)$ and $y \notin S \setminus N(y)$, it follows that $(x, y) \notin \delta_{S \setminus N(y)}$. This is a contradiction.

Case 2.2: $y \notin N(x)$. Using the similar proof as in Case 2.1, we get a contradiction. Hence $(x, y) \in \mathcal{N}$. Therefore $\bigcap\{\delta_K : K \in \mathcal{CPI}(\mathcal{S})\} \subseteq \mathcal{N}$ and the proof is complete. □

As we have known from [21] that the relation $\mathcal{N}$ is the smallest semilattice strongly regular relation on semihypergroups. However, we illustrate by the following example that the similar result in n-ary semihypergroups, for $n \geq 3$, is not necessary true.

Example 4.6. Let $S = \{1, -1, i, -i\}$. Then $(S, \cdot)$ is a commutative semigroup under the usual multiplication $\cdot$ over complex numbers. Define $f : S \times S \times S \to \mathcal{P}^*(S)$ by $f(x, y, z) = \{x \cdot y \cdot z\}$. Then (S, f) is a ternary semihypergroup. Clearly, $\mathcal{CPI}(\mathcal{S}) = \{S\}$. In fact, suppose that A is an arbitrary hyperideal of S.

If $1 \in A$ then $S = f(S, 1, 1) \subseteq f(S, S, 1) \subseteq A$.

If $-1 \in A$ then $S = f(S, -1, -1) \subseteq f(S, S, -1) \subseteq A$.

If $i \in A$ then $S = f(S, -i, i) \subseteq f(S, S, i) \subseteq A$.

If $-i \in A$ then $S = f(S, i, -i) \subseteq f(S, S, -i) \subseteq A$.

Consequently, $S = A$. By Theorem 4.5(ii), we have

$$\mathcal{N} = \delta_S = \{(1,1), (1,-1), (1,i), (1,-i), (-1,1), (-1,-1), (-1,i), (-1,-i), (i,1), (i,-1), (i,i), (i,-i), (-i,1), (-i,-1), (-i,i), (-i,-i)\}.$$

Next, let

$$\rho = \{(1,1),(-1,-1),(i,i),(-i,-i),(1,-1),(-1,1),(i,-i),(-i,i)\}.$$

It is not difficult to show that ρ is a strongly regular relation on S. Furthermore, we have

$$1\,\bar{\bar{\rho}}\, f(1,1,1) = \{1\},\ \ -1\,\bar{\bar{\rho}}\, f(-1,-1,-1) = \{-1\},$$

$$i\,\bar{\bar{\rho}}\, f(i,i,i) = \{-i\} \text{ and } -i\,\bar{\bar{\rho}}\, f(-i,-i,-i) = \{i\}.$$

Since $(S,\cdot)$ is commutative, we get $f(x_1^3) = f(x_{\alpha(1)}^{\alpha(3)})$ and so $f(x_1^3)\ \bar{\bar{\rho}} f(x_{\alpha(1)}^{\alpha(3)})$ for all $\alpha \in S_3$. By Definition 4.2, we conclude that ρ is a semilattice strongly regular relation on S and $\rho \subseteq \mathcal{N}$.

By the previous example, we observe that the relation $\mathcal{N}$ is not necessary the smallest semilattice strongly regular relation on n-ary semihypergroups, for $n \geq 3$, in general. To provide a sufficient condition that makes the above conclusion true, we need to introduce the following definition.

Definition 4.7. Let (S, f) be an n-ary semihypergroup with $n \geq 3$. A semilattice regular relation (semilattice strongly regular relation) ρ on S is called a *$*$-semilattice regular relation ($*$-semilattice strongly regular relation)* if for every $x, y \in S$,

$$x\,\bar{\rho}\, f(x, y^{n-1}) \text{ implies } x\,\bar{\rho}\, f(x^{n-1}, y)$$

$$(x\,\bar{\bar{\rho}}\, f(x, y^{n-1}) \text{ implies } x\,\bar{\bar{\rho}}\, f(x^{n-1}, y)).$$

Example 4.8. Consider a ternary semihypergroup (S, f) given in Example 4.6. Obviously, $-i\ \bar{\bar{\rho}}\ f(-i,-1,-1) = \{-i\}$. Since $f(-i,-i,-1) = \{1\}$ and $(-i,1) \notin \rho$, we conclude that ρ is not a $*$-semilattice strongly regular relation on S.

However, the relation $\mathcal{N}$ is always a $*$-semilattice strongly regular relation on n-ary semihypergroups where $n \geq 3$.

Proposition 4.9. *Let (S, f) be an n-ary semihypergroup with $n \geq 3$. Then the following assertions hold:*

(i) *If K is a completely prime hyperideal of S, then δ_K is a $*$-semilattice strongly regular relation on S.*

(ii) *The relation $\mathcal{N}$ is a $*$-semilattice strongly regular relation on S.*

Proof. From Theorems 4.4 and 4.5, we obtain that δ_K and $\mathcal{N}$ are semilattice strongly regular relations on S.

