Research article

On meromorphic solutions of certain differential-difference equations

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Abstract: In this article, we mainly use Nevanlinna theory to investigate some differential-difference equations. Our results about the existence and the forms of solutions for these differential-difference equations extend the previous theorems given by Wang, Xu and Tu [19].

Keywords: entire functions; differential-difference equations; value distribution; finite-order
Mathematics Subject Classification: 30D35, 39A10

1. Introduction and main results

We assume that the reader is familiar with the basic notions of Nevanlinna theory (see [4, 6, 22]). Recently, a number of papers (including [1–3, 5, 7–21, 23]) have focused on solvability and existence of meromorphic solutions of differential equations or differential-difference equations in complex plane. In 2009, Liu [10] obtained the Fermat type equation \( l(z)^2 + [l(z + c) − l(z)]^2 = 1 \) has a nonconstant entire solution of finite order. In 2012, Liu et al. [11] proved that \( l(z)^2 + l(z + c)^2 = 1 \) has a transcendental entire solution of finite order. In 2018, Zhang [23] obtained the difference equations \( l(z)^2 + [l(z + c) − l(z)]^2 = R(z) \) has no finite order transcendental meromorphic solutions with finitely many poles. In 2020, Wang et al. [18] further discussed the existence and the forms of the solutions for some differential-difference equations, they obtained

**Theorem A.** Let \( c \) be a nonzero constant, \( R(z) \) be a nonzero rational function, and \( \alpha, \beta \in \mathbb{C} \) satisfy \( \alpha^2 − \beta^2 \neq 1 \). Then the following difference equation of Fermat-type

\[ l(z)^2 + [\alpha l(z + c) − \beta l(z)]^2 = R(z), \]

has no finite order transcendental meromorphic solutions with finitely many poles.

**Theorem B.** Let \( c(\neq 0), a(\neq 0), \beta \in \mathbb{C}, \) and \( P(z), Q(z) \) be nonzero polynomials satisfying one of two following cases:

(i) \( \text{deg}_z P(z) \geq 1, \text{deg}_z Q(z) \geq 1; \)
(ii) $P(z)$, $Q(z)$ are two constants and $P^2(\alpha^2 - \beta^2) \neq 1$. Then the following Fermat-type difference equation

$$l(z)^2 + P^2(z)[\alpha l(z) - \beta l(z)]^2 = Q(z),$$

has no transcendental entire solutions with finite order.

For further study, we continue to discuss the existence and the forms of solutions for certain differential-difference equations with more general forms than the previous forms by Liu et al. [10, 11, 18, 23] and obtain the following results.

**Theorem 1.1.** Let $c_j(j = 1, 2, \cdots, m)$ be distinct constants, $a \in \mathbb{C}\setminus\{0\}$, $Q_i \in \mathbb{C}$ ($i = 1, 2, \cdots, m$), $R(z)$ be a nonzero rational function, and $\sum_{i=1}^{m} Q_i(exp^{ac} + exp^{-ac}) \neq 0$. Then the following difference equation

$$l(z)^2 + [Q_1 l(z + c_1) + Q_2 l(z + c_2) + \cdots + Q_m l(z + c_m)]^2 = R(z)$$

has no finite order transcendental meromorphic solutions with finitely many poles.

**Theorem 1.2.** Let $c_j(j = 1, 2, \cdots, m)$ be distinct constants, $a \in \mathbb{C}\setminus\{0\}$, $Q_i \in \mathbb{C}$ ($i = 1, 2, \cdots, m$), and $P(z)$, $Q(z)$ be nonzero polynomials satisfying one of two following cases:

(i) $\text{deg}_z P(z) \geq 1$;

(ii) $P$ is a constant and $P^2[\sum_{i=1}^{m} Q_i exp^{ac} + \sum_{i=1}^{m} Q_i exp^{-ac}] \neq 1$. Then the following difference equation

$$l(z)^2 + P(z)^2[Q_1 l(z + c_1) + Q_2 l(z + c_2) + \cdots + Q_m l(z + c_m)]^2 = Q(z)$$

has no transcendental entire solutions with finite order.

**Theorem 1.3.** Let $c_j(j = 1, 2, \cdots, m)$ be distinct constants, $a \in \mathbb{C}\setminus\{0\}$, $Q_i \in \mathbb{C}$ ($i = 1, 2, \cdots, m$). Let $l(z)$ be a transcendental finite order meromorphic solution of difference-differential equation

$$l'(z)^2 + [Q_1 l(z + c_1) + Q_2 l(z + c_2) + \cdots + Q_m l(z + c_m)]^2 = R(z),$$

where $R(z)$ is a nonzero rational function. If $l(z)$ has finitely many poles, and

$$\sum_{j=1}^{m} c_j^2 Q_j exp^{ac} + \sum_{j=1}^{m} c_j^2 Q_j exp^{-ac} \neq 0,$$

then $R(z)$ is a nonconstant polynomial with $\text{deg}_z R(z) \leq 2$, and

$$\sum_{j=1}^{m} c_j Q_j exp^{ac} + \sum_{j=1}^{m} c_j Q_j exp^{-ac} = 1.$$

Furthermore,

(i) If $R(z)$ is a nonconstant polynomial with $\text{deg}_z R(z) \leq 2$, and $\sum_{j=1}^{m} Q_j \neq 0$, then we have

$$l(z) = \frac{s_1(z)exp^{ac} + s_2(z)exp^{-ac}}{2},$$

where $R(z) = (m_1 + as_1(z))(m_2 - as_2(z)), a \neq 0, b \in \mathbb{C}$ and $a, b, c_j, Q_i$ satisfy $i(Q_1 exp^{ac} + \cdots + Q_m exp^{acm}) = a$ and $i(Q_1 exp^{-ac} + \cdots + Q_m exp^{-acm}) = a$, where $s_j(z) = m_jz + n_j, m_j, n_j \in \mathbb{C}(j = 1, 2)$.

