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Research article

Growth of solutions with $L^{2(p+2)}$-norm for a coupled nonlinear viscoelastic Kirchhoff equation with degenerate damping terms

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Abstract: In this work, we consider a coupled nonlinear viscoelastic Kirchhoff equations with degenerate damping, dispersion and source terms. Under suitable hypothesis, we will prove that when the initial data are large enough (in the energy point of view), the energy grows exponentially and thus so the $L^{2(p+2)}$-norm.

Keywords: viscoelastic equation; exponential growth; degenerate damping term
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1. Introduction

In this paper, we consider the following problem

$$\begin{cases} |u_t|^{\eta} u_{tt} - \Delta u + \int_0^t h_1(t-s)\Delta u(s)ds - \Delta u_{tt} + \left(|u|^k + |v|^l\right)|u_t|^{j-1} u_t \\ \qquad = f_1(u,v), \quad (x,t) \in \Omega \times (0,T), \\ |v_t|^{\eta} v_{tt} - \Delta v + \int_0^t h_2(t-s)\Delta v(s)ds - \Delta v_{tt} + \left(|v|^{\theta} + |u|^{\varrho}\right)|v_t|^{s-1} v_t \\ \qquad = f_2(u,v), \quad (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \qquad x \in \Omega, \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \qquad x \in \Omega, \end{cases} \tag{1.1}$$

where $k, l, \theta, \varrho \geq 0$; $j, s \geq 1$ for $N = 1, 2$, and $0 \leq j, s \leq \frac{N+2}{N-2}$ for $N \geq 3$; and $\eta \geq 0$ for $N = 1, 2$ and $0 < \eta \leq \frac{2}{N-2}$ for $N \geq 3$, $h_i(.) : R^+ \to R^+$ $(i = 1, 2)$ are positive relaxation functions which will be

specified later. $\left(|(.)|^{a}+|(.)|^{b}\right)|(.)_{t}|^{\tau-1}(.)_{t}$ and $-\Delta(.)_{tt}$ are the degenerate damping term and the dispersion term, respectively.
And

$$\begin{cases} f_1(u,v)=a_1|u+v|^{2(p+1)}(u+v)+b_1|u|^p.u.|v|^{p+2}, \\ f_2(u,v)=a_1|u+v|^{2(p+1)}(u+v)+b_1|v|^p.v.|u|^{p+2}. \end{cases} \tag{1.2}$$

It is well known that viscous materials are the opposite of elastic materials which have the capacity to store and dissipate mechanical energy. As the mechanical properties of these viscous substances are of great importance when they appear in many applications of other applied sciences.

Physically, the relationship between the stress and strain history in the beam inspired by Boltzmann theory called viscoelastic damping term, where the kernel of the term of memory is the function h (see [1–9]). If $\eta \geq 0$, this type of problem has been studied by many authors. For more depth, here are some papers that focused on the study of this damping. See for example [10–15]. The effect of the degenerate damping terms often appear in many applications and piratical problems and turns a lot of systems into different problems worth studying.

The well known "Growth" phenomenon is one of the most important phenomena of asymptotic behavior, where many researches omit from its study especially when it comes from the evolution problems. It gives us very important information to know the behavior of equation when time arrives at infinity, it differs from global existence and blow up in both mathematically and in applications point of view.

Recently, the stability, the asymptotic behavior, blowing up and exponential growth of solutions for evolution systems with time degenerate damping has been studied by many authors. See [16–20].

The great importance of the source term with nonlinear functions f_1 and f_2 satisfying appropriate conditions. In physics is that they appear in several issues and theories. Many researchers also touched on this type of problem in several different issues, where the global existence of solutions, stability, blow up and growth of solutions were studied. For more information, the reader is referred to ([21–28]). Recently, If $\gamma = 0, \alpha_1 = 1$ our problem (1.1) has been studied in [27], under some restrictions on the initial datum, standard conditions on relaxation functions, the authors are established the global existence and proved the general decay of solutions.

Based on all of the above, the combination of these terms of damping (Memory term, degenerate damping, dispersion and the source terms) we believe that it constitutes a new problem worthy of study and research, different from the above that we will try to shed light on.

In fact it will be proved that the $L^{2(p+2)}$ norm of the solution grows as an exponential function. An essential tool of the proof is an idea used in the literature, which based on an auxiliary function (which is a small perturbation of the total energy), in order to obtain a differential inequality leads to the exponential growth result provided that under suitable hypothesis.

Our paper is divided into several sections: in the next section we lay down the hypotheses, concepts and lemmas we need. In the third section we prove our main result. Finally, a general conclusion has been drawn up.