(i) Let $x, y \in S$ be such that $x\overline{\overline{\delta_K}} f(x, y^{n-1})$. To show that $x\overline{\overline{\delta_K}} f(x^{n-1}, y)$, let $a \in f(x^{n-1}, y)$. We consider two cases as follows:

Case 1: $x \notin K$. Since $x\overline{\overline{\delta_K}} f(x, y^{n-1})$, by the definition of δ_K, we have $f(x, y^{n-1}) \cap K = \emptyset$. It follows that $f(x, y^{n-1}) \cap (S \setminus K) \neq \emptyset$. Since K is a completely prime hyperideal of S, by Corollary 3.8, we have $S \setminus K$ is a hyperfilter of S. By (F1) in Definition 3.1, we have $x, y \in S \setminus K$. Since $S \setminus K$ is an n-ary subsemihypergroup of S, we obtain $a \in f(x^{n-1}, y) \subseteq S \setminus K$. By the definition of δ_K, we have $x\delta_K a$. Thus $x\overline{\overline{\delta_K}} f(x^{n-1}, y)$.

Case 2: $x \in K$. Since K is a hyperideal of S, by (I1) in Definition 2.1, we obtain $a \in f(x^{n-1}, y) \subseteq K$. Consequently, $x\delta_K a$ and so $x\overline{\overline{\delta_K}} f(x^{n-1}, y)$. From Cases 1 and 2, we conclude that δ_K is a $*$-semilattice strongly regular relation on S.

(ii) Let $x, y \in S$ with $x\overline{\overline{\mathcal{N}}}f(x, y^{n-1})$. Then $N(x) = N(b)$ for all $b \in f(x, y^{n-1})$. To show that $x\overline{\overline{\mathcal{N}}}f(x^{n-1}, y)$, let $c \in f(x^{n-1}, y)$. Since $b \in f(x, y^{n-1}) \cap N(x)$, by (F1) in Definition 3.1 , we have $x, y \in N(x)$. Since $N(x)$ is an n-ary semihypergroup, we have $c \in f(x^{n-1}, y) \subseteq N(x)$. By Lemma 3.10, we get $N(c) \subseteq N(x)$. Conversely, since $f(x^{n-1}, y) \cap N(c) \neq \emptyset$, by (F1) in Definition 3.1 , we have $x \in N(c)$. So $N(x) \subseteq N(c)$. Consequently, $N(x) = N(c)$ and it follows that $x\mathcal{N}c$ for all $c \in f(x^{n-1}, y)$. Therefore $x\overline{\overline{\mathcal{N}}}f(x^{n-1}, y)$ and the proof is complete. □

Theorem 4.10. *Let (S, f) be an n-ary semihypergroup with $n \geq 3$. Let ρ be a $*$-semilattice strongly regular relation on S. Then there exists a collection C of proper completely prime hyperideals of S such that*

$$\rho = \bigcap_{K \in C} \delta_K.$$

Proof. Let $x \in S$. Setting

$$F_x := \left\{ y \in S \, : \, x\bar{\bar{\rho}}f(x^{n-1}, y) \right\}.$$

Since ρ is a $*$-semilattice strongly regular relation, we obtain $x\bar{\bar{\rho}}f(x^n)$. So $x \in F_x$. Firstly, we show that F_x is a hyperfilter of S.

(1) We show that F_x is an n-ary subsemihypergroup of S. Let $y_1^n \in F_x$. Then $f(y_1^n) \subseteq F_x$. Indeed, let $u \in f(y_1^n)$. We will show that $u \in F_x$, i.e., $x\bar{\bar{\rho}}f(x^{n-1}, u)$. Let $v \in f(x^{n-1}, u)$. Since $y_k \in F_x$ for all $k = 1, ..., n$, we have $x\bar{\bar{\rho}}f(x^{n-1}, y_k)$ for all $k = 1, ..., n$. Since ρ is a $*$-semilattice strongly regular relation on S, by associativity and Lemma 4.1, we obtain

$$\begin{aligned} x\bar{\bar{\rho}}f(x^n) \;&\bar{\bar{\rho}}\; f(f(x^{n-1}, y_1), ..., f(x^{n-1}, y_{n-1}), f(x^{n-1}, y_n)) \\ &\bar{\bar{\rho}}\; f(\underbrace{f(x^n), ..., f(x^n)}_{n-1 \text{ times}}, f(y_1^n)) \\ &\bar{\bar{\rho}}\; f(\underbrace{x, ..., x}_{n-1 \text{ times}}, f(y_1^n)). \end{aligned}$$

Hence $x\bar{\bar{\rho}}f(x^{n-1}, f(y_1^n))$. Since $v \in f(x^{n-1}, u) \subseteq f(x^{n-1}, f(y_1^n))$, we have $x\rho v$. So $x\bar{\bar{\rho}}f(x^{n-1}, u)$ and it implies that $u \in F_x$. Thus $f(y_1^n) \subseteq F_x$.

