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Research article

Preventing extinction in *Rastrelliger brachysoma* using an impulsive mathematical model

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Abstract: In this paper, we proposed a mathematical model of the population density of Indo-Pacific mackerel (*Rastrelliger brachysoma*) and the population density of small fishes based on the impulsive fishery. The model also considers the effects of the toxic environment that is the major problem in the water. The developed impulsive mathematical model was analyzed theoretically in terms of existence and uniqueness, positivity, and upper bound of the solution. The obtained solution has a periodic behavior that is suitable for the fishery. Moreover, the stability, permanence, and positive of the periodic solution are investigated. Then, we obtain the parameter conditions of the model for which Indo-Pacific mackerel conservation might be expected. Numerical results were also investigated to confirm our theoretical results. The results represent the periodic behavior of the population density of the Indo-Pacific mackerel and small fishes. The outcomes showed that the duration and quantity of fisheries were the keys to prevent the extinction of Indo-Pacific mackerel.

Keywords: impulsive model; Indo-Pacific mackerel; stability; harvesting control; fisheries
Mathematics Subject Classification: 92-10, 34A37, 37G15

1. Introduction

One of the most important foods for human is fish. It is about 35% of protein humans consumed. The South Pacific jack mackerel (JM) *(Trachurus murphyi)* is one of the most important fish for consumption in the world [6]. The Atlantic Mackerel *(Scomber scombrus)* is one of the most fish species in the North Atlantic which is widely spawned to produce human's food [10]. The India mackerel *(Rastrilliger kangurta)* [4] and Spanish mackerel *(Scomberomorus commerson)* [17] are also the fish production using for human consumption. In the Asia-Pacific region, Indo-Pacific mackerel *(Rastrelliger brachysoma)* or short mackerel is a fish indigenous in this region. This species spread almost in coasts and islands in the Gulf of Thailand and the Andaman sea [5]. It is about 14–20 centimeters long. It can spawn about $20,000$ eggs at one time. Short mackerel is a pelagic fish that is highly economically important in the region because it is cheap and tasty [12]. It is therefore widely consumed in this region. So, there is high demand in South East Asia, resulting in numerous fisheries.

In 1995, the Department of Fisheries, Ministry of Agriculture and Cooperatives, Thailand reported that short mackerels spawn two periods which are January to March and June to August each year. Moreover, they spawn more than 20 meters below sea level. Recently, the Department of Fisheries reports that the number of catching short mackerel drastically decreased every year. In 2014, on average 145,000 tons were caught per year 70,000 tons were caught in 2015, then 31,000 tons, 25,000 tons and 17,700 tons in 2016–2018, respectively [18]. The major problem is catching fish in the spawning season. The economic value loss caused by catching one ton of Indo-Pacific mackerel fry is about seven to eight million baht. The government proposed a law to protect fry from fisheries in 2015.

The mathematical model is used for preparing to predict future phenomena in real-world problems. The model allows simulating any possibility. Many mathematical models have been used to describe the dynamic of the fish population. Bueno et al. reviewed the mathematical model for managing the carrying capacity of aquaculture activities in lakes [3]. The model of fish population dynamic was proposed by Hallam et al. by using the ordinary differential equations [7]. The model studied the changes in lipid and structure components which considered the energy demand and available energy. Khatun et al. proposed the mathematical model to studied renewable fishery management [11]. The two new second-order characteristic scheme was proposed to solve an age-structured population model with nonlinear diffusion and reaction [15] which used to consider the environmental and spatial region effect. Raymond et al. proposed the model to describe the dynamics of a two-prey one predator system in fishery [21].

In addition, the toxicity in the sea is one of the problems which affect the population of short mackerels and their food. Bergami et al. studied the effect of plastic pollution in the marine ecosystem especially nanoplastics [2] to the plankton species which is the food of many fishes in the marine. Hashiguchi et al. identified and evaluated the toxicity of palm oil mill which affected the plankton species [8]. The antifouling compound zinc pyrithione (ZPT) was studied the effect on the natural planktonic communities by Hjorth et al. [9]. They found evidence of the diverse effect of ZPT on marine plankton. Kaur et al. proposed the dynamical model to study the effect of toxicity to the plankton system [19]. Randall and Tsui studied the effect of ammonia in the aquatic environment on the central nervous system of fish [20].

To prevent the extinction of Indo-Pacific mackerels, we propose a mathematical model to find a way to control the catching of the Indo-Pacific mackerels by using the impulsive model. The novelty of this work is the prey-predator model of Indo-Pacific mackerels (short mackerels) and their foods which composes the toxic environment and impulsive fisheries. The impulsive model is a suitable model for fisheries because the fishermen are allowed to harvest fish on some periods for preventing the extinction of fish. The proposed model can be forecast the population of short mackerel and short mackerel food (plankton and small fishes). We analyze the model to consider the behavior of the model solution. The numerical simulations are used to verify the theoretical results.

The rest of this paper is organized as follows: The modification model is proposed in Section 2, and model analysis is presented in Section 3. In Section 4, numerical simulations are showed, followed by conclusions in Section 5.

2. Model modification

Now, an impulsive mathematical model was proposed in which the control of the extinction of the short mackerel by considering biological control, toxic environment, and impulsive fisheries are as follows:

For $t \neq nT$,

$$\frac{dx_1}{dt} = a_1 x_1 \left(1 - \frac{x_1}{k_1}\right) - \frac{b x_1 x_2}{k_2 + x_1} - d_1 x_1 - u_1 x_1^2, \tag{2.1}$$

$$\frac{dx_2}{dt} = a_2 x_2 \left(1 - \frac{x_2}{k_3}\right) + \frac{r b x_1 x_2}{k_2 + x_1} - d_2 x_2 - u_2 x_2^2. \tag{2.2}$$

For $t = nT$,

$$x_1(t^+) = (1 - \gamma) x_1(t), \tag{2.3}$$

$$x_2(t^+) = (1 - \omega) x_2(t), \tag{2.4}$$

where $x_1(t)$ is the density of small fishes (short mackerel food) population at time t, $x_2(t)$ is the density of the short mackerel population at time t, T is the period of impulsive fisheries, γ is the negative effect of fisheries on the density of small fishes population, and ω is the negative effect of fisheries on the density of short mackerel population with $n \in \mathbb{Z}_+$, $\mathbb{Z}_+ = \{1, 2, 3, ...\}$, $0 < \gamma < 1$, and $0 < \omega < 1$. All parameters in the model are non-negative.

Equation (2.1) expresses the rate of change of the population density of short mackerel food (small fishes). The small fishes increase by the logistic function with the growth rate a_1 and the carrying capacity k_1. On the other hand, the density of them decreases due to being hunted by short mackerel with rate b, natural death with the rate d_1, and toxic death with the rate u_1.

Equation (2.2) expresses the rate of change of the population density of the short mackerel *(Rastrelliger brachysoma)*. The short mackerels increase by the logistic function with the growth rate a_2 and the carrying capacity k_3. On the other hand, the density of them increases due to hunting small fishes with the rate r, natural death with the rate d_2, and toxic death with the rate u_2.

3. Model analysis

Let

$$V : \mathbb{R}_+ \times \mathbb{R}_+^2 \to \mathbb{R}_+, \tag{3.1}$$

where $\mathbb{R}_+ = [0, \infty), \mathbb{R}_+^2 = \{X \in \mathbb{R}^2 : X = (x_1, x_2), x_1 \geq 0, x_2 \geq 0\}$. The map defined by the right hand side of system (2.1)–(2.4) is denoted by $F = (F_1, F_2)$.

Definition 3.1 ([1]). *The function V defined in* (3.1) *is said to belong to class* V_0 *if*

(a) V is continuous in $(nT, (n+1)T] \times \mathbb{R}_+^2 \to \mathbb{R}_+$ *and for each* $X \in \mathbb{R}_+^2, n \in \mathbb{Z}_+$, $\lim_{(t,Y)\to(nT^+,X)} V(t, Y) = V(nT^+, X)$ *exists.*

(b) V is locally Lipschitzian in X.

Definition 3.2 ([1]). *Suppose $V \in V_0$. For $(t, X) \in (nT, (n+1)T] \times \mathbb{R}^2_+$, the upper right derivative of $V(t, X)$ with respect to system* (2.1)–(2.4) *is defined by*

$$D^+V(t, X) = \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, X + hF(t, X)) - V(t, X)]$$

where $F = (F_1, F_2)$.

The solution of system (2.1)–(2.4), $X(t) = (x_1(t), x_2(t))$, is assumed to be a piecewise continuous function. It means that, $X(t)$ is continuous on $(nT, (n+1)T]$, $n \in \mathbb{Z}_+$ and $\lim_{t \to nT^+} X(t) = X(nT^+)$ exists. By the smoothness properties of F, the system (2.1)–(2.4) has a unique solution.

