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Research article

Semilinear viscous Moore-Gibson-Thompson equation with the derivative-type nonlinearity: Global existence versus blow-up

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Abstract: In this paper, we study global existence and blow-up of solutions to the viscous Moore-Gibson-Thompson (MGT) equation with the nonlinearity of derivative-type $|u_t|^p$. We demonstrate global existence of small data solutions if $p > 1 + 4/n$ $(n \leq 6)$ or $p \geq 2 - 2/n$ $(n \geq 7)$, and blow-up of nontrivial weak solutions if $1 < p \leq 1 + 1/n$. Deeply, we provide estimates of solutions to the nonlinear problem. These results complete the recent works for semilinear MGT equations by [4].

Keywords: Moore-Gibson-Thompson equation; derivative-type nonlinearity; global existence of small data solution; decay estimate; blow-up
Mathematics Subject Classification: 35A01, 35B40, 35B44, 35G25

1. Introduction

In this work, we are going to investigate the following Cauchy problem for viscous Moore-Gibson-Thompson (MGT) equation with the derivative-type nonlinearity on the right-hand side:

$$\begin{cases} \tau u_{ttt} + u_{tt} - \Delta u - (\delta + \tau)\Delta u_t = |u_t|^p, & x \in R^n,\ t > 0, \\ (u, u_t, u_{tt})(0, x) = (u_0, u_1, u_2)(x), & x \in R^n, \end{cases} \tag{1.1}$$

where $\tau > 0$ denotes the thermal relaxation from the Cattaneo-Maxwell law of heat conduction, $\delta > 0$ is the diffusivity of sound which consists of viscous coefficients in Navier-Stokes equation (this is the reason for the terminology "viscous"), and the power $p > 1$. The present paper is a continuation of the papers Chen-Ikehata [4] and Chen-Palmieri [6]. Our aim in this work is to derive some conditions with respect to p for global existence and blow-up of solutions to the nonlinear Cauchy problem (1.1).

It is well-known that the MGT equation is actually the linearized model for the wave propagation in viscous thermally relaxing fluids. Physically, the MGT equation is one of the most important models in the field of acoustic waves. There are numerous applications, for examples, high-intensity ultrasonic waves have been applied in medical imaging and therapy, ultrasound cleaning and welding. We refer the readers for their applications in the literatures Abramov [1], Dreyer-Krauss-Bauer-Riedlinger [9], Kaltenbacher-Landes-Hoffelner-Simkovics [13] and references therein.

We begin with presenting some backgrounds of the MGT equation. As we all know, the Kuznetsov equation is the fundamental model in nonlinear acoustics, where the Fourier law of heat conduction is used in the modeling. In order to eliminate the paradox for the infinite signal speed paradox in wave's propagation, Jordan [11] introduced the Cattaneo-Maxwell law of heat conduction instead of the Fourier law of heat conduction in acoustic wave's modeling. Then, it leads to the mathematical model of the MGT equation

$$\tau u_{ttt} + u_{tt} - \Delta u - (\delta + \tau)\Delta u_t = 0,$$

where $u = u(t, x)$ stands for the acoustic velocity potential due to the application of irrotational flow in the modeling. The parameters τ and δ represent the thermal relaxation and the diffusivity of sound, respectively.

For the research of the MGT equation, Kaltenbacher-Lasiecka-Marchand [12], Marchand-McDevitt-Triggiani [14] and Conejero-Lizama-Rodenas [8] gave an important classification of the MGT equation with respect to the value of δ: the viscous case $\delta > 0$, the inviscid case $\delta = 0$ and the chaotic case $\delta < 0$. Later, there are a lot of works of the MGT equation in different frameworks, e.g. inverse problems, controllability, long-time behaviors with attractor, initial (boundary) value problems, hereditary fluids, etc. Among them, we will introduce precisely the Cauchy problem since it relates to our aim model (1.1). The first work for the Cauchy problem of the viscous MGT equation has been done by Pellicer-Said-Houari [16], where they found decay properties by using energy methods in the Fourier space and asymptotic expansions of characteristic roots. Later, sharp decay estimates, and asymptotic profiles for large-time ($t \to \infty$) or the small thermal relaxation ($\tau \to 0$) even a singular layer were developed by Chen-Ikehata [4] and Chen [2], in which they used WKB analysis associated with the Fourier analysis delicately. Recently, inviscid limits for $\delta \to 0$ for the Cauchy problem has been investigated by Chen [3] with the aid of suitable energy methods.