2. Preliminaries

We prove the exponential growth of solutions under the following suitable assumptions.
(**A1**) $h_i : \mathbb{R}_+ \to \mathbb{R}_+$ are a differentiable and decreasing functions such that

$$h_i(t) \geq 0 \quad , \quad 1 - \int_0^\infty h_i\,(s)\,ds = l_i > 0, \;\; i = 1, 2. \tag{2.1}$$

(**A2**) There exists a constants $\xi_1, \xi_2 > 0$ such that

$$h_i'\,(t) \leq -\xi_i h_i\,(t) \quad , \quad t \geq 0, \;\; i = 1, 2. \tag{2.2}$$

Theorem 2.1. *Assume (2.1) and (2.2) holds. Let*

$$\begin{cases} -1 < p < \dfrac{4-n}{n-2}, & n \geq 3; \\ p \geq -1, & n = 1, 2 \end{cases} \tag{2.3}$$

Then for any initial data

$$(u_0, u_1, v_0, v_1) \in \mathcal{H},$$

the problem (1.1) has a unique solution, for some $T > 0$

$$u \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega)),$$

where

$$\mathcal{H} \;=\; H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega).$$

In the next theorem we give the global existence result, its proof based on the potential well depth method in which the concept of so-called stable set appears, where we show that if we restrict our initial data in the stable set, then our local solution obtained is global in time, We will make use of arguments in [15].

Theorem 2.2. *Suppose that (2.1), (2.2) and (2.3) holds. If* $u_0, v_0 \in H_0^1(\Omega)$, $u_1, v_1 \in L^2(\Omega)$

$$\rho\Big(\frac{2(p+2)}{(p+1)l}E(0)\Big)^{p+1} < 1, \tag{2.4}$$

where $\rho > 0$ *is a constant. Then the local solution* (u, v) *is global in time.*

To achieve our goal, we need the following lemmas.

Lemma 2.1. *There exists a function* $F(u, v)$ *such that*

$$\begin{aligned} F(u, v) &= \frac{1}{2(\rho+2)}\left[u f_1(u, v) + v f_2(u, v)\right] \\ &= \frac{1}{2(\rho+2)}\left[a_1|u+v|^{2(p+2)} + 2b_1|uv|^{p+2}\right] \geq 0, \end{aligned}$$

where

$$\frac{\partial F}{\partial u} = f_1(u, v), \quad \frac{\partial F}{\partial v} = f_2(u, v),$$

we take $a_1 = b_1 = 1$ for convenience.

Lemma 2.2. *[2] There exist two positive constants c_0 and c_1 such that*

$$\frac{c_0}{2(\rho+2)}\left(|u|^{2(p+2)} + |v|^{2(p+2)}\right) \le F(u,v) \le \frac{c_1}{2(\rho+2)}\left(|u|^{2(\rho+2)} + |v|^{2(p+2)}\right). \tag{2.5}$$

Now, we define the energy functional

Lemma 2.3. *Assume (2.1), (2.2) and (2.3) hold, let (u, v) be a solution of (1.1), then $E(t)$ is non-increasing, that is*

$$\begin{aligned} E(t) &= \frac{1}{\eta+2}\Big[\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\Big] + \frac{1}{2}\Big[\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\Big] \\ &\quad + \frac{1}{2}\Big[\Big(1 - \int_0^t h_1(s)ds\Big)\|\nabla u\|_2^2 + \Big(1 - \int_0^t h_2(s)ds\Big)\|\nabla v\|_2^2\Big] \\ &\quad + \frac{1}{2}\Big[(h_1 o \nabla u)(t) + (h_2 o \nabla v)(t)\Big] - \int_\Omega F(u,v)dx, \end{aligned} \tag{2.6}$$

satisfies

$$\begin{aligned} E'(t) &\le \frac{1}{2}\Big[(h_1' o \nabla u)(t) + (h_2' o \nabla v)(t)\Big] - \frac{1}{2}\Big[h_1(t)\|\nabla u\|_2^2 + h_2(t)\|\nabla v\|_2^2\Big] \\ &\quad - \int_\Omega (|u|^k + |v|^l)|u_t|^{j+1}dx - \int_\Omega (|v|^\theta + |u|^\varrho)|v_t|^{s+1}dx \\ &\le 0. \end{aligned} \tag{2.7}$$

Proof. By multiplying $(1.1)_1, (1.1)_2$ by u_t, v_t and integrating over Ω, we get

$$\begin{aligned} &\frac{d}{dt}\Big\{\frac{1}{\eta+2}\|u_t\|_{\eta+2}^{\eta+2} + \frac{1}{\eta+2}\|v_t\|_{\eta+2}^{\eta+2} + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\nabla v_t\|_2^2 \\ &\quad + \frac{1}{2}\Big(1 - \int_0^t h_1(s)ds\Big)\|\nabla u\|_2^2 + \frac{1}{2}\Big(1 - \int_0^t h_2(s)ds\Big)\|\nabla v\|_2^2 \\ &\quad + \frac{1}{2}(h_1 o \nabla u)(t) + \frac{1}{2}(h_2 o \nabla v)(t) - \int_\Omega F(u,v)dx\Big\} \\ &= -\int_\Omega (|u|^k + |v|^l)|u_t|^{j+1}dx - \int_\Omega (|v|^\theta + |u|^\varrho)|v_t|^{s+1}dx \\ &\quad + \frac{1}{2}(h_1' o \nabla u) - \frac{1}{2}h_1(t)\|\nabla u\|_2^2 + \frac{1}{2}(h_2' o \nabla v) - \frac{1}{2}h_2(t)\|\nabla v\|_2^2, \end{aligned} \tag{2.8}$$