Lemma 3.3. *Suppose $X(t) = (x_1(t), x_2(t))$ is a solution of the system* (2.1)–(2.4) *with the initial value $X(0^+) \geq 0$. Then the solution $X(t) \geq 0$ for all $t \geq 0$.*

Proof. The solution $x_1(t)$ with non-negative initial value can be negative when the slope of $x_1(t)$ at 0 is negative. So, the proof of this Lemma can be expressed as follows:

Case $t \neq nT$.

Consider the Eq (2.1)

$$\frac{dx_1}{dt} = a_1 x_1 \left(1 - \frac{x_1}{k_1}\right) - \frac{b x_1 x_2}{k_2 + x_1} - d_1 x_1 - u_1 x_1^2.$$

Whenever $x_1(t) = 0$, the slope of $x_1(t)$ can be described by $\frac{dx_1}{dt} = 0$. This means that $x_1(t)$ cannot be negative. So, $x_1(t)$ is a non-negative solution.

Consider the Eq (2.2)

$$\frac{dx_2}{dt} = a_2 x_2 \left(1 - \frac{x_2}{k_3}\right) + \frac{r b x_1 x_2}{k_2 + x_1} - d_2 x_2 - u_2 x_2^2.$$

Whenever $x_2(t) = 0$, the slope of $x_2(t)$ can be described by $\frac{dx_2}{dt} = 0$. This means that $x_2(t)$, cannot be negative. So, $x_2(t)$ is a non-negative solution.

Case $t = nT$.

Consider the Eq (2.3)

$$x_1(t^+) = (1 - \gamma) x_1(t).$$

Since $x_1 \geq 0$ and condition $0 < \gamma < 1$, we have that $x_1(t^+) \geq 0$.

Consider the Eq (2.4)

$$x_2(t^+) = (1 - \omega) x_2(t).$$

Since $x_2 \geq 0$ and condition $0 < \omega < 1$, we have that $x_2(t^+) \geq 0$. □

Lemma 3.4. *The solution $X(t) = (x_1(t), x_2(t))$ has upper bound i.e. $x_1(t) \leq M$ and $x_2(t) \leq M$, for sufficiently large t, provided that*

$$d_2 > \frac{br}{k_2}. \tag{3.2}$$

Proof. We let $V(t) = x_1(t) + x_2(t)$,

$$M_1 = \max\left(a_1x_1\left(1 - \frac{x_1}{k_1}\right) - u_1x_1^2\right) = \frac{a_1^2k_1}{4(a_1 + u_1k_1)},$$

$$M_2 = \max\left(a_2x_2\left(1 - \frac{x_2}{k_3}\right) - u_2x_2^2\right) = \frac{a_2^2k_3}{4(a_2 + u_2k_3)},$$

and

$$M_3 = \sup\left(\frac{rbx_1}{1 + k_2x_1}\right) = \frac{rb}{k_2}.$$

Consider $t \neq nT$, we choose $\hat{c} > 0$ and

$$\hat{c} = \min\{d_1, d_2 - M_3\}.$$

Then,

$$\begin{aligned}
D^+V + \hat{c}V =& \frac{dx_1}{dt} + \frac{dx_2}{dt} + \hat{c}x_1 + \hat{c}x_2 \\
=& a_1x_1\left(1 - \frac{x_1}{k_1}\right) - \frac{bx_1x_2}{1 + k_2x_1} - d_1x_1 - u_1x_1^2 + a_2x_2\left(1 - \frac{x_2}{k_3}\right) \\
&+ \frac{rbx_1x_2}{1 + k_2x_1} - d_2x_2 - u_2x_2^2 + \hat{c}x_1 + \hat{c}x_2 \\
\leq& (\hat{c} - d_1)x_1 + (\hat{c} + M_3 - d_2)x_2 + M_1 + M_2 \\
\leq& M_1 + M_2 \\
\equiv& M_0.
\end{aligned}$$

Hence $D^+V + \hat{c}V \leq M_0$.

Consider $t = nT$,

$$\begin{aligned}
V(nT^+) &= x_1(nT^+) + x_2(nT^+) \\
&= (1 - \gamma)x_1(nT) + (1 - \omega)x_2(nT) \\
&= x_1(nT) + x_2(nT) - \gamma x_1(nT) - \omega x_2(nT) \\
&\leq V(nT).
\end{aligned}$$

By Lemma 2.2 of Liu et al. [16] we obtain that, for $t \in (nT, (n + 1)T]$,

$$\begin{aligned}
V(t) &\leq V(0)e^{-\hat{c}t} + \int_0^t M_0e^{-\hat{c}(t-s)}ds \\
&\leq V(0)e^{-\hat{c}t} + M_0\left(\frac{1}{\hat{c}} - \frac{e^{-\hat{c}t}}{\hat{c}}\right) \\
&< \frac{M_0}{\hat{c}} \equiv M \quad \text{as} \quad t \to \infty.
\end{aligned}$$

Since there exists $M > 0$ such that $x_1(t) \leq M$ and $x_2(t) \leq M$ for sufficiently large t thus $V(t)$ is uniformly ultimately bounded. □

3.1. The reduced impulsive system

The reduced impulsive system of system (2.1)–(2.4) when the population density of small fishes is zero ($x_1 = 0$) is:

$$\begin{aligned}\frac{dx_2}{dt} &= Bx_2 - Ax_2^2, \quad t \neq nT &(3.3)\\ x_2(nT^+) &= (1-\omega)x_2(nT), \quad t = nT &(3.4)\\ x_2(0^+) &= x_{20} &(3.5)\end{aligned}$$

where $A \equiv \frac{a_2}{k_3} + u_2 > 0$ and $B \equiv a_2 - d_2$.

We obtain $B > 0$ if

$$a_2 > d_2. \tag{3.6}$$

The solution of Eq (3.3) is

$$\frac{1}{x_2(t)} = \frac{A}{B} + ce^{-Bt} \tag{3.7}$$

where c is arbitrary constant.

By the conditions (3.4), (3.6) and x_2 is increasing function, a periodic solution of system (3.3)–(3.5) is

$$\frac{1}{\tilde{x}_2(t)} = \frac{A}{B} + \frac{\omega A e^{-B(t-nT)}}{B(1-\omega-e^{-BT})}, \qquad t \in (nT, (n+1)T] \tag{3.8}$$

with $\tilde{x}_2(0^+) = \frac{B(1-\omega-e^{-BT})}{A(1-e^{-BT})} > 0$.

Therefore, the system (3.3)–(3.5) has the positive solution

$$\frac{1}{x_2(t)} = \left(\frac{1}{x_{20}} - \frac{A}{B} - \frac{\omega A}{B(1-\omega-e^{-BT})}\right)e^{-Bt} + \frac{1}{\tilde{x}_2(t)}, \qquad t \in (nT, (n+1)T]. \tag{3.9}$$

Lemma 3.5. *The periodic solution $\tilde{x}_2(t)$ of system* (3.3)–(3.5) *exists and $x_2(t) \to \tilde{x}_2(t)$ as $t \to \infty$ for all solution $x_2(t)$ of system* (3.3)–(3.5)*. Hence,*

$$(0, \tilde{x}_2(t)) = \left(0, \frac{B(1-\omega-e^{-BT})}{\omega A e^{-B(t-nT)} + A(1-\omega-e^{-BT})}\right), \quad t \in (nT, (n+1)T]$$

is a periodic solution of the original system (2.1)–(2.4) *at the zero density of small fishes for $t \in (nT, (n+1)T]$ and*

$$\tilde{x}_2(nT^+) = \tilde{x}_2(0^+) = \frac{B(1-\omega-e^{-BT})}{A(1-e^{-BT})}, \qquad n \in \mathbb{Z}_+.$$

Theorem 3.6. *Suppose*

$$T_{min} < T < T_{max}, \tag{3.10}$$

$$a_1 > d_1 + \frac{bB}{k_2 A}, \tag{3.11}$$

and

$$\ln\left(\frac{1}{1-\gamma}\right) > \frac{b}{k_2 A}\ln\left(\frac{1}{1-\omega}\right), \tag{3.12}$$

where

$$T_{min} \equiv \frac{1}{B}\ln\left(\frac{1}{1-\omega}\right),$$

and

$$T_{max} \equiv \frac{1}{a_1 - d_1 - \dfrac{bB}{k_2A}}\left[\ln\left(\frac{1}{1-\gamma}\right) - \frac{b}{k_2A}\ln\left(\frac{1}{1-\omega}\right)\right].$$

Then the solution $(0, \tilde{x}_2(t))$ *of the system* (2.1)–(2.4) *is locally asymptotically stable.*

Proof. Here, we focus on a small perturbation $(v_1(t), v_2(t))$ from the point $(0, \tilde{x}_2(t))$:

$$\begin{aligned} x_1(t) &= v_1(t), \\ x_2(t) &= \tilde{x}_2(t) + v_2(t). \end{aligned}$$

Then,

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \Phi(t)\begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix}, \quad 0 < t < T$$

where $\Phi(t)$ satisfies

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} a_1 - d_1 - \dfrac{b\tilde{x}_2(t)}{k_2} & 0 \\ * & B - 2A\tilde{x}_2(t) \end{pmatrix}\Phi(t)$$

with the identity matrix $\Phi(0) = I$.