Let us turn to the semilinear MGT equations with power nonlinearities $|u|^p$ or $|u_t|^p$. The first study of this field is the initial value problem with the power-type nonlinearity for

$$\tau u_{ttt} + u_{tt} - \Delta u - (\delta + \tau)\Delta u_t = |u|^p.$$

For the inviscid case $\delta = 0$, Chen-Palmieri [6] proved blow-up of energy solutions if $1 < p \leq p_0(n)$, where $p_0(n)$ denotes the Strauss exponent

$$p_0(1) := \infty \text{ and } p_0(n) := \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n - 1)} \text{ for } n \geq 2.$$

It means that the inviscid model is wave-like. For the inviscid case $\delta > 0$, Chen-Ikehata [4] derived some conditions for global existence of small data lower-order Sobolev solution, and blow-up of weak solutions. Additionally, they also got some decay estimates of solution to the nonlinear problem, which indicates the viscous model is diffusion wave-like. Simultaneously, the initial value problem with the derivative-type nonlinearity for

$$\tau u_{ttt} + u_{tt} - \Delta u - (\delta + \tau)\Delta u_t = |u_t|^p.$$

catches some attentions. Since the time-derivative u_t appears in the nonlinear term, some new difficulties come. The new paper Chen-Fino [7] studied blow-up result for the inviscid case $\delta = 0$, and it shows that every nontrivial energy solution blows up providing that $1 < p \leq p_1(n)$, where $p_1(n)$ denotes the Glassey exponent

$$p_1(1) := \infty \ \text{ and } \ p_1(n) := \frac{n+1}{n-1} \ \text{ for } \ n \geq 2.$$

Nevertheless, from the authors' knowledge, concerning the MGT equation with nonlinearity $|u_t|^p$, the global existence and the blow-up phenomenon for the viscous case $\delta > 0$ are still open. It motivates us to study the Cauchy problem for nonlinear MGT Eq (1.1). The inviscid case and the viscous case in the Cauchy problem (1.1) are quite different. For our consideration $\delta > 0$, the damping term $-\delta\Delta u_t$ is effective to bring some decay estimates of solutions. For this reason, we can expect the range of p for global existence to be larger. However, the admissible range is not clear yet. What's more, because of $|u_t|^p$ on the right-hand side, higher regularity of solution and initial data are expected. Our approaches are motived by the study of semilinear damped waves, e.g., Chen-Palmieri [5] for global existence and blow-up ideas.

To finish the introduction, we next will state two results that are global existence of small data solutions and blow-up of solutions, individually. Their proofs will be given in the rest sections.

Theorem 1.1. *Let $\tau > 0$ and $\delta > 0$. Let us assume initial data that $u_\ell \in H^{2-\ell} \cap L^1$ such that their values are small in the sense of $\|u_\ell\|_{H^{2-\ell}\cap L^1} \leq \varepsilon$ for $\ell = 0, 1, 2$ with a small constant $\varepsilon > 0$. Moreover, we assume $p \geq 2$, and $p \leq n/(n-2)$ for $n \geq 3$. Let us take*

$$\begin{aligned} &p > 1 + \frac{4}{n} \ \textit{ for } \ 2 \leq n \leq 6, \\ &p \geq 2 - \frac{2}{n} \ \textit{ for } \ n \geq 7. \end{aligned} \tag{1.2}$$

Then, there is a uniquely global energy solution

$$u \in C([0,\infty), H^2) \cap C^1([0,\infty), H^1) \cap C^2([0,\infty), L^2)$$

to the nonlinear viscous MGT Eq (1.1)*. The solution and its derivatives satisfy*

$$\|u(t)\|_{L^2} \leq \begin{cases} C\varepsilon(\ln(e+t))^{\frac{1}{2}} & \textit{if } \ n = 2, \\ C\varepsilon(1+t)^{\frac{1}{2}-\frac{n}{4}} & \textit{if } \ n \geq 3, \end{cases}$$

and

$$\|\partial_t^\ell u(t)\|_{L^2} \leq C\varepsilon(1+t)^{\frac{1}{2}-\frac{\ell}{2}-\frac{n}{4}} \ \textit{ if } \ \ell = 1, 2,$$

$$\| \, |D|^{2-\ell}\partial_t^\ell u(t)\|_{L^2} \leq C\varepsilon(1+t)^{-\frac{1}{2}-\frac{n}{4}} \quad \text{if } \ell = 0,1,2.$$

Here the pseudo-differential operator $|D|^{2-\ell}$ is defined according to its symbol $|\xi|^{2-\ell}$.

