Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(1): 467–477.
DOI: 10.3934/math.2022030
Received: 25 July 2021
Accepted: 09 October 2021
Published: 13 October 2021

Research article

On (fuzzy) pseudo-semi-normed linear spaces

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Abstract: In this paper, we introduce the notion of pseudo-semi-normed linear spaces, following the concept of pseudo-norm which was presented by Schaefer and Wolff, and illustrate their relationship. On the other hand, we introduce the concept of fuzzy pseudo-semi-norm, which is weaker than the notion of fuzzy pseudo-norm initiated by Nădăban. Moreover, we give some examples which are according to the commonly used t-norms. Finally, we establish norm structures of fuzzy pseudo-semi-normed spaces and provide (fuzzy) topological spaces induced by (fuzzy) pseudo-semi-norms, and prove that the (fuzzy) topological spaces are (fuzzy) Hausdorff.

Keywords: pseudo-norm; pseudo-semi-norm; fuzzy pseudo-norm; fuzzy pseudo-semi-norm; fuzzy pseudo-semi-norm linear space
Mathematics Subject Classification: 47A30, 54A21, 54A40

1. Introduction and preliminaries

In 1981, Katsaras [11] introduced the notion of fuzzy topological vector space by assuming that the fuzzy topology of such a space contained all the constant fuzzy sets. Later, fuzzy semi-normed spaces and fuzzy normed spaces were investigated by Katsaras [12]. In 1988, Morsi [15] provided a different method for introducing fuzzy pseudo-metric topologies and fuzzy pseudo-normed topologies on vector spaces, and showed them equivalent to Katsaras-type. Afterwards, Felbin [9], Cheng and Mordeson [7], Bag and Samanta [2–5], proposed other concepts of fuzzy norms, respectively. Recently, Nădăban and Dzitac [17] introduced a generated fuzzy norm, by replacing the "*min*" of condition (N4) with a general form, and obtained some decomposition theorems for fuzzy norms into a family of semi-norms. Moreover, motivated by the work of Alege and Romaguera [1], Nădăban, in 2016 [16], proposed the notion of fuzzy pseudo-norm, and obtained a characterization of metrizable topological linear spaces in terms of a fuzzy F-norm.

On the other hand, Das and Das [8] constructed a fuzzy topology in a fuzzy normed linear space, which was proved to be fuzzy Hausdorff. Afterwards, many researchers devoted to providing some properties of these fuzzy topologies [10, 19, 20, 23–25], and it became a hot topic in developing fuzzy functional analysis and its applications.

In this paper, on one hand, following the notion of pseudo-norm defined by Schaefer and Wolff [21], we introduce a concept of pseudo-semi-norm, by replacing the condition (N2) with (NWP2), i.e $N_{wp}(x,t) = 0$ for all $t > 0$ if $x = \theta$, where θ is a zero element. On the other hand, following the notion of fuzzy pseudo-normed linear spaces defined by Nădăban, we further introduce a new concept of fuzzy pseudo-semi-norm according to general t-norm. Also we give some examples with respect to $*_M, *_P$ and $*_L$, respectively. Finally, we obtain (fuzzy) topologies induced by (fuzzy) pseudo-semi-normed linear spaces, and prove that they are (fuzzy) Hausdorff.

Throughout this paper, X always denotes a non-empty set, the letters $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$ always denote the set of real numbers, of positive real numbers and of complex numbers, respectively. From now, the scalar field $\mathbb{K}$ means either the field $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. A *pseudo-semi-norm* on a linear space X is a real function $\|\cdot\| : X \to \mathbb{R}$ satisfying the following conditions: $\forall x, y \in X$ and for all $\lambda \in \mathbb{K}$ with $|\lambda| \le 1$,
(NPS1) $\|x\| \ge 0$;
(NPS2) $\|x\| = 0$ if $x = \theta$, where θ is a zero element of X;
(NPS3) $\|\lambda x\| \le \|x\|$;
(NPS4) $\|x + y\| \le \|x\| + \|y\|$.

If a pseudo-semi-norm also satisfies (NPS5): $\|x\| = 0$ implies $x = \theta$, then it is a *pseudo-norm* [21].

A pseudo(-semi)-normed space is a pair $(X, \|\cdot\|)$ such that $\|\cdot\|$ is a pseudo(-semi)-norm on X.

Particularly, if $\|\cdot\|$ satisfies (NPS1),(NPS2),(NPS4),(NPS5) and (NPS6): $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{K}$, then it is a norm .

Apparently, the condition (NPS6) is weaker than (NPS3). Hence, each norm is a pseudo-norm.

Remark 1.2. It is obvious that each (pseudo-)norm is a pseudo(-semi)-norm, but we show that the converse is not true as the following examples show:

Example 1.3. Let $(X, \|\cdot\|)$ be a linear space, where $X = R^2$. Define $\|\cdot\| : X \to \mathbb{R}$ by $\|x\| = \frac{|x_1|}{1+|x_1|} + \frac{|x_2|}{2(1+|x_2|)}$ for all $x = (x_1, x_2) \in X$. Then $(X, \|\cdot\|)$ is a pseudo(-semi)-normed space.