we obtain (2.6) and (2.7). □

3. Main result

In this section, we prove the exponential growth of solution with $L^{2(p+2)}$-norm of problem (1.1). First, we define the functional

$$\mathbb{H}(t) = -E(t) = -\frac{1}{\eta+2}\Big[\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\Big] - \frac{1}{2}\Big[\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\Big]$$

$$
\begin{aligned}
&-\frac{1}{2}\Big[\Big(1-\int_0^t h_1(s)ds\Big)\|\nabla u\|_2^2+\Big(1-\int_0^t h_2(s)ds\Big)\|\nabla v\|_2^2\Big]\\
&-\frac{1}{2}\Big[(h_1 o\nabla u)(t)+(h_2 o\nabla v)(t)\Big]\\
&+\frac{1}{2(p+2)}[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}].
\end{aligned} \tag{3.1}
$$

Theorem 3.1. *Assume (2.1), (2.2), and (2.3) hold, and suppose that $E(0) < 0$, and*

$$
2(p+2) > \max\Big\{k+j+1; l+j+1; \theta+s+1; \varrho+s+1; \frac{\eta+2}{\eta+1}\Big\}. \tag{3.2}
$$

Then the solution of problem (1.1) growth exponentially.

Proof. From (2.6), we have

$$
E(t) \le E(0) \le 0. \tag{3.3}
$$

Therefore

$$
\mathbb{H}'(t) = -E'(t) \;\; \ge \;\; \int_\Omega (|u|^k+|v|^l)|u_t|^{j+1}dx + \int_\Omega (|v|^\theta+|u|^\varrho)|v_t|^{s+1}dx, \tag{3.4}
$$

hence

$$
\begin{aligned}
\mathbb{H}'(t) &\ge \int_\Omega (|u|^k+|v|^l)|u_t|^{j+1}dx \ge 0\\
\mathbb{H}'(t) &\ge \int_\Omega (|v|^\theta+|u|^\varrho)|v_t|^{s+1}dx \ge 0.
\end{aligned} \tag{3.5}
$$

By (3.1) and (2.5), we have

$$
\begin{aligned}
0 \le \mathbb{H}(0) \le \mathbb{H}(t) &\le \frac{1}{2(p+2)}\Big(\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}\Big)\\
&\le \frac{c_1}{2(p+2)}\Big(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\Big).
\end{aligned} \tag{3.6}
$$

We set

$$
\begin{aligned}
\mathcal{K}(t) &= \mathbb{H}(t)+\frac{\varepsilon}{\eta+1}\int_\Omega \Big[u|u_t|^\eta u_t + v|v_t|^\eta v_t\Big]dx\\
&+\varepsilon\int_\Omega\Big[\nabla u_t\nabla u+\nabla v_t\nabla v\Big]dx,
\end{aligned} \tag{3.7}
$$

where $\varepsilon > 0$ to be assigned later.
By multiplying $(1.1)_1, (1.1)_2$ by u, v and with a derivative of (3.7), we get

$$
\mathcal{K}'(t) = \mathbb{H}'(t)+\frac{\varepsilon}{\eta+1}(\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2})+\varepsilon(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2)
$$

$$+\underbrace{\varepsilon\int_\Omega \nabla u\int_0^t g(t-s)\nabla u(s)dsdx}_{J_1}+\underbrace{\varepsilon\int_\Omega \nabla v\int_0^t h(t-s)\nabla v(s)dsdx}_{J_2}$$
$$-\underbrace{\varepsilon\int_\Omega (|u|^k+|v|^l)|u_t|^{j-1}u_t.udx}_{J_3}-\underbrace{\varepsilon\int_\Omega (|v|^\theta+|u|^\varrho)|v_t|^{s-1}v_t.vdx}_{J_4}$$
$$-\varepsilon(\|\nabla u\|_2^2+\|\nabla v\|_2^2)+\underbrace{\varepsilon\Big[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}\Big]}_{J_5}. \tag{3.8}$$

We have

$$\begin{aligned}J_1 &= \varepsilon\int_0^t h_1(t-s)ds\int_\Omega \nabla u.(\nabla u(s)-\nabla u(t))dxds+\varepsilon\int_0^t h_1(s)ds\|\nabla u\|_2^2\\ &\geq \frac{\varepsilon}{2}\int_0^t h_1(s)ds\|\nabla u\|_2^2-\frac{\varepsilon}{2}(h_1o\nabla u).\end{aligned} \tag{3.9}$$

$$\begin{aligned}J_2 &= \varepsilon\int_0^t h_2(t-s)ds\int_\Omega \nabla v.(\nabla v(s)-\nabla v(t))dxds+\varepsilon\int_0^t h_2(s)ds\|\nabla v\|_2^2\\ &\geq \frac{\varepsilon}{2}\int_0^t h_2(s)ds\|\nabla v\|_2^2-\frac{\varepsilon}{2}(h_2o\nabla v).\end{aligned} \tag{3.10}$$

