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Research article

A class of hypersurfaces in $\mathbb{E}_s^{n+1}$ satisfying $\Delta\vec{H} = \lambda\vec{H}$

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Abstract: A nondegenerate hypersurface in a pseudo-Euclidean space $\mathbb{E}_s^{n+1}$ is called to have proper mean curvature vector if its mean curvature $\vec{H}$ satisfies $\Delta\vec{H} = \lambda\vec{H}$ for a constant λ. In 2013, Arvanitoyeorgos and Kaimakamis conjectured [1]: any hypersurface satisfying $\Delta\vec{H} = \lambda\vec{H}$ in pseudo-Euclidean space $\mathbb{E}_s^{n+1}$ has constant mean curvature. This paper will give further support evidences to this conjecture by proving that a linear Weingarten hypersurface M_r^n in $\mathbb{E}_s^{n+1}$ satisfying $\Delta\vec{H} = \lambda\vec{H}$ has constant mean curvature if M_r^n has diagonalizable shape operator with less than seven distinct principal curvatures.

Keywords: proper mean curvature vector; linear Weingarten hypersurfaces; principal curvatures
Mathematics Subject Classification: 53C40, 53C42

1. Introduction

A submanifold in a pseudo-Euclidean space $\mathbb{E}_s^{n+1}$ is called to have proper mean curvature vector if its mean curvature vector $\vec{H}$ satisfies a geometric equation

$$\Delta\vec{H} = \lambda\vec{H} \tag{1.1}$$

for a constant λ.

Equation (1.1) was first appeared in Chen's paper [4] in 1988 where surfaces of $\mathbb{E}^3$ satisfying Eq (1.1) were classified. In [6], Chen proved that a submanifold in $\mathbb{E}^{n+1}$ satisfying Eq (1.1) is a biharmonic submanifold or 1-type submanifold or null 2-type submanifold. In [10], Defever proved that every hypersurface of $\mathbb{E}^4$ satisfying Eq (1.1) has constant mean curvature. And recently, Fu [14] also got the same conclusion for hypersurface of $\mathbb{E}^{n+1}$ satisfying Eq (1.1) with three distinct principal curvatures. Noting that when $\lambda = 0$, Eq (1.1) becomes the classical biharmonic submanifolds equation, biharmonic submanifolds have received great attention in recent years, see [9, 12, 13]. Other results about submanifolds satisfying Eq (1.1) have been obtained by Chen [7, 8], or see a survey in book [9].

The study of submanifolds in $\mathbb{E}_s^{n+1}$ satisfying Eq (1.1) was originated by Ferrández and Lucas [11], and the complete classification of surfaces M_r^2 $(r = 0, 1)$ in $\mathbb{E}_1^3$ was given. In [2] Arvanitoyeorgos, Defever and Kaimakamis proved that if hypersurface $M_r^3 (r = 0, 1, 2, 3)$ of $\mathbb{E}_s^4$ satisfying Eq (1.1) has diagonalizable shape operator, then it has constant mean curvature. Also, Arvanitoyeorgos, Kaimakamis and Magid [3] got the same conclusion for nondegenerate Lorentz hypersurface M_1^3 of $\mathbb{E}_1^4$ satisfying Eq (1.1). Based on these results, Arvanitoyeorgos and Kaimakamis [1] in 2013 proposed the following conjecture:

"Any hypersurface of $\mathbb{E}_s^{n+1}$ satisfying $\Delta\vec{H} = \lambda\vec{H}$ has constant mean curvature."

In recent years, Liu and Yang in [18] proved that the conjecture holds for hypersurfaces with at most two distinct principal curvatures in $\mathbb{E}_s^{n+1}$, and for the Lorentz hypersurfaces with at most three distinct principal curvatures in $\mathbb{E}_1^{n+1}$, see [17]. We mention here that, Liu and Yang in [16] gave the following result.

Theorem 1.1. *Let M_r^n be a nondegenerate hypersurface in $\mathbb{E}_s^{n+1}$ $(n \geq 4)$ satisfying $\Delta\vec{H} = \lambda\vec{H}$. If M_r^n has diagonalizable Weingarten operator with at most three distinct principal curvatures, then the mean curvature of M_r^n is constant.*

As the number of different principal curvatures increases, this problem becomes more and more complicated. In this paper, we will deal with hypersurfaces with at most six distinct principal curvatures. Meanwhile, we will be concerned with linear Weingarten hypersurfaces (satisfying $aR + bH = c$, where R is the scalar curvature, H is the mean curvature and a, b, c are constants such that $a^2 + b^2 \neq 0$), which is a natural generalization of hypersurfaces with constant mean curvature and constant scalar curvature.

Precisely, we will consider linear Weingarten hypersurfaces of $\mathbb{E}_s^{n+1}$ satisfying Eq (1.1) and with less than seven distinct principal curvatures. Our result gives further support evidence to Arvanitoyeorgos and Kaimakamis's conjecture [1].

Theorem 1.2 (Main theorem)**.** *Let M_r^n be a nondegenerate linear Weingarten hypersurface of $\mathbb{E}_s^{n+1}$ satisfying $\Delta\vec{H} = \lambda\vec{H}$ with less than seven distinct principal curvatures. If M_r^n has diagonalizable shape operator, then M_r^n must have constant mean curvature and constant scalar curvature.*

Naturally, we could get a conclusion:

Corollary 1.3. *Let M_r^6 be a nondegenerate linear Weingarten hypersurface in $\mathbb{E}_s^7$ satisfying $\Delta\vec{H} = \lambda\vec{H}$. If M_r^6 has diagonalizable shape operator, M_r^6 must have constant scalar curvature and constant mean curvature.*

Note that the *linear Weingarten* submanifolds include all submanifolds with *constant scalar curvature* (for $b = 0$). Hence we get a conclusion:

Corollary 1.4. *Let M_r^n be a nondegenerate hypersurface in $\mathbb{E}_s^{n+1}$ satisfying $\Delta\vec{H} = \lambda\vec{H}$ with less than seven distinct principal curvatures. If M_r^n has constant scalar curvature and diagonalizable shape operator, then M_r^n must have constant mean curvature.*

This paper is organized as follow. We give some important formulas for linear Weingarten hypersurfaces satisfying Eq (1.1) in Section 2. Under the condition that the mean curvature H is not constant, we give an important proposition concerning the principal curvature of linear Weingarten hypersurfaces in $\mathbb{E}_s^{n+1}$ in Section 3. The detail proof of main Theorem 1.2 is given in Section 4.