It is trivial to verify the conditions (NPS1), (NPS2) and (NPS5). We will check the conditions (NPS3) and (NPS4) as follows:

(NPS3) Let $x = (x_1, x_2) \in X, \lambda \in \mathbb{K}$ and $|\lambda| \le 1$. We have

$$\|\lambda x\| = \frac{|\lambda x_1|}{1+|\lambda x_1|} + \frac{|\lambda x_2|}{2(1+|\lambda x_2|)} \le \frac{|\lambda||x_1|}{1+|\lambda||x_1|} + \frac{|\lambda||x_2|}{2(1+|\lambda||x_2|)} \le \frac{|x_1|}{1+|x_1|} + \frac{|x_2|}{2(1+|x_2|)} = \|x\|$$

(NPS4) Let $x = (x_1, x_2), y = (y_1, y_2) \in X$. We have

$$\|x + y\| = \frac{|x_1+y_1|}{1+|x_1+y_1|} + \frac{|x_2+y_2|}{2(1+|x_2+y_2|)} \le \frac{|x_1|+|y_1|}{1+|x_1|+|y_1|} + \frac{|x_2|+|y_2|}{2(1+|x_2|+|y_2|)}$$

and

$$\frac{|x_1|+|y_1|}{1+|x_1|+|y_1|} + \frac{|x_2|+|y_2|}{2(1+|x_2|+|y_2|)} \le \frac{|x_1|}{1+|x_1|} + \frac{|x_2|}{2(1+|x_2|)} + \frac{|y_1|}{1+|y_1|} + \frac{|y_2|}{2(1+|y_2|)}.$$

It follows that $\|x + y\| \le \|x\| + \|y\|$.

However, it is not a norm. Indeed, set $x_0 = (1, 0), \lambda_0 = \frac{1}{2}$. We have $\|\lambda_0 x_0\| = \frac{1}{3}$ and $|\lambda_0|\|x_0\| = \frac{1}{4}$. So $\|\lambda_0 x_0\| \ne |\lambda_0|\|x_0\|$.

Example 1.4. Let $(X, \|\cdot\|)$ be a linear space, where $X = R^n$. Define $\|\cdot\| : X \to \mathbb{R}$ by $\|x\| = |x_n|$ for all $x = (x_1, x_2, \ldots, x_n) \in X$. Then $(X, \|\cdot\|)$ is a pseudo-semi-normed space, but it is not a pseudo-normed space.

It is trivial to verify the conditions (NPS1) and (NPS2). We will check the conditions (NPS3) and (NPS4) as follows:

(NPS3) Let $x = (x_1, x_2, \ldots, x_n) \in X, \lambda \in \mathbb{K}$ and $|\lambda| \leq 1$. We have

$$\|\lambda x\| = |\lambda x_n| = |\lambda||x_n| \leq |x_n| = \|x_n\|$$

(NPS4) Let $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in X$. We have

$$\|x + y\| = |x_n + y_n| \leq |x_n| + |y_n| = \|x\| + \|y\|$$

However, since $\|x\| = |x_n| = 0$ does not imply $x = \theta$, it is not a pseudo-norm.

In addition to following sections, we will recall some basic concepts on triangular norms and fuzzy normed linear spaces.

Definition 1.5. [14] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a *continuous triangular norm* (briefly t-norm) if it satisfies the following conditions:
(T1) $*$ is associative and commutative;
(T2) $*$ is continuous;
(T3) $a * 1 = a$ for all $a \in [0, 1]$;
(T4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

The following are the three basic t-norms: minimum, usual product and Lukasiewicz t-norm, which are given by, respectively: $a *_M b = \min\{a, b\}$, $a *_P b = ab$ and $a *_L b = \max\{0, a + b - 1\}$, for all $a, b \in [0, 1]$.

Definition 1.6. [2] Let X be a linear space over $\mathbb{K}$. A fuzzy set N of $X \times \mathbb{R}$ is called a *fuzzy norm* on X if it satisfies the following conditions: $\forall x, y \in X$ and $\lambda \in \mathbb{K}$,
(FN1) $N(x, t) = 0$ for all $t \in \mathbb{R}$ with $t \leq 0$;
(FN2) $N(x, t) = 1$ for all $t \in \mathbb{R}^+$ if and only if $x = \theta$, where θ is a zero element of X;
(FN3) $N(\lambda x, t) = N(x, \frac{t}{|\lambda|})$ for all $t \in \mathbb{R}, \lambda \neq 0$;
(FN4) $N(x + y, t + s) \geq \min\{N(x, t), N(y, s)\}$ for all $x, y \in X, s, t \in \mathbb{R}$;
(FN5) $\lim_{t\to+\infty} N(x, t) = 1$.
The pair (X, N) is called to be a *fuzzy normed space linear space*. Obviously, if N is a fuzzy norm, then $N(x, \cdot)$ is non-decreasing for all $x \in X$.

2. Fuzzy pseudo-semi-normed spaces

Definition 2.1. Let X be a linear space over $\mathbb{K}$ and $*$ be a continuous t-norm. A fuzzy set N_{ps} of $X \times \mathbb{R}$ is called a *fuzzy pseudo-semi-norm* on X if it satisfies the following conditions: $\forall x, y \in X$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$,
(FNPS1) $N_{ps}(x, t) = 0$ for all $t \in \mathbb{R}$ with $t \leq 0$;
(FNPS2) $N_{ps}(x, t) = 1$ for all $t \in \mathbb{R}^+$ if $x = \theta$, where θ is a zero element of X;
(FNPS3) $N_{ps}(\lambda x, t) \geq N_{ps}(x, t)$ for all $t \in \mathbb{R}$;

(FNPS4) $N_{ps}(x+y,t+s) \geq N_{ps}(x,t) * N_{ps}(y,s)$ for all $x,y \in X, s,t \in \mathbb{R}$;
(FNPS5) $\lim_{t\to+\infty} N_{ps}(x,t) = 1$.
The triple $(X, N_{ps}, *)$ is called to be a *fuzzy pseudo-semi-normed linear space*.