(ii) If $R(z)$ is a nonzero constant, and $\sum_{j=1}^{m} Q_j \neq 0$, then

$$l(z) = \frac{n_1 exp^{ac} + n_2 exp^{-ac}}{2},$$

where $R(z) = (m_1 + as_1(z))(m_2 - as_2(z)), a \neq 0, b \in \mathbb{C}$ and $a, b, c_j, Q_i$ satisfy $i(Q_1 exp^{ac} + \cdots + Q_m exp^{acm}) = a$ and $i(Q_1 exp^{-ac} + \cdots + Q_m exp^{-acm}) = a$, where $s_j(z) = m_jz + n_j, m_j, n_j \in \mathbb{C}(j = 1, 2)$. 

AIMS Mathematics  Volume 6, Issue 9, 10343–10354.
where $R(z) = -a^2 n_1 n_2, a \neq 0, b \in \mathbb{C}$.

**Theorem 1.4.** Let $c_j (j = 1, 2, \cdots, m)$ be distinct constants, $a \in \mathbb{C} \setminus \{0\}, \varrho_i \in \mathbb{C}$ ($i = 1, 2, \cdots, m$). Let $l(z)$ be a transcendental meromorphic solution of the following difference-differential equation

\[
l''(z)^2 + [\varrho_1 l(z + c_1) + \varrho_2 l(z + c_2) + \cdots + \varrho_m l(z + c_m)]^2 = R(z),
\]

where $R(z)$ is a nonzero rational function.

(i) If $\sum_{i=1}^m \varrho_i \exp a_{ij} + \sum_{i=1}^m \varrho_i \exp -a_{ij} \neq 0$, then (1.4) has no finite order transcendental meromorphic solution with finitely many poles.

(ii) If $\sum_{j=1}^m \varrho_j \exp a_{ij} \neq 2a$, $\sum_{j=1}^m \varrho_j \exp -a_{ij} \neq 2a$, and (1.4) has a finite order transcendental meromorphic solution $l(z)$ with finitely many poles, then $R(z)$ is a constant. Furthermore if $\sum_{i=1}^m \varrho_i \neq 0$, then we have

\[
l(z) = \frac{t_1 \exp a_{ij} + t_2 \exp -a_{ij}}{2},
\]

where $a, b, t_1, t_2, \varrho_i, c_j$ satisfy $\sum_{i=1}^m \varrho_i \exp a_{ij} + \sum_{i=1}^m \varrho_i \exp -a_{ij} = 0, R(z) = a^t t_1 t_2, b \in \mathbb{C}$.

2. Preliminary lemmas

The following two lemmas play an important role in the proof of our results.

**Lemma 2.1.** ([22]) Suppose that $f_1, f_2, \cdots, f_n (n \geq 2)$ are meromorphic functions and $g_1, g_2, \cdots, g_n$ are entire functions satisfying the following conditions:

(i) $\sum_{j=1}^n f_j \exp g_j \equiv 0$;

(ii) $g_j - g_k$ are not constants for $1 \leq j < k \leq n$;

(iii) For $1 \leq j \leq n, 1 \leq h < k \leq n$, $T(r, f_j) = o(T(r, \exp g_k - g_i))(r \to \infty, r \notin E)$, where $E$ is a set of $r \in (0, \infty)$ with finite linear measure.

Then $f_j \equiv 0 (j = 1, 2, \cdots, m)$.

**Lemma 2.2.** ([22]) Let $l(z)$ be a meromorphic function of finite order $\rho(l)$. Write

\[
l(z) = c_k z^k + c_{k+1} z^{k+1} + \cdots, (c_k \neq 0),
\]

near $z = 0$ and let $\{a_1, a_2, \cdots\}$ and $\{b_1, b_2, \cdots\}$ be the zeros and poles of $l$ in $\mathbb{C} \setminus \{0\}$, respectively. Then

\[
l(z) = z^k \exp^{\varrho(z)} \frac{P_1(z)}{P_2(z)},
\]

where $P_1(z)$ and $P_2(z)$ are the canonical products of $l$ formed with the non-null zeros and poles of $l$, respectively, and $Q(z)$ is a polynomial of degree $\leq \rho(l)$.

3. Proof of Theorem 1.1

Suppose that (1.1) has a finite order transcendental meromorphic solution $l(z)$ with finitely many poles. Rewriting (1.1) as follows

\[
(l(z) + l_{ij} l(z + c_1) + l_{12} l(z + c_2) + \cdots + l_{1m} l(z + c_m))(l(z) - l_{ij} l(z + c_1) + l_{12} l(z + c_2) + \cdots + l_{1m} l(z + c_m)) = R(z).
\]

\[\text{(3.1)}\]
Substituting (3.4) into (3.5), we have

\[ l(z) + i(q_1 l(z + c_1) + q_2 l(z + c_2) + \cdots + q_m l(z + c_m)) = R_1(z) \exp^{p(z)}, \quad (3.2) \]

and

\[ l(z) - i(q_1 l(z + c_1) + q_2 l(z + c_2) + \cdots + q_m l(z + c_m)) = R_2(z) \exp^{-p(z)}, \quad (3.3) \]

where \( R_1(z), R_2(z) \) are two nonzero rational functions such that \( R_1(z) R_2(z) = R(z) \), and \( p(z) \) is a nonconstant polynomial. (3.2) and (3.3) imply that

\[ l(z) = \frac{R_1(z) \exp^{p(z)} + R_2(z) \exp^{-p(z)}}{2}, \quad (3.4) \]

and

\[ q_1 l(z + c_1) + q_2 l(z + c_2) + \cdots + q_m l(z + c_m) = \frac{R_1(z) \exp^{p(z)} - R_2(z) \exp^{-p(z)}}{2i}. \quad (3.5) \]