2. A necessary result

Let $x : M_r^n \to \mathbb{E}_s^{n+1}$ be an isometric immersion. ∇ and $\bar{\nabla}$ are the Levi-Civita connections of M_r^n and $\mathbb{E}_s^{n+1}$. Let vector fields X and Y be tangent to M_r^n and ξ be a unite normal vector of M_r^n satisfying $\langle \xi, \xi \rangle = \varepsilon = \pm 1$.

Recall the mean curvature vector $\vec{H}$ is given by $\vec{H} = H\vec{\xi}$, where H is the mean curvature satisfying

$$H = \frac{1}{n}\varepsilon \operatorname{tr} A \tag{2.1}$$

for the shape operator A. For arbitrary vector fields X, Y, Z are tangent to M_r^n, the Gauss equation is given by

$$R(X, Y)Z = \langle AY, Z \rangle AX - \langle AX, Z \rangle AY, \tag{2.2}$$

where $R(X, Y)Z$ is defined as

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.3}$$

The Codazzi equation is given by

$$(\nabla_X A)Y = (\nabla_Y A)X, \tag{2.4}$$

where

$$(\nabla_X A)Y = \nabla_X (AY) - A(\nabla_X Y).$$

It is well known that R, H and the squared length of the second fundamental form B satisfy

$$R = n^2 H^2 - B. \tag{2.5}$$

According to [5], we know M_r^n satisfies Eq (1.1) if and only if it satisfies equations

$$\Delta H + \varepsilon H \operatorname{tr} A^2 = \lambda H, \tag{2.6}$$

$$A(\nabla H) = -\frac{n}{2}\varepsilon H(\nabla H), \tag{2.7}$$

where the Laplace operator Δ is defined by

$$\Delta f = -\sum_{i=1}^{n} \varepsilon_i \left(e_i e_i - \nabla_{e_i} e_i \right) f, \tag{2.8}$$

for a local orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ satisfying $\langle e_i, e_i \rangle = \varepsilon_i = \pm 1$.

Let $\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_{ij}^k e_k$. According to Eqs (2.2) and (2.3), by a direct calculations about $< R(e_i, e_j)e_k, e_n >$ and $< R(e_j, e_k)e_i, e_n >$, we have

$$\begin{aligned} \omega_{ij}^k(\varepsilon_j \omega_{jj}^n - \varepsilon_k \omega_{kk}^n) &= \omega_{ji}^k(\varepsilon_i \omega_{ii}^n - \varepsilon_k \omega_{kk}^n), \\ \omega_{jk}^i(\varepsilon_k \omega_{kk}^n - \varepsilon_i \omega_{ii}^n) &= \omega_{kj}^i(\varepsilon_j \omega_{jj}^n - \varepsilon_i \omega_{ii}^n). \end{aligned} \tag{2.9}$$

In the following, we assume H is not constant and give a necessary proposition concerning the principal curvatures of linear Weingarten hypersurface in $\mathbb{E}_s^{n+1}$, which is critical to prove Theorem 1.2.

Proposition 2.1. Let M_r^n be a nondegenerate linear Weigartem hypersurface in $\mathbb{E}_s^{n+1}$ satisfying $\Delta\vec{H} = \lambda\vec{H}$ with diagonalizable Weingarten operator and less than seven distinct principal curvatures. If H is not constant, then all the principal curvatures of M_r^n are functions with one variable.

Proof. According to the condition that H is not constant, we know that there exists at least one point p_0 satisfying $\operatorname{grad} H(p_0) \neq 0$. And we will work on an neighborhood of p_0. From Eq (2.7), it is to know that ∇H is an eigenvector of A and $-n\varepsilon H/2$ is the corresponding principal curvature. Therefore, we let $e_n = \frac{\nabla H}{|\nabla H|}$. Under the assumption that the Weingarten operator of M_r^n is diagonalizable, there exits a suitable orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ satisfying

$$A = diag\,(\lambda_1, \lambda_2, \cdots, \lambda_n), \tag{2.10}$$

where λ_i are the principal curvatures and

$$\lambda_n = -n\varepsilon H/2. \tag{2.11}$$

It follows from (2.1) that

$$\sum_{i=1}^{n-1} \lambda_i = -3\lambda_n. \tag{2.12}$$

We have assumed that the mean curvature of M_r^n is not constant, the Weingarten condition $aR + bH = c$ implies that $a \neq 0$, and from (2.11) we have

$$R = -\frac{b}{a}H + \frac{c}{a} = \frac{2b\varepsilon}{na}\lambda_n + \frac{c}{a}. \tag{2.13}$$

Furthermore, from (2.5) and (2.11), we have

$$R = n^2H^2 - B = 3\lambda_n^2 - \sum_{i=1}^{n-1} \lambda_i^2. \tag{2.14}$$

Equations (2.13) and (2.14) force that

$$\sum_{i=1}^{n-1} \lambda_i^2 = 3\lambda_n^2 + c_1\lambda_n - c_2, \tag{2.15}$$

where $c_1 = -2b\varepsilon/na, c_2 = c/a$. Since e_n is parallel to ∇H, we have

$$e_n(H) \neq 0, \quad e_i(H) = 0. \tag{2.16}$$

According to (2.11), Eq (2.16) is equivalent to

$$e_n(\lambda_n) \neq 0, \quad e_i(\lambda_n) = 0 \tag{2.17}$$

for $i = 1, 2, \cdots, n-1$. We note that λ_n is a smooth function depending only on one variable t.