If a fuzzy pseudo-semi-norm with aspect to $*_M$ also satisfies (FNPS2*): $N_{ps}(x,t) = 1$ for all $t \in \mathbb{R}^+$ implies $x = \theta$, then it is a *fuzzy pseudo-norm* [16].

Remark 2.2. (1) It is easy to see that $N_{ps}(-x,t) = N_{ps}(x,t), \forall x \in X, t \in \mathbb{R}$.
(2) For all $x \in X$, $N_{ps}(x,\cdot)$ is non-decreasing.
(3) Every fuzzy normed linear space is a fuzzy pseudo-(semi-)normed linear space with respect to $*_M$.

Indeed, let $t > s > 0$. By (NPS4), we have $N_{ps}(x,t) = N_{ps}(x, s+(t-s)) \geq N_{ps}(x,s) * N_{ps}(\theta, t-s) = N_{ps}(x,s) * 1 = N_{ps}(x,s)$ for all $x \in X$. Thus, Remark 2.2 (2) holds.

Furthermore, if $(X, N_{ps}, *)$ is a fuzzy normed linear space, we only check (FNPS3) in the following cases:

Case 1: Suppose that $\lambda = 0$. By (FNPS1), we have $N_{ps}(\lambda x,t) = N_{ps}(\theta,t) = 1$ for all $t \in \mathbb{R}^+$. Thus $N_{ps}(\lambda x,t) \geq N_{ps}(x,t)$ for all $x \in X, t \in \mathbb{R}$.

Case 2: For all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$ and $\lambda \neq 0$, by Remark 2.2 (2), we have $N_{ps}(x, \frac{t}{|\lambda|}) \geq N_{ps}(x,t)$. From (FN3), it implies that $N_{ps}(\lambda x,t) = N_{ps}(x, \frac{t}{|\lambda|})$. Hence, $N_{ps}(\lambda x,t) \geq N_{ps}(x,t)$.

Example 2.3. Let X be a linear space over $\mathbb{K}$ and $||\cdot||$ be a pseudo-semi-norm. Define a fuzzy set N_{ps}: $X \times \mathbb{R} \to [0,1]$ by

$$N_{ps}(x,t) = \begin{cases} \frac{t}{t+||x||}, & t > 0; \\ 0, & t \leq 0. \end{cases}$$

for all $x \in X$. Then $(X, N_{ps}, *_M)$ is a fuzzy pseudo-semi-normed space.

It is trivial to verify (FNPS1),(FNPS2) and (FNPS5).

We need to verify the conditions (FNPS3) and (FNPS4), respectively.

(FNPS3): We will distinguish in the following cases:

Case 1: Suppose that $t \leq 0$. It implies that $N_{ps}(\lambda x,t) = N_{ps}(x,t) = 0$.

Case 2: For all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$, by (NPS3), we have $N_{ps}(\lambda x,t) = \frac{t}{t+||\lambda x||} \geq \frac{t}{t+||x||} = N_{ps}(x,t)$ for all $x \in X$.

(FNPS4): We will distinguish in following cases:

Case 1: Suppose that $t \leq 0$ or $s \leq 0$. It follows that $N_{ps}(x,t) = 0$ or $N_{ps}(x,s) = 0$ for all $x \in X$, then $N_{ps}(x,t) *_M N_{ps}(y,s) = 0$. Hence, $N_{ps}(x+y,t+s) \geq N_{ps}(x,t) *_M N_{ps}(y,s)$.

Case 2: For any $x,y \in X, t,s > 0$, without loss of generality, suppose that $s||x|| \geq t||y||$, namely, $N_{ps}(x,t) \leq N_{ps}(y,s)$, that is $N_{ps}(x,t) *_M N_{ps}(y,s) = N_{ps}(x,t)$. By (NPS4), we have $N_{ps}(x+y,t+s) = \frac{t+s}{t+s+||x+y||} \geq \frac{t+s}{t+s+||x||+||y||}$. Since $s||x|| \geq t||y||$, it follows that $\frac{t+s}{t+s+||x||+||y||} \leq \frac{t+s}{t+s+||x||+\frac{s}{t}||x||} = \frac{t}{t+||x||}$. Thus, we can deduce that $N_{ps}(x+y,t+s) \geq N_{ps}(x,t) *_M N_{ps}(y,s)$.

Example 2.4. Let $X = R^2$ be a linear space over $\mathbb{R}$ and $||\cdot||$ be a pseudo-semi-norm. Define a fuzzy set N_{ps}: $X \times \mathbb{R} \to [0,1]$ by

$$N_{ps}(x,t) = \begin{cases} \frac{t^2}{(t+|x_1|)(t+|x_2|)}, & t > 0; \\ 0, & t \leq 0. \end{cases}$$

for all $x = (x_1, x_2) \in X$. Then $(X, N_{ps}, *_P)$ is a fuzzy pseudo-semi-normed space.

It is trivial to verify (FNPS1),(FNPS2) and (FNPS5).

We need to verify the conditions (FNPS3) and (FNPS4), respectively.

(FNPS3): We will distinguish in the following cases:

Case 1: Suppose that $t \leq 0$. It implies that $N_{ps}(\lambda x, t) = N_{ps}(x, t) = 0$.

Case 2: For all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$, by (NPS3), we have $N_{ps}(\lambda x, t) = \frac{t^2}{(t+|\lambda||x_1|)(t+|\lambda||x_2|)} \geq \frac{t^2}{(t+|x_1|)(t+|x_2|)} = N_{ps}(x, t)$ for all $x \in X$.