Therefore, the matrix of fundamental solution is

$$\Phi(t) = \begin{pmatrix} \exp\int_0^t \left(a_1 - d_1 - \dfrac{b\tilde{x}_2(s)}{k_2}\right)ds & 0 \\ ** & \exp\int_0^t (B - 2A\tilde{x}_2(s))\,ds \end{pmatrix}.$$

Note that the terms (*) and (**) are not necessary to be calculated because the further analysis does not require these terms.

Linearization of Eqs (2.3)–(2.4) yields

$$\begin{pmatrix} v_1(nT^+) \\ v_2(nT^+) \end{pmatrix} = \begin{pmatrix} 1-\gamma & 0 \\ 0 & 1-\omega \end{pmatrix}\begin{pmatrix} v_1(nT) \\ v_2(nT) \end{pmatrix}.$$

The eigenvalues of the following matrix W,

$$W = \begin{pmatrix} 1-\gamma & 0 \\ 0 & 1-\omega \end{pmatrix}\Phi(T),$$

are

$$\begin{aligned} \lambda_1 &= (1-\gamma)\exp\left((a_1 - d_1)T - \frac{b}{k_2A}\left[\ln(1-\omega) + BT\right]\right), \\ \lambda_2 &= (1-\omega)\exp\left(-BT - 2\ln(1-\omega)\right). \end{aligned}$$

Since $0 < \gamma < 1, 0 < \omega < 1$, and the conditions (3.6), (3.10)–(3.12) hold, it implies that

$$\frac{1}{B}\ln\left(\frac{1}{1-\omega}\right) < T < \frac{1}{a_1 - d_1 - \dfrac{bB}{k_2 A}}\left[\ln\left(\frac{1}{1-\gamma}\right) - \frac{b}{k_2 A}\ln\left(\frac{1}{1-\omega}\right)\right].$$

Therefore, $|\lambda_1| < 1$ and $|\lambda_2| < 1$. We can conclude that the solution $(0, \tilde{x_2}(t))$ of the system (2.1)–(2.4) is locally asymptotically stable by Floquet theory. The proof is completed. □

3.2. Permanence of system

Definition 3.7 ([1]). *The system* (2.1)–(2.4) *is permanent if the solution is bounded. That is, there are positive constants* $\bar{a}$ *and* $\bar{b}$ *and a finite time* t_0 *such that for all solution with the positive initial values* $x_1(0^+) > 0$ *and* $x_2(0^+) > 0$

$$\bar{a} \le x_1(t) \le \bar{b},$$
$$\bar{a} \le x_2(t) \le \bar{b},$$

for all $t > t_0$.

Theorem 3.8. *The system* (2.1)–(2.4) *is permanent if inequalities* (3.2), (3.6), (3.11), (3.12) *hold and the following conditions:*

$$T > T_{max} \tag{3.13}$$

and

$$A > B + M_3 \tag{3.14}$$

are satisfied.

Proof. By Lemma 3.4, there is a constant $M > 0$ such that the solution $x_1(t) \le M$ and $x_2(t) \le M$ for sufficiently large t.

Since $\dfrac{rbx_1x_2}{1+k_2x_1} \ge 0$, Eq (2.2) implies that

$$\frac{dx_2}{dt} = a_2x_2\left(1 - \frac{x_2}{k_3}\right) + \frac{rbx_1x_2}{k_2 + x_1} - d_2x_2 - u_2x_2^2 \ge Bx_2 - Ax_2^2, \quad t \ne nT$$
$$x_2(nT^+) = (1-\omega)x_2(nT), \quad t = nT$$

then we obtain

$$x_2(t) > \tilde{x}_2(t) - \epsilon$$

for some positive ϵ and large enough t.

Hence,

$$x_2(t) > \frac{B(1-\omega-e^{-BT})}{A(1-e^{-BT})} - \epsilon \equiv m_1$$

for large enough t.

Hence, the remaining proof is the system has a lower bound. That is, there exists a positive constant m_2 such that $x(t) > m_2$. To obtain the result, we let

$$\hat{M}_1 = a_1\left(1 - \frac{m_3}{k_1}\right) - d_1 - u_1 m_3,$$

for some $m_3 > 0$.

Next, there are two steps as follows:

Step 1. Prove by contradiction that there is t_1 such that $x_1(t_1) \geq m_3$. Suppose that $x_1(t) < m_3$ for all $t > 0$. From Eqs (2.2) and (2.4), we get

$$\begin{aligned} \frac{dx_2}{dt} &= a_2x_2\left(1 - \frac{x_2}{k_3}\right) + \frac{rbx_1x_2}{k_2 + x_1} - d_2x_2 - u_2x_2^2, \quad t \neq nT \\ &\leq a_2x_2\left(1 - \frac{x_2}{k_3}\right) + M_3x_2 - d_2x_2 - u_2x_2^2 \\ &= (B + M_3)x_2 - Ax_2^2, \\ & \qquad x_2(t^+) = (1-\omega)x_2(t), \quad t = nT. \end{aligned}$$

Let us consider the comparison system

$$\frac{dY}{dt} = (B + M_3)Y - AY^2, \quad t \neq nT \tag{3.15}$$

$$Y(t^+) = (1-\omega)Y(t), \quad t = nT \tag{3.16}$$

and

$$Y(0^+) = x_2(0^+). \tag{3.17}$$

Hence,

$$\frac{1}{\tilde{Y}(t)} = \frac{A}{(B+M_3)} + \frac{\omega A e^{-(B+M_3)(t-nT)}}{(B+M_3)(1-\omega-e^{-(B+M_3)T})}, \qquad t \in (nT, (n+1)T] \tag{3.18}$$

is a positive periodic solution of system (3.15)–(3.17) with

$$\frac{1}{Y(0^+)} = \frac{A}{(B+M_3)} + \frac{\omega A}{(B+M_3)(1-\omega-e^{-(B+M_3)T})} > 0.$$

The system (3.15)–(3.17) has a positive solution

$$\frac{1}{Y(t)} = \left(-\frac{\omega A}{(B+M_3)(1-\omega-e^{-(B+M_3)T})} + \frac{1}{Y(0^+)} - \frac{A}{(B+M_3)}\right)e^{-(B+M_3)t} + \frac{1}{\tilde{Y}(t)}, \tag{3.19}$$

with $t \in (nT, (n+1)T]$ and $\frac{1}{Y(t)} \to \frac{1}{\tilde{Y}(t)}$ as $t \to \infty$, where

$$\frac{1}{\tilde{Y}(t)} = \frac{\omega A e^{-(B+M_3)(t-nT)}}{(B+M_3)(1-\omega-e^{-(B+M_3)T})} + \frac{A}{(B+M_3)}.$$

By the comparison theorem in [14], we obtain that

$$x_2(t) \leq Y(t).$$

Now, we consider (2.1)

$$\begin{aligned}\frac{dx_1}{dt} &= a_1x_1\left(1-\frac{x_1}{k_1}\right)-\frac{bx_1x_2}{k_2+x_1}-d_1x_1-u_1x_1^2\\ &\geq \left(a_1\left(1-\frac{m_3}{k_1}\right)-\frac{bx_2}{k_2}-d_1-u_1m_3\right)x_1\\ &= \left(\hat{M}_1-\frac{bx_2}{k_2}\right)x_1.\end{aligned}$$

Since $x_2(t) \leq Y(t)$, there is a $T^* > 0$ such that,

$$x_2(t) \leq Y(t) < \tilde{Y}(t)+\epsilon_1, \quad t \neq nT, t \geq T^*$$

for a sufficiently small $\epsilon_1 > 0$.

Therefore,

$$\frac{dx_1}{dt} > \left(\hat{M}_1-\frac{b}{k_2}\left(\tilde{Y}(t)+\epsilon_1\right)\right)x_1, \tag{3.20}$$

for $t \neq nT, t \geq T^*$ and

$$x_1(t^+) = (1-\gamma)x_1(t), \tag{3.21}$$

for $t = nT, t \geq T^*$.

Letting $N \in \mathbb{Z}_+$ and $NT \geq T^*$, and integrating over $(nT,(n+1)T], n \geq N$, we get

$$\begin{aligned}x_1((n+1)T) &\geq x_1(nT)(1-\gamma)\exp\left(\int_{nT}^{(n+1)T}\left(\hat{M}_1-\frac{b}{k_2}\left(\tilde{Y}(t)+\epsilon_1\right)\right)dt\right)\\ &= x_1(nT)(1-\gamma)\exp\left(\left(\hat{M}_1-\frac{b}{k_2}\epsilon_1-\frac{b(B+M_3)}{k_2A}\right)T+\frac{b}{k_2A}\ln\left(\frac{1}{1-\omega}\right)\right)\\ &= x_1(nT)\eta,\end{aligned}$$

where

$$\eta \equiv (1-\gamma)\exp\left(\left(\hat{M}_1-\frac{b}{k_2}\epsilon_1-\frac{b(B+M_3)}{k_2A}\right)T+\frac{b}{k_2A}\ln\left(\frac{1}{1-\omega}\right)\right).$$

Consider

$$\ln\eta = \ln(1-\gamma)+\left(\hat{M}_1-\frac{b}{k_2}\epsilon_1-\frac{b(B+M_3)}{k_2A}\right)T+\frac{b}{k_2A}\ln\left(\frac{1}{1-\omega}\right).$$

For sufficiently small $\epsilon_1 > 0$,

$$\ln\eta \approx -\ln\left(\frac{1}{1-\gamma}\right)+\left(\hat{M}_1-\frac{b(B+M_3)}{k_2A}\right)T+\frac{b}{k_2A}\ln\left(\frac{1}{1-\omega}\right).$$

Since $\hat{M}_1 < a_1-d_1$ and (3.11) is satisfied, a small positive m_3 is chosen so that $\ln\eta > 0$.