Substituting $H = -2\varepsilon\lambda_n/n$ into Eq (2.6), from (2.8), (2.15) and (2.17), we get

$$\varepsilon_n e_n e_n(\lambda_n) = e_n(\lambda_n)(\sum_{i=1}^{n-1} \varepsilon_i \omega_{ii}^n) + \varepsilon\lambda_n(4\lambda_n^2 + c_1\lambda_n - c_2) - \lambda\lambda_n. \tag{2.18}$$

Since $\lambda_n = \lambda_n(t)$ and $e_n(\lambda_n) \neq 0$, from (2.18) we have $\sum_{i=1}^{n-1} \varepsilon_i \omega_{ii}^n$ is a function with the variable t, we denote it by $f_1(t)$, that is

$$\sum_{i=1}^{n-1} \varepsilon_i \omega_{ii}^n = \frac{\varepsilon_n e_n e_n(\lambda_n) - \varepsilon\lambda_n(4\lambda_n^2 + c_1\lambda_n - c_2) + \lambda\lambda_n}{e_n(\lambda_n)} \triangleq f_1(t). \tag{2.19}$$

It follows from (2.17) that $[e_i, e_j](H) = 0$, which forces to

$$\omega_{ij}^n = \omega_{ji}^n, \tag{2.20}$$

for $i, j = 1, 2, \ldots, n-1$ and $i \neq j$. By simple calculations about $\nabla_{e_k}\langle e_i, e_i\rangle = 0$ and $\nabla_{e_k}\langle e_i, e_j\rangle = 0$, we have

$$\omega_{ki}^i = 0, \quad \varepsilon_j \omega_{ki}^j + \varepsilon_i \omega_{kj}^i = 0, \tag{2.21}$$

for $i, j, k = 1, 2, 3, \ldots, n$ and $i \neq j \neq k$. Putting $X = e_i$ and $Y = e_j$, Codazzi equation (2.4) gives

$$e_i(\lambda_j)e_j + \sum_{k=1}^{n} \omega_{ij}^k(\lambda_j - \lambda_k)e_k = e_j(\lambda_i)e_i + \sum_{k=1}^{n} \omega_{ji}^k(\lambda_i - \lambda_k)e_k$$

for $i \neq j$. So we get

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \tag{2.22}$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j, \tag{2.23}$$

for $i, j, k = 1, 2, \ldots, n$ and $i \neq j \neq k$.

In the case $\lambda_j = \lambda_n$ for $j = 1, \cdots, n-1$, putting $i = n$ into (2.22) we get $e_n(\lambda_n) = e_n(\lambda_j) = 0$, which is contradict to (2.17), so we get $\lambda_j \neq \lambda_n$, $j = 1, \cdots, n-1$.

For $j = n$, $i \neq n$ in (2.22) we have

$$e_i(\lambda_n) = (\lambda_i - \lambda_n)\omega_{ni}^n.$$

From (2.17) and $\lambda_i \neq \lambda_n$ for $i = 1, \cdots, n-1$, we have $\omega_{ni}^n = 0$, which together with the second equation of (2.21) forces that $\omega_{nn}^i = 0$, $i = 1, \cdots, n$.

For $j = n$, $k, i \neq n$ in (2.23) we have

$$(\lambda_i - \lambda_n)\omega_{ki}^n = (\lambda_k - \lambda_n)\omega_{ik}^n,$$

which together with (2.20) yields $\omega_{ij}^n = 0$ for $i, j = 1, 2, \cdots, n-1$ and $i \neq j$.

From Eq (2.3), we have

$$\begin{aligned}
&< R(e_n, e_i)e_n, e_i > \\
=&\varepsilon_i[e_n(\omega^i_{in}) + \sum_{k=1}^{n}\omega^k_{in}\omega^i_{nk} - e_i(\omega^i_{nn}) - \sum_{k=1}^{n}\omega^k_{nn}\omega^i_{ik} - \sum_{k=1}^{n}\omega^k_{ni}\omega^i_{kn} + \sum_{k=1}^{n}\omega^k_{in}\omega^i_{kn}] \\
=&\varepsilon_i e_n(\omega^i_{in}) + \varepsilon_i(\omega^i_{in})^2.
\end{aligned} \tag{2.24}$$

From Eq (2.2), we have

$$< R(e_n, e_i)e_n, e_i >= -\varepsilon_n\varepsilon_i\lambda_n\lambda_i. \tag{2.25}$$

Hence Eqs (2.24) and (2.25) force to

$$e_n(\varepsilon_i\omega^n_{ii}) = \varepsilon_n(\varepsilon_i\omega^n_{ii})^2 + \lambda_n\lambda_i, \tag{2.26}$$

for $i = 1, 2, \cdots, n-1$. Summing both sides of Eq (2.26) and from (2.12) and (2.19), we have

$$\sum_{i=1}^{n-1}(\varepsilon_i\omega^n_{ii})^2 = \varepsilon_n e_n(f_1) + 3\varepsilon_n\lambda_n^2 \triangleq f_2(t). \tag{2.27}$$

Putting $i = n$ into (2.22) and according to (2.21), we have

$$e_n(\lambda_i) = \varepsilon_n\varepsilon_i(\lambda_i - \lambda_n)\omega^n_{ii}. \tag{2.28}$$

Similarly, taking sum of Eq (2.28) and from (2.12) and (2.19), we have

$$\sum_{i=1}^{n-1}\lambda_i\varepsilon_i\omega^n_{ii} = \lambda_n f_1 - 3\varepsilon_n e_n(\lambda_n) \triangleq g_1(t). \tag{2.29}$$

Multiplying on both sides of Eq (2.26) by $\varepsilon_i\omega^n_{ii}$ and Summing for $i = 1, 2, \cdots, n-1$, then from (2.27) and (2.29) we obtain

$$\sum_{i=1}^{n-1}(\varepsilon_i\omega^n_{ii})^3 = \frac{1}{2}\varepsilon_n e_n(f_2) - \varepsilon_n\lambda_n g_1 \triangleq f_3(t). \tag{2.30}$$

Differentiating (2.29) along e_n, from (2.28) and (2.26), we have

$$e_n(g_1) = 2\sum_{i=1}^{n-1}\varepsilon_n\lambda_i(\varepsilon_i\omega^n_{ii})^2 - \sum_{i=1}^{n-1}\varepsilon_n\lambda_n(\varepsilon_i\omega^n_{ii})^2 + \sum_{i=1}^{n-1}\lambda_n\lambda_i^2.$$