(FNPS4): In [4], the authors proved that $N_{ps}(x + y, t + s) \geq N_{ps}(x, t) *_P N_{ps}(y, s)$ for all $x, y \in \mathbb{R}$.

Example 2.5. Let X be a linear space over $\mathbb{K}$ and $\|\cdot\|$ be a pseudo-semi-norm. Define a fuzzy set N_{ps}: $X \times \mathbb{R} \to [0, 1]$ by

$$N_{ps}(x, t) = \begin{cases} 1, & \|x\| < t; \\ 0, & \|x\| \geq t. \end{cases}$$

for all $x \in X, t \in \mathbb{R}$. Then $(X, N_{ps}, *_L)$ is a fuzzy pseudo-semi-normed space.

It is trivial to verify (FNPS1), (FNPS2) and (FNPS5).

We need to verify the conditions (FNPS3) and (FNPS4), respectively.

(FNPS3): We will distinguish in the following cases:

Case 1: Suppose that $N_{ps}(x, t) = 0$. It is obvious that $N_{ps}(\lambda x, t) \geq N_{ps}(x, t)$.

Case 2: Suppose that $N_{ps}(x, t) = 1$, that is $\|x\| < t$. For all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$, by (NPS3), we have $\|\lambda x\| \leq \|x\| < t$. Thus, $N_{ps}(\lambda x, t) = 1$, namely, $N_{ps}(\lambda x, t) = N_{ps}(x, t)$.

(FNPS4): Since

$$N_{ps}(x, t) + N_{ps}(y, s) - 1 = \begin{cases} -1, & \|x\| \geq t, \|y\| \geq s; \\ 0, & \|x\| \geq t, \|y\| < s \text{ or } \|x\| < t, \|y\| \geq s; \\ 1, & \|x\| < t, \|y\| < s. \end{cases}$$

it follows that

$$\max\{0, N_{ps}(x, t) + N_{ps}(y, s) - 1\} = \begin{cases} 1, & \|x\| < t, \|y\| < s; \\ 0, & \text{otherwise}. \end{cases}$$

namely, $N_{ps}(x, t) *_L N_{ps}(y, s) = 0$ or 1.

We will distinguish in the following cases:

Case 1: Suppose that $N_{ps}(x, t) *_L N_{ps}(y, s) = 0$. It is easy to show that $N_{ps}(x + y, t + s) \geq N_{ps}(x, t) *_L N_{ps}(y, s)$ for all $x, y \in \mathbb{R}$.

Case 2: For all $\|x\| < t, \|y\| < s$, by (NPS4), we have $\|x+y\| \leq \|x\|+\|y\| < t+s$, then $N_{ps}(x+y, t+s) = 1$. Thus, $N_{ps}(x + y, t + s) = N_{ps}(x, t) *_L N_{ps}(y, s)$.

The following example shows that not every fuzzy pseudo-semi-normed linear space is a fuzzy pseudo-normed linear space.

Example 2.6. Let $X = R^n$ be a linear space over $\mathbb{R}$ and $\|\cdot\|$ be a pseudo-semi-norm. Define a fuzzy set N_{ps}: $X \times \mathbb{R} \to [0, 1]$ by

$$N_{ps}(x, t) = \begin{cases} \frac{t}{t+\lim_{n\to+\infty} |x_n|}, & t > 0; \\ 0, & t \leq 0. \end{cases}$$

for all $x = (x_1, x_2, \ldots, x_n) \in X$. Then $(X, N_{ps}, *_M)$ is a fuzzy pseudo-semi-normed space.

It is trivial to verify (FNPS1), (FNPS2) and (FNPS5).

We need to verify the conditions (FNPS3) and (FNPS4), respectively.

(FNPS3): We will distinguish in the following cases:

Case 1: Suppose that $t \le 0$. We have $N_{ps}(\lambda x, t) = N_{ps}(x, t) = 0$.

Case 2: For all $t > 0$, by (NPS3), we have

$$N_{ps}(\lambda x, t) = \frac{t}{t+\lim_{n\to+\infty}|\lambda x_n|} \ge \frac{t}{t+\lim_{n\to+\infty}|x_n|} = N_{ps}(x, t),$$

for all $\forall x \in X$ and $\lambda \in \mathbb{K}$ with $|\lambda| \le 1$.

(FNPS4): We will distinguish in the following cases:

Case 1: Suppose that $t \le 0$ or $s \le 0$. It follows that $N_{ps}(x, t) = 0$ or $N_{ps}(y, s) = 0$, that is $N_{ps}(x, t) *_M N_{ps}(y, s) = 0$. Thus, $N_{ps}(x + y, t + s) \ge N_{ps}(x, t) *_M N_{ps}(y, s)$ for all $x, y \in \mathbb{R}$.

Case 2: For all $x, y \in \mathbb{R}, s, t > 0$, without loss of generality, suppose that $s|x_n| \ge t|y_n|$, namely, $N_{ps}(x, t) \le N_{ps}(y, s)$, that is $N_{ps}(x, t) *_M N_{ps}(y, s) = N_{ps}(x, t)$. Then, we have

$$N_{ps}(x + y, t + s) = \frac{t+s}{t+s+\lim_{n\to+\infty}|x_n+y_n|} \ge \frac{t+s}{t+s+\lim_{n\to+\infty}|x_n|+\lim_{n\to+\infty}|y_n|}.$$

Since $s|x_n| \ge t|y_n|$, it follows that $\frac{t+s}{t+s+|x_n|+|y_n|} \le \frac{t+s}{t+s+|x_n|+\frac{s}{t}|x_n|} = \frac{t}{t+|x_n|}$. Thus, we can deduce that $N_{ps}(x + y, t + s) \ge N_{ps}(x, t) *_M N_{ps}(y, s)$.

