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Research article

Landweber iterative regularization method for reconstructing the unknown source of the modified Helmholtz equation

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Abstract: This paper considers the inverse problem of determining an unknown source which depends only one spatial variable on modified Helmholtz equation. This problem is well known to be severely ill-posed, the solution (if it exists) does not depend continuously on the data. Landweber iterative regularization method is used to solve this inverse source problem. The Hölder type error estimates are obtained between the exact solution and regularization solutions under an *a priori* and an *a posteriori* regularization parameters choice rules, respectively. Numerical examples are provided to show the effectiveness of the proposed method.

Keywords: modified Helmholtz equation; ill-posed problem; posteriori regularization parameter choice rule; Landweber iterative method
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1. Introduction

In recent years, the Helmholtz equation usually appears in a lot of branches of science and engineering. It is used for describing electromagnetic field, wave propagation, vibration of a structure, scattering and so on [1,2]. The modified Helmholtz equation or the Yukawa equation which is refered to [3] appears in implicit marching schemes for Debye-H*ü*ckel theory, in the heat equation and in the linearization of the Poisson-Boltzmann equation. For the inverse problem of the modified Helmholtz equation, there are much different numerical methods to solve the Cauchy problems of the modified Helmholtz equation, such as the method of fundamental solutions(MFS) [4], Tikhonov type regulation [5], the iteration regularization method [6], Quasi-boundary regularization [7,8], Quasi-reversibility and truncation methods [9–11], and so on. For reconstructing the unknown source of the modified Helmholtz equation, in [12], the authors used the simplified Tikhonov regularization method to reconstruct an unknown source which depended only on one variable. In [13], the authors used the quasi-reversibility regularization method to reconstruct the unknown source in an unbound domain. However, in [12,13], the choice of regularization parameters is given by the *a priori* rule. *A priori* choice of the regularization parameter relies on the *a priori* bound E of the unknown solution. Meanwhile, the *a priori* bound E cannot be known exactly in reality, it will lead to the bad regularized solution by using a wrong constant E.

In this paper, two choices of the regularization parameter is given, i.e., the *priori* choice and the *posterior* choice. In addition, we compare the effectiveness between the the *posterior* choice rule and the *priori* choice rule. To the authors' knowledge, there were few papers to identify the unknown source on the modified Helmholtz equation by posteriori regularization parameter choice rules on half unbounded domain.

In this work, we consider the following inverse problem: to find a pair of functions $(u(x,y), f(x))$ which satisfy

$$\begin{cases} \Delta u(x,y) - k^2 u(x,y) = f(x), & -\infty < x < \infty, 0 < y < +\infty, \\ u(x,0) = 0, & -\infty < x < \infty, \\ u(x,y)\mid_{y\longrightarrow\infty} \textit{bounded}, & -\infty < x < \infty, \\ u(x,1) = g(x), & -\infty < x < \infty, \end{cases} \tag{1.1}$$

where $f(x)$ is the unknown source, the constant k is the large wave number, $u(x,1) = g(x)$ is given function in $L^2(R)$. We use $u(x,1) = g(x)$ to get $f(x)$. However, in practice, data is affected by measurement errors meaning that we have to consider a function $g^\delta(x)$ which is merely in $L^2(\mathbb{R})$ satisfies

$$\|g^\delta - g\|_{L^2(\mathbb{R})} \leq \delta, \tag{1.2}$$

where $\|\cdot\|$ denotes $L^2(\mathbb{R})$ norm and $\delta > 0$ is a noise level.

This paper is organized as follows. In Section 2, we present the Landweber iterative regularization method. The convergence estimates under an *a priori* and an *a posteriori* choice rules will be given in Section 3. In Section 4, numerical examples are given to show the effectiveness of our method. We will have a brief conclusion in Section 5.

2. Landweber iterative regularization method

Before giving the main results of this paper, we first give some useful lemmas.