(2) We show that F_x satisfies (F1) in Definition 3.1. Let $y_1^n \in S$ such that $f(y_1^n) \cap F_x \neq \emptyset$. Then there exists $v \in f(y_1^n)$ and $v \in F_x$. Then $x\bar{\bar{\rho}}f(x^{n-1}, v)$. First of all, we show that $x\bar{\bar{\rho}}f(x^{n-1}, f(y_1^n))$. Let $a \in f(x^{n-1}, f(y_1^n))$. Then $a \in f(x^{n-1}, d)$ for some $d \in f(y_1^n)$. Since $f(y_1^n)\bar{\bar{\rho}}f(y_1^n)$ and $v, d \in f(y_1^n)$, we have $v\rho d$. Since ρ is a strongly regular relation, we obtain $f(x^{n-1}, v)\bar{\bar{\rho}}f(x^{n-1}, d)$. Since $x\bar{\bar{\rho}}f(x^{n-1}, v)$, we obtain $x\bar{\bar{\rho}}f(x^{n-1}, d)$. Since $a \in f(x^{n-1}, d)$, we have $x\rho a$. Thus $x\bar{\bar{\rho}}f(x^{n-1}, f(y_1^n))$. Next, we show that $y_k \in F_x$ for all $k = 1, ..., n$. For any $1 \leq k \leq n$, we have

$$\begin{aligned} x \;&\bar{\bar{\rho}}\; f(x^{n-1}, f(y_1^n)), \text{ since } \rho \text{ is semilattice}, \\ &\bar{\bar{\rho}}\; f(f(x^{n-1}, y_k), y_1^{k-1}, y_{k+1}^n), \\ &\qquad \text{since } \rho \text{ is semilattice and } y_i\bar{\bar{\rho}}f([y_i]^n) \text{ for all } i = 1, ..., k-1, k+1, ..., n, \\ &\bar{\bar{\rho}}\; f(f(x^{n-1}, y_k), f([y_1]^n), ..., f([y_{k-1}]^n), f([y_{k+1}]^n), ..., f([y_n]^n)), \\ &\qquad \text{since } \rho \text{ is semilattice}, \end{aligned}$$

$$
\begin{aligned}
&\bar{\bar{\rho}} \quad \underbrace{f(...f(f(...f(f(x^{n-1}, f(y_1^n)), [y_1]^{n-1}), ...)}_{f \text{ appears } n+1 \text{ terms}}, [y_{k-1}]^{n-1}), [y_{k+1}]^{n-1}), ..., [y_n]^{n-1}),\\
&\quad \text{since } x\bar{\bar{\rho}} f(x^{n-1}, f(y_1^n)),\\
&\bar{\bar{\rho}} \quad \underbrace{f(...f(f(...f(x, [y_1]^{n-1}), ...)}_{f \text{ appears } n-1 \text{ terms}}, [y_{k-1}]^{n-1}), [y_{k+1}]^{n-1}), ..., [y_n]^{n-1}),\\
&\quad \text{since } x\bar{\bar{\rho}} f(x^n),\\
&\bar{\bar{\rho}} \quad \underbrace{f(...f(f(...f(f(x^n), [y_1]^{n-1}), ...)}_{f \text{ appears } n \text{ terms}}, [y_{k-1}]^{n-1}), [y_{k+1}]^{n-1}), ..., [y_n]^{n-1}),\\
&\quad \text{since } \rho \text{ is semilattice},\\
&\bar{\bar{\rho}} \quad f(x, \underbrace{f(x, y_1^{k-1}, y_{k+1}^n), ..., f(x, y_1^{k-1}, y_{k+1}^n)}_{f \text{ appears } n-1 \text{ terms}}).
\end{aligned}
$$

Since ρ is a strongly regular relation, we obtain

$$
\begin{aligned}
f(x, [y_k]^{n-1}) \quad &\bar{\bar{\rho}} \quad f(f(x, \underbrace{f(x, y_1^{k-1}, y_{k+1}^n), ..., f(x, y_1^{k-1}, y_{k+1}^n)}_{f \text{ appears } n-1 \text{ terms}}), [y_k]^{n-1}),\\
&\quad\ \text{since } \rho \text{ is semilattice},\\
&\bar{\bar{\rho}} \quad f(f(x^{n-1}, x), [f(y_1^n)]^{n-1})\\
&= \quad f(x^{n-1}, f(x, [f(y_1^n)]^{n-1})), \text{ since } x\bar{\bar{\rho}} f(x^n),\\
&\bar{\bar{\rho}} \quad f([f(x^n)]^{n-1}, f(x, [f(y_1^n)]^{n-1}))\\
&= \quad f([f(x, x^{n-1})]^{n-1}, f(x, [f(y_1^n)]^{n-1})), \text{ since } \rho \text{ is semilattice},\\
&\bar{\bar{\rho}} \quad f(f(x^{n-1}, x), [f(x^{n-1}, f(y_1^n))]^{n-1}), \text{ since } f(x^{n-1}, f(y_1^n))\, \bar{\bar{\rho}}\, x,\\
&\bar{\bar{\rho}} \quad f(f(x^{n-1}, x), x^{n-1})\\
&\bar{\bar{\rho}} \quad x.
\end{aligned}
$$