From (3.8), we find

$$\begin{aligned}\mathcal{K}'(t) &\geq \mathbb{H}'(t)+\frac{\varepsilon}{\eta+1}(\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2})+\varepsilon(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2)\\ &\quad-\varepsilon\Big[\Big(1-\frac{1}{2}\int_0^t h_1(s)ds\Big)\|\nabla u\|_2^2+\Big(1-\frac{1}{2}\int_0^t h_2(s)ds\Big)\|\nabla v\|_2^2\Big]\\ &\quad-\frac{\varepsilon}{2}(h_1o\nabla u)-\frac{\varepsilon}{2}(h_2o\nabla v)-J_3-J_4+J_5.\end{aligned} \tag{3.11}$$

At this point, we use Young's inequality, for $\delta>0$

$$XY\leq \frac{\delta^\alpha X^\alpha}{\alpha}+\frac{\delta^{-\beta}X^\beta}{\beta},\quad \alpha,\beta>0,\quad \frac{1}{\alpha}+\frac{1}{\beta}=1, \tag{3.12}$$

we get, for $\delta_1,\delta_2>0$

$$\begin{aligned}\Big|u|u_t|^{j-1}u_t\Big| &\leq \frac{\delta_1^{j+1}}{j+1}|u|^{j+1}+\frac{j}{j+1}\delta_1^{-(\frac{j+1}{j})}|u_t|^{j+1},\\ \Big|v|v_t|^{s-1}v_t\Big| &\leq \frac{\delta_2^{s+1}}{s+1}|v|^{s+1}+\frac{s}{s+1}\delta_2^{-(\frac{s+1}{s})}|v_t|^{s+1}.\end{aligned} \tag{3.13}$$

Hence, we have

$$J_3 \leq \varepsilon\frac{\delta_1^{j+1}}{j+1}\int_\Omega(|u|^k+|v|^l)|u|^{j+1}dx+\varepsilon\frac{j\delta_1^{-(\frac{j+1}{j})}}{j+1}\int_\Omega(|u|^k+|v|^l)|u_t|^{j+1}dx,$$

$$J_4 \leq \varepsilon\frac{\delta_2^{s+1}}{s+1}\int_\Omega(|v|^\theta+|u|^\varrho)|v|^{s+1}dx+\varepsilon\frac{s\delta_2^{-(\frac{s+1}{s})}}{s+1}\int_\Omega(|v|^\theta+|u|^\varrho)|v_t|^{s+1}dx. \tag{3.14}$$

Therefore, using (3.5) and by setting δ_1, δ_1 so that,

$$\frac{j\delta_1^{-(\frac{j+1}{j})}}{j+1}=\frac{\kappa}{2},\quad \frac{s\delta_2^{-(\frac{s+1}{s})}}{s+1}=\frac{\kappa}{2},$$

substituting in (3.11), we get

$$\begin{aligned}
\mathcal{K}'(t) \geq\ & [1-\varepsilon\kappa]\mathbb{H}'(t)+\frac{\varepsilon}{\eta+1}\Big(\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2}\Big)\\
&-\varepsilon\Big[\Big(1-\frac{1}{2}\int_0^t h_1(s)ds\Big)\|\nabla u\|_2^2+\Big(1-\frac{1}{2}\int_0^t h_2(s)ds\Big)\|\nabla v\|_2^2\Big]\\
&+\varepsilon\Big(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\Big)-\frac{\varepsilon}{2}(h_1 o\nabla u)-\frac{\varepsilon}{2}(h_2 o\nabla v)\\
&-\varepsilon C_1(\kappa)\int_\Omega(|u|^k+|v|^l)|u|^{j+1}dx\\
&-\varepsilon C_2(\kappa)\int_\Omega(|v|^\theta+|u|^\varrho)|v|^{s+1}dx+J_5,
\end{aligned} \tag{3.15}$$

where

$$C_1(\kappa):=\Big(\frac{2j}{\kappa(j+1)}\Big)^{j+1}\frac{1}{j+1},\quad C_2(\kappa):=\Big(\frac{2s}{\kappa(s+1)}\Big)^{s+1}\frac{1}{s+1}, \tag{3.16}$$

we have

$$\begin{aligned}
\int_\Omega(|u|^k+|v|^l)|u|^{j+1}dx &= \|u\|_{k+j+1}^{k+j+1}+\int_\Omega|v|^l|u|^{j+1}dx,\\
\int_\Omega(|v|^\theta+|u|^\varrho)|v|^{s+1}dx &= \|v\|_{\theta+s+1}^{\theta+s+1}+\int_\Omega|u|^\varrho|v|^{s+1}dx.
\end{aligned} \tag{3.17}$$