Lemma 2.1. *For* $0 \leq i \leq 1$ *and* $n \geq 1$, *there holds*

$$(1-i)^n i \leq \frac{1}{n+1},$$

$$\frac{1-(1-i)^n}{i} \leq n. \tag{2.1}$$

Proof. Denote $\phi(i) = (1-i)^n i$ *and* $\varphi(i) = 1 - (1-i)^n - ni$,
then

$$\phi'(i) = -n(1-i)^{n-1} i + (1-i)^n,$$

$$\varphi'(i) = n(1-i)^{n-1} - n.$$

Setting $\phi'(i) = 0$, we have

$$i = \frac{1}{n+1}.$$

Note $\phi(0) = \phi(1) = 0$, $\phi(i)$ has an unique maximal value at $i = \frac{1}{n+1}$. Therefore,

$$\phi(i) = (1-i)^n i \leq \frac{1}{n+1}.$$

Because of $\varphi'(i) \leq 0$ and $\varphi(0) = 0$, we can get $\varphi(i) \leq 0$. □

Lemma 2.2. *For* $0 \leq i \leq 1$, $0 < \alpha \leq 1$ *and* $n \geq 1$, *we can obtain:*

$$(1-i)^n i^\alpha \leq (n+1)^{-\alpha},$$

$$\frac{1-(1-i)^n}{i^\alpha} \leq n^\alpha.$$

Proof. Due to (2.1), we can obtain

$$(1-i)^n i^\alpha \leq [(1-i)^n i]^\alpha \leq (n+1)^{-\alpha},$$

$$\frac{1-(1-i)^n}{i^\alpha} \leq [\frac{1-(1-i)^n}{i}]^\alpha \leq n^\alpha.$$

□

Lemma 2.3. *For* $t \geq 1$, *there holds:*

$$\frac{1}{1-e^{-\sqrt{t}}} \leq 2 \tag{2.2}$$

and

$$\frac{1}{2t} \leq \frac{1-e^{-\sqrt{t}}}{t} \leq \frac{1}{t}. \tag{2.3}$$

Using the Fourier transform, we obtain the solution of problem (1.1) as follows:

$$\widehat{f}(\xi) = -\frac{\xi^2+k^2}{1-e^{-\sqrt{\xi^2+k^2}}}\widehat{g}(\xi). \tag{2.4}$$

Then

$$f(x) = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi x}\frac{\xi^2+k^2}{1-e^{-\sqrt{\xi^2+k^2}}}\widehat{g}(\xi)d\xi. \tag{2.5}$$

Due to (2.4), we define operator $\widehat{A} : \widehat{f} \to \widehat{g}$ and obtain

$$\widehat{Af}(\xi) = -\frac{1-e^{-\sqrt{\xi^2+k^2}}}{\xi^2+k^2}\widehat{f}(\xi). \tag{2.6}$$

The operator $\widehat{A}$ is a linear self adjoint compact operator, so $\widehat{A} = \widehat{A}^*$. Due to (2.6), we know

$$\widehat{A} = \widehat{A}^* = -\frac{1-e^{-\sqrt{\xi^2+k^2}}}{\xi^2+k^2}. \tag{2.7}$$

When $\xi \to \infty$, $(\xi^2+k^2)/(1-e^{-\sqrt{\xi^2+k^2}})$ in (2.4) or (2.5) can be seen as an amplification factor of $\widehat{g}(\xi)$. Thus the exact data function $\widehat{g}(\xi)$ must decay rapidly. However, $g(x)$ can be measured with error. When we use the data function $g(x)$ with error to reconstruct the unknown source $f(x)$, this error can destroy the solution. So we will use the regularization method to deal with the ill-posed problem. Assume a priori bound of the unknown source $f(x)$ as follows:

$$\|f(\cdot)\|_p \le E, \quad p > 0, \tag{2.8}$$

where $E > 0$ is a constant and $\|f(\cdot)\|_p$ is defined as follows:

$$\|f(\cdot)\|_p = \Big(\int_{-\infty}^{\infty} (k^2+\xi^2)^p \mid \widehat{f}(\xi) \mid^2 d\xi\Big)^{\frac{1}{2}}. \tag{2.9}$$

Theorem 2.1. *If* $\|f(\cdot)\|_p \le E,\ p > 0$, *then*

$$\|f(\cdot)\| \le C_1 E^{\frac{2}{p+2}} \|g(\cdot)\|^{\frac{p}{p+2}},$$

where $C_1 := 2^{\frac{p}{p+2}}$.