Remark 1.1. *Differently from Theorem 5.1 in Chen-Ikehata [4], our result allows to assume nontrivial data u_0 and u_1. Moreover, our solution's space is more precise since we also control first-order as well as second-order time-derivative of solution.*

Remark 1.2. *More importantly, we derive a new global existence condition* (1.2)*, which is larger than those in Chen-Ikehata [4] (we refer the readers to Example 5.1 in their paper). It mainly indicates the influence of derivative-type nonlinear term $|u_t|^p$. Benefit from the viscous effect, we can prove some global existence results.*

Theorem 1.2. *Let $\tau > 0$ and $\delta > 0$. Let us assume that $u_0, u_1, u_2 \in L^1$ so that*

$$\int_{R^n} u_0(x)dx > 0 \ \text{ as well as } \ \int_{R^n} (u_1(x) + \tau u_2(x))dx > 0.$$

Then, the global nontrivial weak solution to the nonlinear viscous MGT Eq (1.1) *with nonlinearity of derivative-type does not exist provided*

$$1 < p \leq 1 + \frac{1}{n} \ \text{ for all } \ n \geq 1. \tag{1.3}$$

Remark 1.3. *Differently from the result in Theorem 6.1 of Chen-Ikehata [4], we do not need to assume vanishing first and second data. It provides more opportunities for the consideration of semilinear MGT equations.*

Remark 1.4. *The blow-up result for the viscous semilinear MGT equation carrying $|u|^p$ in Chen-Ikehata [4] holds if $1 < p < \infty$ for $n = 1$ and $1 < p \leq (n+1)/(n-1)$ for $n \geq 2$. The blow-up range in the consideration of $|u_t|^p$ that is* (1.3) *becomes smaller since some "dissipative effects" from the nonlinearity $|u_t|^p$.*

Remark 1.5. *Let us consider the semilinear MGT equation with derivative-type nonlinear term $|u_t|^p$. The blow-up range of p in the inviscid case is $1 < p \leq p_1(n)$ in Chen-Palmieri [**?**]. However, due to the damping influence of viscous term $-\delta\Delta u_t$, the blow-up range is weakened as* (1.3).

2. Global existence of solutions: Proof of Theorem 1.1

2.1. Preliminaries and main tool

Firstly, let us recall some estimates of solution and its derivatives for the linear problem that is the MGT equation

$$\begin{cases} \tau v_{ttt} + v_{tt} - \Delta v - (\delta+\tau)\Delta v_t = 0, & x \in R^n, \ t > 0, \\ (v, v_t, v_{tt})(0,x) = (v_0, v_1, v_2)(x), & x \in R^n. \end{cases} \tag{2.1}$$

Recalling the operator $|D|^s$, we get $f \in \dot{H}^s$ if and only if $|D|^s f \in L^2$. According to Propositions 2.1 and 2.2 in Chen [2], or Theorems 2.1 and 2.2 in Chen-Ikehata [4], we recall that

$$\|v(t)\|_{L^2} \leq C\mathcal{D}_n(t)\|(v_0, v_1, v_2)\|_{(L^2\cap L^1)^3},$$

$$\|\,|D|^2 v(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}-\frac{n}{4}}\|(v_0,v_1,v_2)\|_{(\dot{H}^2\cap L^1)\times(\dot{H}^1\cap L^1)\times(L^2\cap L^1)},$$

with the time-dependent coefficients

$$\mathcal{D}_n(t) := \begin{cases} (\ln(e+t))^{\frac{1}{2}} & \text{if } n=2, \\ (1+t)^{\frac{1}{2}-\frac{n}{4}} & \text{if } n\geq 3. \end{cases}$$