We get

$$\eta \equiv (1-\gamma)\exp\left(\left(\hat{M}_1 - \frac{b}{k_2}\epsilon_1 - \frac{b(B+M_3)}{k_2 A}\right)T + \frac{b}{k_2 A}\ln\left(\frac{1}{1-\omega}\right)\right) > 1. \tag{3.22}$$

We observe that $x_1((n+k)T) \geq x_1(nT)\eta^k \to \infty$ as $k \to \infty$ which contradicts the boundedness of $x_1(t)$. Therefore, there exists $t_1 > 0$ such that $x_1(t_1) \geq m_3$.

Step 2. The proof is completed if $x_1(t) \geq m_3$ for all $t > t_1$. Otherwise, $x_1(t) < m_3$ for some $t > t_1$. Setting $t^* = \inf_{t>t_1}\{x_1(t) < m_3\}$. There are two cases as follows:

Case 1: $t^* = n_1 T$ for some $n_1 \in Z_+$. That is $x_1(t) \geq m_3$ for $t \in (t_1, t^*]$ and $x_1(t^*) = m_3$ by continuity of the solution $x_1(t)$.

Since $x_1(t) < M$ and $m_1 < x_2(t) < M$ for some positive M and m_1 with sufficiently large t, we can choose $\bar{M} > 0$ and $\bar{m}_1 > 0$ so that

$$x_1(t) < \bar{M} \text{ and } \bar{m}_1 < x_2(t) < \bar{M}$$

and

$$\hat{M}_1 < \frac{b}{k_2}\bar{M}, \tag{3.23}$$

such that

$$\frac{1}{\bar{m}_1} > \left|\frac{1}{x_2(t^{*+})} - \frac{A}{(B+M_3)} - \frac{\omega A}{(B+M_3)(1-\omega-e^{-(B+M_3)T})}\right| - \omega. \tag{3.24}$$

Then, we choose $n_2, n_3 \in \mathbb{Z}_+$ such that

$$n_2 T > \frac{1}{(B+M_3)}\ln\left(\frac{\frac{1}{\bar{m}_1}+\omega}{\epsilon_1}\right) \tag{3.25}$$

and

$$(1-\gamma)^{n_2}\exp((n_2+1)\eta_1 T)\eta^{n_3} > 1, \tag{3.26}$$

where

$$\eta_1 \equiv \hat{M}_1 - \frac{b}{k_2}\bar{M} < 0.$$

Define $T' = n_2 T + n_3 T$. There exists $t_2 \in (t^*, t^* + T']$ so that $x_1(t_2) > m_3$.

Otherwise, considering Eq (3.19) with

$$\frac{1}{Y(t^{*+})} = \frac{1}{x_2(t^{*+})},$$

we obtain

$$\frac{1}{Y(t)} = \left(-\frac{\omega A}{(B+M_3)(1-\omega-e^{-(B+M_3)T})} + \frac{1}{Y(t^{*+})} - \frac{A}{(B+M_3)}\right)e^{-(B+M_3)(t-t^*)} + \frac{1}{\tilde{Y}(t)}$$

for $t \in (nT, (n+1)T]$ where $n_1 \leq n \leq n_1 + n_2 + n_3$.

For $n_2 T \leq t - t^* \leq T'$, we have

$$
\begin{aligned}
\left|\frac{1}{Y(t)}-\frac{1}{\tilde{Y}(t)}\right| &= \left|-\frac{\omega A}{(B+M_3)(1-\omega-e^{-(B+M_3)T})}+\frac{1}{Y(t^{*+})}-\frac{A}{(B+M_3)}\right| e^{-(B+M_3)(t-t^*)} \\
&= \left|-\frac{\omega A}{(B+M_3)(1-\omega-e^{-(B+M_3)T})}+\frac{1}{x_2(t^{*+})}-\frac{A}{(B+M_3)}\right| e^{-(B+M_3)(t-t^*)} \\
&< \left(\frac{1}{\bar{m}_1}+\omega\right)e^{-(B+M_3)(t-t^*)} \\
&< \left(\frac{1}{\bar{m}_1}+\omega\right)e^{-(B+M_3)n_2T} \\
&< \epsilon_1.
\end{aligned}
$$

Since the condition (3.14), we get

$$
|Y(t)-\tilde{Y}(t)| < \frac{|Y(t)-\tilde{Y}(t)|}{|Y(t)\tilde{Y}(t)|} = \left|\frac{1}{Y(t)}-\frac{1}{\tilde{Y}(t)}\right| < \epsilon_1.
$$

Then,

$$
x_2(t) \le Y(t) < \tilde{Y}(t)+\epsilon_1.
$$

According to Step 1, we obtain

$$
\begin{aligned}
x_1(t^*+T') &= x_1(n_1T+n_2T+n_3T) \\
&\ge x_1(t^*+n_2T)\eta^{n_3}.
\end{aligned}
$$

From Eq (2.1), we have

$$
\begin{aligned}
\frac{dx_1}{dt} &= a_1x_1\left(1-\frac{x_1}{k_1}\right)-\frac{bx_1x_2}{k_2+x_1}-d_1x_1-u_1x_1^2 \\
&\ge \left(a_1\left(1-\frac{m_3}{k_1}\right)-\frac{bx_2}{k_2}-d_1-u_1m_3\right)x_1 \\
&= \left(\hat{M}_1-\frac{bx_2}{k_2}\right)x_1 \\
&\ge \left(\hat{M}_1-\frac{b}{k_2}\bar{M}\right)x_1 \\
&= \eta_1x_1, \quad t\ne nT \\
x_1(t^+) &= (1-\gamma)x_1(t), \quad t=nT.
\end{aligned}
\tag{3.27}
$$

Integrating inequality (3.27) over $[t^*, t^*+n_2T]$, we have

$$
\begin{aligned}
x_1(t^*+n_2T) &\ge x_1(t^*)(1-\gamma)^{n_2}\exp\left(\int_{n_1T}^{n_1T+n_2T}\eta_1dt\right) \\
&\ge m_3(1-\gamma)^{n_2}\exp(n_2\eta_1T) \\
&\ge m_3(1-\gamma)^{n_2}\exp((n_2+1)\eta_1T),
\end{aligned}
$$

hence,

$$\begin{aligned} x_1(t^* + T') &\geq x_1(t^* + n_2T)\eta^{n_3} \\ &\geq m_3(1-\gamma)^{n_2}\exp((n_2+1)\eta_1 T)\eta^{n_3} \\ &> m_3. \end{aligned}$$

It is in contradiction to the definition of m_3. Therefore, there exists $t_2 \in (t^*, t^* + T']$ so that $x_1(t_2) > m_3$.

Now, we define $\tilde{t} = \inf_{t>t^*}\{x_1(t) > m_3\}$. This means that, $x_1(t) < m_3$ for $t \in (t^*, \tilde{t})$ and $x_1(\tilde{t}) = m_3$ by the continuity of $x_1(t)$. Then, we choose $p \in \mathbb{Z}_+$ so that $p \leq n_2 + n_3$ and $t^* + pT \geq \tilde{t}$, and suppose $t \in (t^* + (p-1)T, t^* + pT]$. By inequality (3.27), we get

$$\begin{aligned} x_1(t) &\geq x_1(t^{*+})(1-\gamma)^{p-1}\exp((p-1)\eta_1 T)\exp(\eta_1(t-(t^*+(p-1)T))) \\ &= x_1(t^*)(1-\gamma)^{p}\exp((p-1)\eta_1 T)\exp(\eta_1(t-(t^*+(p-1)T))) \\ &= m_3(1-\gamma)^p\exp(\eta_1(t-t^*)) \\ &\geq m_3(1-\gamma)^{n_2+n_3}\exp(\eta_1 pT) \\ &\geq m_3(1-\gamma)^{n_2+n_3}\exp((n_2+n_3)\eta_1 T). \end{aligned}$$

Since $\eta_1 < 0$ and $p \leq n_2 + n_3$.

Let

$$\bar{m}_2 = m_3(1-\gamma)^{n_2+n_3}\exp((n_2+n_3)\eta_1 T).$$

Thus, $x_1(t) \geq \bar{m}_2$ for $t \in (t^*, \tilde{t})$. Similarly, we use $\tilde{t}$ instead of t^*. Then, we will obtain $x_1(t) \geq \bar{m}_2$ for all sufficiently large t.