However, the statement is not true which $\lim_{n\to+\infty}|x_n| = 0$ implies $x = \theta$. Thus it is not a fuzzy pseudo-normed linear space with respect to $*_M$.

Furthermore, we will present some decomposition theorems for fuzzy pseudo-semi-norms, and construct a fuzzy pseudo-semi-normed space from the family of fuzzy pseudo-semi-norms.

Theorem 2.7. *Let $(X, N_{ps}, *)$ be a fuzzy pseudo-semi-normed linear space. Define $\|x\|_0 = \bigwedge\{t > 0 : N_{ps}(x, t) = 1\}, \forall x \in X$. Then $\|x\|_0$ is a pseudo-semi-norm on X.*

Proof. It is trivial to verify (NPS1) and (NPS2).

We need to verify the conditions (NPS3) and (NPS4).

(NPS3): First, we have $\{t > 0 : N_{ps}(x, t) = 1\} \subset \{t > 0 : N_{ps}(x, t) \le 1\}$. Then, it implies that $\|x\|_0 = \bigwedge\{t > 0 : N_{ps}(x, t) = 1\} \ge \bigwedge\{t > 0 : N_{ps}(x, t) \le 1\}$. By (FNPS3), we have $N_{ps}(\lambda x, t) \ge N_{ps}(x, t)$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \le 1$. It follows that $\{t > 0 : N_{ps}(\lambda x, t) = 1\} = \{t > 0 : N_{ps}(x, t) \le 1\}$, that is, $\|\lambda x\|_0 = \bigwedge\{t > 0 : N_{ps}(\lambda x, t) = 1\} = \bigwedge\{t > 0 : N_{ps}(x, t) \le 1\}$. Hence, $\|x\|_0 \ge \|\lambda x\|_0$ for all $x \in X$, $\lambda \in \mathbb{K}$ with $|\lambda| \le 1$.

(NPS4): From the definition of $\|x\|_0$, it is easy to see $N_{ps}(x, \|x\|_0 + \frac{\varepsilon}{2}) = 1$ and $N_{ps}(y, \|y\|_0 + \frac{\varepsilon}{2}) = 1$ for all $x, y \in X, \varepsilon > 0$. By (FNPS4), we have

$$N_{ps}(x + y, \|x\|_0 + \|y\|_0 + \varepsilon) \ge N_{ps}(x, \|x\|_0 + \tfrac{\varepsilon}{2}) * N_{ps}(y, \|y\|_0 + \tfrac{\varepsilon}{2}) = 1 * 1 = 1.$$

Thus $N_{ps}(x + y, \|x\|_0 + \|y\|_0 + \varepsilon) = 1$, it follows that $\|x + y\|_0 \le \|x\|_0 + \|y\|_0 + \varepsilon$. By the arbitrariness of ε, we have $\|x + y\|_0 \le \|x\|_0 + \|y\|_0$. □

Theorem 2.8. *Let $(X, N_{ps}, *)$ be a fuzzy pseudo-semi-normed linear space. Define $\|x\|_\alpha = \bigwedge\{t > 0 : N_{ps}(x, t) > 1 - \alpha\}, \forall x \in X$. Then the following statements hold:*

(1) $\{\|x\|_\alpha : \alpha \in (0, 1)\}$ is non-increasing with respect to α.

(2) $\{\|x\|_\alpha : \alpha \in (0, 1)\}$ is a left-continuous function on $\alpha \in (0, 1)$.

Furthermore, $\{\|x\|_\alpha : \alpha \in (0, 1)\}$ is a continuous function on $\alpha \in (0, 1)$ if N_{ps} is strictly increasing.

(3) $\{\|x\|_\alpha : \alpha \in (0, 1)\}$ is a pseudo-semi-norm family corresponding to the fuzzy pseudo-semi-norm N_{ps} on X.

Proof. (1) Case 1: If $x = \theta$, it is evident.

Case 2: Let $x \neq \theta$, for all $\alpha, \beta \in (0, 1)$, $\alpha < \beta$, by Remark 2.2 (2), we have

$$\{t > 0 : N_{ps}(x, t) > 1 - \alpha\} \subset \{t > 0 : N_{ps}(x, t) > 1 - \beta\},$$

that is $\bigwedge\{t > 0 : N_{ps}(x, t) > 1 - \alpha\} \geq \bigwedge\{t > 0 : N_{ps}(x, t) > 1 - \beta\}$. Thus $\|x\|_\alpha \geq \|x\|_\beta$. Hence, $\{\|x\|_\alpha : \alpha \in (0, 1)\}$ is non-increasing.

(2) First, from Theorem 2.8 (1), it is clear that $\|x\|_{\alpha-\varepsilon} \geq \|x\|_\alpha$ for all $0 < \varepsilon < \alpha, 0 < \alpha < 1$. Thus $\lim_{\varepsilon \to 0^+} \|x\|_{\alpha-\varepsilon} \geq \|x\|_\alpha$. Additionally, we claim that $\lim_{\varepsilon \to 0^+} \|x\|_{\alpha-\varepsilon} \leq \|x\|_\alpha$ for all $\alpha \in (0, 1), x \in X$. Otherwise, assume that $\lim_{\varepsilon \to 0^+} \|x\|_{\alpha-\varepsilon} > \|x\|_\alpha$. Then, there exists $t_0 > 0$ such that $\lim_{\varepsilon \to 0^+} \|x\|_{\alpha-\varepsilon} > t_0 > \|x\|_\alpha$. It implies that $\|x\|_{\alpha-\varepsilon} > t_0 > \|x\|_\alpha$ for all $0 < \varepsilon < \alpha$. Since $t_0 > \|x\|_\alpha$ and $\|x\|_{\alpha-\varepsilon} > t_0$, by the definition of $\|x\|_\alpha$, we have $1 - \alpha < N_{ps}(x, t_0)$ and $N_{ps}(x, t_0) \leq 1 - \alpha + \varepsilon$. By the arbitrariness of α, we have $N_{ps}(x, t_0) \leq 1 - \alpha$, which is a contradiction.