Proof. Using the Hölder inequality, we obtain

$$\begin{aligned}
\|f(\cdot)\|^2 = \|\widehat{f}(\cdot)\|^2 &= \int_{-\infty}^{\infty} \Big(\frac{\xi^2+k^2}{1-e^{-\sqrt{\xi^2+k^2}}}\widehat{g}(\xi)\Big)^2 d\xi \\
&= \int_{-\infty}^{\infty} (\widehat{g}(\xi))^{\frac{4}{p+2}} \Big(\frac{\xi^2+k^2}{1-e^{-\sqrt{\xi^2+k^2}}}\Big)^2 (\widehat{g}(\xi))^{\frac{2p}{p+2}} d\xi \\
&\le \Big(\int_{-\infty}^{\infty} (\widehat{g}(\xi))^{\frac{4}{p+2}} \Big(\frac{\xi^2+k^2}{1-e^{-\sqrt{\xi^2+k^2}}}\Big)^2\Big)^{\frac{p+2}{2}} d\xi\Big)^{\frac{2}{p+2}} \Big(\int_{-\infty}^{\infty} \Big((\widehat{g}(\xi))^{\frac{2p}{p+2}}\Big)^{\frac{p+2}{p}} d\xi\Big)^{\frac{p}{p+2}} \\
&= \Big(\int_{-\infty}^{\infty} (\widehat{g}(\xi))^2 \Big(\frac{\xi^2+k^2}{1-e^{-\sqrt{\xi^2+k^2}}}\Big)^{p+2} d\xi\Big)^{\frac{2}{p+2}} \Big(\int_{-\infty}^{\infty} ((\widehat{g}(\xi))^2 d\xi\Big)^{\frac{p}{p+2}} \\
&= \Big(\int_{-\infty}^{\infty} (\widehat{f}(\xi))^2 \Big(\frac{\xi^2+k^2}{1-e^{-\sqrt{\xi^2+k^2}}}\Big)^{p} d\xi\Big)^{\frac{2}{p+2}} \|\widehat{g}(\cdot)\|^{\frac{2p}{p+2}} \\
&\le \sup_{\xi\in R} \Big(\frac{1}{1-e^{-\sqrt{\xi^2+k^2}}}\Big)^{\frac{2p}{p+2}} \Big(\int_{-\infty}^{\infty} (\widehat{f}(\xi))^2 (\xi^2+k^2)^p d\xi\Big)^{\frac{2}{p+2}} \|g(\cdot)\|^{\frac{2p}{p+2}} \\
&\le 2^{\frac{2p}{p+2}} E^{\frac{4}{p+2}} \|g(\cdot)\|^{\frac{2p}{p+2}}.
\end{aligned}$$

So we obtain

$$\|f(\cdot)\| \le C_1 E^{\frac{2}{p+2}} \|g(\cdot)\|^{\frac{p}{p+2}},$$

where $C_1 := 2^{\frac{p}{p+2}}$. We finish the proof of the Theorem 2.1.

We can get the regularization solution of (1.1) by using Landweber iterative method. The Landweber iterative method is as follows: the equation $\widehat{Af} = \widehat{g}$ can be written in the form $\widehat{f} = (I - a\widehat{A}^*\widehat{A})\widehat{f} + a\widehat{A}^*\widehat{g}$ for some $a > 0$. Then iterate this equation, i.e.,

$$\widehat{f}^{0,\delta}(\xi) := 0,\ \widehat{f}^{m,\delta}(\xi) = (I - a\widehat{A}^*\widehat{A})\widehat{f}^{m-1,\delta}(\xi) + a\widehat{A}^*\widehat{g}^{\delta}(\xi),\ m = 1,2,3,\cdots, \tag{2.10}$$

where m is iterative step number, and it is selected as regularization parameter. a is relaxation factor, which satisfies $0 < a < \frac{1}{\|\widehat{A}\|^2}$. Since $\widehat{A}$ is self adjoint operator, we obtain

$$\widehat{f}^{m,\delta}(\xi) = a\sum_{k=0}^{m-1}(I - a\widehat{A}^2)^k\widehat{A}\widehat{g}^\delta(\xi). \tag{2.11}$$

Using (2.7), we get

$$\widehat{f}^{m,\delta}(\xi) = R_m\widehat{g}^\delta(\xi) = \frac{1-(1-a\widehat{A}^2)^m}{\widehat{A}}\widehat{g}^\delta(\xi). \tag{2.12}$$

Then

$$f^{m,\delta}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi x}\frac{1-(1-a\widehat{A}^2)^m}{\widehat{A}}\widehat{g}^\delta(\xi)d\xi. \tag{2.13}$$

Before giving the error estimation between the regularization solution and the exact solution, we give the uniqueness of problem (1.1), i.e., through using $u(x,1) = g(x)$, we can uniquely determine a pair $(u(x,y), f(x))$.