Since ρ is a $*$-semilattice strongly regular relation on S, we have $x\bar{\bar{\rho}} f(x^{n-1}, y_k)$. So $y_k \in F_x$ for all $k = 1, 2, ..., n$. This follows that F_x satisfies the condition (F1) in Definition 3.1. By (1) and (2), we conclude that F_x is a hyperfilter of S. By Corollary 3.8, we have $S \setminus F_x = \emptyset$ or $S \setminus F_x$ is a completely prime hyperideal of S. If $S \setminus F_x = S$ then, since $F_x \subseteq S$, we have $F_x = \emptyset$. It is impossible. Thus $S \setminus F_x$ is a proper completely prime hyperideal of S. Define

$$C := \{S \setminus F_x : x \in S \text{ and } S \setminus F_x \text{ is a proper completely prime hyperideal of } S\}.$$

Next, we show that $\rho = \bigcap_{x \in S} \delta_{S \setminus F_x}$. Let $(a, b) \in \rho$ and $x \in S$. We consider the following two cases.

Case 1: $a \notin S \setminus F_x$. Then $a \in F_x$. We obtain $x\bar{\bar{\rho}} f(x^{n-1}, a)$. Since $(a, b) \in \rho$, we obtain $f(x^{n-1}, a)\bar{\bar{\rho}} f(x^{n-1}, b)$. Then $x\bar{\bar{\rho}} f(x^{n-1}, b)$ and so $b \in F_x$. It follows that $b \notin S \setminus F_x$. Thus $(a, b) \in \delta_{S \setminus F_x}$.

Case 2: $a \in S \setminus F_x$. If $b \notin S \setminus F_x$ then, using the similar proof as in Case 1, we have $a \notin S \setminus F_x$, which is a contradiction. So $b \in S \setminus F_x$. It follows that $(a, b) \in \delta_{S \setminus F_x}$. Thus $\rho \subseteq \bigcap_{x \in S} \delta_{S \setminus F_x}$. Conversely, let $(a, b) \in \bigcap_{x \in S} \delta_{S \setminus F_x}$. Then $(a, b) \in \delta_{S \setminus F_a}$. Since $a \in F_a$, we have $a \notin S \setminus F_a$. Since $(a, b) \in \delta_{S \setminus F_a}$, we have $b \notin S \setminus F_a$. So $b \in F_a$. By the definition of F_a, we have $a\bar{\bar{\rho}} f(a^{n-1}, b)$. On the other hand, by using the similar process, we have $b\bar{\bar{\rho}}f(b^{n-1},a)$. Then

$$\begin{aligned} a &\ \bar{\bar{\rho}}\ f(a^{n-1},b) \\ &\ \bar{\bar{\rho}}\ f(a^{n-1},f(b^{n-1},a)) \\ &\ \bar{\bar{\rho}}\ f(b^{n-1},f(a^{n})),\ \text{since } \rho \text{ is semilattice}, \\ &\ \bar{\bar{\rho}}\ f(b^{n-1},a),\ \text{since } a\bar{\bar{\rho}}f(a^{n}), \\ &\ \bar{\bar{\rho}}\ b. \end{aligned}$$

It implies that $(a,b)\in\rho$. Therefore $\bigcap\limits_{x\in S}\delta_{S\setminus F_x}\subseteq\rho$ and the proof is complete. □

Corollary 4.11. *Let (S,f) be an n-ary semihypergroup with $n\geq 3$. Then the relation $\mathcal{N}$ is the least $*$-semilattice strongly regular relation on S.*

Proof. Let ρ be a $*$-semilattice strongly regular relation on S. By Theorem 4.10, there exists a collection C of proper completely prime hyperideals of S such that $\rho=\bigcap\limits_{K\in C}\delta_K$. By Theorem 4.5, we have $\mathcal{N}=\bigcap\limits_{K\in\mathcal{CPI}(\mathcal{S})}\delta_K$. Since $C\subseteq\mathcal{CPI}(\mathcal{S})$, we obtain that

$$\mathcal{N}=\bigcap_{K\in\mathcal{CPI}(\mathcal{S})}\delta_K\subseteq\bigcap_{K\in C}\delta_K=\rho.$$

□

5. Complete $*$-semilattice strongly regular relations

As we have known from [21] that the relation $\mathcal{N}$ is not the smallest semilattice congruence on ordered semihypergroups in general. The following example show that the result is also true in case of the $*$-semilattice strongly regular relation on ordered n-ary semihypergroups with $n\geq 3$.

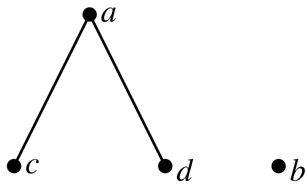
Example 5.1. Let $S=\{a,b,c,d\}$. Define $f:S\times S\times S\to\mathcal{P}^*(S)$ by $f(x_1,x_2,x_3)=(x_1\circ x_2)\circ x_3$, for all $x_1,x_2,x_3\in S$, where $\circ$ is defined by the following table.

$\circ$	a	b	c	d
a	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a\}$
b	$\{a,d\}$	$\{b\}$	$\{a,d\}$	$\{a,d\}$
c	$\{a,d\}$	$\{a,d\}$	$\{c\}$	$\{a,d\}$
d	$\{a\}$	$\{a,d\}$	$\{a,d\}$	$\{d\}$

By applying Example 1 in [31], (S,f) is a ternary semihypergroup. Define a partial order $\leq$ on S by

$$\leq:=\{(a,a),(b,b),(c,a),(c,c),(d,a),(d,d)\}.$$

Such relation can be presented by the following diagram.