Case 2: $t^* \neq nT$ for all $n \in \mathbb{Z}_+$. That is $x_1(t) \geq m_3$ for $t \in [t_1, t^*)$ and $x_1(t^*) = m_3$. We assume that $t^* \in (\bar{n}_1 T, (\bar{n}_1+1)T)$, for some $\bar{n}_1 \in \mathbb{Z}_+$. We can consider this into two subcases.

Case 2.1: $x_1(t) \leq m_3$ for all $t \in (t^*, (\bar{n}_1+1)T]$. Suppose that there is $t_2' \in [(\bar{n}_1+1)T, (\bar{n}_1+1)T + T']$ so that $x_1(t_2') > m_3$. Otherwise, considering Eq (3.19) with

$$\frac{1}{Y((\bar{n}_1+1)T^+)} = \frac{1}{x_2((\bar{n}_1+1)T^+)}.$$

For $t \in (nT, (n+1)T]$, $\bar{n}_1 + 1 \leq n \leq \bar{n}_1 + 1 + n_2 + n_3$, we obtain

$$\frac{1}{Y(t)} = \left(\frac{1}{Y((\bar{n}_1+1)T^+)} - \frac{A}{(B+M_3)} - \frac{\omega A}{(B+M_3)(1-\omega-e^{-(B+M_3)T})}\right)e^{-(B+M_3)(t-(\bar{n}_1+1)T)} + \frac{1}{\tilde{Y}(t)}.$$

In a similar way to Case 1, for $n_2T \leq t - t^*$, we get

$$\left|Y(t) - \tilde{Y}(t)\right| < \epsilon_1.$$

Thus,

$$x_2(t) \leq Y(t) < \tilde{Y}(t) + \epsilon_1.$$

Since $n_2T \leq (\bar{n}_1 + 1 + n_2)T - t^*$, we get

$$\begin{aligned} x_1((\bar{n}_1+1+n_2)T) &\geq x_1(t^*)(1-\gamma)^{n_2}\exp(\eta_1((\bar{n}_1+1+n_2)T - t^*)) \\ &\geq m_3(1-\gamma)^{n_2}\exp(\eta_1((\bar{n}_1+1+n_2)T - \bar{n}_1 T)) \\ &\geq m_3(1-\gamma)^{n_2}\exp((n_2+1)\eta_1 T). \end{aligned}$$

Then,

$$\begin{aligned}x_1((\bar{n}_1+1+n_2+n_3)T) &\geq x_1((\bar{n}_1+1+n_2)T)\eta^{n_3}\\ &\geq m_3(1-\gamma)^{n_2}\exp((n_2+1)\eta_1 T)\eta^{n_3}\\ &> m_3.\end{aligned}$$

It is in contradiction to the definition of m_3. Thus, there exists $t_2' \in [(\bar{n}_1+1)T, (\bar{n}_1+1)T+T']$ so that $x_1(t_2') > m_3$.

Now, we define $\bar{t} = \inf_{t>t^*}\{x_1(t) > m_3\}$. Thus, $x_1(t) \leq m_3$ for $t \in [t^*, \bar{t})$, and $x_1(\bar{t}) = m_3$. We choose $p' \in Z_+$ such that $p' \leq n_2+n_3+1$ and suppose $t \in (\bar{n}_1 T + (p'-1)T, \bar{n}_1 T + p'T]$. From inequality (3.27), we get

$$\begin{aligned}x_1(t) &\geq x_1((\bar{n}_1 T + (p'-1)T)^+)\exp(\eta_1(t-(\bar{n}_1 T + (p'-1)T)))\\ &= x_1(\bar{n}_1 T + (p'-1)T)(1-\gamma)\exp(\eta_1(t-(\bar{n}_1 T + (p'-1)T)))\\ &\geq x_1(t^*)(1-\gamma)^{p'-1}\exp(\eta_1(t-t^*))\\ &\geq m_3(1-\gamma)^{p'-1}\exp(\eta_1(t-t^*)).\end{aligned}$$

Since $\eta_1 < 0$ and $t - t^* \leq p'T$. Then,

$$x_1(t) \geq m_3(1-\gamma)^{n_2+n_3}\exp((n_2+n_3+1)\eta_1 T).$$

Let

$$m_2 = m_3(1-\gamma)^{n_2+n_3}\exp((n_2+n_3+1)\eta_1 T).$$

Therefore, $x_1(t) \geq m_2$ for $t \in (t^*, \bar{t})$. We do the same way by using $\bar{t}$ instead of t^*. Then, we will obtain $x_1(t) \geq m_2$ for all sufficiently large t.

Case 2.2: There exists $t'' \in (t^*, (\bar{n}_1+1)T]$ so that $x_1(t'') > m_3$. Define $\underline{t} = \inf_{t>t^*}\{x_1(t) > m_3\}$. Hence, $x_1(t) < m_3$ for $t \in [t^*, \underline{t})$, and $x_1(\underline{t}) = m_3$. For $t \in [t^*, \underline{t})$, inequality (3.27) holds, we get

$$\begin{aligned}x_1(t) &\geq x_1(t^*)\exp\left(\int_{t^*}^{t}\eta_1 dt\right)\\ &= m_3\exp(\eta_1(t-t^*))\\ &\geq m_3\exp(\eta_1 T)\\ &> m_2,\end{aligned}$$

since $t < \bar{n}_1 T + T < t^* + T$. For $t > \underline{t}$, we can do the same way since $x_1(\underline{t}) \geq m_3$. Since $m_2 < \bar{m}_2 < m_3$, we can conclude that $x_1(t) \geq m_2$ for $t \geq t_1$. The proof is completed. □

3.3. *The positive periodic solution*

Now, we carry out the conditions to guarantee the positive periodic solution of the system (2.1)–(2.4) near the periodic solution $(0, \tilde{x}_2)$. For more convenience, we change the variables, and then the new system is shown as follows:

$$\frac{dx_1}{dt} = a_2 x_1\left(1-\frac{x_1}{k_3}\right) + \frac{rbx_1x_2}{k_2+x_2} - d_2x_1 - u_2x_1^2, \tag{3.28}$$

$$\frac{dx_2}{dt} = a_1x_2\left(1-\frac{x_2}{k_1}\right)-\frac{bx_1x_2}{k_2+x_2}-d_1x_2-u_1x_2^2, \tag{3.29}$$

for $t \neq nT$ with

$$x_1(nt^+) = (1-\omega)x_1(t), \quad t = nT, \tag{3.30}$$
$$x_2(nt^+) = (1-\gamma)x_2(t), \quad t = nT. \tag{3.31}$$

Let

$$F_1(x_1,x_2) = a_2x_1\left(1-\frac{x_1}{k_3}\right)+\frac{rbx_1x_2}{k_2+x_2}-d_2x_1-u_2x_1^2,$$
$$F_2(x_1,x_2) = a_1x_2\left(1-\frac{x_2}{k_1}\right)-\frac{bx_1x_2}{k_2+x_2}-d_1x_2-u_1x_2^2.$$

According to Lakmeche and Arini [13],

$$\Theta_1(x_1,x_2) = (1-\omega)x_1, \Theta_2(x_1,x_2) = (1-\gamma)x_2, \zeta(t) = (\tilde{x}_2(t),0)^T, X_0 = (\tilde{x}_2(\tau_0),0)^T, \tau_0 = T_{max},$$