Substituting (3.4) into (3.5), we have

\[ \exp^{p(z)} (i q_1 R_1(z + c_1) \exp^{p(z + c_1)} - p(z)) + i q_2 R_1(z + c_2) \exp^{p(z + c_2)} - p(z) + \cdots + i q_m R_1(z + c_m) \exp^{p(z + c_m)} - p(z) - R_1(z) = 0, \quad (3.6) \]

By Lemma 2.1 and (3.6), we have

\[ i q_1 R_1(z + c_1) \exp^{p(z + c_1)} - p(z) + i q_2 R_1(z + c_2) \exp^{p(z + c_2)} - p(z) + \cdots + i q_m R_1(z + c_m) \exp^{p(z + c_m)} - p(z) = 0, \quad (3.7) \]

and

\[ i q_1 R_2(z + c_1) \exp^{-p(z + c_1)} + i q_2 R_2(z + c_2) \exp^{-p(z + c_2)} + \cdots + i q_m R_2(z + c_m) \exp^{-p(z + c_m)} + R_2(z) = 0. \quad (3.8) \]

Since \( R_1(z), R_2(z) \) are two nonzero rational functions and that \( l(z) \) is of finite order, we obtain that \( p(z) \) is a polynomial of degree one. If \( \deg_z p(z) \geq 2 \), then we obtain that \( \deg_z [p(z + c_j) - p(z + c_j)] \geq 1 \). Hence, we have \( T(r, i q_j R_j(z + c_j)) = S(r, \exp^{p(z + c_j)} - p(z + c_j)) \), Lemma 2.1 and (3.7) imply that \( R_1(z) \equiv 0 \). This is impossible. By the similar method as above, we also have \( R_2(z) \equiv 0 \), a contradiction. So we have \( \deg_z p(z) = 1 \). Set \( p(z) = az + b, a \neq 0, b \in \mathbb{C} \). By (3.7) and (3.8), we have

\[ \lim_{|z| \to \infty} i q_1 \frac{R_1(z + c_1)}{R_1(z)} \exp^{p(z + c_1)} - p(z) + \cdots + q_m \frac{R_1(z + c_m)}{R_1(z)} \exp^{p(z + c_m)} - p(z) = i(q_1 \exp^{zc_1} + \cdots + q_m \exp^{zc_m}) = 1, \]

and

\[ \lim_{|z| \to \infty} i q_1 \frac{R_2(z + c_1)}{R_2(z)} \exp^{-p(z + c_1)} + \cdots + q_m \frac{R_2(z + c_m)}{R_2(z)} \exp^{-p(z + c_m)} = i(q_1 \exp^{-zc_1} + \cdots + q_m \exp^{-zc_m}) = -1. \]

Thus, it yields that \( \sum_{i=1}^{m} q_i (\exp^{zc_1} + \exp^{-zc_1}) = 0 \), this is a contradiction with the assumption of Theorem 1.1. Hence, Theorem 1.1 holds.
4. Proof of Theorem 1.2

If \(l(z)\) is a transcendental entire solution with finite order of (1.2), then by the similar method as the proof of Theorem 1.1, we have

\[
l(z) = \frac{Q_1(z)\exp^{p(z)} + Q_2(z)\exp^{-p(z)}}{2},
\]

and

\[
q_1l(z + c_1) + q_2l(z + c_2) + \cdots + q_m l(z + c_m) = \frac{Q_1(z)\exp^{p(z)} - Q_2(z)\exp^{-p(z)}}{2iP(z)},
\]

where \(p(z)\) is a nonconstant polynomial and \(Q_1(z)Q_2(z) = Q(z), Q_1(z), Q_2(z)\) are nonzero polynomials.

Together (4.1) with (4.2), we have

\[
\begin{align*}
exp^{p(z)}(i\varphi_1 P(z)Q_1(z + c_1)\exp^{p(z+c_1)-p(z)} + i\varphi_2 P(z)Q_1(z + c_2)\exp^{p(z+c_2)-p(z)} \\
+ \cdots + i\varphi_m P(z)Q_1(z + c_m)\exp^{p(z+c_m)-p(z)} - Q_1(z)) + \\
exp^{-p(z)}(i\varphi_1 P(z)Q_2(z + c_1)\exp^{p(z-c_1)-p(z)} + i\varphi_2 P(z)Q_2(z + c_2)\exp^{p(z-c_2)-p(z)} \\
+ \cdots + i\varphi_m P(z)Q_2(z + c_m)\exp^{p(z-c_m)-p(z)} + Q_2(z)) = 0.
\end{align*}
\]