Furthermore, from (2.15) and (2.27) we have

$$\sum_{i=1}^{n-1}\lambda_i(\varepsilon_i\omega^n_{ii})^2 = \frac{1}{2}\{\varepsilon_n e_n(g_1) + \lambda_n f_2 - \varepsilon_n\lambda_n(3\lambda_n^2 + c_1\lambda_n - c_2)\} \triangleq g_2(t). \tag{2.31}$$

Multiplying on the Eq (2.26) by $(\varepsilon_i\omega^n_{ii})^2$ and summing for i, we get

$$\frac{1}{3}\sum_{i=1}^{n-1}e_n((\varepsilon_i\omega^n_{ii})^3) = \sum_{i=1}^{n-1}\varepsilon_n(\varepsilon_i\omega^n_{ii})^4 + \sum_{i=1}^{n-1}\lambda_n\lambda_i(\varepsilon_i\omega^n_{ii})^2.$$

According to (2.30) and (2.31), we obtain

$$\sum_{i=1}^{n-1}(\varepsilon_i\omega_{ii}^n)^4 = \frac{1}{3}\varepsilon_n e_n(f_3) - \varepsilon_n\lambda_n g_2 \triangleq f_4(t). \tag{2.32}$$

In the same way, we can get

$$\sum_{i=1}^{n-1}(\varepsilon_i\omega_{ii}^n)^5 = \frac{1}{4}\varepsilon_n e_n(f_4) - \lambda_n g_3 \triangleq f_5(t). \tag{2.33}$$

Based on the discussions above, from (2.19), (2.27), (2.30), (2.32) and (2.33), we conclude that

$$\sum_{i=1}^{n-1}(\varepsilon_i\omega_{ii}^n)^k = f_k(t), \quad for \quad k = 1, \cdots 5. \tag{2.34}$$

Next, taking into account Eq (2.23) and the second Eq of (2.21) for distinct principal curvatures $\lambda_i, \lambda_j, \lambda_k$ $(i, j, k = 1, 2, \cdots, n-1)$, we have

$$\varepsilon_i\omega_{jk}^i(\lambda_k - \lambda_i) = \varepsilon_j\omega_{ki}^j(\lambda_i - \lambda_j) = \varepsilon_k\omega_{ij}^k(\lambda_j - \lambda_k), \tag{2.35}$$

$$\omega_{ij}^k\omega_{ji}^k + \omega_{jk}^i\omega_{kj}^i + \omega_{ik}^j\omega_{ki}^j = 0. \tag{2.36}$$

It follows from (2.9) and (2.21) that

$$\omega_{jk}^i(\varepsilon_i\varepsilon_k\omega_{kk}^n - \omega_{ii}^n) = \omega_{ki}^j(\varepsilon_i\varepsilon_j\omega_{ii}^n - \omega_{jj}^n) = \omega_{ij}^k(\varepsilon_j\varepsilon_k\omega_{jj}^n - \omega_{kk}^n). \tag{2.37}$$

Since $\lambda_i \neq \lambda_j$, it follows from (2.21), (2.22) that

$$\omega_{ji}^j = \omega_{jj}^i = 0. \tag{2.38}$$

Computing $\langle R(e_i, e_j)e_i, e_j\rangle$ by (2.3) and (2.2), we can get

$$e_i(\omega_{ji}^j) + \sum_{k=1}^{n}\omega_{ji}^k\omega_{ik}^j - e_j(\omega_{ii}^j) - \sum_{k=1}^{n}\omega_{ii}^k\omega_{jk}^j - \sum_{k=1}^{n}(\omega_{ij}^k - \omega_{ji}^k)\omega_{ki}^j = -\varepsilon_i\lambda_i\lambda_j, \tag{2.39}$$

for $\lambda_i \neq \lambda_j$, $i, j = 1, 2, \cdots, n-1$.

From (2.21), (2.36) and (2.38), Eq (2.39) becomes

$$\varepsilon_n\omega_{ii}^n\omega_{jj}^n - 2\varepsilon_k\sum_{k=1,k\neq l(i,j)}^{n-1}\omega_{ij}^k\omega_{ji}^k = -\varepsilon_i\varepsilon_j\lambda_i\lambda_j \tag{2.40}$$

for $\lambda_i \neq \lambda_j$, where $l(i, j)$ stands for the indexes satisfying $\lambda_{l(i,j)} = \lambda_i$ or λ_j.

Since $\lambda_j \neq \lambda_n$, $j = 1, 2, \cdots, n-1$, there are at most five distinct principal curvatures except λ_n. And we assume the principal curvatures are $\lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e$ in sequence and the corresponding multiplicities are n_1, n_2, n_3, n_4, n_5 $(\sum_{i=1}^{5} n_i = n-1)$. Equations (2.35) and (2.37) imply that $\lambda_i \neq \lambda_j$ is equivalent to $\omega_{ii}^n \neq \omega_{jj}^n$ $(i, j = 1, 2, \cdots, n-1)$.

In the following, let g be the number of distinct principal curvatures.

Firstly, $g = 6$. In this case, $\varepsilon_a\omega^n_{aa}, \varepsilon_b\omega^n_{bb}, \cdots, \varepsilon_e\omega^n_{ee}$ are different from each other. Five equations in (2.34) could transfer into

$$n_1(\varepsilon_a\omega^n_{aa})^k + n_2(\varepsilon_b\omega^n_{bb})^k + n_3(\varepsilon_c\omega^n_{cc})^k + n_4(\varepsilon_d\omega^n_{dd})^k + n_5(\varepsilon_e\omega^n_{ee})^k = f_k \tag{2.41}$$

for $k = 1, 2, \cdots, 5$. Differentiating (2.41) with respect to e_i for any $i = 1, 2, \cdots, n-1$, and from $e_i(f_k) = 0$ for $k = 1, 2, \cdots, 5$, we can get a homogeneous equation system with five equations and five unknowns $e_i(\varepsilon_j\omega^n_{jj})$ for $j = a, b, c, d, e$ and $k = 1, 2, \cdots, 5$

$$\begin{aligned} n_1(\varepsilon_a\omega^n_{aa})^{k-1}e_i(\varepsilon_a\omega^n_{aa}) + n_2(\varepsilon_b\omega^n_{bb})^{k-1}e_i(\varepsilon_b\omega^n_{bb}) + n_3(\varepsilon_c\omega^n_{cc})^{k-1}e_i(\varepsilon_c\omega^n_{cc}) \\ +n_4(\varepsilon_d\omega^n_{dd})^{k-1}e_i(\varepsilon_d\omega^n_{dd}) + n_5(\varepsilon_e\omega^n_{ee})^{k-1}e_i(\varepsilon_e\omega^n_{ee}) = 0. \end{aligned} \tag{2.42}$$