Then $(S, f, \leq)$ is an ordered ternary semihypergroup. Moreover, we have $N(b) = \{b\}$ and $N(a) = S = N(c) = N(d)$. It follows that

$$\mathcal{N} := \{(a,a),(a,c),(a,d),(b,b),(c,a),(c,c),(c,d),(d,a),(d,c),(d,d)\}.$$

It is not difficult to show that

$$\rho = \{(a,a),(a,d),(b,b),(c,c),(d,a),(d,d)\}$$

is a $*$-semilattice strongly regular relation on S and $\rho \subseteq \mathcal{N}$.

Next, we introduce the concept of complete $*$-semilattice (strongly) regular relations on ordered n-ary semihypergroups, where $n \geq 3$, that generalizes Definition 4.1 in [21].

Definition 5.2. *Let S be an ordered n-ary semihypergroup with $n \geq 3$. A $*$-semilattice regular relation ($*$-semilattice strongly regular relation) ρ on S is said to be complete if for every $x, y \in S$,*

$$x \leq y \textit{ implies } x\bar{\rho} f(x^{n-1}, y)$$

$$(x \leq y \textit{ implies } x\bar{\bar{\rho}} f(x^{n-1}, y)).$$

Note that Theorem 4.4, Theorem 4.5 and Proposition 4.9 are also true for ordered n-ary semihypergroups where $n \geq 3$.

Proposition 5.3. *Let $(S, f, \leq)$ be an ordered n-ary semihypergroup with $n \geq 3$. Then the relation $\mathcal{N}$ is a complete $*$-semilattice strongly regular relation on S.*

Proof. Firstly, let $x, y \in S$ such that $x \leq y$. We show that $x\overline{\overline{\mathcal{N}}} f(x^{n-1}, y)$. Let $a \in f(x^{n-1}, y)$. Since $x \leq y$, by Lemma 3.10, we have $y \in N(y) \subseteq N(x)$. Then $x, y \in N(x)$. Since $N(x)$ is an n-ary subsemihypergroup, we obtain $a \in f(x^{n-1}, y) \subseteq N(x)$. Hence $N(a) \subseteq N(x)$. Conversely, since $f(x^{n-1}, y) \cap N(a) \neq \emptyset$, by (F1), we obtain $x \in N(a)$. By Lemma 3.10, we get $N(x) \subseteq N(a)$. So $N(x) = N(a)$. It follows that $x\mathcal{N}a$. Thus $x\overline{\overline{\mathcal{N}}} f(x^{n-1}, y)$. □

Theorem 5.4. *Let $(S, f, \leq)$ be an ordered n-ary semihypergroup with $n \geq 3$. Let ρ be a complete $*$-semilattice strongly regular relation on S. Then there exists a collection C of proper completely prime hyperideals of S such that*

$$\rho = \bigcap_{K \in C} \delta_K.$$

Proof. Let $x \in S$. Define
$$F_x := \left\{ y \in S \,:\, x\bar{\bar{\rho}} f(x^{n-1}, y) \right\}.$$
From Theorem 4.10, it remains to show that the set F_x satisfies (F2) in Definition 3.1. Let $y \in F_x$ and $z \in S$ such that $y \leq z$. Since $y \in F_x$, we have $x\bar{\bar{\rho}} f(x^{n-1}, y)$. Since $y \leq z$ and ρ is a complete $*$-semilattice strongly regular relation on S, we have $y\bar{\bar{\rho}} f(y^{n-1}, z)$. Since ρ is a $*$-semilattice strongly regular relation on S, we have $y\bar{\bar{\rho}} f(y^n)$. Next, we consider

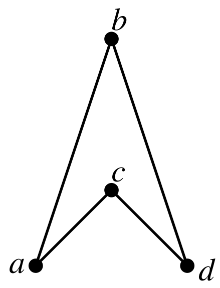
$$\begin{aligned}
f(x^{n-1}, z) &= f(x^{n-2}, x, z), \text{ since } x\bar{\bar{\rho}} f(x^{n-1}, y),\\
&\bar{\bar{\rho}}\ f(x^{n-2}, f(x^{n-1}, y), z), \text{ since } y\bar{\bar{\rho}} f(y^n),\\
&\bar{\bar{\rho}}\ f(x^{n-2}, f(x^{n-1}, f(y^n)), z)\\
&\bar{\bar{\rho}}\ f(x^{n-2}, f(f(x^{n-1}, y), y^{n-1}), z), \text{ since } x\bar{\bar{\rho}} f(x^{n-1}, y),\\
&\bar{\bar{\rho}}\ f(x^{n-2}, f(x, y^{n-1}), z)\\
&= f(x^{n-1}, f(y^{n-1}, z)), \text{ since } y\bar{\bar{\rho}} f(y^{n-1}, z),\\
&\bar{\bar{\rho}}\ f(x^{n-1}, y)\\
&\bar{\bar{\rho}}\ x.
\end{aligned}$$

Consequently, $z \in F_x$ and the proof is complete. □

Corollary 5.5. *Let $(S, f, \leq)$ be an ordered n-ary semihypergroup with $n \geq 3$. Then $\mathcal{N}$ is the smallest complete $*$-semilattice strongly regular relation on S.*

Proof. The proof is similar to Corollary 4.11. □

Example 5.6. Consider the ternary semihypergroup (S, f) given in Example 5.1. Define a partial order $\leq$ on S by
$$\leq := \{(a, a), (a, b), (a, c), (b, b), (c, c), (d, b), (d, c), (d, d)\}.$$
Such relation can be presented by the following diagram.