By Lemma 2.1 and \(p(z)\) is a nonconstant polynomial, we have

\[
\begin{align*}
i\varphi_1 P(z)Q_1(z + c_1)\exp^{p(z+c_1)-p(z)} + i\varphi_2 P(z)Q_1(z + c_2)\exp^{p(z+c_2)-p(z)} \\
+ \cdots + i\varphi_m P(z)Q_1(z + c_m)\exp^{p(z+c_m)-p(z)} - Q_1(z) = 0,
\end{align*}
\]

and

\[
\begin{align*}
i\varphi_1 P(z)Q_2(z + c_1)\exp^{p(z-c_1)-p(z)} + i\varphi_2 P(z)Q_2(z + c_2)\exp^{p(z-c_2)-p(z)} \\
+ \cdots + i\varphi_m P(z)Q_2(z + c_m)\exp^{p(z-c_m)-p(z)} + Q_2(z) = 0.
\end{align*}
\]

If \(\deg p(z) \geq 2\), then we have that \(\deg[p(z+c_1)-p(z+c_1)] \geq 1\). Hence, we have \(T(r, i\varphi P(z)Q_1(z + c_1)) = S(r, \exp^{p(z+c_1)-p(z+c_1)})\), Lemma 2.1 and (4.4) imply that \(Q_1(z) \equiv 0\). A contradiction. By the similar method as above, we also obtain that \(Q_2(z) \equiv 0\), this is also impossible. Hence, \(\deg p(z) = 1\). Let \(p(z) = az + b, a \neq 0, b \in \mathbb{C}\). (4.4) and (4.5) imply that

\[
\begin{align*}
i\varphi_1 P(z)Q_1(z + c_1)\exp^{p(z+c_1)-p(z)} + i\varphi_2 P(z)Q_1(z + c_2)\exp^{p(z+c_2)-p(z)} \\
+ \cdots + i\varphi_m P(z)Q_1(z + c_m)\exp^{p(z+c_m)-p(z)} = Q_1(z),
\end{align*}
\]

and

\[
\begin{align*}
i\varphi_1 P(z)Q_2(z + c_1)\exp^{p(z-c_1)-p(z)} + i\varphi_2 P(z)Q_2(z + c_2)\exp^{p(z-c_2)-p(z)} \\
+ \cdots + i\varphi_m P(z)Q_2(z + c_m)\exp^{p(z-c_m)-p(z)} = -Q_2(z).
\end{align*}
\]

By this, we have

\[
\begin{align*}
P(z)\left[q_1^2 Q(z + c_1) + q_2^2 Q(z + c_2) + \cdots + q_m^2 Q(z + c_m) + \\
q_1q_2 Q_1(z + c_1)Q_2(z + c_2)\exp^{p_{c_1-c_2}} + \cdots + \\
q_1q_m Q_1(z + c_1)Q_2(z + c_m)\exp^{p_{c_1-c_m}} + \cdots + \\
q_1q_{m-1} Q_1(z + c_{m-1})Q_2(z + c_m)\exp^{p_{c_1-c_m-1}}\right] = Q(z).
\end{align*}
\]

Set \(\deg P(z) = p\) and \(\deg Q(z) = q\), then \(p \geq 0, q \geq 0\) and \(p, q \in \mathbb{N}_+\). Next we divided the following proof into four cases:
**Case 1.** $p \geq 1$ and $\sum_{i=1}^{m} q_{i} e^{\rho_{i}} \sum_{i=1}^{m} q_{i} e^{-\rho_{i}} = 0$. If $q \geq 1$, by comparing the order both sides of (4.6), we have $2p + q - 1 \leq q$, that is, $p \leq \frac{1}{2}$, this is impossible. If $q = 0$, that is, $Q(z)$ is a constant. Hence, by (4.6), we have $Q(z) = 0$, a contradiction.

**Case 2.** $p \geq 1$ and $\sum_{i=1}^{m} q_{i} e^{\rho_{i}} \sum_{i=1}^{m} q_{i} e^{-\rho_{i}} \neq 0$. If $q \geq 1$, by comparing the order both sides of (4.6), we have $2p + q = q$, that is, $p = 0$, a contradiction. If $q = 0$, that is, $Q(z)$ is a constant. Hence, by (4.6), we have $P(z)$ is a constant, this is impossible.

**Case 3.** $p = 0$ and $\sum_{i=1}^{m} q_{i} e^{\rho_{i}} \sum_{i=1}^{m} q_{i} e^{-\rho_{i}} = 0$. That is, $P(z) = K(\neq 0)$. If $q \geq 1$, we have $q - 1 = q$, this is impossible. If $q = 0$, we have $Q(z) \equiv 0$. A contradiction.

**Case 4.** $p = 0$ and $\sum_{i=1}^{m} q_{i} e^{\rho_{i}} \sum_{i=1}^{m} q_{i} e^{-\rho_{i}} \neq 0$. If $q \geq 1$, set $P(z) = K(\neq 0)$, $Q(z) = b_{q} z^{q} + b_{q-1} z^{q-1} + \cdots + b_{0}$, $b_{q} \neq 0, b_{q-1}, \cdots, b_{0}$ are constants. By comparing the coefficients of $z^{q}$ both sides of (4.6), we have

$$K^{2} [\sum_{i=1}^{m} q_{i} e^{\rho_{i}} \sum_{i=1}^{m} q_{i} e^{-\rho_{i}}] = 1.$$  \hspace{1cm} (4.7)

This is a contradiction with the condition of Theorem 1.2. If $q = 0$, then $K^{2} [\sum_{i=1}^{m} q_{i} e^{\rho_{i}} \sum_{i=1}^{m} q_{i} e^{-\rho_{i}}] = 1$, this is impossible.