Theorem 2.2. *Let $u_i(x,y) \in C^{2,2}((-\infty,+\infty)\times(0,1])$, $f_i(x) \in L^2(\mathbb{R})(i = 1,2)$. If $(u_i(x,y), f_i(x))(i = 1,2)$ satisfy* (1.1)*, then $(u_1(x,y), f_1(x)) = (u_2(x,y), f_2(x))$.*

Proof. Put $\hat{u}(x,y) = u_1(x,y) - u_2(x,y)$, $\hat{f} = f_1(x) - f_2(x)$, then $\hat{u}(x,y)$ satisfy

$$\begin{cases} \Delta\hat{u}(x,y) - k^2\hat{u}(x,y) = \hat{f}(x), & -\infty < x < \infty, 0 < y < +\infty, \\ \hat{u}(x,0) = 0, & -\infty < x < \infty, \\ \hat{u}(x,1) = 0, & -\infty < x < \infty. \end{cases} \tag{2.14}$$

Using (2.4) or (2.5), we know $\hat{f}(x) = 0$. Then using (2.14), we know $\hat{u} = 0$. Thus the uniqueness is proved.

3. Error estimate under two parameters choice

In this section, we will give two error estimates with a priori parameter choice rule and posterior parameter choice rule, respectively.

3.1. The error estimate with a priori parameter choice

Theorem 3.1. *Let $f(x)$ given by* (2.5) *be the exact solution of problem* (1.1)*. Let $f^{m,\delta}(x)$ given by* (2.13) *be the regularization Landweber iterative approximation solution. Choosing the regularization parameter $m = [r]$, where*

$$r = (\frac{E}{\delta})^{\frac{4}{p+2}}, \tag{3.1}$$

then we have an error estimate:

$$\|f^{m,\delta}(\cdot) - f(\cdot)\| \le C_2E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, \tag{3.2}$$

where $[r]$ denotes the largest integer less than or equal to r, and $C_2 = \sqrt{a} + (\frac{4p}{a})^{\frac{p}{4}}$ is positive constant depending on a, p.

Proof. By the triangle inequality and Parseval formula, we know

$$\begin{aligned}\|f^{m,\delta}(\cdot)-f(\cdot)\| &= \|\widehat{f^{m,\delta}}(\cdot)-\widehat{f}(\cdot)\|\\ &\le \|\widehat{f^{m,\delta}}(\cdot)-\widehat{f^{m}}(\cdot)\|+\|\widehat{f^{m}}(\cdot)-\widehat{f}(\cdot)\|.\end{aligned}$$

We firstly give an estimate for the first term. From condition (1.2) and (2.12), we have

$$\begin{aligned}\|\widehat{f^{m,\delta}}(\cdot)-\widehat{f^{m}}(\cdot)\|^2 &= \|\frac{1-(1-a\widehat{A}^2)^m}{\widehat{A}}\widehat{g}^{\delta}(\xi)-\frac{1-(1-a\widehat{A}^2)^m}{\widehat{A}}\widehat{g}(\xi)\|\\ &= \|\frac{1-(1-a\widehat{A}^2)^m}{\widehat{A}}(\widehat{g}^{\delta}(\xi)-\widehat{g}(\xi))\|^2\\ &\le \sup_{\xi\in\mathbb{R}} B(\xi)^2\delta^2,\end{aligned}$$

where $B(\xi) := \frac{1-(1-a\widehat{A}^2)^m}{\widehat{A}}$.
By using Bernolli inequality, we get

$$\frac{1-(1-a\widehat{A}^2)^m}{\widehat{A}} \le \sqrt{am}, \tag{3.3}$$

i.e.,

$$B(\xi) \le \sqrt{am}. \tag{3.4}$$

So

$$\|\widehat{f^{m,\delta}}(\cdot)-\widehat{f^{m}}(\cdot)\| \le \sqrt{am}\delta. \tag{3.5}$$