By Young's inequality, we find for $\delta_3, \delta_4 > 0$

$$\begin{aligned}
\int_\Omega|v|^l|u|^{j+1}dx &\leq \frac{l}{l+j+1}\delta_3^{(\frac{l+j+1}{l})}\|v\|_{l+j+1}^{l+j+1}+\frac{j+1}{l+j+1}\delta_3^{-(\frac{l+j+1}{l})}\|u\|_{l+j+1}^{l+j+1},\\
\int_\Omega|u|^\varrho|v|^{s+1}dx &\leq \frac{\varrho}{\varrho+s+1}\delta_4^{(\frac{\varrho+s+1}{\varrho})}\|u\|_{\varrho+s+1}^{\varrho+s+1}+\frac{s+1}{\varrho+s+1}\delta_4^{-(\frac{\varrho+s+1}{\varrho})}\|v\|_{\varrho+s+1}^{\varrho+s+1}.
\end{aligned} \tag{3.18}$$

Hence

$$\begin{aligned}
\int_\Omega(|u|^k+|v|^l)|u|^{j+1}dx \leq\ & \|u\|_{k+j+1}^{k+j+1}+\frac{l}{l+j+1}\delta_3^{(\frac{l+j+1}{l})}\|v\|_{l+j+1}^{l+j+1}\\
&+\frac{(j+1)}{l+j+1}\delta_3^{-(\frac{l+j+1}{l})}\|u\|_{l+j+1}^{l+j+1},
\end{aligned}$$

$$\int_\Omega (|v|^\theta + |u|^\varrho)|v|^{s+1}dx \leq \|v\|_{\theta+s+1}^{\theta+s+1} + \frac{\varrho}{\varrho+s+1}\delta_4^{(\frac{\varrho+s+1}{\varrho})}\|u\|_{\varrho+s+1}^{\varrho+s+1} + \frac{(s+1)}{\varrho+s+1}\delta_4^{-(\frac{\varrho+s+1}{\varrho})}\|v\|_{\varrho+s+1}^{\varrho+s+1}. \tag{3.19}$$

By using (3.6) and (3.2), since $2(p+2) > k+j+1$, we have from the embedding $L^{2(p+2)}(\Omega) \hookrightarrow L^{k+j+1}(\Omega)$,

$$\|u\|_{k+j+1}^{k+j+1} \leq C\|u\|_{2(p+2)}^{k+j+1} \leq \left(\|u\|_{2(p+2)}^{2(p+2)}\right)^{\frac{k+j+1}{2(p+2)}}, \tag{3.20}$$

since $0 < \frac{k+j+1}{2(p+2)} < 1$, to find by using the algebraic inequality

$$B^\varsigma \leq (B+1) \leq (1+\frac{1}{b})(B+b), \quad \forall B > 0, \quad 0 < \varsigma < 1, \quad b > 0, \tag{3.21}$$

$$\left(\|u\|_{2(p+2)}^{2(p+2)}\right)^{\frac{k+j+1}{2(p+2)}} \leq K(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(0)), \tag{3.22}$$

where $K = 1 + \frac{1}{\mathbb{H}(0)}$.
Similarly, by (3.2) we get

$$\begin{aligned}
\|v\|_{k+j+1}^{k+j+1} &\leq \left(\|v\|_{2(p+2)}^{2(p+2)}\right)^{\frac{k+j+1}{2(p+2)}} \leq K(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(0)), \\
\|v\|_{\theta+s+1}^{\theta+s+1} &\leq \left(\|v\|_{2(p+2)}^{2(p+2)}\right)^{\frac{k+j+1}{2(p+2)}} \leq K(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(0)), \\
\|u\|_{\theta+s+1}^{\theta+s+1} &\leq \left(\|u\|_{2(p+2)}^{2(p+2)}\right)^{\frac{k+j+1}{2(p+2)}} \leq K(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(0)).
\end{aligned} \tag{3.23}$$

Hence, by fixed $\delta_3, \delta_4 > 0$, and (3.19), gives

$$\begin{aligned}
&\int_\Omega (|u|^k + |v|^l)|u|^{j+1}dx \\
&\leq M_1\Big(1 + \frac{l\delta_3^{(\frac{l+j+1}{l})}}{l+j+1} + \frac{(j+1)\delta_3^{-(\frac{l+j+1}{l})}}{l+j+1}\Big)\Big(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)\Big), \\
&\int_\Omega (|v|^\theta + |u|^\varrho)|v|^{s+1}dx \\
&\leq M_2\Big(1 + \frac{\varrho\delta_4^{(\frac{\varrho+s+1}{\varrho})}}{\varrho+s+1} + \frac{(s+1)\delta_4^{-(\frac{\varrho+s+1}{\varrho})}}{\varrho+s+1}\Big)\Big(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)\Big),
\end{aligned} \tag{3.24}$$

for some constants $M_1, M_2 > 0$.