Since $\varepsilon_j\omega^n_{jj}$ are mutually different, it follows that the determinant

$$\begin{vmatrix} n_1 & n_2 & n_3 & n_4 & n_5 \\ n_1\varepsilon_a\omega^n_{aa} & n_2\varepsilon_b\omega^n_{bb} & n_3\varepsilon_c\omega^n_{cc} & n_4\varepsilon_d\omega^n_{dd} & n_5\varepsilon_e\omega^n_{ee} \\ n_1(\varepsilon_a\omega^n_{aa})^2 & n_2(\varepsilon_b\omega^n_{bb})^2 & n_3(\varepsilon_c\omega^n_{cc})^2 & n_4(\varepsilon_d\omega^n_{dd})^2 & n_5(\varepsilon_e\omega^n_{ee})^2 \\ n_1(\varepsilon_a\omega^n_{aa})^3 & n_2(\varepsilon_b\omega^n_{bb})^3 & n_3(\varepsilon_c\omega^n_{cc})^3 & n_4(\varepsilon_d\omega^n_{dd})^3 & n_5(\varepsilon_e\omega^n_{ee})^3 \\ n_1(\varepsilon_a\omega^n_{aa})^4 & n_2(\varepsilon_b\omega^n_{bb})^4 & n_3(\varepsilon_c\omega^n_{cc})^4 & n_4(\varepsilon_d\omega^n_{dd})^4 & n_5(\varepsilon_e\omega^n_{ee})^4 \end{vmatrix} \neq 0,$$

which together with (2.42) forces that

$$e_i(\varepsilon_j\omega^n_{jj}) = 0,$$

for $i, j = 1, 2, \cdots, n-1$. Moreover, from

$$e_ie_n(\varepsilon_j\omega^n_{jj}) - e_ne_i(\varepsilon_j\omega^n_{jj}) = [e_i, e_n](\varepsilon_j\omega^n_{jj}) = \sum_{k=1}^{n-1}(\omega^k_{in} - \omega^k_{ni})e_k(\varepsilon_j\omega^n_{jj}),$$

we get

$$e_ie_n(\varepsilon_j\omega^n_{jj}) = 0,$$

for $i, j = 1, 2, \cdots, n-1$. From (2.26) we have

$$\lambda_ne_i(\lambda_j) = e_ie_n(\varepsilon_j\omega^n_{jj}) - 2\varepsilon_n\varepsilon_j\omega^n_{jj}e_i(\varepsilon_j\omega^n_{jj}) = 0, \tag{2.43}$$

hence

$$e_i(\lambda_j) = 0,$$

for $i, j = 1, 2, \cdots, n-1$, which means that λ_i is a function with only one variable t for any i.

Secondly, we consider the case $g \leq 5$. In this case, except λ_n, there are up to four distinct principal curvatures. It is very similar discussion to above, the number of different $\varepsilon_i\omega^n_{ii}$ for $i = 1, \cdots, n$ is up to four. If four ones of $\varepsilon_i\omega^n_{ii}$ are different, we only consider Eq (2.41) for $k = 1, \cdots, 4$. A similar discussion to $g = 6$ yields the conclusion of Proposition 2.1. If less than four ones of $\varepsilon_i\omega^n_{ii}$ are different, the discussion is quite similar. We prove Proposition 2.1. □

3. The proof of Theorem 1.2

In this part, we begin with the assumption that the mean curvature of M_r^n is not constant, then we will deduce a contradiction. Our proof of Theorem 1.2 are divided into the following four cases according to g.

Firstly, $g = 6$. In this case, we assume that the principal curvatures are $\lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e, \lambda_n$ and the corresponding multiplicities are $n_1, n_2, n_3, n_4, n_5, 1$ with $\sum_{i=1}^{5} n_i = n-1$. According to Eqs (2.35)–(2.37), we divide into the following three cases:

(i) Precisely three ω_{ij}^k are not zero for different i, j, k. We assume $\omega_{ab}^c \neq 0, \omega_{ab}^d \neq 0$ and $\omega_{ab}^e \neq 0$. From Eqs (2.35) and (2.37), we have

$$\frac{\varepsilon_a\omega_{aa}^n - \varepsilon_b\omega_{bb}^n}{\lambda_a - \lambda_b} = \frac{\varepsilon_a\omega_{aa}^n - \varepsilon_j\omega_{jj}^n}{\lambda_a - \lambda_j} = \frac{\varepsilon_b\omega_{bb}^n - \varepsilon_j\omega_{jj}^n}{\lambda_b - \lambda_j},$$

for $j = c, d, e$. It follows from Proposition 2.1 that λ_i and ω_{ii}^n depend on a parameter t, so there are two smooth functions $\alpha(t)$ and $\beta(t)$ satisfying

$$\varepsilon_i\omega_{ii}^n = \alpha\lambda_i + \beta, \tag{3.1}$$

for $i = 1, 2, \cdots, n-1$. Differentiating Eq (3.1) along e_n and using (2.26) and (2.28), we have

$$e_n(\alpha) = \varepsilon_n\{(\lambda_n(\alpha^2 + 1) + \alpha\beta\}, \tag{3.2}$$

$$e_n(\beta) = \varepsilon_n\beta(\lambda_n\alpha + \beta). \tag{3.3}$$

Differentiating Eq (2.12) along e_n and using (2.28), we have

$$3e_n(\lambda_n) = \sum_{i=1}^{n-1} \varepsilon_n\varepsilon_i(\lambda_n - \lambda_i)\omega_{ii}^n. \tag{3.4}$$

Substituting (3.1) into (3.4) and from (2.12), (2.15) we have

$$3e_n(\lambda_n) = \varepsilon_n\{\alpha(-6\lambda_n^2 - c_1\lambda_n + c_2) + (n+2)\beta\lambda_n\}. \tag{3.5}$$