Moreover, the following estimates hold for $\ell = 1,2$:

$$\begin{aligned}
\|\partial_t^\ell v(t)\|_{L^2} &\leq C(1+t)^{\frac{1}{2}-\frac{\ell}{2}}\|(v_0,v_1,v_2)\|_{(\dot{H}^{\ell-1})^2\times L^2},\\
\|\partial_t^\ell v(t)\|_{L^2} &\leq C(1+t)^{\frac{1}{2}-\frac{\ell}{2}-\frac{n}{4}}\|(v_0,v_1,v_2)\|_{(\dot{H}^{\ell-1}\cap L^1)^2\times(L^2\cap L^1)},\\
\|\,|D|^{2-\ell}\partial_t^\ell v(t)\|_{L^2} &\leq C(1+t)^{-\frac{1}{2}}\|(v_0,v_1,v_2)\|_{(\dot{H}^1)^2\times L^2},\\
\|\,|D|^{2-\ell}\partial_t^\ell v(t)\|_{L^2} &\leq C(1+t)^{-\frac{1}{2}-\frac{n}{4}}\|(v_0,v_1,v_2)\|_{(\dot{H}^1\cap L^1)^2\times(L^2\cap L^1)}.
\end{aligned}$$

Our main tool to treat nonlinear term is based on the Gagliardo-Nirenberg inequality. Let us state a more general form (fractional case).

Proposition 2.1. *(Fractional Gagliardo-Nirenberg inequality, [10]) Let $p, p_0, p_1 \in (1,\infty)$ and $\kappa \in [0,s)$ with $s\in(0,\infty)$. Then, concerning $f \in L^{p_0}\cap \dot{H}^s_{p_1}$ the next estimate holds:*

$$\|f\|_{\dot{H}^\kappa_p} \leq C\|f\|^{1-\beta}_{L^{p_0}}\|f\|^{\beta}_{\dot{H}^s_{p_1}},$$

where

$$\beta = \left(\frac{1}{p_0}-\frac{1}{p}+\frac{\kappa}{n}\right)\Big/\left(\frac{1}{p_0}-\frac{1}{p_1}+\frac{s}{n}\right) \text{ and } \beta\in\left[\frac{\kappa}{s},1\right].$$

2.2. *Treatment for the nonlinear terms*

To begin with, let us construct an evolution solution space as

$$\mathcal{X}(T) := C([0,T],H^2)\cap C^1([0,T],H^1)\cap C^2([0,T],L^2),$$

and define the norm of it by

$$\begin{aligned}
\|u\|_{\mathcal{X}(T)} := \sup_{0\leq t\leq T}\Bigg((\mathcal{D}_n(t))^{-1}\|u(t)\|_{L^2} &+ \sum_{\ell=1,2}(1+t)^{\frac{\ell}{2}+\frac{n}{4}-\frac{1}{2}}\|\partial_t^\ell u(t)\|_{L^2}\\
&+\sum_{\ell=0,1,2}(1+t)^{\frac{1}{2}+\frac{n}{4}}\|\,|D|^{2-\ell}\partial_t^\ell u(t)\|_{L^2}\Bigg).
\end{aligned}$$

Next, we will control the nonlinear part $\|\,|u_t(t)|^p\|_{L^2}$ and $\|\,|u_t(t)|^p\|_{L^1}$ since the estimates of the linear part. Let us apply Proposition 2.1 to get

$$\begin{aligned}
\|\,|u_t(t)|^p\|_{L^1} &\leq C\|u_t(t)\|^{(1-\beta_1)p}_{L^2}\|u_t(t)\|^{\beta_1 p}_{\dot{H}^1},\\
\|\,|u_t(t)|^p\|_{L^2} &\leq C\|u_t(t)\|^{(1-\beta_2)p}_{L^2}\|u_t(t)\|^{\beta_2 p}_{\dot{H}^1},
\end{aligned}$$

since $\| |u_t(t)|^p\|_{L^q} \le C\|u_t(t)\|^p_{L^{pq}}$, where we selected

$$\beta_1 = n\left(\frac{1}{2}-\frac{1}{p}\right) \in [0,1], \quad \beta_2 = \frac{n}{2}\left(1-\frac{1}{p}\right) \in [0,1],$$

so that $p \ge 2$, and $p \le n/(n-2)$ for $n \ge 3$. From the definition of the evolution space $\mathcal{X}(T)$, we have the estimates

$$\| |u_t(t)|^p\|_{L^1} \le C(1+t)^{\frac{n}{4}-\frac{np}{4}}\|u\|^p_{\mathcal{X}(T)},$$
$$\| |u_t(t)|^p\|_{L^2} \le C(1+t)^{-\frac{np}{4}}\|u\|^p_{\mathcal{X}(T)}.$$