Now, for $0 < a < 1$, from (3.1)

$$J_5 = \varepsilon[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}] = \varepsilon a\Big[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\Big]$$

$$
\begin{aligned}
&+\frac{2\varepsilon(p+2)(1-a)}{\eta+2}(\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2})\\
&+\varepsilon(p+2)(1-a)(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2)\\
&+\varepsilon(p+2)(1-a)(1-\int_0^t g(s)ds)\|\nabla u\|_2^2\\
&+\varepsilon(p+2)(1-a)(1-\int_0^t h(s)ds)\|\nabla v\|_2^2\\
&-\varepsilon(p+2)(1-a)((h_1 o\nabla u)+(h_2 o\nabla v))\\
&+\varepsilon 2(p+2)(1-a)\mathbb{H}(t). \qquad (3.25)
\end{aligned}
$$

Substituting in (3.15), and by using (2.5), we get

$$
\begin{aligned}
\mathcal{K}'(t) \;\geq\; & \Big\{1-\varepsilon\kappa\Big\}\mathbb{H}'(t)+\varepsilon\Big\{(p+2)(1-a)+1\Big\}(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2)\\
&+\varepsilon\Big\{\frac{2\varepsilon(p+2)(1-a)}{\eta+2}+\frac{1}{\eta+1}\Big\}(\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2})\\
&+\varepsilon\Big\{(p+2)(1-a)\Big(1-\int_0^t h_1(s)ds\Big)-\Big(1-\frac{1}{2}\int_0^t h_1(s)ds\Big)\Big\}\|\nabla u\|_2^2\\
&+\varepsilon\Big\{(p+2)(1-a)\Big(1-\int_0^t h_2(s)ds\Big)-\Big(1-\frac{1}{2}\int_0^t h_2(s)ds\Big)\Big\}\|\nabla v\|_2^2\\
&+\varepsilon\Big\{(p+2)(1-a)-\frac{1}{2}\Big\}(h_1 o\nabla u+h_2 o\nabla v)\\
&+\varepsilon\Big\{c_0 a-\Big(M_3C_1(\kappa)+M_4C_2(\kappa)\Big)\Big\}\Big(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\Big)\\
&+\varepsilon\Big\{(2(p+2)(1-a)-\Big(M_3C_1(\kappa)+M_4C_2(\kappa)\Big)\Big\}\mathbb{H}(t), \qquad (3.26)
\end{aligned}
$$

where

$$
\begin{aligned}
M_3 &:= M_1\Big(1+\frac{l\delta_3^{(\frac{l+j+1}{l})}}{l+j+1}+\frac{(j+1)\delta_3^{-(\frac{l+j+1}{l})}}{l+j+1}\Big)>0\\
M_4 &:= M_2\Big(1+\frac{\varrho\delta_4^{(\frac{\varrho+s+1}{\varrho})}}{\varrho+s+1}+\frac{(s+1)\delta_4^{-(\frac{\varrho+s+1}{\varrho})}}{\varrho+s+1}\Big)>0.
\end{aligned}
$$

In this stage, we take $a > 0$ small enough so that

$$\lambda_1 = (p+2)(1-a)-1>0,$$

and we assume

$$\max\Big\{\int_0^\infty h_1(s)ds,\int_0^\infty h_2(s)ds\Big\}<\frac{(p+2)(1-a)-1}{((p+2)(1-a)-\frac{1}{2})}=\frac{2\lambda_1}{2\lambda_1+1}, \qquad (3.27)$$

gives

$$\lambda_2 \;=\; \Big\{\Big((p+2)(1-a)-1\Big)-\int_0^t h_1(s)ds\Big((p+2)(1-a)-\frac{1}{2}\Big)\Big\}>0,$$

$$\lambda_3 = \left\{\Big((p+2)(1-a)-1\Big)-\int_0^t h_2(s)ds\Big((p+2)(1-a)-\frac{1}{2}\Big)\right\}>0,$$

then we choose κ so large that

$$\begin{aligned}\lambda_4 &= ac_0-\Big(M_3C_1(\kappa)+M_4C_2(\kappa)\Big)>0,\\ \lambda_5 &= 2(p+2)(1-a)-\Big(M_3C_1(\kappa)+M_4C_2(\kappa)\Big)>0.\end{aligned}$$

Finally, we fixed κ, a, and we appoint ε small enough so that

$$\lambda_6 = 1-\varepsilon\kappa>0,$$

and, from (3.7)

$$\begin{aligned}\mathcal{K}(t) &\le \frac{1}{2(p+2)}[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}]\\ &\le \frac{c_1}{2(p+2)}[\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}].\end{aligned} \tag{3.28}$$

Thus, for some $\beta>0$, estimate (3.26) becomes

$$\begin{aligned}\mathcal{K}'(t) \ge\ & \beta\Big\{\mathbb{H}(t)+\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2}+\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2+\|\nabla u\|_2^2+\|\nabla v\|_2^2\\ &+(h_1o\nabla u)+(h_2o\nabla v)+\|u\|_{2(p+2)}^{2(p+2)}+\|u\|_{2(p+2)}^{2(p+2)}\Big\}.\end{aligned} \tag{3.29}$$

By (2.5), for some $\beta_1>0$, we obtain

$$\begin{aligned}\mathcal{K}'(t) \ge\ & \beta_1\Big\{\mathbb{H}(t)+\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2}+\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2+\|\nabla u\|_2^2+\|\nabla v\|_2^2\\ &+(h_2o\nabla u)+(h_2o\nabla v)+\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}\Big\}.\end{aligned} \tag{3.30}$$

and

$$\mathcal{K}(t)\ge\mathcal{K}(0)>0,\quad t>0. \tag{3.31}$$

Next, using Holder's and Young's inequalities, we have

$$\begin{aligned}\Big|\int_\Omega(u|u_t|^\eta u_t+v|v_t|^\eta v_t)dx\Big| \le\ & C\Big[\|u\|_{2(p+2)}^{\theta}+\|u_t\|_{\eta+2}^{\mu}\\ &+\|v\|_{2(p+2)}^{\theta}+\|v_t\|_{\eta+2}^{\mu}\Big],\end{aligned} \tag{3.32}$$

where $\frac{1}{\mu}+\frac{1}{\theta}=1$.