Substituting (3.1) into (2.18) and from (2.12) we get

$$e_ne_n(\lambda_n) = \varepsilon_n\{e_n(\lambda_n)[-3\lambda_n\alpha + (n-1)\beta] + \varepsilon\lambda_n(4\lambda_n^2 + c_1\lambda_n - c_2) - \lambda\lambda_n\}. \tag{3.6}$$

Differentiating (3.5) along e_n and using (3.2), (3.3), then comparing to (3.6), we eliminate $e_ne_n(\lambda_n)$ as follows

$$2(n-4)\beta e_n(\lambda_n) = -6(\varepsilon_n + 2\varepsilon)\lambda_n^3 + (-c_1\lambda_n + c_2)(\varepsilon_n + 3\varepsilon)\lambda_n + 3\lambda\lambda_n. \tag{3.7}$$

Furthermore, eliminating $e_n(\lambda_n)$ between (3.5) and (3.7), we get

$$\begin{aligned} &2(n-4)\beta\{\alpha(-6\lambda_n^2 - c_1\lambda_n + c_2) + (n+2)\beta\lambda_n\} = \\ &\quad 3\varepsilon_n\{-6(\varepsilon_n+2\varepsilon)\lambda_n^3 + (-c_1\lambda_n + c_2)(\varepsilon_n + 3\varepsilon)\lambda_n + 3\lambda\lambda_n\}. \end{aligned} \tag{3.8}$$

Similarly, differentiating Eq (3.7) along e_n and from (3.5), (3.6), (3.7), we have

$$\{72(\varepsilon_n + 2\varepsilon)\lambda_n^4 - 18(\varepsilon_n + 2\varepsilon)(-c_1\lambda_n + c_2)\lambda_n^2 + a_1\}\alpha - (a_2\lambda_n^3 + a_3\lambda_n)\beta = 0, \tag{3.9}$$

where $a_i = a_i(\varepsilon_n, \varepsilon, \lambda, n)$ are given by

$$\begin{aligned} a_1 &= (-c_1\lambda_n + c_2)\{(-c_1\lambda_n + c_2)(\varepsilon_n + 3\varepsilon) + 3\lambda\}, \\ a_2 &= 36(\varepsilon_n + 2\varepsilon) + 24(n-4)\varepsilon, \\ a_3 &= \{2(n-1)\varepsilon_n + 18\varepsilon\}(-c_1\lambda_n + c_2) + 18\lambda. \end{aligned} \tag{3.10}$$

Differentiating Eq (3.9) along e_n and from (3.2), (3.3), (3.5), (3.7), we get

$$\begin{aligned} &\{288(\varepsilon_n + 2\varepsilon)\lambda_n^3 - 36(-c_1\lambda_n + c_2)(\varepsilon_n + 2\varepsilon)\lambda_n\} \cdot 2(n-4)\alpha\varepsilon_n \\ &\times \{\alpha(-6\lambda_n^2 - c_1\lambda_n + c_2) + (n+2)\beta\lambda_n\} \\ &+ \{72(\varepsilon_n + 2\varepsilon)\lambda_n^4 - 18(\varepsilon_n + 2\varepsilon)(-c_1\lambda_n + c_2)\lambda_n^2 + a_1\}\varepsilon_n\{\varepsilon_n(\alpha^2 + 1) + \alpha\beta\} \\ &- 3(3a_2\lambda_n^2 + a_3)\{-6(\varepsilon_n + 2\varepsilon)\lambda_n^3 + (-c_1\lambda_n + c_2)(\varepsilon_n + 3\varepsilon)\lambda_n + 3\lambda\lambda_n\} \\ &- (a_2\lambda_n^3 + a_3\lambda_n)\varepsilon_n\beta(\lambda_n\alpha + \beta) = 0. \end{aligned} \tag{3.11}$$

For convenience, Eq (3.11) could be rewritten as

$$q_1(\lambda_n)\alpha^2 + q_2(\lambda_n)\alpha\beta + q_3(\lambda_n)\beta^2 + q_4(\lambda_n) = 0, \tag{3.12}$$

where

$$\begin{aligned} q_1(\lambda_n) &= \{288(\varepsilon_n + 2\varepsilon)\lambda_n^3 - 36(-c_1\lambda_n + c_2)(\varepsilon_n + 2\varepsilon)\lambda_n\} \\ &\quad \times 2(n-4)(-6\lambda_n^2 - c_1\lambda_n + c_2) \\ &\quad + \{72(\varepsilon_n + 2\varepsilon)\lambda_n^4 - 18(\varepsilon_n + 2\varepsilon)(-c_1\lambda_n + c_2)\lambda_n^2 + a_1\}\lambda_n, \\ q_2(\lambda_n) &= \{288(\varepsilon_n + 2\varepsilon)\lambda_n^3 - 36(-c_1\lambda_n + c_2)(\varepsilon_n + 2\varepsilon)\lambda_n\} \cdot 2(n-4)(n+2)\lambda_n \\ &\quad + \{72(\varepsilon_n + 2\varepsilon)\lambda_n^4 - 18(\varepsilon_n + 2\varepsilon)(-c_1\lambda_n + c_2)\lambda_n^2 + a_1\} \\ &\quad - (a_2\lambda_n^3 + a_3\lambda_n)\lambda_n, \\ q_3(\lambda_n) &= -(a_2\lambda_n^3 + a_3\lambda_n), \\ q_4(\lambda_n) &= \{72(\varepsilon_n + 2\varepsilon)\lambda_n^4 - 18(\varepsilon_n + 2\varepsilon)(-c_1\lambda_n + c_2)\lambda_n^2 + a_1\}\lambda_n \\ &\quad - 3\varepsilon_n(3a_2\lambda_n^2 + a_3)\{-6(\varepsilon_n + 2\varepsilon)\lambda_n^3 + (-c_1\lambda_n + c_2)(\varepsilon_n + 3\varepsilon)\lambda_n + 3\lambda\lambda_n\}. \end{aligned} \tag{3.13}$$