2.3. Proof for global existence of small data solutions

Let us define a suitable operator $\mathcal{N}$ such that

$$\mathcal{N}: \; u(t) \in \mathcal{X}(T) \mapsto \mathcal{N}u(t) := u^{\text{lin}}(t) + u^{\text{non}}(t),$$

where $u^{\text{lin}}(t)$ is the solution to the linear MGT equation, and $u^{\text{non}}(t)$ denotes the nonlinear part that

$$u^{\text{non}}(t) := \int_0^t K_2(t-s,x) * |u_t(s)|^p ds,$$

from the Duhamel's principle with the kernel function $K_2(t,x)$ for the third data of the viscous MGT equation. Obviously,

$$\|u^{\text{lin}}\|_{\mathcal{X}(T)} \le C\|(u_0,u_1,u_2)\|_{(H^2\cap L^1)\cap(H^1\cap L^1)\cap(L^2\cap L^1)} \le C\varepsilon$$

because the definition of $\mathcal{X}(T)$ comes from the estimates for the linear problem.

We now control $u^{\text{non}}(t)$ in $\mathcal{X}(T)$ by several parts:

- For the estimate in the L^2 norm, we arrive at

$$\begin{aligned}\|u^{\text{non}}(t)\|_{L^2} &\le C\int_0^t \mathcal{D}_n(t-s)\| |u_t(s)|^p\|_{L^2\cap L^1} ds\\ &\le C\mathcal{D}_n(t)\|u\|^p_{\mathcal{X}(T)}\int_0^{t/2}(1+s)^{\frac{n}{4}-\frac{np}{4}}ds + C(1+t)^{\frac{n}{4}-\frac{np}{4}}\|u\|^p_{\mathcal{X}(T)}\int_{t/2}^t \mathcal{D}_n(t-s)ds\\ &\le C\mathcal{D}_n(t)\|u\|^p_{\mathcal{X}(T)},\end{aligned}$$

  since the assumption on p that

$$p > 1+\frac{4}{n} \;\Rightarrow\; \frac{n}{4}-\frac{np}{4} < -1,$$

  and the fact that

$$\int_{t/2}^t (1+t-s)^{\frac{1}{2}-\frac{n}{4}}ds \le \begin{cases} C(1+t)^{\frac{3}{2}-\frac{n}{4}} & \text{if } n \le 5,\\ C\ln(e+t) & \text{if } n = 6,\\ C & \text{if } n \ge 7.\end{cases}$$

Precisely, we used

$$(1+t)^{\frac{n}{4}-\frac{np}{4}}\int_{t/2}^{t}\mathcal{D}_n(t-s)ds\leq C\mathcal{D}_n(t),$$

provided that

$$p>1+\frac{4}{n}\ \text{ for }\ n\leq 6,\quad p\geq 2-\frac{2}{n}\ \text{ for }\ n\geq 7.$$

Moreover, one can easily get

$$\begin{aligned}\|\partial_t^{\ell}u^{\text{non}}(t)\|_{L^2}&\leq C\int_0^{t/2}(1+t-s)^{\frac{1}{2}-\frac{\ell}{2}-\frac{n}{4}}\|\,|u_t(s)|^p\|_{L^2\cap L^1}ds+C\int_{t/2}^{t}(1+t-s)^{\frac{1}{2}-\frac{\ell}{2}}\|\,|u_t(s)|^p\|_{L^2}ds\\&\leq C(1+t)^{\frac{1}{2}-\frac{\ell}{2}-\frac{n}{4}}\|u\|^p_{\mathcal{X}(T)}+C(1+t)^{-\frac{np}{4}+\frac{3}{2}-\frac{\ell}{2}}\|u\|^p_{\mathcal{X}(T)}\\&\leq C(1+t)^{\frac{1}{2}-\frac{\ell}{2}-\frac{n}{4}}\|u\|^p_{\mathcal{X}(T)},\end{aligned}$$

for $\ell=1,2$, where we used $p>1+4/n$ again.