Hence, Theorem 1.2 holds.

**5. Proof of Theorem 1.3**

Suppose that (1.3) has a finite order transcendental meromorphic solution $l(z)$ with finitely many poles. Rewriting (1.3) as follows

\[
\begin{align*}
(l'(z) + i(q_{1} l(z + c_{1}) + q_{2} l(z + c_{2}) + \cdots + q_{m} l(z + c_{m}))) & \cdot (l' - i(q_{1} l(z + c_{1}) + q_{2} l(z + c_{2}) + \cdots + q_{m} l(z + c_{m}))) = R(z),
\end{align*}
\]  \hspace{1cm} (5.1)

Since $l(z)$ has finitely many poles, and $R(z)$ is a nonzero rational function, then $l'(z) + i(q_{1} l(z + c_{1}) + q_{2} l(z + c_{2}) + \cdots + q_{m} l(z + c_{m}))$ and $l'(z) - i(q_{1} l(z + c_{1}) + q_{2} l(z + c_{2}) + \cdots + q_{m} l(z + c_{m}))$ both have finitely many poles and zeros. Hence, by Lemma 2.2, (5.1) can be written as

\[
\begin{align*}
l'(z) + i(q_{1} l(z + c_{1}) + q_{2} l(z + c_{2}) + \cdots + q_{m} l(z + c_{m})) &= R_{1}(z) e^{p(z)},
\end{align*}
\]  \hspace{1cm} (5.2)

and

\[
\begin{align*}
l'(z) - i(q_{1} l(z + c_{1}) + q_{2} l(z + c_{2}) + \cdots + q_{m} l(z + c_{m})) &= R_{2}(z) e^{-p(z)},
\end{align*}
\]  \hspace{1cm} (5.3)

where $R_{1}(z), R_{2}(z)$ are two nonzero rational functions such that $R_{1}(z) R_{2}(z) = R(z)$, and $p(z)$ is a nonconstant polynomial. (5.2) and (5.3) imply that

\[
l'(z) = \frac{R_{1}(z) e^{p(z)} + R_{2}(z) e^{-p(z)} }{2},
\]  \hspace{1cm} (5.4)