By applying Example 1 in [31], $(S, f, \leq)$ is an ordered ternary semihypergroup. There are 4 complete $*$-semilattice strongly regular relations as follows:

$$\begin{aligned}
\sigma_1 &= \{(a, a), (a, d), (b, b), (c, c), (d, a), (d, d)\},\\
\sigma_2 &= \{(a, a), (a, b), (a, d), (b, a), (b, b), (b, d), (c, c), (d, a), (d, b), (d, d)\},\\
\sigma_3 &= \{(a, a), (a, c), (a, d), (b, b), (c, a), (c, c), (c, d), (d, a), (d, c), (d, d)\},\\
\sigma_4 &= \{(a, a), (a, b), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, a), (c, b), (c, c),
\end{aligned}$$

$(c,d),(d,a),(d,b),(d,c),(d,d)\}.$

Furthermore, we have $N(b) = \{b\}$, $N(c) = \{c\}$ and $N(a) = N(d) = S$. It follows that $\mathcal{N} := \{(a,a),(a,d),(b,b),(c,c),(d,a),(d,d)\} = \sigma_1$. Obviously, $\mathcal{N}$ is the smallest complete $*$-semilattice strongly regular relation on S.

6. Decomposition of (completely) prime hyperideals of ordered n-ary semihypergroups

In this section, we show that every completely prime hyperideal on an ordered n-ary semihypergroups can be decomposable into its $\mathcal{N}$-classes. But the result is not true in the case of prime hyperideals, see Example 6.6. This leads to answer the open problem given in [31]. However, we will provide a sufficient condition that makes the above conclusion true for case of prime hyperideals. Note that Theorems 4.4 and 4.5 are also true for every ordered n-ary semihypergroup with $n \geq 2$.

Theorem 6.1. *Let $(S, f, \leq)$ be an ordered n-ary semihypergroup with $n \geq 2$. Let K be a completely prime hyperideal of S. Then*

$$K = \bigcup \{(a)_{\mathcal{N}} : a \in K\}.$$

Proof. Firstly, we show that $\bigcup\{(a)_{\mathcal{N}} : a \in K\} \subseteq K$. Let $b \in (a)_{\mathcal{N}}$ for some $a \in K$. By Theorem 4.5, we have $(a,b) \in \mathcal{N} = \bigcap\{\delta_A : A \in \mathcal{CPI}(\mathcal{S})\}$. So $(a,b) \in \delta_K$. Since $a \in K$, we get $b \in K$. Thus $\bigcup\{(a)_{\mathcal{N}} : a \in K\} \subseteq K$. As we have known that $a \in (a)_{\mathcal{N}}$ for all $a \in K$, so $K \subseteq \bigcup\{(a)_{\mathcal{N}} : a \in K\}$. Therefore $K = \bigcup\{(a)_{\mathcal{N}} : a \in K\}$. □

To solve the problem raised by Tang and Davvaz in [31], we firstly prove the following lemmas.

Lemma 6.2. *Let $(S, f, \leq)$ be an ordered n-ary semihypergroup with $n \geq 2$. Let K be a nonempty subset of S satisfying the following property:*

$$(\odot)\ \textit{either } f(x_1^n) \subseteq K \textit{ or } f(x_1^n) \subseteq S \setminus K \textit{ for all } x_1^n \in S.$$

For any $1 \leq j \leq n$, if a nonempty subset $S \setminus K$ is a prime j-hyperideal of S then K is a j-hyperfilter of S.

Proof. The proof is similar to the proof of the converse of Theorem 3.7 by applying the property $(\odot)$. □

Remark 6.3. For case $n = 2$, we obtain that the notion of prime hyperideal on S satisfying the property $(\odot)$ coincides with the notion of prime hyperideal on ordered semihypergroups introduced by Kehayopulu (see Definition 2.5 in [21]).

The following results, Lemma 6.4 and Theorem 6.5, generalize Proposition 2.5 and Theorem 2.10 in [21].

Lemma 6.4. *Let $(S, f, \leq)$ be an ordered n-ary semihypergroup with $n \geq 2$. Then the following statements hold:*

(i) *For any $K \in \mathcal{P}^*(S)$ satisfying the property $(\odot)$, K is a hyperfilter of S if and only if either $S \setminus K = \emptyset$ or $S \setminus K$ is a prime hyperideal of S.*

(ii) $\mathcal{N} = \bigcap\{\delta_K : K \in \mathcal{PH}(\mathcal{S})\}$ *where* $\mathcal{PH}(\mathcal{S})$ *is the set of all prime hyperideals of* S *satisfying the property* $(\odot)$.