- For the estimate in the $\dot{H}^{2-\ell}$ norm for $\ell=0,1,2$, we actually can obtain

$$\begin{aligned}\|\partial_t^{\ell}u^{\text{non}}(t)\|_{\dot{H}^{2-\ell}}&\leq C\int_0^{t/2}(1+t-s)^{-\frac{1}{2}-\frac{n}{4}}\|\,|u_t(s)|^p\|_{L^2\cap L^1}ds+C\int_{t/2}^{t}(1+t-s)^{-\frac{1}{2}}\|\,|u_t(s)|^p\|_{L^2}ds\\&\leq C(1+t)^{-\frac{1}{2}-\frac{n}{4}}\|u\|^p_{\mathcal{X}(T)}+(1+t)^{-\frac{np}{4}+\frac{1}{2}}\|u\|^p_{\mathcal{X}(T)}\\&\leq C(1+t)^{-\frac{1}{2}-\frac{n}{4}}\|u\|^p_{\mathcal{X}(T)},\end{aligned}$$

where we used the assumption $p>1+4/n$.

- In conclusion, we claim that

$$\|u^{\text{non}}\|_{\mathcal{X}(T)}\leq C\|u\|^p_{\mathcal{X}(T)},$$

if the power p fulfills our assumption (1.2), $p\geq 2$, and $p\leq n/(n-2)$ for $n\geq 3$.

All in all, we assert

$$\|\mathcal{N}u\|_{\mathcal{X}(T)}\leq C\varepsilon+C\|u\|^p_{\mathcal{X}(T)}.\tag{2.2}$$

By contraction argument, for small $\varepsilon>0$, it holds that $\mathcal{N}u\in\mathcal{X}(T)$ and the solution u globally exists. One may see detail in Section 3 of Palmieri [15].

To show the uniqueness of solution, we take $u,\bar{u}\in\mathcal{X}(T)$ for the same initial data. Therefore, we get $\mathcal{N}u,\mathcal{N}\bar{u}\in\mathcal{X}(T)$ from the last step. It is clear that

$$\|\mathcal{N}u-\mathcal{N}\bar{u}\|_{\mathcal{X}(T)}=\left\|\int_0^t K_2(t-s,x)*(|u_t(s)|^p-|\bar{u}_t(s)|^p)\,ds\right\|_{\mathcal{X}(T)}.$$

Again, we can use the estimates stated in Section 2.1, and repeat all techniques in the last step to estimate

$$\|u_t(t)-\bar{u}_t(t)\|_{L^{kp}},\ \ \|u_t(t)\|_{L^{kp}},\ \ \|\bar{u}_t(t)\|_{L^{kp}},$$

for $k = 1, 2$. By following the same procedure of proving (2.2), it concludes

$$\|\mathcal{N}u - \mathcal{N}\bar{u}\|_{\mathcal{X}(T)} \leq C\|u - \bar{u}\|_{\mathcal{X}(T)} \left(\|u\|^{p-1}_{\mathcal{X}(T)} + \|\bar{u}\|^{p-1}_{\mathcal{X}(T)}\right),$$

provided that $p \geq 2$, and $p \leq n/(n-2)$ for $n \geq 3$, moreover the hypothesis (1.2) holds. Finally, by employing the well-known Banach's fixed point argument, we can get global existence of unique energy solution. Our proof is complete.

3. Blow-up of solutions: Proof of Theorem 1.2

Let us introduce a test function for spatial variables

$$\varphi(x) := \langle x\rangle^{-n-2} \text{ for all } n \geq 1,$$

where $\langle x\rangle := \sqrt{1+|x|^2}$ as the Japanese bracket. Moreover, we define another test function for time variable

$$\eta(t) := \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \text{decreasing} & \text{if } \frac{1}{2} \leq t \leq 1, \\ 0 & \text{if } t \geq 1. \end{cases}$$

Then, we take a crucial functional motived by [17] that

$$J_R := \int_0^\infty \int_{R^n} |u_t|^p \varphi_R(x)\eta_R(t)dxdt,$$

where $\varphi_R(x) := \varphi(R^{-1}K^{-1}x)$ for some $K \geqslant 1$ to be determined later, and $\eta_R(t) := \eta(R^{-1}t)$ with $R \gg 1$ as a large parameter. Hereafter, we denote $u = u(t, x)$ for briefness.