We take $\mu=(\eta+2)$, to get

$$\theta=\frac{(\eta+2)}{(\eta+1)}\le 2(p+2).$$

Subsequently, by using (3.2) and (3.21) , we obtain

$$\begin{aligned} \|u\|_{2(p+2)}^{\frac{\eta+2}{(\eta+1)}} &\leq K(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)) \\ \|v\|_{2(p+2)}^{\frac{\eta+2}{(\eta+1)}} &\leq K(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \quad \forall t \geq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left| \int_\Omega (u|u_t|^\eta u_t + v|v_t|^\eta v_t)dx \right| \\ &\leq c_{13}\Big\{\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \mathbb{H}(t)\Big\}. \end{aligned} \tag{3.33}$$

Hence,

$$\begin{aligned} \mathcal{K}(t) &= \Big(\mathbb{H}(t) + \frac{\varepsilon}{\eta+1}\int_\Omega (u|u_t|^\eta u_t + v|v_t|^\eta v_t)dx + \varepsilon\int_\Omega (\nabla u_t \nabla u + \nabla v_t \nabla v)dx\Big) \\ &\leq c\Big(\mathbb{H}(t) + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 \\ &\quad +(h_1 o \nabla u) + (h_2 o \nabla v) + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\Big). \end{aligned} \tag{3.34}$$

From (3.29) and (3.34), gives

$$\mathcal{K}'(t) \geq \lambda \mathcal{K}(t), \tag{3.35}$$

where $\lambda > 0$, this depends only on β and c.
by integration of (3.35), we obtain

$$\mathcal{K}(t) \geq \mathcal{K}(0)e^{(\lambda t)}, \forall t > 0. \tag{3.36}$$

From (3.7) and (3.28), we have

$$\mathcal{K}(t) \leq \frac{c_1}{2(p+2)}[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}]. \tag{3.37}$$

By (3.36) and (3.37), we have

$$\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \geq Ce^{(\lambda t)}, \forall t > 0.$$

Hence, we conclude that the solution in the $L^{2(p+2)}$– norm is growths exponentially. This completes the proof.

□

4. Conclusions

The purpose of this work was to study when the initial data are large enough, the energy grows exponentially with $L^{2(p+2)}$-norm of solutions for a coupled nonlinear viscoelastic Kirchhoff equations with degenerate damping, dispersion and source terms. This type of problem is frequently found in some mathematical models in applied sciences. Especially in the theory of viscoelasticity. What interests us in this current work is the combination of these terms of damping (degenerate damping, dispersion and source terms), which dictates the emergence of these terms in the problem. In the next work, we will try to using the same method with same problem. But in added of other damping (Balakrishnan-Taylor damping and Logarithmic terms).

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Conflict of interest

This work does not have any conflicts of interest.

References

1. E. Piskin, S. Boulaaras, H. Kandemir, B. B. Cherif, M. Biomy, On a couple of nonlocal singular viscoelastic equations with damping and general source terms: blow-up of solutions, *J. Funct. Space.*, **2021** (2021), 9914386. doi: 10.1155/2021/9914386.
2. D. R. Bland, *The theory of linear viscoelasticity*, Mineola: Courier Dover Publications, 2016.
3. S. Boulaaras, A. Choucha, D. Ouchenane, B. Cherif, Blow up of solutions of two singular nonlinear viscoelastic equations with general source and localized frictional damping terms, *Adv. Differ. Equ.*, **2020** (2020), 310. doi: 10.1186/s13662-020-02772-0.
4. S. Boulaaras, A. Draifia, Kh. Zennir, General decay of nonlinear viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping and logarithmic nonlinearity, *Math. Method. Appl. Sci.*, **42** (2019), 4795–4814. doi: 10.1002/mma.5693.
5. M. M. Cavalcanti, D. Cavalcanti, J. Ferreira, Existence and uniform decay for nonlinear viscoelastic equation with strong damping, *Math. Method. Appl. Sci.*, **24** (2001), 1043–1053. doi: 10.1002/mma.250.
6. A. Choucha, D. Ouchenane, Kh. Zennir, B. Feng, Global well-posedness and exponential stability results of a class of Bresse-Timoshenko-type systems with distributed delay term, *Math. Method. Appl. Sci.*, 2020, 1–26. doi: 10.1002/mma.6437.
7. A. Choucha, D. Ouchenane, S. Boulaaras, Well posedness and stability result for a thermoelastic laminated Timoshenko beam with distributed delay term, *Math. Method. Appl. Sci.*, **43** (2020), 9983–10004. doi: 10.1002/mma.6673.
8. A. Choucha, D. Ouchenane, S. Boulaaras, Blow-up of a nonlinear viscoelastic wave equation with distributed delay combined with strong damping and source terms, *J. Nonlinear Funct. Anal.*, **2020** (2020), 31. doi: 10.23952/jnfa.2020.31.
9. A. Choucha, S. Boulaaras, D. Ouchenane, S. Beloul, General decay of nonlinear viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping, logarithmic nonlinearity and distributed delay terms, *Math. Method. Appl. Sci.*, **44** (2021), 5436–5457. doi: 10.1002/mma.7121.
10. A. Choucha, S. M. Boulaaras, D. Ouchenane, B. B. Cherif, M. Abdalla, Exponential stability of swelling porous elastic with a viscoelastic damping and distributed delay term, *J. Funct. Space.*, **2021** (2021), 5581634. doi: 10.1155/2021/5581634.
11. B. D. Coleman, W. Noll, Foundations of linear viscoelasticity, *Rev. Mod. Phys.*, **33** (1961), 239. doi: 10.1103/RevModPhys.33.239.
12. V. Georgiev, G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source term, *J. Differ. Equations*, **109** (1994), 295–308. doi: 10.1006/jdeq.1994.1051.