In the same way, (3.8) and (3.9) could be rewritten as

$$p_1(\lambda_n)\beta^2 + p_2(\lambda_n)\alpha\beta + p_3(\lambda_n) = 0, \tag{3.14}$$

$$h_1(\lambda_n)\alpha + h_2(\lambda_n)\beta = 0, \tag{3.15}$$

where

$$\begin{aligned} p_1(\lambda_n) &= 2(n-4)(n+2)\lambda_n, \\ p_2(\lambda_n) &= 2(n-4)(-6\lambda_n^2 - c_1\lambda_n + c_2), \\ p_3(\lambda_n) &= -3\varepsilon_n\{-6(\varepsilon_n + 2\varepsilon)\lambda_n^3 + (-c_1\lambda_n + c_2)(\varepsilon_n + 3\varepsilon)\lambda_n + 3\lambda\lambda_n\}, \\ h_1(\lambda_n) &= 72(\varepsilon_n + 2\varepsilon)\lambda_n^4 - 18(\varepsilon_n + 2\varepsilon)(-c_1\lambda_n + c_2)\lambda_n^2 + a_1, \\ h_2(\lambda_n) &= -(a_2\lambda_n^3 + a_3\lambda_n). \end{aligned} \tag{3.16}$$

Combining (3.12), (3.14) with (3.15) to eliminate α, we have

$$P_1\beta^2 + P_2 = 0, \tag{3.17}$$

$$Q_1\beta^2 + Q_2 = 0. \tag{3.18}$$

Moreover, combining (3.17) with (3.18) to eliminate β^2, we obtain

$$P_2Q_1 - P_1Q_2 = 0, \tag{3.19}$$

where P_i, Q_i $(i = 1, 2)$ are given by

$$\begin{aligned}
P_1 &= h_2^2 q_1 - h_1h_2q_2 + h_1^2 q_3 = l_{11}\lambda_n^{11} + \cdots,\\
P_2 &= h_1^2 q_4 = l_{21}\lambda_n^{13} + \cdots,\\
Q_1 &= h_1^2 p_1 - h_1h_2p_2 = l_{31}\lambda_n^{9} + \cdots,\\
Q_2 &= h_1^2 p_3 = l_{41}\lambda_n^{11} + \cdots,
\end{aligned}$$

where $l_{i1} = l_{i1}(\varepsilon, \varepsilon_n, n)$ $(i = 1, 2, 3, 4)$. Substituting P_i and Q_i into Eq (3.19), we could get

$$\sum_{i=0}^{22} L_i\lambda_n^i = 0, \tag{3.20}$$

where $L_i = L_i(\varepsilon, \varepsilon_n, n)$ are constant, and the coefficient L_{22} is given by

$$\begin{aligned}
L_{22} &= l_{21}l_{31} - l_{11}l_{41}\\
&= 72^2(\varepsilon_n + 2\varepsilon)^2 a_2^2\{2(n-4)(n+2)(\varepsilon_n + 2\varepsilon)(72 + 54a_2\varepsilon_n)\\
&+ 18\varepsilon_n(\varepsilon_n + 2\varepsilon)^2[3456(n-4) - 72]\}\\
&- 72^2(\varepsilon_n + 2\varepsilon)^2 \times 72a_2(\varepsilon_n + 2\varepsilon) \times 18\varepsilon_n(\varepsilon_n + 2\varepsilon)\times\\
&\{576(\varepsilon_n + 2\varepsilon)(n-4)(n+2) + 72(\varepsilon_n + 2\varepsilon) - a_2\}\\
&- 72^2(\varepsilon_n + 2\varepsilon)^2 \times 72a_2(\varepsilon_n + 2\varepsilon)\\
&\times 12(n-4)(\varepsilon_n + 2\varepsilon)(72 + 54\varepsilon_n a_2)\\
&+ 72^4(\varepsilon_n + 2\varepsilon)^4 \times 18a_2\varepsilon_n(\varepsilon_n + 2\varepsilon).
\end{aligned} \tag{3.21}$$

Since $\varepsilon = \pm 1$, $\varepsilon_n = \pm 1$, we consider L_{22} in the following

- $\varepsilon = 1$, $\varepsilon_n = 1$, in this case

$$a_2 = 108 + 24(n-4),$$

and

$$\begin{aligned}
L_{22} =& 46656 \times \{4478976(n+2)(n-4)^4 + 60715008(n+2)(n-4)^3\\
&+ 80621568(n-4)^3 - 209392128(n+2)(n-4)^2 + 712157184(n-4)^2\\
&- 1763596800(n+2)(n-4) + 1531809792(n-4) + 90699264\}\\
=& 46656 \times [4478976(n-4)^2 - 209392128](n+2)(n-4)^2\\
&+ 46656 \times [60715008(n-4)^2 - 1763596800](n+2)(n-4)\\
&+ 46656 \times \{80621568(n-4)^3 + 712157184(n-4)^2\\
&+ 1531809792(n-4) + 90699264\}.
\end{aligned} \tag{3.22}$$

So $L_{22} > 0$ if $n \geq 100$, and $L_{22} \neq 0$ if $1 \leq n \leq 100$ with the tool of MATLAB.

• $\varepsilon = 1$, $\varepsilon_n = -1$, in this case

$$a_2 = -36 - 24(n-4),$$

and

$$\begin{aligned} L_{22} = 5184 \times \{&746496(n+2)(n-4)^4 + 6635520(n+2)(n-4)^3 \\ &+ 12317184(n-4)^3 - 8087040(n+2)(n-4)^2 + 34805376(n-4)^2 \\ &- 22021632(n+2)(n-4) + 22254912(n-4) + 3359232\}. \end{aligned}$$

In the same way with the first case, we also get $L_{22} \neq 0$.

• $\varepsilon = -1$, $\varepsilon_n = 1$, in this case

$$a_2 = 36 + 24(n-4),$$

and

$$\begin{aligned} L_{22} = 5184 \times \{&-1492992(n+2)(n-4)^4 - 6801408(n+2)(n-4)^3 \\ &+ 62705664(n-4)^3 + 75893377(n+2)(n-4)^2 + 186625000(n-4)^2 \\ &+ 21648384(n+2)(n-4) + 138848256(n-4)\}. \end{aligned}$$

In the same way with the first case, we also get $L_{22} \neq 0$.