and

\[
q_{1} l(z + c_{1}) + q_{2} l(z + c_{2}) + \cdots + q_{m} l(z + c_{m}) = \frac{R_{1}(z) e^{p(z)} - R_{2}(z) e^{-p(z)}}{2i}.
\]  \hspace{1cm} (5.5)
(5.5) implies that
\[
\varrho_1 l'(z + c_1) + \varrho_2 l'(z + c_2) + \cdots + \varrho_m l'(z + c_m) = \frac{A_1(z)\exp^{pl(z)} - B_1(z)\exp^{-pl(z)}}{2i}, \tag{5.6}
\]
where \(A_1(z) = R'_1 + R_1(z)p'\) and \(B_1(z) = R'_2 - R_2(z)p'\). Substituting (5.4) into (5.6), we have
\[
\exp^{pl(z)}(i\varrho_1 R_1(z + c_1)\exp^{p(z+c_1) - pl(z)} + i\varrho_2 R_1(z + c_2)\exp^{p(z+c_2) - pl(z)} + \cdots + i\varrho_m R_1(z + c_m)\exp^{p(z+c_m) - pl(z)} - A_1(z)) +
\exp^{-pl(z)}(i\varrho_1 R_2(z + c_1)\exp^{p(z+c_1) - pl(z)} + i\varrho_2 R_2(z + c_2)\exp^{p(z+c_2) - pl(z)} + \cdots + i\varrho_m R_2(z + c_m)\exp^{p(z+c_m) - pl(z)} + B_1(z)) = 0. \tag{5.7}
\]
Together Lemma 2.1 with (5.7), we have
\[
i\varrho_1 R_1(z + c_1)\exp^{p(z+c_1) - pl(z)} + i\varrho_2 R_1(z + c_2)\exp^{p(z+c_2) - pl(z)} + \cdots + i\varrho_m R_1(z + c_m)\exp^{p(z+c_m) - pl(z)} - A_1(z) = 0, \tag{5.8}
\]
and
\[
i\varrho_1 R_2(z + c_1)\exp^{p(z-c_1) - pl(z)} + i\varrho_2 R_2(z + c_2)\exp^{p(z-c_2) - pl(z)} + \cdots + i\varrho_m R_2(z + c_m)\exp^{p(z-c_m) - pl(z)} + B_1(z) = 0. \tag{5.9}
\]
Since \(R_1(z), R_2(z)\) are two nonzero rational functions and \(l(z)\) is of finite order, by the similar method as the proof of Theorem 1.1, we have \(\deg p(z) = 1\). Let \(p(z) = az + b, a \neq 0, b \in \mathbb{C}\). Substituting \(p(z), A_1(z), B_1(z)\) into (5.8) and (5.9), as \(z \to \infty\), we have
\[
\lim_{|z| \to \infty} i\varrho_1 R_1(z + c_1)\exp^{p(z+c_1) - pl(z)} + \cdots + i\varrho_m R_1(z + c_m)\exp^{p(z+c_m) - pl(z)} = i(\varrho_1 \exp^{ac_1} + \cdots + \varrho_m \exp^{ac_m}) = \frac{R'_1(z)}{R_1(z)} + a = a,
\]
and
\[
\lim_{|z| \to \infty} i\varrho_1 R_2(z + c_1)\exp^{p(z-c_1) - pl(z)} + \cdots + i\varrho_m R_2(z + c_m)\exp^{p(z-c_m) - pl(z)} = i(\varrho_1 \exp^{-ac_1} + \cdots + \varrho_m \exp^{-ac_m}) = -\frac{R'_2(z)}{R_2(z)} + a = a.
\]
That is
\[
i\varrho_1 \exp^{ac_1} + \cdots + \varrho_m \exp^{ac_m} = a, \quad i\varrho_1 \exp^{-ac_1} + \cdots + \varrho_m \exp^{-ac_m} = a. \tag{5.10}
\]
According to (5.8), (5.9) and (5.10), we have
\[
i\varrho_1 \exp^{ac_1}(R_1(z + c_1) - R_1(z)) + i\varrho_2 \exp^{ac_2}(R_1(z + c_2) - R_1(z)) + \cdots + i\varrho_m \exp^{ac_m}(R_1(z + c_m) - R_1(z)) = R_1'(z), \tag{5.11}
\]
and
\[
i\varrho_1 \exp^{-ac_1}(R_2(z + c_1) - R_2(z)) + i\varrho_2 \exp^{-ac_2}(R_2(z + c_2) - R_2(z)) + \cdots + i\varrho_m \exp^{-ac_m}(R_2(z + c_m) - R_2(z)) = -R_2'(z). \tag{5.12}
\]
If \(R_1(z), R_2(z)\) are two nonzero constants, then (5.11) and (5.12) hold and \(R_1(z)R_2(z) = R(z)\) is a constant.
We next consider the case that \(R_1(z), R_2(z)\) are two nonzero rational functions. If \(R_1(z)\) has a pole of multiplicity \(v\) at \(z_0\), by (5.11), we know that there exists at least on index \(l_1 \in \{1, 2, \cdots, m\}\) such that
$z_0 + c_i$ is a pole of $R_1(z)$ of multiplicity $v + 1$, following the above step, we know $R_1(z)$ has a sequence of poles
\[
\{\tau_n = z_0 + c_i + \cdots + c_{n_a} : n = 1, 2, \cdots \}.
\]
Hence, we have $\lambda(\frac{1}{R_1(z)}) \geq 1$, this is impossible. So $R_1(z)$ is a polynomial. Using the same method as above, we know that $R_2(z)$ is also a polynomial. If $R_1(z)$ is a nonconstant polynomial with $\deg R_1(z) \geq 2$. Let $R_1(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0$, then
\[
R_1'(z) = na_nz^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots ,
\]
(5.13)
\[
R_1(z + c_m) - R_1(z) = na_ne_mz^{n-1} + (a_nC_m^2 - n + 1)a_{n-1}c_m)z^{n-2} + \cdots ,
\]
(5.14)
where $i = 1, 2$. Substituting (5.13) and (5.14) into (5.11) and (5.12), comparing the coefficients of $z^{n-1}, z^{n-2}$, we have $\sum_{j=1}^{m} ic_jQ_j\exp^ {acj} = 1$, $\sum_{j=1}^{m} c_j^2Q_j\exp^{-acj} = 0$ and $\sum_{j=1}^{m} ic_jQ_j\exp^{-acj} = -1$, $\sum_{j=1}^{m} c_j^2Q_j\exp^{-acj} = 0$, a contradiction with $\sum_{j=1}^{m} c_j^2Q_j\exp^{-acj} \neq 0$. Hence, $\deg R_1(z) \leq 1$. So $\deg R_1(z) = \deg R_1(z)R_2(z) \leq 2$.

(i) If $R(z)$ is a nonconstant polynomial with $\deg R(z) \leq 2$, then by (5.4), we have
\[
l(z) = \frac{s_1(z)\exp^{ac+b} + s_2(z)\exp^{-ac+b}}{2} + \theta ,
\]
(5.15)
where $s_j(z) = m_jz + n_j, m_j, n_j \in \mathbb{C}, (j = 1, 2)$ and $\theta \in \mathbb{C}$.

**Case 1.** If $\deg R(z) = 2$, then $m_j \neq 0, j = 1, 2$. If $\sum_{i=1}^{m} Q_i \neq 0$, substituting (5.15) into (5.5), we have $\theta \equiv 0, R(z) = (m_1 + as_1(z))(m_2 - as_2(z))$. Hence, we have
\[
l(z) = \frac{s_1(z)\exp^{ac+b} + s_2(z)\exp^{-ac+b}}{2} ,
\]
\[
R(z) = (m_1 + as_1(z))(m_2 - as_2(z)), a \neq 0, b \in \mathbb{C}.
\]

**Case 2.** If $\deg R(z) = 1$, then one of $m_1, m_2$ is zero, we can assume that $m_1 = 0$. Substituting (5.15) into (5.5), we have $R_1(z)$ is a constant and $R_2(z)$ is a polynomial of degree one. Using the same method as case 1, we have $\theta \equiv 0$. Hence, we obtain that
\[
l(z) = \frac{s_1(z)\exp^{ac+b} + s_2(z)\exp^{-ac+b}}{2} ,
\]
\[
R(z) = (m_1 + as_1(z))(m_2 - as_2(z)), a \neq 0, b \in \mathbb{C}.
\]