Now we estimate the second term. By using (2.6), (2.8) and (2.9), we have

$$\begin{aligned}\|\widehat{f^{m}}(\cdot)-\widehat{f}(\cdot)\|^2 &= \|\frac{1-(1-a\widehat{A}^2)^m}{\widehat{A}}\widehat{g}(\xi)-\widehat{A}^{-1}\widehat{g}(\xi)\|^2\\ &= \|\frac{(1-a\widehat{A}^2)^m}{\widehat{A}}\widehat{g}(\xi)\|^2\\ &= \|(1-a\widehat{A}^2)^m(k^2+\xi^2)^{-\frac{p}{2}}(k^2+\xi^2)^{\frac{p}{2}}\widehat{f}(\xi)\|^2\\ &\le \sup_{\xi\in\mathbb{R}} D(\xi)^2E^2,\end{aligned}$$

where $D(\xi) := (1-a(-\frac{1-e^{-\sqrt{\xi^2+k^2}}}{\xi^2+k^2})^2)^m(k^2+\xi^2)^{-\frac{p}{2}}$.
Using Lemma 2.2 and (2.3), we have

$$D(\xi) \le (1-\frac{a}{4(\xi^2+k^2)^2})^m(k^2+\xi^2)^{-\frac{p}{2}}.$$

Let $\xi^2+k^2=t$,

$$F(t) = (1-\frac{a}{(2t)^2})^m t^{-\frac{p}{2}}. \tag{3.6}$$

Let t_0 satisfy $F'(t_0)=0$, and we easily obtain

$$t_0 = (\frac{a(4m+p)}{4p})^{\frac{1}{2}}. \tag{3.7}$$

Thus

$$F(t_0) = (1 - \frac{p}{4m+p})^m (\frac{a(4m+p)}{4p})^{-\frac{p}{4}} \leq (\frac{4p}{a(m+1)})^{\frac{p}{4}},$$

i.e.,

$$F(t) \leq (\frac{4p}{a})^{\frac{p}{4}} (m+1)^{-\frac{p}{4}}. \tag{3.8}$$

Thus we obtain

$$D(\xi) \leq (\frac{4p}{a})^{\frac{p}{4}} (m+1)^{-\frac{p}{4}}. \tag{3.9}$$

Hence

$$\|\widehat{f^m}(\cdot) - \widehat{f}(\cdot)\| \leq (\frac{4p}{a})^{\frac{p}{4}} (m+1)^{-\frac{p}{4}} E. \tag{3.10}$$

Combining (3.5) and (2.12), we choose $m = [r]$, and get

$$\|\widehat{f^{m,\delta}}(\cdot) - \widehat{f}(\cdot)\| \leq C_2 E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, \tag{3.11}$$

where $C_2 = \sqrt{a} + (\frac{4p}{a})^{\frac{p}{4}}$.
We complete the proof of Theorem 3.1.

3.2. The error estimate with a posteriori parameter choice

We consider the *a posteriori* parameter choice in the Morozov's discrepancy, and construct regularization solution sequences $f^{m,\delta}(x)$ by Landweber iterative regularization method. Assume $\tau > 1$ is given a fixed constant. Stop the algorithm at the first occurrence of $m = m(\delta) \in \mathbb{N}_0$ with

$$\|\widehat{Af^{m,\delta}}(\cdot) - \widehat{g^{\delta}}(\cdot)\| \leq \tau\delta, \tag{3.12}$$

where $\|\widehat{g^{\delta}}\| \geq \tau\delta$ is constant.

Lemma 4.1. *Let* $\rho(m) = \|\widehat{Af^{m,\delta}}(\cdot) - \widehat{g^{\delta}}(\cdot)\|$, *then we have the following conclusions:*
(a) $\rho(m)$ *is a continuous function;*
(b) $\lim_{m\to 0^+} \rho(m) = \|\widehat{g^{\delta}}\|$;
(c) $\lim_{m\to +\infty} \rho(m) = 0$;
(d) $\rho(m)$ *is a strictly decreasing function, for any* $m \in (0, +\infty)$.

The proof is very obvious and we omit it here.

Lemma 4.2. *For fixed* $\tau > 1$, *combining Landweber's iteration method with stopping rule* (3.12)*, we can obtain the regularization parameter* $m = m(\delta, \widehat{g^{\delta}}) \in \mathbb{N}_0$ *satisfies:*

$$m \leq (\frac{2(p+2)}{a})(\frac{E}{(\tau-1)\delta})^{\frac{4}{p+2}}. \tag{3.13}$$