and

$$\begin{aligned}
\frac{\partial\Phi_1(\tau_0,X_0)}{\partial\tau} &= \frac{\partial\tilde{x}_2(\tau_0,X_0)}{\partial t}\\
&= \frac{\omega A\exp(-B\tau_0)\tilde{x}_2^2(\tau_0,X_0)}{1-\omega-\exp(-B\tau_0)} > 0,\\
\frac{\partial\Phi_1(\tau_0,X_0)}{\partial x_1} &= \exp\left(\int_0^{\tau_0}\frac{\partial F_1(\zeta(s))}{\partial x_1}ds\right) > \frac{1}{1-\omega} > 0,\\
\frac{\partial\Phi_1(\tau_0,X_0)}{\partial x_2} &= \int_0^{\tau_0}\left[\exp\left(\int_\upsilon^{\tau_0}\frac{\partial F_1\zeta(s)}{\partial x_1}ds\right)\frac{\partial F_1(\zeta(\upsilon))}{\partial x_2}\ \exp\left(\int_0^{\upsilon}\frac{\partial F_2(\zeta(s))}{\partial x_2}ds\right)\right]d\upsilon\\
&= \int_0^{\tau_0}\left[\exp\left(\int_\upsilon^{\tau_0}(B-2A\tilde{x}_2(s))ds\right)\frac{rb\tilde{x}_2(\upsilon)}{k_2}\ \exp\left(\int_0^{\upsilon}\left(a_1-d_1-\frac{b\tilde{x}_2(s)}{k_2}\right)ds\right)\right]d\upsilon,\\
\frac{\partial\Phi_2(\tau_0,X_0)}{\partial x_2} &= \exp\left(\int_0^{\tau_0}\frac{\partial F_2(\zeta(s))}{\partial x_2}ds\right)\\
&= \exp\left(\int_0^{\tau_0}\left(a_1-d_1-\frac{b\tilde{x}_2(s)}{k_2}\right)ds\right),\\
\frac{\partial^2\Phi_2(\tau_0,X_0)}{\partial x_1\partial x_2} &= \int_0^{\tau_0}\left[\exp\left(\int_\upsilon^{\tau_0}\frac{\partial F_2(\zeta(s))}{\partial x_2}ds\right)\frac{\partial^2 F_2(\zeta(\upsilon))}{\partial x_1\partial x_2}\ \exp\left(\int_0^{\upsilon}\frac{\partial F_2(\zeta(s))}{\partial x_2}ds\right)\right]d\upsilon\\
&= \frac{-b\tau_0}{k_2(1-\gamma)} < 0,
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2\Phi_2(\tau_0,X_0)}{\partial x_2^2} &= \int_0^{\tau_0}\exp\left(\int_{\upsilon}^{\tau_0}\frac{\partial F_2(\zeta(s))}{\partial x_2}ds\right)\frac{\partial^2 F_2(\zeta(\upsilon))}{\partial x_2^2}\exp\left(\int_0^{\upsilon}\frac{\partial F_2(\zeta(s))}{\partial x_2}ds\right)d\upsilon \\
&+\int_0^{\tau_0}\left[\exp\left(\int_{\upsilon}^{\tau_0}\frac{\partial F_2(\zeta(s))}{\partial x_2}ds\right)\frac{\partial^2 F_2(\zeta(\upsilon))}{\partial x_1\partial x_2}\right] \\
&\times\left[\int_0^{\upsilon}\exp\left(\int_{\theta}^{\upsilon}\frac{\partial F_1(\zeta(s))}{\partial x_1}ds\right)\frac{\partial F_1(\zeta(\theta))}{\partial x_2}\ \exp\left(\int_0^{\theta}\frac{\partial F_2(\zeta(s))}{\partial x_2}ds\right)d\theta\right]d\upsilon \\
&=\int_0^{\tau_0}\left(\frac{2b\tilde{x}_2(\upsilon)}{k_2^2}-\frac{2a_1}{k_1}-2u_1\right)\exp\left(\int_0^{\tau_0}\left(a_1-d_1-\frac{b\tilde{x}_2(s)}{k_2}\right)ds\right)d\upsilon \\
&-\frac{b}{k_2}\int_0^{\tau_0}\left[\exp\left(\int_{\upsilon}^{\tau_0}\left(a_1-d_1-\frac{b\tilde{x}_2(s)}{k_2}\right)ds\right)\right] \\
&\times\left[\int_0^{\upsilon}\exp\left(\int_{\theta}^{\upsilon}(B-2A\tilde{x}_2(s))ds\right)\frac{rb\tilde{x}_2(\theta)}{k_2}\ \exp\left(\int_0^{\theta}\left(a_1-d_1-\frac{b\tilde{x}_2(s)}{k_2}\right)ds\right)d\theta\right]d\upsilon,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\Phi_2(\tau_0,X_0)}{\partial x_2\partial\tau} &= \frac{\partial F_2(\zeta(\tau_0))}{\partial x_2}\exp\left(\int_0^{\tau_0}\frac{\partial F_2(\zeta(s))}{\partial x_2}ds\right) \\
&=\left(a_1-d_1-\frac{b\tilde{x}_2(\tau_0)}{k_2}\right)\exp\left(\int_0^{\tau_0}\left(a_1-d_1-\frac{b\tilde{x}_2(s)}{k_2}\right)ds\right) \\
&=\frac{1}{1-\gamma}\left(a_1-d_1-\frac{bB(1-\omega-\exp(-B\tau_0))}{\beta_5}\right),
\end{aligned}
$$

where

$$\beta_5 = k_2\omega A\exp(-B\tau_0)+k_2A(1-\omega-\exp(-B\tau_0)).$$

Now, we can compute

$$
\begin{aligned}
d_0' &= 1-\left(\frac{\partial\Theta_2}{\partial x_2}\frac{\partial\Phi_2}{\partial x_2}\right)_{(\tau_0,X_0)} \\
&= 1-(1-\gamma)\exp\left(\int_0^{\tau_0}\left(a_1-d_1-\frac{b\tilde{x}_2(s)}{k_2}\right)ds\right),
\end{aligned}
$$

where τ_0 is the root of $d_0'=0$. Note that $d_0'>0$ if $T<T_{max}$ and $d_0'<0$ if $T>T_{max}$.

$$a_0' = 1-\left(\frac{\partial\Theta_1}{\partial x_1}\frac{\partial\Phi_1}{\partial x_1}\right)_{(\tau_0,X_0)}$$

$$= 1 - (1-\omega)\exp\left(\int\limits_0^{\tau_0} (B - 2A\tilde{x}_2(s))ds\right).$$

Note that $a_0' > 0$ if $T > T_{min}$.

$$\begin{aligned}
b_0' &= -\frac{\partial \Theta_1}{\partial x_1}\frac{\partial \Phi_1(\tau_0, X_0)}{\partial x_2} \\
&= -(1-\omega)\int\limits_0^{\tau_0} \exp\left(\int\limits_{\upsilon}^{\tau_0} (B - 2A\tilde{x}_2(s))ds\right)\frac{rb\tilde{x}_2(\upsilon)}{k_2}\exp\left(\int\limits_0^{\upsilon} (a_1 - d_1 - \frac{rb\tilde{x}_2(s)}{k_2})ds\right)d\upsilon \\
&< 0,
\end{aligned}$$

$$\begin{aligned}
G^* =& -(a_1 - d_1) + bB\left(\frac{1-\omega-\exp(-B\tau_0)}{\beta_5}\right) \\
&+ \frac{b\tau_0(1-\omega)\omega\exp(-B\tau_0)}{k_2\left(1 - \left((1-\omega)\exp(-B\tau_0) + \frac{1}{1-\omega}\right)\right)} \\
&\times \frac{B^2(1-\omega-\exp(-B\tau_0))}{A[\omega\exp(-B\tau_0) + (1-\omega-\exp(-B\tau_0))]^2},
\end{aligned}$$

$$H^* = 2(1-\gamma)\frac{b_0'}{a_0'}\frac{\partial^2\Phi_2}{\partial x_1 \partial x_2} - (1-\gamma)\frac{\partial^2 \Phi_2}{\partial x_2^2}.$$

Note that $G^* < 0$ if

$$a_1 k_2\left(\frac{a_2}{k_3} + u_2\right) > b(a_2 - d_2), \tag{3.32}$$

and $H^* > 0$ if

$$k_2^2\left(\frac{a_1}{k_1} + u_1\right)\left(\frac{a_2}{k_3} + u_2\right) > b(a_2 - d_2). \tag{3.33}$$

Thus, $G^*H^* < 0$, and by Lakmeche and Arini [13], we obtain the following theorem.

Theorem 3.9. *If all conditions* (3.2), (3.6), (3.11), (3.12), (3.14), (3.32), (3.33) *and* $T > T_{max} > T_{min}$ *hold, then the system* (3.28)–(3.31) *has a positive periodic solution which is supercritical.*

4. Numerical results and interpretation

In this section, the numerical results of the system (2.1)–(2.4) are carried out by using ode15 package in MATLAB to confirm the analysis of solutions.

The numerical simulations of the (2.1)–(2.4) are computed by using the parameters and initial conditions given in Table 1.

The remaining parameters $\gamma = 0.9, \omega = 0.1, T = 0.1$ are used to simulate the solutions as shown in Figures 1. All parameters are satisfied the conditions in Theorem 3.6. The trend of solution is close to a limit cycle $(0, \tilde{x}_2)$ as proved.

The initial situation starts with 10 units in both of population density of small fishes $x_1(0)$ and short mackerel $x_2(0)$. After that, the population of small fishes continues to decrease and tends to zero due to the short period and high quantity of fisheries. However, the population of shot mackerel decreases in the beginning then it tends to a closed orbit between 0.2593–0.2881 because of the low rate of toxic death and high capacity of hunting with the same period of fisheries.

The computer simulations of the system (2.1)–(2.4) with setting parameters in Table 1 and $\gamma = 0.5, \omega = 0.2, T = 0.5$ are shown in Figure 2. For this case, we have that all parameters are satisfied the conditions in Theorem 3.8. The solution is permanent as proved.

The initial situation starts with 10 units in both of population density of small fishes $x_1(0)$ and short mackerel $x_2(0)$. After that, both small fishes and shot mackerel decrease until tending to a range of 0.0987–0.1974 and 0.2347–0.2934, respectively. The fisherman catches enough of them for the economy and they remain alive in the system. The long period of fisheries, low-frequency fisheries, and the appropriate number of fishing are the most of factors in persistence.