Furthermore, from Theorem 2.8 (2), we will prove that N_{ps} is right-continuous if N_{ps} is strictly increasing, i.e. $\lim_{\varepsilon \to 0^+} \|x\|_{\alpha+\varepsilon} = \|x\|_\alpha$ for all $\alpha \in (0, 1)$. It is easy to see that $\lim_{\varepsilon \to 0^+} \|x\|_{\alpha+\varepsilon} \leq \|x\|_\alpha$, then we only prove that $\lim_{\varepsilon \to 0^+} \|x\|_{\alpha+\varepsilon} \geq \|x\|_\alpha$. Otherwise, suppose that $\lim_{\varepsilon \to 0^+} \|x\|_{\alpha+\varepsilon} < \|x\|_\alpha$. For all $\|x\|_{\alpha+\varepsilon} < t < \|x\|_\alpha$, by the definition of $\|x\|_\alpha$, we have $1 - \alpha - \varepsilon < N_{ps}(x, t) \leq 1 - \alpha$. By the arbitrariness of α, it follows that $N_{ps}(x, t) = 1 - \alpha$. Since N_{ps} is strictly increasing, thus, it is a contradiction.

(3) It is trivial to verify (NPS1), (NPS2) and (NPS3), then we only check (NPS4). Indeed, by (FNPS4), we have

$$\begin{aligned}
\|x\|_\alpha + \|y\|_\alpha &= \bigwedge\{t > 0 : N_{ps}(x, t) > 1 - \alpha\} + \bigwedge\{t > 0 : N_{ps}(y, s) > 1 - \alpha\} \\
&= \bigwedge\{t + s > 0 : N_{ps}(x, t) > 1 - \alpha, N_{ps}(y, s) > 1 - \alpha\} \\
&= \bigwedge\{t + s > 0 : N_{ps}(x, t) *_M N_{ps}(y, s) > 1 - \alpha\} \\
&\geq \bigwedge\{t + s > 0 : N_{ps}(x + y, t + s) > 1 - \alpha\} \\
&= \|x + y\|_\alpha,
\end{aligned}$$

for all $x, y \in X$, $\alpha \in (0, 1)$. □

Following Theorem 3.8 [16], we have the following proposition:

Proposition 2.9. *Let $\{\|x\|_\alpha : \alpha \in (0, 1)\}$ be a pseudo-semi-norm family linear space, which is continuous and non-decreasing. Define*

$$N_{ps}(x, t) = \begin{cases} \bigvee\{\alpha \in (0, 1) : \|x\|_\alpha < t\}, & t > 0; \\ 0, & t \leq 0 \text{ or } \{\alpha \in (0, 1) : \|x\|_\alpha < t\} = \emptyset. \end{cases}$$

Then $(X, N_{ps}, *)$ is a fuzzy pseudo-semi-normed linear space, where $*$ is a continuous t-norm.

3. (Fuzzy) topology on fuzzy pseudo-semi-normed linear space

According to Bag and Samanta [2] investigated the connection which the fuzzy metric could be induced by the fuzzy norm, Nădăban and Dzitac [17] obtained that $\mathcal{P} = \{p_\alpha(x)\}_{\alpha \in (0,1)}$ is an ascending family of semi-norms. In addition, Das and Das [8] defined a fuzzy topology on the fuzzy normed linear space. We will obtain the fuzzy pseudo-metric induced by the pseudo-semi-norm in following section. Firstly, we will recall some notions and results related to fuzzy pseudo-metrics.

Definition 3.1. [18] A triple $(X, M_{p_k}, *)$ is called a *fuzzy pseudo-metric space* if X is an arbitrary nonempty set, $*$ is a continuous t-norm and M: $X \times X \times [0, +\infty) \to [0, 1]$ is a map satisfying the following conditions: $\forall x, y, z \in X$ and $t, s \geq 0$,
(FPM1) $M(x, y, 0) = 0$;
(FPM2) $M(x, x, t) = 1$ for all $t > 0$;
(FPM3) $M(x, y, t) = M(y, x, t)$;
(FPM4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(FPM5) The function $M(x, y, \cdot) : [0, +\infty) \to [0, 1]$ is left-continuous;
(FPM6) $\lim_{t \to +\infty} M(x, y, t) = 1$.
The map M is called a *fuzzy pseudo-metric*.

Definition 3.2. Let $(X, N_{ps}, *)$ be a fuzzy pseudo-semi-normed linear space. For all $x \in X, 0 < \alpha < 1, t > 0$, a set $B(x, r, t) = \{y \in X : N_{ps}(x - y, t) > 1 - \alpha\}$ is called an *open ball*.

Definition 3.3. [6, 8] A *fuzzy topology* on a set X is a family $\mathcal{T}$ of fuzzy subsets of X satisfying the following conditions:
(FT1) The fuzzy subsets $\underline{0}, \underline{1}$ are in $\mathcal{T}$;
(FT2) $\mathcal{T}$ is closed under finite intersection of fuzzy subsets;
(FT3) $\mathcal{T}$ is closed under arbitrary union of fuzzy subsets.
The pair $(X, \mathcal{T})$ is called a *fuzzy topological space*.