13. K. Agre, M. A. Rammaha, Systems of nonlinear wave equations with damping and source terms, *Differ. Integral Equ.*, **19** (2006), 1235–1270.
14. J. M. Ball, Remarks on blow-up and nonexistence theorems for nonlinear evolutions equation, *Quart. J. Math.*, **28** (1977), 473–486. doi: 10.1093/qmath/28.4.473.
15. L. He, On decay and blow-up of solutions for a system of equations, *Appl. Anal.*, **100** (2019), 2449–2477. doi: 10.1080/00036811.2019.1689562.
16. G. Kirchhoff, *Vorlesungen uber Mechanik*, Leipzig: Tauber, 1883.
17. S. Boulaaras, A. Choucha, P. Agarwal, M. Abdalla, S. A. Idris, Blow-up of solutions for a quasilinear system with degenerate damping terms, *Adv. Differ. Equ.*, **2021** (2021), 446. doi: 0.1186/s13662-021-03609-0.
18. W. Liu, General decay and blow-up of solution for a quasilinear viscoelastic problem with nonlinear source, *Nonlinear Anal. Theor.*, **73** (2010), 1890–1904. doi: 10.1016/j.na.2010.05.023.
19. F. Mesloub, S. Boulaaras, General decay for a viscoelastic problem with not necessarily decreasing kernel, *J. Appl. Math. Comput.*, **58** (2018), 647–665. doi: 10.1007/s12190-017-1161-9.
20. N. Mezouar, S. Boulaaras, Global existence and decay of solutions for a class of viscoelastic Kirchhoff equation, *Bull. Malays. Math. Sci. Soc.*, **43** (2020), 725–755. doi: 10.1007/s40840-018-00708-2.
21. A. Choucha, S. Boulaaras, Asymptotic behavior for a viscoelastic Kirchhoff equation with distributed delay and Balakrishnan–Taylor damping, *Bound. Value Probl.*, **2021** (2021), 77.
22. D. Ouchenane, S. Boulaaras, F. Mesloub, General decay for a class of viscoelastic problem with not necessarily decreasing kernel, *Appl. Anal.*, **98** (2019), 1677–1693. doi: 10.1080/00036811.2018.1437421.
23. E. Piskin, S. Boulaaras, N. Irkil, Qualitative analysis of solutions for the p-Laplacian hyperbolic equation with logarithmic nonlinearity, *Math. Method. Appl. Sci.*, **44** (2021), 4654–4672. doi: 10.1002/mma.7058.
24. H. T. Song, D. S. Xue, Blow up in a nonlinear viscoelastic wave equation with strong damping, *Nonlinear Anal. Theor.*, **109** (2014), 245–251. doi: 10.1016/j.na.2014.06.012.
25. H. T. Song, C. K. Zhong, Blow-up of solutions of a nonlinear viscoelastic wave equation, *Nonlinear Anal. Real*, **11** (2010), 3877–3883. doi: 10.1016/j.nonrwa.2010.02.015.
26. H. Yang, S. Fang, F. Liang, M. Li, A general stability result for second order stochastic quasi-linear evolution equations with memory, *Bound. Value Probl.*, **2020** (2020), 62. doi: 10.1186/s13661-020-01359-8.
27. S. T. Wu, General decay of energy for a viscoelastic equation with damping and source terms, *Taiwanese J. Math.*, **16** (2012), 113–128. doi: 10.11650/twjm/1500406531.
28. K. Zennir, Exponential growth of solutions with L_p-norm of a nonlinear viscoelastic hyperbolic equation, *J. Nonlinear Sci. Appl.*, **6** (2013), 252–262. doi: 10.22436/jnsa.006.04.03.