The computer simulations of the system (2.1)–(2.4) with the parametric values $\gamma = 0.7, \omega = 0.3, T = 1.5$ and the remaining parameters as in Table 1 are shown in Figure 3. All parameters in this case are satisfied the conditions in Theorem 3.9. The system has a positive periodic solution as proved.

The initial situation start with 10 units both of population density of small fishes $x_1(0)$ and short mackerel $x_2(0)$. Then both small fishes and shot mackerel decrease until tending to oscillatory in a narrow range of 0.0592 – 0.1974 and 0.2054 – 0.2934, respectively. The decreased population of small fishes and shot mackerel occurs when the period of fisheries starts while the population of them back hits the peak when fisherman do not allow to fishing like a periodic behavior. The suitable amount and period of fisheries like this case to prevent the extinction of small fishes and short mackerel and to prevent the damage of economic are expected situation.

The computer simulations of the system (2.1)–(2.4) with the parametric values in Table 1 and $\gamma = 0.5, \omega = 0.2$ which focused on changes in the period of impulsive fisheries (T) and the negative effect of fisheries on the density of short mackerel population (ω) with $\gamma = 0.1, T = 0.2$ are shown in Figure 4 and Figure 5 respectively. The results indicated that the densities of the short mackerel were in periodic fashions. Moreover, the different values of T and ω provided the different highest densities of $x_2(t)$.

Table 1. Parameter values.

parameter	value
a_1	20
a_2	20 [18]
b	0.1
d_1	0.2
d_2	0.4
k_1	0.2
k_2	0.6
k_3	0.3
r	0.5
u_1	0.1
u_2	0.1
$x_1(0)$	10
$x_2(0)$	10

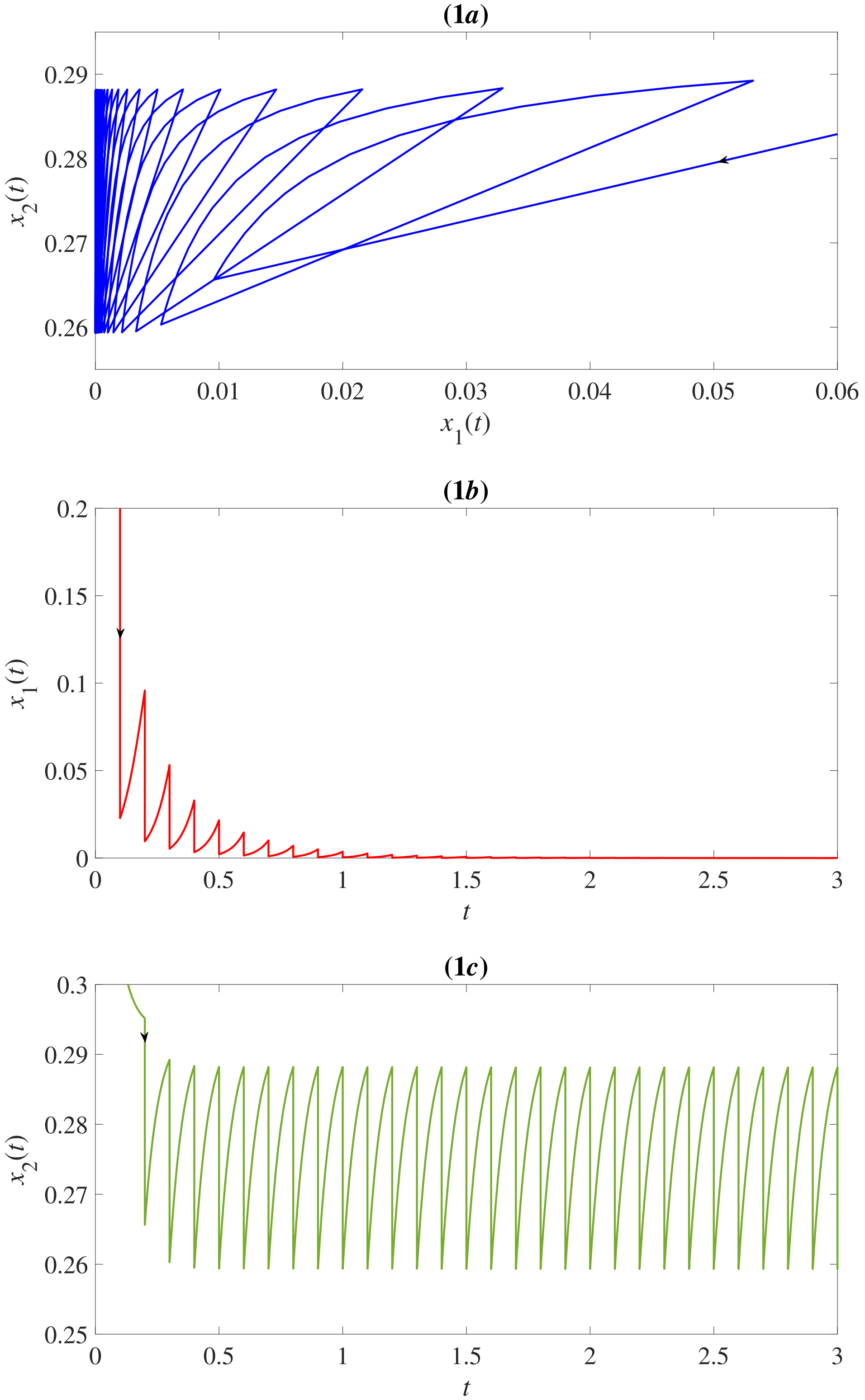

Figure 1. Simulation results of system (2.1)–(2.4). The parametric values are chosen to satisfy the conditions in Theorem 3.6. (1a) The phase-portrait of (x_1, x_2). (1b) The population time series of small fishes (x_1) tending to zero. (1c) The population time series of short mackerel (x_2) exhibiting positive pulsation. The solution goes toward the periodic solution $(0, \tilde{x}_2)$ as time passed.

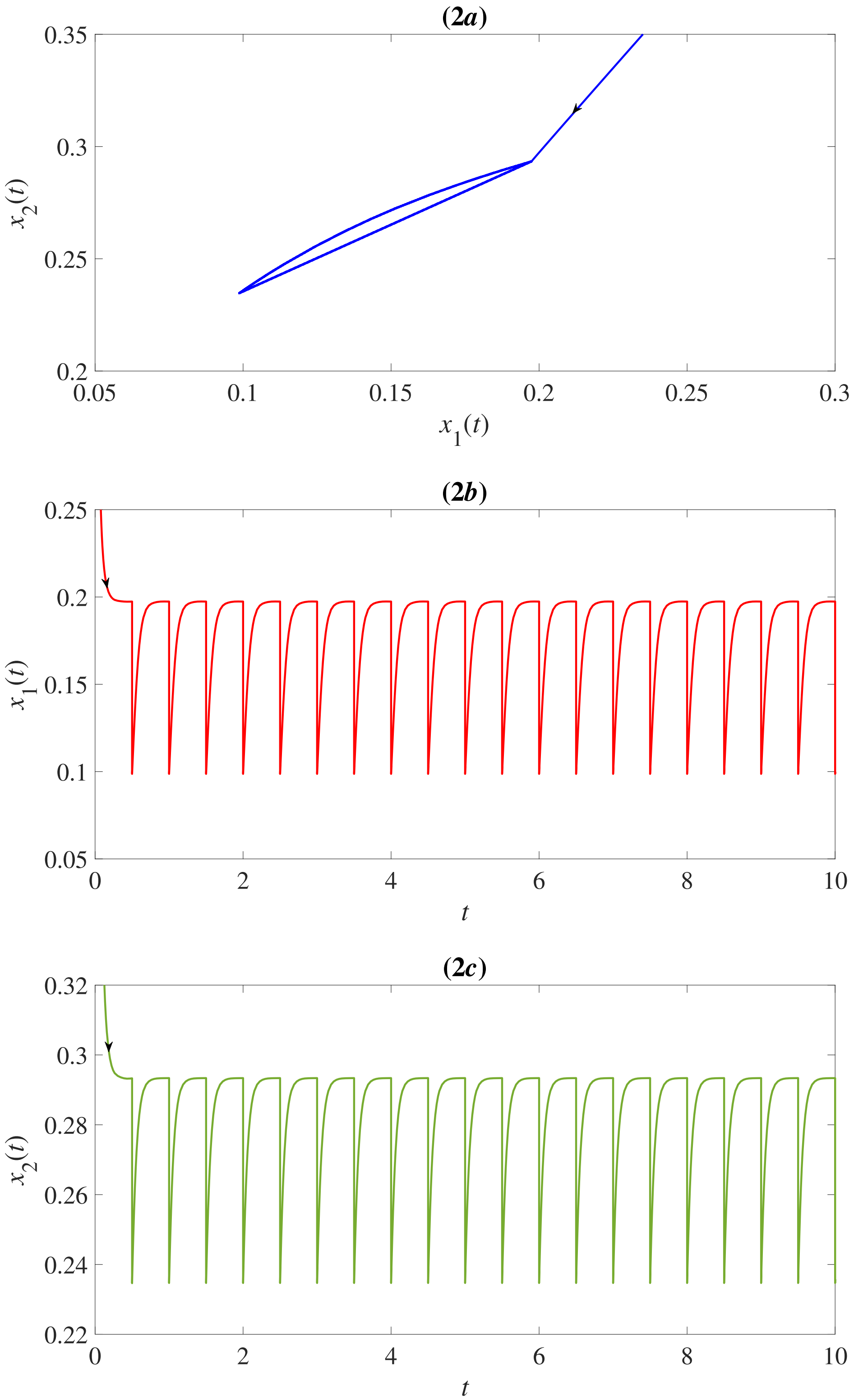

Figure 2. Simulation results of system (2.1)–(2.4). The parametric values are chosen to satisfy the conditions in Theorem 3.8. (2a) The phase-portrait of (x_1, x_2). (2b), (2c) The population time series of small fishes (x_1) and short mackerel (x_2) showing boundedness. The solution of the system is permanent.