Definition 3.4. [8] A fuzzy topological space $(X, \mathcal{T})$ is said to be *fuzzy Hausdorff* if for $x, y \in X$ and $x \neq y$, there exist $\mu, \eta \in \mathcal{T}$ with $\mu(x) = \eta(y) = 1$ and $\mu \cap \eta = \emptyset$.

Theorem 3.5. *Suppose $(X, N_{ps}, *)$ is a fuzzy pseudo-semi-normed linear space such that satisfies* (FNPS6) :$(\forall)x \in X$, $N(x, \cdot)$ *is left-continuous. Define a mapping* M: $X \times X \times [0, +\infty) \to [0, 1]$ *by* $M(x, y, t) = N_{ps}(x - y, t)$ *for all* $x, y \in X$ *and* $t \geq 0$. *Then* $(X, M, *)$ *is a fuzzy pseudo-metric space.*

Proof. It is trivial to prove that $(X, M, *)$ satisfies (FPM1), (FPM2), (FPM5) and (FPM6). We verify conditions (FPM3) and (FPM4) as follows:

(FPM3): By Remark 2.2 (1), we have $M(x, y, t) = N_{ps}(x - y, t) = N_{ps}(y - x, t) = M(y, x, t)$.

(FPM4): By (FNPS4), we have $M(x, z, t + s) = N_{ps}(x - z, t + s) = N_{ps}(x - y + y - z, t + s) \geq N_{ps}(x - y, t) * N_{ps}(y - z, s) = M(x, y, t) * M(y, z, s)$. □

Proposition 3.6. *Let $(X, N_{ps}, *)$ be a fuzzy pseudo-semi-normed linear space. Define a family of subsets of X by*

$\mathcal{T}_{N_{ps}} = \{V \subset X : x \in V$ if and only if there exist $t > 0, r \in (0, 1)$ such that $B(x, r, t) \subset V\}$.

Then the following statements hold:
(1) $\mathcal{T}_{N_{ps}}$ is a topology on X.
(2) $(X, \mathcal{T}_{N_{ps}})$ is Hausdorff if $*$ satisfies (T5)*:* $\bigvee_{a \in (0,1)} a * a = 1$ *and N_{ps} satisfies* (FNPS7)*:* $N_{ps}(x, t) > 0$ *for all* $t > 0$ *implies* $x = \theta$.

Proof. (1) We will prove $\mathcal{T}_{N_{ps}}$ is a topology on X in the following steps:

Step 1: It is clear that $\emptyset, X \in \mathcal{T}_{N_{ps}}$.

Step 2: Let $V_1, V_2 \in \mathcal{T}_{N_{ps}}$. For any $x \in V_1 \cap V_2$, by the definition of $\mathcal{T}_{N_{ps}}$, there exist $t_i > 0, r_i \in (0, 1)$, such that $B(x, r_i, t_i) \subset V_i$, where $i = 1, 2$. Taking $r = \min\{r_i : i = 1, 2\}, t = \max\{t_i : i = 1, 2\}$, it follows that $1 - r \geq 1 - r_i$ and $B(x, r, t) \subset V_i, i = 1, 2$. Thus, $B(x, r, t) \subset V_1 \cap V_2$.

Step 3: Let $V_\gamma \in \mathcal{T}_{N_{ps}}$, $\gamma \in \Gamma$, where Γ is an index set. For any $x \in \bigcup_{\gamma\in\Gamma} V_\gamma$, we have $x \in V_{\gamma_0}$ for some $\gamma_0 \in \Gamma$. Since $V_{\gamma_0} \in \mathcal{T}_{N_{ps}}$, by the definition of $\mathcal{T}_{N_{ps}}$, there exist $t > 0, r \in (0, 1)$, such that $B(x, r, t) \subset V_{\gamma_0} \subset \bigcup_{\gamma\in\Gamma} V_\gamma$. Hence, $\bigcup_{\gamma\in\Gamma} V_\gamma \in \mathcal{T}_{N_{ps}}$.

(2) Let $x, y \in X, x \neq y$. Then there exists $t_0 > 0$, such that $N_{ps}(x - y, t_0) < 1$. Otherwise, suppose that $N_{ps}(x - y, t_0) = 1$ for all $t > 0$. By (FNSP7), we have $x - y = \theta$, namely $x = y$, which is a contradiction. Set $r = N(x - y, t_0)$. By (T5), there is $r_0 \in (0, 1)$, such that $r_0 * r_0 > r$. Thus, we have $B(x, 1 - r_0, \frac{t}{2}) \cap B(y, 1 - r_0, \frac{t}{2}) = \emptyset$. Otherwise, suppose that $B(x, 1 - r_0, \frac{t}{2}) \cap B(y, 1 - r_0, \frac{t}{2}) \neq \emptyset$. Then there exists $z \in B(x, 1 - r_0, \frac{t}{2}) \cap B(y, 1 - r_0, \frac{t}{2})$, that is, $z \in B(x, 1 - r_0, \frac{t}{2})$ and $z \in B(y, 1 - r_0, \frac{t}{2})$, which implies that $N_{ps}(x - z, \frac{t}{2}) > r_0$ and $N_{ps}(y - z, \frac{t}{2}) > r_0$. By (FNPS4), we have $N_{ps}(x - y, t) \geq N_{ps}(x - z, \frac{t}{2}) * N_{ps}(y - z, \frac{t}{2}) > r_0 * r_0 > r$, which is a contradiction. □