• $\varepsilon = -1$, $\varepsilon_n = -1$, in this case

$$a_2 = -108 - 24(n-4),$$

and

$$\begin{aligned} L_{22} = 46659 \times \{&-4478976(n+2)(n-4)^4 - 60715008(n+2)(n-4)^3 \\ &- 80621568(n-4)^3 + 209392128(n+2)(n-4)^2 - 739031040(n-4)^2 \\ &- 195507303(n+2)(n-4) - 1638645216(n-4) - 272097792\}. \end{aligned}$$

In the same way with the first case, we also get $L_{22} \neq 0$.

Therefore λ_n is constant and so $H = -2\varepsilon\lambda_n/n$ is a constant, this contradicts the assumption that the mean curvature is not constant.

(ii) Precisely two ones of ω_{ij}^k are not zero for different i, j, k. And we assume $\omega_{ab}^c \neq 0, \omega_{ab}^d \neq 0$ and $\omega_{ij}^k = 0$. In this case, it follows from (2.40) that

$$\varepsilon_i \varepsilon_n \omega_{ii}^n \omega_{ee}^n = -\varepsilon_e \lambda_i \lambda_e, \tag{3.23}$$

for $i = 1, \cdots, n_1 + n_2 + n_3 + n_4$. Similarly, from Eqs (2.37) and (2.35), we have

$$\varepsilon_i \omega_{ii}^n = \alpha\lambda_i + \beta, \tag{3.24}$$

where α and β are smooth functions satisfying (3.2) and (3.3). From (3.23) and (3.24), we get

$$\alpha\varepsilon_n \omega_{ee}^n = -\varepsilon_e \lambda_e. \tag{3.25}$$

According to (3.25), (3.24) and (3.23), we get

$$\beta\lambda_e = 0. \tag{3.26}$$

This forces that ω_{ee}^n and λ_e are entirely determined by α and β. Substituting (3.24)–(3.26) into (3.4), and differentiating it along e_n, from (2.26), (2.28), (3.2), (3.3) we know λ_n is a constant by similar discussion as case (i), which is also a contradiction.

(iii) Precisely one of ω_{ij}^k (i, j, k are different) is not zero. We assume $\omega_{ab}^c \neq 0$ and $\omega_{ij}^k = 0$ for all other different triplets i, j, k. In this case, according to (2.40), we have

$$\varepsilon_i \varepsilon_n \omega_{ii}^n \omega_{jj}^n = -\varepsilon_j \lambda_i \lambda_j, \tag{3.27}$$

$i = 1, \cdots, n_1 + n_2 + n_3$ and $j = n_3 + 1, \cdots, n - 1$. It is very similar to Case (ii), we know that λ_n is constant, which gives also a contradiction.

Secondly, considering $g = 5$, in this case we assume the principal curvatures are $\lambda_a, \lambda_b, \lambda_c, \lambda_d = \lambda_e, \lambda_n$ and the corresponding multiplicities are $n_1, n_2, n_3, n_4 + n_5, 1$ ($\sum_{i=1}^5 n_i = n - 1$). According to Eqs (2.35)–(2.37), we only consider the case of existing one, and two ones of ω_{ij}^k are not zero. A similar discussion to $g = 6$ gives the conclusion.

Thirdly, $g = 4$, in this case, we assume the principal curvatures are $\lambda_a, \lambda_b, \lambda_c = \lambda_d = \lambda_e, \lambda_n$ and the corresponding multiplicities are $n_1, n_2, n_3 + n_4 + n_5, 1$ ($\sum_{i=1}^5 n_i = n - 1$). According to Eqs (2.35)–(2.37), we only consider the case of existing one ω_{ij}^k are not zero. A similar discussion can gives the conclusion.

At last, $g < 4$. We could refer to the proof of Liu and Yang [16] with adding additional condition $aR + bH = c$.

In conclusion, we have proved that mean curvature H is constant, and so the scalar curvature is constant as well because of the linear Weingarten condition $aR + bH = c$. We complete the proof of Theorem 1.2. □

Interestingly, hypersurfaces satisfying $\Delta\vec{H} = \lambda\vec{H}$ have constant mean curvature and constant scalar curvature under some geometric assumptions. At the end of this paper, we supply two examples of such hypersurfaces satisfying $\Delta\vec{H} = \lambda\vec{H}$.

Example 3.1. A hypersurface $M^n = \mathbb{R}_v^r \times \mathbb{H}_{s-v-1}^{n-r}(c)$ in $\mathbb{E}_s^{n+1}$ for $c < 0$. It is easy to check that the hypersurface M^n has two distinct principal curvature

$$\lambda_1 = \ldots = \lambda_r = 0, \qquad \lambda_{r+1} = \ldots = \lambda_n = \sqrt{-c}.$$

The mean curvatue H and the scalar curvature R of this hypersurface M^n are constants.

Example 3.2. ([15]) A hypersurface M^n in $\mathbb{E}_1^{n+1}$ is given by

$$\begin{aligned} & f(t, s, x_1, u_2, \ldots, u_{n-2}) \\ = & \left(\frac{s}{2}(t^2 + \sum_{j=1}^{n-2} u_j^2) + \phi + a_1 u_1^2 + a_2 u_2^2 + a_3 u_3^2 + a_4 \sum_{i=4}^{n-2} u_i^2, \right. \\ & \quad u_1(s + 2a_1), \quad st, \quad u_2(s + 2a_2), \quad u_3(s + 2a_3), \\ & \quad u_4(s + 2a_4), \quad u_5(s + 2a_4), \quad \ldots \quad u_{n-2}(s + 2a_4), \\ & \left. \quad a_1 u_1^2 + a_2 u_2^2 + a_3 u_3^2 + a_4 \sum_{i=4}^{n-2} u_i^2 + \frac{s}{2}(t^2 + \sum_{j=1}^{n-2} u_j^2) + \phi - s \right), \end{aligned}$$

where $\phi = \phi(s)$ is a smooth function such that $2\phi' - 1 > 0$, $s, t, u_1, u_2, \ldots, u_{n-2} \in \mathbb{R}$, $2a_i + s > 0$ and a_i are all distinct constants for $i = 1, 2, 3, 4$. In Remark 3 of [15], the author showed that M^n has constant mean curvature, and thus has constant scalar curvature by the linear Weingarten assumption.

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Conflict of interest

The authors declare no conflict of interest.

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