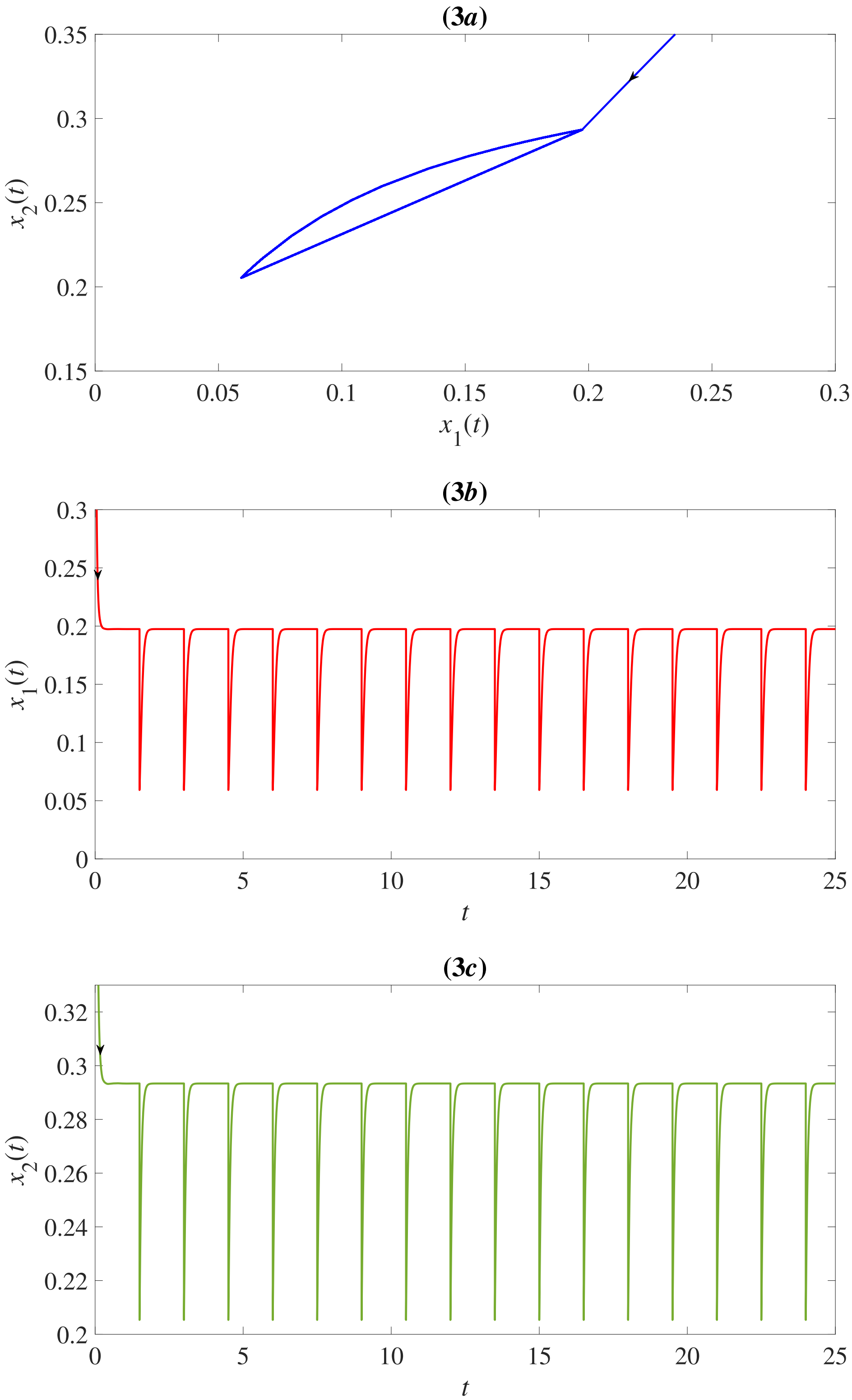

Figure 3. Simulation results of system (2.1)–(2.4). The parametric values are chosen to satisfy the conditions in Theorem 3.9. (3a) The phase-portrait of (x_1, x_2). (3b), (3c) The population time series of small fishes (x_1) and short mackerel (x_2) exhibiting positive oscillation. The system has a positive periodic solution.

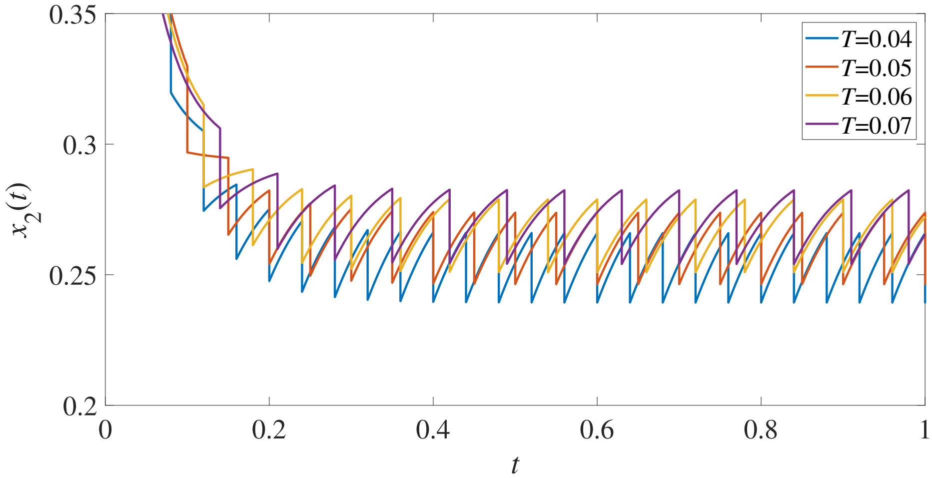

Figure 4. Simulation results of short mackerel density focused on changes in the period of impulsive fisheries (T).

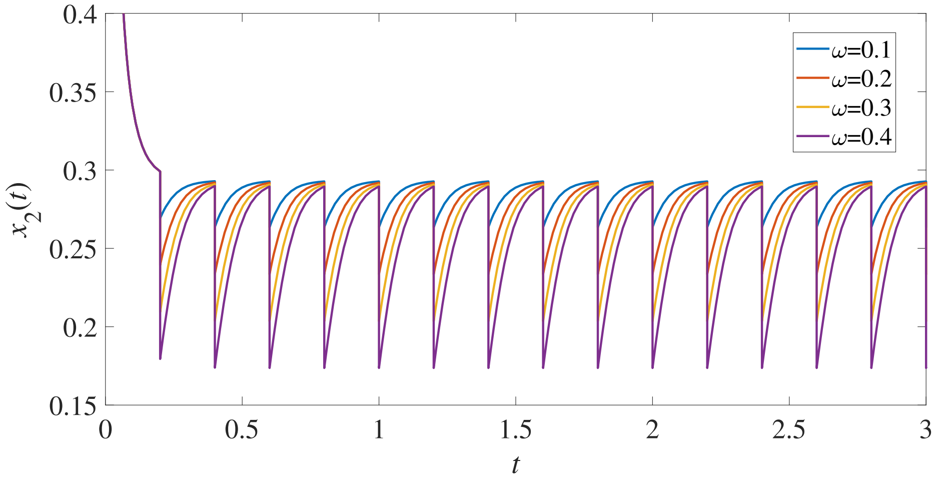

Figure 5. Simulation results of short mackerel density focused on changes in the negative effect of fisheries on the density of short mackerel population (ω).

5. Discussion and conclusions

We propose the modification mathematical model to forecast the population density dynamic of Indo-Pacific mackerel *(Rastrelliger brachysoma)* or short mackerel. The proposed model is utilized to control the population of short mackerel by considering the decrease population affected by catching, toxic, and natural death. The suitable period T and the quantity affecting the decreasing rate of small fishes population γ and the decreasing rate of short mackerel population ω are essential things to maintain short mackerel population and small fish population without extinction. The numerical results show the periodic behavior of the density population. Moreover, there are enough short mackerel for fishermen and humans for a long time.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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