(ii) If $R(z)$ is a nonzero constant, by (5.4), we have
\[
l(z) = \frac{n_1\exp^{az+b} + n_2\exp^{-az+b}}{2} + d ,
\]
(5.16)
where $n_1, n_2 \in \mathbb{C}$ and $d \in \mathbb{C}$. Substituting (5.16) into (5.5), we have $d = 0, R(z) = -a^2n_1n_2$. Hence, Theorem 1.3 holds.
6. Proof of Theorem 1.4

Suppose that (1.4) has a finite order transcendental meromorphic solution \( l(z) \) with finitely many poles. Rewriting (1.4) as follows

\[
\begin{align*}
(l''(z) + i(q_1 l(z + c_1) + q_2 l(z + c_2) + \cdots + q_m l(z + c_m)) l''(z) - \\
\quad i(q_1 l(z + c_1) + q_2 l(z + c_2) + \cdots + q_m l(z + c_m)) = R(z).
\end{align*}
\]  
(6.1)

Since \( l(z) \) has finitely many poles, \( R(z) \) is a nonzero rational function, then \( l''(z) + i(q_1 l(z + c_1) + q_2 l(z + c_2) + \cdots + q_m l(z + c_m)) \) and \( l''(z) - i(q_1 l(z + c_1) + q_2 l(z + c_2) + \cdots + q_m l(z + c_m)) \) both have finitely many poles and zeros. Hence, we can rewrite (6.1) as follows

\[
\begin{align*}
l''(z) + i(q_1 l(z + c_1) + q_2 l(z + c_2) + \cdots + q_m l(z + c_m)) = R_1(z) \exp^{p(z)},
\end{align*}
\]  
(6.2)

and

\[
\begin{align*}
l''(z) - i(q_1 l(z + c_1) + q_2 l(z + c_2) + \cdots + q_m l(z + c_m)) = R_2(z) \exp^{-p(z)},
\end{align*}
\]  
(6.3)

where \( R_1(z), R_2(z) \) are two nonzero rational functions such that \( R_1(z) R_2(z) = R(z) \), and \( p(z) \) is a nonconstant polynomial. By (6.2) and (6.3), we obtain

\[
l''(z) = \frac{R_1(z) \exp^{p(z)} + R_2(z) \exp^{-p(z)}}{2},
\]  
(6.4)

and

\[
\begin{align*}
q_1 l(z + c_1) + q_2 l(z + c_2) + \cdots + q_m l(z + c_m) = \frac{R_1(z) \exp^{p(z)} - R_2(z) \exp^{-p(z)}}{2i}.
\end{align*}
\]  
(6.5)

(6.5) implies that

\[
\begin{align*}
q_1 l''(z + c_1) + q_2 l''(z + c_2) + \cdots + q_m l''(z + c_m) = \frac{A_2(z) \exp^{p(z)} - B_2(z) \exp^{-p(z)}}{2i},
\end{align*}
\]  
(6.6)

where \( A_2(z) = A_1' + A_1(z) p' \) and \( B_2(z) = B_1' - B_1(z) p' \). Together (6.4) with (6.6), we obtain that

\[
\begin{align*}
ex^{p(z)} (i q_1 R_1(z + c_1) \exp^{p(z+c_1) - p(z)} + i q_2 R_1(z + c_2) \exp^{p(z+c_2) - p(z)} \\
+ \cdots + i q_m R_1(z + c_m) \exp^{p(z+c_m) - p(z)} - A_2(z)) + \\
ex^{-p(z)} (i q_1 R_2(z + c_1) \exp^{-p(z+c_1) + p(z)} + i q_2 R_2(z + c_2) \exp^{-p(z+c_2) + p(z)} \\
+ \cdots + i q_m R_2(z + c_m) \exp^{-p(z+c_m) + p(z)} + B_2(z)) = 0.
\end{align*}
\]  
(6.7)

Lemma 2.1 and (6.7) imply that

\[
\begin{align*}
i q_1 R_1(z + c_1) \exp^{p(z+c_1) - p(z)} + i q_2 R_1(z + c_2) \exp^{p(z+c_2) - p(z)} \\
+ \cdots + i q_m R_1(z + c_m) \exp^{p(z+c_m) - p(z)} - A_2(z) = 0,
\end{align*}
\]  
(6.8)

and

\[
\begin{align*}
i q_1 R_2(z + c_1) \exp^{-p(z+c_1) + p(z)} + i q_2 R_2(z + c_2) \exp^{-p(z+c_2) + p(z)} \\
+ \cdots + i q_m R_2(z + c_m) \exp^{-p(z+c_m) + p(z)} + B_2(z) = 0.
\end{align*}
\]  
(6.9)
Since \( R_1(z), R_2(z) \) are two nonzero rational functions and \( l(z) \) is of finite order, using the similar method as the proof of Theorem 1.1, we know that \( p(z) \) is a polynomial of degree one. Let \( p(z) = az + b, a \neq 0, b \in \mathbb{C} \). Substituting \( p(z), A_2(z), B_2(z) \) into (6.8) and (6.9), and as \( z \to \infty \), we have

\[
\lim_{|z| \to \infty} iq_1^{R_1(z+c_1)}e^{p(z+c_1)-p(z)} + \cdots + q_m^{R_1(z+c_m)}e^{p(z+c_m)-p(z)} = i(q_1 \exp^{ac_1} + \cdots + q_m \exp^{ac_m}) = \frac{A_1(z)}{R_1(z)} + a^2 = a^2,
\]

and

\[
\lim_{|z| \to \infty} iq_1^{R_2(z+c_1)}e^{p(z+c_1)-p(z)} + \cdots + q_m^{R_2(z+c_m)}e^{p(z+c_m)-p(z)} = i(q_1 \exp^{-ac_1} + \cdots + q_m \exp^{-ac_m}) = -\frac{B_1(z)}{R_2(z)} - a^2 = -a^2,
\]

that is

\[
i(q_1 \exp^{ac_1} + \cdots + q_m \exp^{ac_m}) = a^2, \quad i(q_1 \exp^{-ac_1} + \cdots + q_m \exp^{-ac_m}) = -a^2.
\] (6.10)

So, we have \( \sum_{i=1}^{m} q_i \exp^{ac_i} + \sum_{i=1}^{m} q_i \exp^{-ac_i} = 0 \).