Proposition 3.7. *Let* $(X, N_{ps}, *)$ *be a fuzzy pseudo-semi-normed linear space. Define a family of subsets of* X *by* $\mathcal{T}^*_{N_{ps}} = \{\mu \in I^X : \forall x \in \text{supp}\mu$ and $r \in (0, 1)$ there exist $\varepsilon > 0$, such that $x + B_\varepsilon \cap \widetilde{r} \subset \mu\}$, *where the fuzzy real number* $\widetilde{r}$*:* $\mathbb{R} \to I$ *is given as follows: For all* $s \in \mathbb{R}$, $\widetilde{r}(s) = 1$ *when* $s < r$, *and* $\widetilde{r}(s) = 0$ *when* $s \geq r$. *Then the following statements hold:*
(1) $\mathcal{T}^*_{N_{ps}}$ *is a fuzzy topology on* X.
(2) $(X, \mathcal{T}^*_{N_{ps}})$ *is fuzzy Hausdorff if* N_{ps} *satisfies* (FNPS7*)*:* $\forall x \neq \theta$, *there is* $t_x > 0$, *such that* $N_{ps}(x, t_x) = 0$.

Proof. (1) We will prove $\mathcal{T}^*_{N_{ps}}$ is a fuzzy topology on X in the following steps:

Step 1: It is clear that $\underline{0}, \underline{1} \in \mathcal{T}^*_{N_{ps}}$.

Step 2: Let $\mu_1, \mu_2 \in \mathcal{T}^*_{N_{ps}}$, and $(\mu_1 \cap \mu_2)(x) > r > 0$. It follows that $\mu_1(x) > r > 0$ and $\mu_2(x) > r > 0$. By the definition of $\mathcal{T}^*_{N_{ps}}$, there exist $\varepsilon_i > 0, i = 1, 2$, such that $x + B_{\varepsilon_1} \cap \widetilde{r} \subset \mu_1$ and $x + B_{\varepsilon_2} \cap \widetilde{r} \subset \mu_2$. Taking $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. It follows that $N_{ps}(x, \varepsilon) \leq N_{ps}(x, \varepsilon_1)$ and $N_{ps}(x, \varepsilon) \leq N_{ps}(x, \varepsilon_2)$. Thus, $x + B_\varepsilon \cap \widetilde{r} \subset x + B_{\varepsilon_1} \cap \widetilde{r}$ and $x + B_\varepsilon \cap \widetilde{r} \subset x + B_{\varepsilon_2} \cap \widetilde{r}$. which implies that $x + B_\varepsilon \cap \widetilde{r} \subset \mu_1 \cap \mu_2$. Thus, $\mu_1 \cap \mu_2 \in \mathcal{T}^*_{N_{ps}}$.

Step 3: Let $\mu_\gamma \in \mathcal{T}^*_{N_{ps}}$, $\gamma \in \Gamma$, and $(\bigcup_{\gamma\in\Gamma} \mu_\gamma)(x) > r > 0$, where Γ is an index set. Then there exist $\gamma_0 \in \Gamma$, such that $\mu_0(x) > r > 0$. Thus, there is some $\varepsilon_0 > 0$, such that $x + B_{\varepsilon_0} \cap \widetilde{r} \subset \bigcup_{\gamma\in\Gamma} \mu_\gamma$. Hence, $\bigcup_{\gamma\in\Gamma} \mu_\gamma \in \mathcal{T}^*_{N_{ps}}$.

(2) Let $x, y \in X, x \neq y$. By (FNPS7*), there exists $t_0 > 0$, such that $N_{ps}(x - y, t_0) = 0$. Set $0 < \varepsilon < t_0$, we claim that $(x + B_{\frac{\varepsilon}{2}}) \cap (y + B_{\frac{\varepsilon}{2}}) = \emptyset$. Otherwise, suppose that $(x + B_{\frac{\varepsilon}{2}}) \cap (y + B_{\frac{\varepsilon}{2}}) \neq \emptyset$. There exists $z \in X$ such that $(x + B_{\frac{\varepsilon}{2}}) \cap (y + B_{\frac{\varepsilon}{2}})(z) > 0$. Then, $(x + B_{\frac{\varepsilon}{2}})(z) > 0$ and $(y + B_{\frac{\varepsilon}{2}})(z) > 0$. By (FNSP4) and Remark 2.2 (1), we have

$$\begin{aligned} N_{ps}(x - y, t_0) &\geq N_{ps}(x - z, \frac{t_0}{2}) * N_{ps}(z - y, \frac{t_0}{2}) \\ &= N_{ps}(x - z, \frac{t_0}{2}) * N_{ps}(y - z, \frac{t_0}{2}) \\ &\geq N_{ps}(x - z, \frac{\varepsilon}{2}) * N_{ps}(y - z, \frac{\varepsilon}{2}) \\ &\geq 0 * 0 = 0, \end{aligned}$$

which is a contradiction. □

4. Conclusions

In this paper, firstly, we introduce the notion of pseudo-semi-norm. Moreover, we take definition of a fuzzy pseudo-norm on a linear space in its general form, and present some examples of fuzzy pseudo-semi-normed spaces. In Section 3, we construct (fuzzy) topologies which were induced by (fuzzy) pseudo-semi-norms, and show that these spaces are Hausdoff.

Acknowledgments

The author thanks the editor and the referees for constructive and pertinent suggestions, which have improved the quality of the manuscript greatly.

Conflict of interest

The author declares that he has no competing interest.

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