Proof. As we have known that every completely prime hyperideal of S is always a prime hyperideal of S. By Corollary 3.8 and Lemma 6.2, the proof of (i) is complete. The proof of (ii) is similar to the proof of Theorem 4.5(ii) by applying (i). □

Using the similar proof of Theorem 6.1, by using Lemma 6.4(ii), we obtain the result as follows.

Theorem 6.5. *Let* $(S, f \leq)$ *be an ordered* n*-ary semihypergroup with* $n \geq 2$. *Let* K *be a prime hyperideal of* S *satisfying the property* $(\odot)$. *Then* $K = \bigcup\{(a)_{\mathcal{N}} : a \in K\}$.

The following example shows that a prime hyperideal of S that does not satisfy the property $(\odot)$ cannot be decomposable into its $\mathcal{N}$-classes.

Example 6.6. From the ordered n-ary semihypergroup $(S, f, \leq)$ given in Example 2.4, we obtain that the sets $A_2 = \{y, z\}$ and $A_3 = \{w, x, z\}$ are prime 2-hyperideals of S. Furthermore, A_2, A_3 are also prime hyperideals of S. Clearly, A_3 satisfies the property $(\odot)$ but A_2 does not satisfy the property $(\odot)$ since there exists $w \in S$ such that $f(w^n) \nsubseteq A_2$ and $f(w^n) \nsubseteq S \setminus A_2$. On the other hand, we can verify that $N(w) = N(x) = N(z) = S$ and $N(y) = \{y\}$. Hence

$$\mathcal{N} = \{(w, w), (w, x), (w, z), (x, w), (x, x), (x, z), (y, y), (z, w), (z, x), (z, z)\}.$$

Then $(w)_{\mathcal{N}} = \{w, x, z\} = (x)_{\mathcal{N}} = (z)_{\mathcal{N}}$ and $(y)_{\mathcal{N}} = \{y\}$. Clearly,

$$A_2 = \{y, z\} \neq S = (y)_{\mathcal{N}} \cup (z)_{\mathcal{N}}$$

and

$$A_3 = \{w, x, z\} = (w)_{\mathcal{N}} \cup (x)_{\mathcal{N}} \cup (z)_{\mathcal{N}}.$$

Remark 6.7. If we set $n = 2$ then the problem given in [31] is answered by Theorem 6.5 and Example 6.6.

7. Conclusions

In this paper, we introduced the concept of j-hyperfilters of ordered n-ary semihypergroups, where a positive integer $1 \leq j \leq n$ and $n \geq 2$, and discussed their related properties. Such notion can be considered as a generalization of left and right hyperfilters of ordered semihypergroups. The interesting properties of j-hyperfilters by means of completely prime j-hyperideals were established. Furthermore, we defined and investigated the semilattice strongly regular relation $\mathcal{N}$ on ordered n-ary semihypergroups by the hyperfilters. In particular, we illustrated by counterexample that the relation $\mathcal{N}$ is not necessary the least semilattice strongly regular relation on ordered n-ary semihypergroups where $n \geq 3$. Meanwhile, we introduced the concept of complete $*$-semilattice strongly regular relations on ordered n-ary semihypergroups, where $n \geq 3$, and proved that $\mathcal{N}$ is the smallest complete $*$-semilattice strongly regular relation. Finally, we provided some necessary conditions to show that every prime hyperideal can be decomposable into its $\mathcal{N}$-classes. This result answers the related problem posed by Tang and Davvaz in [31].

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Conflict of interest

The authors declare no conflict of interest.

References

1. S. M. Anvariyeh, S. Momeni, *n*-ary hypergroups associated with *n*-ary relations, *Bull. Korean Math. Soc.*, **50** (2013), 507–524.
2. Y. Cao, Chain decompositions of ordered semigroups, *Semigroup Forum*, **65** (2002), 83–106. doi: 10.1007/s002330010120.
3. T. Changphas, B. Davvaz, Properties of hyperideals in ordered semihypergroups, *Ital. J. Pure Appl. Math.*, **33** (2014), 425–432.
4. P. Corsini, *Prolegomena of hypergroup theory*, 1 Ed., Tricesimo: Aviani Editore, 1993.
5. P. Corsini, V. Leoreanu-Fotea, *Applications of hyperstructure theory*, Advances Mathematics, Dordrecht: Kluwer Academic Publishers, 2003. doi: 10.1007/978-1-4757-3714-1.
6. I. Cristea, M. Ştefănescu, Binary relations and reduced hypergroups, *Discrete Math.*, **308** (2008), 3537–3544. doi: 10.1016/j.disc.2007.07.011.
7. I. Cristea, Ştefănescu, Hypergroups and *n*-ary relations, *Eur. J. Combin.*, **31** (2010), 780–789. doi: 10.1016/j.ejc.2009.07.005.
8. J. Daengsaen, S. Leeratanavalee, B. Davvaz, On minimal and maximal hyperideals in *n*-ary semihypergroups, *Mathematics*, **8** (2020), 1656. doi: 10.3390/math8101656.
9. J. Daengsaen, S. Leeratanavalee, Regularities in ordered n-ary semihypergroups, *Mathematics*, **9** (2021), 1857. doi: 10.3390/math9161857.
10. B. Davvaz, *Semihypergroup theory*, 1 Ed., London: Academic Press, 2016. doi: 10.1016/C2015-0-06681-3.
11. B. Davvaz, P. Corsini, T. Changphas, Relationship between ordered semihypergroups and ordered semigroups by using pseudoorder, *Eur. J. Combin.*, **44** (2015). 208–217. doi: 10.1016/j.ejc.2014.08.006.
12. B. Davvaz, W. A. Dudek, S. Mirvakili, Neutral elements, fundamental relations and *n*-ary hypersemigroups, *Internat. J. Algebra Comput.*, **19** (2009), 567–583. doi: 10.1142/S0218196709005226.
13. B. Davvaz, W. A. Dudek, T. Vougiouklis, A generalization of *n*-ary algebraic systems, *Commun. Algebra*, **37** (2009), 1248–1263. doi: 10.1080/00927870802466835.
14. B. Davvaz, T. Vougiouklis, *n*-ary hypergroups, *Iran. J. Sci. Technol.*, **30** (2006), 165–174.