(i) If \( \sum_{i=1}^{m} q_i \exp^{ac_i} + \sum_{i=1}^{m} q_i \exp^{-ac_i} \neq 0 \), this is a contradiction with \( \sum_{i=1}^{m} q_i \exp^{ac_i} + \sum_{i=1}^{m} q_i \exp^{-ac_i} = 0 \). Hence, Theorem 1.4 (i) holds.

(ii) If \( \sum_{j=1}^{m} ic_j q_j \exp^{ac_j} \neq 2a \) and \( \sum_{j=1}^{m} ic_j q_j \exp^{-ac_j} \neq 2a \). By (6.8)–(6.10), we have

\[
iq_1 \exp^{ac_1}(R_1(z + c_1) - R_1(z)) + iq_2 \exp^{ac_2}(R_1(z + c_2) - R_1(z)) + \cdots + iq_m \exp^{ac_m}(R_1(z + c_m) - R_1(z)) = R_1''(z) + 2aR_1'(z),
\] (6.11)

and

\[
iq_1 \exp^{-ac_1}(R_2(z + c_1) - R_2(z)) + iq_2 \exp^{-ac_2}(R_2(z + c_2) - R_2(z)) + \cdots + iq_m \exp^{-ac_m}(R_2(z + c_m) - R_2(z)) = -R_2''(z) + 2aR_2'(z).
\] (6.12)

If \( R_1(z), R_2(z) \) are two nonzero rational functions, using the similar method as the proof of Theorem 1.3, we know that \( R_i(z) \) is a polynomial. If \( \deg_z R_i(z) \geq 2 \). Let \( R_i(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \), then

\[
R_i'(z) = na_n z^{n-1} + (n-1)a_{n-1} z^{n-2} + \cdots ,
R_i''(z) = n(n-1)a_n z^{n-2} + (n-1)(n-2)a_{n-2} z^{n-3} + \cdots ,
R_i(z + c_m) - R_i(z) = na_n c_m z^{n-1} + (a_n C_n c_m + (n-1)a_{n-1} c_m) z^{n-2} + (a_n C_n c_m + a_{n-1} C_{n-1} c_m) z^{n-3} + \cdots ,
\] (6.13)

where \( i = 1, 2 \). Substituting (6.13) into (6.11) and (6.12), comparing the coefficients of \( z^{n-1}, z^{n-2}, \) we have \( \sum_{j=1}^{m} ic_j q_j \exp^{ac_j} = 2a \), \( \sum_{j=1}^{m} c_j^2 q_j \exp^{ac_j} = 2 \) and \( \sum_{j=1}^{m} ic_j q_j \exp^{-ac_j} = 2a \), \( \sum_{j=1}^{m} c_j^2 q_j \exp^{-ac_j} = -2 \), a contradiction. Hence, \( \deg_z R_i(z) \leq 1 \).

If \( \deg_z R_i(z) = 1 \), then (6.11) and (6.12) imply that \( \sum_{j=1}^{m} ic_j q_j \exp^{ac_j} = 2a \) and \( \sum_{j=1}^{m} ic_j q_j \exp^{-ac_j} = 2a \), a contradiction. Hence, \( R_1(z), R_2(z) \) are two nonzero constants, \( R(z) = R_1(z)R_2(z) \) is a constant. By (6.5), we have

\[
l(z) = \frac{t_1 \exp^{az+b} + t_2 \exp^{-(az+b)}}{2} + P(z),
\]
where \( a \neq 0, b \in \mathbb{C}, t_1, t_2 \in \mathbb{C} \setminus \{0\} \) and \( P(z) \) is a polynomial of degree one. Since \( \sum_{i=1}^{m} \varrho_i \neq 0 \), then by (6.5), we have \( P(z) \equiv 0 \). So, we have
\[
l(z) = \frac{t_1 \exp(az+b) + t_2 \exp(-az-b)}{2},
\]
(6.14)
where \( \sum_{i=1}^{m} \varrho_i \exp(ac_i) + \sum_{i=1}^{m} \varrho_i \exp(-ac_i) = 0, b \in \mathbb{C}, R(z) = a^4 t_1 t_2 \). Hence, Theorem 1.4 holds.

Acknowledgments

We thank the referee(s) for reading the manuscript very carefully and making a number of valuable and kind comments which improved the presentation. The work was supported by the NNSF of China (No.10771121, 11401387), the NSF of Zhejiang Province, China (No. LQ14A010007), the NSFC Tianyuan Mathematics Youth Fund (No. 11226094), the NSF of Shandong Province, China (No. ZR2012AQ020 and No. ZR2010AM030) and the Fund of Doctoral Program Research of Shaoxing College of Art and Science (20135018).

Conflict of interest

The authors declare that none of the authors have any competing interests in the manuscript.

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