Let us assume, by contradiction, that the function u is a global weak solution for any $t \geqslant 0$. Therefore, it yields

$$\begin{aligned} J_R &= \int_0^\infty \int_{R^n} (\tau u_{ttt} + u_{tt} - \Delta u - (\delta+\tau)\Delta u_t)(\varphi_R(x)\eta_R(t))dxdt \\ &= \int_0^\infty \int_{R^n} u_t \left(\tau\varphi_R(x)\partial_t^2\eta_R(t) - \varphi_R(x)\partial_t\eta_R(t) - \Delta\varphi_R(x)\psi_R(t) - (\delta+\tau)\Delta\varphi_R(x)\eta_R(t)\right)dxdt \\ &\quad - \psi(0)\int_{R^n} u_0(x)\Delta\varphi_R(x)dx - \int_{R^n}(u_1(x) + \tau u_2(x))\varphi_R(x)dx, \end{aligned}$$

where $\psi(t)$ is the compactly supported primitive of $\eta(t)$ from $[t, \infty]$ that is

$$\psi(t) := \int_t^\infty \eta(\zeta)d\zeta.$$

By recalling our assumption on initial data that

$$\int_{R^n} u_0(x)dx > 0 \text{ and } \int_{R^n}(u_1(x) + \tau u_2(x))dx > 0,$$

we can get

$$J_R < \int_0^\infty \int_{R^n} |u_t| \Big| \tau R^{-2} \varphi_R(x) \eta_R''(t) - R^{-1} \varphi_R(x) \eta_R'(t) - R^{-2} K^{-2} \varphi_R''(x) \psi_R(t) - (\delta + \tau) R^{-2} K^{-2} \varphi_R''(x) \eta_R(t) \Big| \, \mathrm{d}x\mathrm{d}t. \tag{3.1}$$

We set $K = 1$ if $1 < p < 1 + 1/n$. Using Hölder's inequality, changes of variables $t \mapsto R^{-1}t$ and $x \mapsto R^{-1}x$ one may find

$$J_R < CR^{\frac{-p'+n+1}{p'}} J_R^{\frac{1}{p}} \leq \frac{1}{p} J_R + CR^{-p'+n+1},$$

where p' denotes the Hölder conjugate of p, namely,

$$J_R < CR^{-\frac{p}{p-1}+n+1}. \tag{3.2}$$

The condition on p such that $1 < p < 1 + 1/n$ implies negativity of the power for R in the estimate (3.2). Thus, by letting $R \gg 1$, the contradiction follows. For another, when $p = 1 + 1/n$, the functional J_R is uniformly bounded from the previous computations, which means

$$u_t \in L^p([0, \infty) \times R^n).$$

For this reason, we gain

$$\lim_{R \to \infty} \int_0^\infty \int_{R^n} u_t(t, x) \varphi_R(x) \partial_t \psi_R(t) dx dt = 0,$$

with the aid of $\psi_R(t) = 1$ in $[0, R/2]$. From (3.1) again, one immediately gets

$$0 \leq \lim_{R \to \infty} J_R < CK^{n-2p'}.$$

It leads to the contradiction for sufficiently large $K \gg 1$ since $n - 2p' = n - (n + 1) = -1 < 0$. We now finish the proof.

4. Conclusions

In this paper, we not only demonstrated global existence of small data solution to the semilinear MGT equation with derivative-type nonlinearity, but also derived blow-up of solution under some conditions for the exponent p. Although the critical exponent (i.e. the threshold for global existence and blow-up) is still unknown, we gave a possible range of critical exponent, namely, $1 + 1/n \leqslant p_{crit}(n) \leq 1 + 4/n$. So far the critical exponent of this model is still open.

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Conflict of interest

The authors declare that they have no competing interests.

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