15. W. A. Dudek, I. Groździńska, On ideals in regular n-semigroups, *Mat. Bilten*, **4**(1980), 35–44.
16. L. Fuchs, *Partially ordered algebraic systems*, 1 Ed., London: Pergemon Press, 1963.
17. A. M. W. Glass, *Partially ordered groups*, 1 Ed., Singapore: World Scientific, 1999.
18. D. Heidari, B. Davvaz, On ordered hyperstructures, *UPB Sci. Bull. Ser. A, Appl. Math. Phys.*, **73** (2011), 85–96.
19. S. Hobanthad, On 0-minimal $(0, 2)$-bi-hyperideals of semihypergroups, *Thai J. Math.*, **19** (2021), 399–405.
20. T. Kumduang, S. Leeratanavalee, Menger hyperalgebras and their representations, *Commun. Algebra*, **49** (2021), 1513–1533. doi: 10.1080/00927872.2020.1839089.
21. N. Kehayopulu,On semilattice congruences on hypersemigroups and on ordered hypersemigroups, *Eur. J. Pure Appl. Math.*, **11** (2018), 476–492. doi: 10.29020/nybg.ejpam.v11i2.3266.
22. N. Kehayopulu, G. Lepouras, M. Tsingelis, On right regular and right duo ordered semigroups, *Math. Japonica*, **2** (1997), 311–315.
23. N. Kehayopulu, M. Tsingelis, On ordered semigroups which are semilattices of simple and regular semigroups, *Commun. Algebra*, **41** (2015), 3252–3260. doi: 10.1080/00927872.2012.682674.
24. N. Kehayopulu, M. Tsingelis, On the decomposition of prime ideals of ordered semigroups into their $\mathcal{N}$-classes, *Semigroup Forum*, **47** (1993), 393–395. doi: 10.1007/BF02573777.
25. S. K. Lee, S. S Lee, Left (right) filters on po-semigroups, *Korean J. Math.*, **8** (2000), 43–45.
26. V. Leoreanu-Fotea, About the simplifiable cyclic semihypergroups, *Ital. J. Pure Appl. Math.*, **7** (2000), 69–76.
27. V. Leoreanu-Fotea, B. Davvaz, n-hypergroups and binary relations, *Eur. J. Combin.*, **29** (2008), 1207–1218. doi: 10.1016/j.ejc.2007.06.025.
28. F. Marty, Sur une generalization de la notion de group, *Proc. 8th Congress Math. Scandenaves*, Stockholm, Sweden, 1934, 45–49.
29. S. Omidi, B. Davvaz, A short note on the relation $\mathcal{N}$ in ordered semihypergroups, *GU. J. Sci.*, **29** (2016), 659–662.
30. P. Pornsurat, P. P. Ayutthaya, B. Pibaljommee, Prime i-ideals in ordered n-ary semigroups, *Mathematics*, **9** (2021), 491. doi: 10.3390/math9050491.
31. J. Tang, B. Davvaz, Study on Green's relations in ordered semihypergroups, *Soft Comput.*, **24** (2020), 11189–11197. doi: 10.1007/s00500-020-05035-y.
32. J. Tang, B. Davvaz, Y. Luo, Hyperfilters and fuzzy hyperfilters of ordered semihypergroups, *J. Intell. Fuzzy Syst.*, **29** (2015), 75–84. doi: 10.3233/IFS-151571.
33. J. Tang, X. Feng, B. Davvaz, X. Y. Xie, A further study on ordered regular equivalence relations in ordered semihypergroups, *Open Math.*, **16** (2018), 168–184. doi: 10.1515/math-2018-0016.
34. Q. S. Zhu, Z. Popovic, Decompositions of ordered semigroups into a chain of k-Archimedean ordered semigroups, *Filomat*, **30** (2016), 2771–2780. doi: 10.2298/FIL1610771Q.