Proof. From (2.12), we have the following representation:

$$R_m\widehat{g} = \frac{1 - (1 - a\widehat{A}^2)^m}{\widehat{A}} \widehat{g}(\xi), \tag{3.14}$$

and

$$\|\widehat{AR_m g} - g\|^2 = \|\widehat{A}R_m\widehat{g} - \widehat{g}\|^2 = \|(1 - (1 - a\widehat{A}^2)^m)\widehat{g}(\xi) - \widehat{g}(\xi)\|^2$$

$$= \|(-(1 - a\widehat{A}^2)^m)\widehat{g}(\xi)\|^2$$
$$= (1 - a\widehat{A}^2)^{2m}\widehat{g}^2(\xi).$$

From $|1 - a\widehat{A}^2| < 1$, we conclude that $\|\widehat{A}R_{m-1} - I\| \leq 1$.
It is easily to see that m is the minimum value and satisfies

$$\|\widehat{A}R_m\widehat{g}^\delta - \widehat{g}^\delta\| = \|\widehat{Af}^{m,\delta} - \widehat{g}^\delta\| \leq \tau\delta.$$

Hence

$$\begin{aligned}\|\widehat{A}R_{m-1}\widehat{g} - \widehat{g}\| &\geq \|\widehat{A}R_{m-1}\widehat{g}^\delta - \widehat{g}^\delta\| - \|(\widehat{A}R_{m-1} - I)(\widehat{g} - \widehat{g}^\delta)\| \\ &\geq \tau\delta - \|\widehat{A}R_{m-1} - I\|\delta \\ &\geq (\tau - 1)\delta.\end{aligned}$$

On the other hand, using (2.8), we obtain

$$\begin{aligned}\|\widehat{A}R_{m-1}\widehat{g}(\xi) - \widehat{g}(\xi)\| &= \|(1 - (1 - a\widehat{A}^2)^{m-1})\widehat{g}(\xi) - \widehat{g}(\xi)\| \\ &= \| - (1 - a\widehat{A}^2)^{m-1}\widehat{g}(\xi)\| \\ &= \|(1 - a\widehat{A}^2)^{m-1}\widehat{A}(\xi^2 + k^2)^{-\frac{p}{2}}\widehat{f}(\xi)(\xi^2 + k^2)^{\frac{p}{2}}\| \\ &\leq \sup_{\xi\in\mathbb{R}} |(1 - a\widehat{A}^2)^{m-1}\widehat{A}(\xi^2 + k^2)^{-\frac{p}{2}}|E.\end{aligned}$$

Let

$$T(\xi) := (1 - a\widehat{A}^2)^{m-1}\widehat{A}(\xi^2 + k^2)^{-\frac{p}{2}}, \tag{3.15}$$

so

$$(\tau - 1)\delta \leq T(\xi)E. \tag{3.16}$$

Using (2.3), we have

$$T(\xi) \leq (1 - a(\frac{1}{2t})^2)^{m-1}t^{-\frac{p}{2}-1}. \tag{3.17}$$

Let

$$G(t) = (1 - a(\frac{1}{2t})^2)^{m-1}t^{-\frac{p}{2}-1}, \tag{3.18}$$

suppose t_* satisfy $G'(t_*) = 0$, we easily get

$$t_* = (\frac{a(4m + p - 2)}{4(p + 2)})^{\frac{1}{2}}, \tag{3.19}$$

so

$$\begin{aligned}G(t_*) &= (1 - \frac{4(p + 2)}{4m + p - 2})^{m-1}(\frac{a(4m + p - 2)}{4(p + 2)})^{-\frac{p+2}{4}} \\ &\leq (\frac{2(p + 2)}{ma})^{\frac{p+2}{4}}.\end{aligned} \tag{3.20}$$

Then

$$T(\xi) \leq (\frac{2(p + 2)}{ma})^{\frac{p+2}{4}}. \tag{3.21}$$

Combining (3.16) with (3.21), we obtain

$$m \leq (\frac{2(p+2)}{a})(\frac{E}{(\tau-1)\delta})^{\frac{4}{p+2}}.$$

The proof of Lemma is completed.

Theorem 4.3. *Let $f(x)$ given by* (2.5) *be the exact solution of problem* (1.1)*. Let $f^{m,\delta}(x)$ given by* (2.13) *be the regularization Landweber iterative approximation solution. The regularization parameter is chosen with stopping rule* (3.12)*, then we have the following error estimate:*

$$\|f^{m,\delta}(\cdot) - f(\cdot)\| \leq (C_1(\tau+1)^{\frac{2}{p+2}} + C_3)E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, \tag{3.22}$$

where $C_3 = (2(p+2))(\frac{1}{\tau-1})^{\frac{2}{p+2}}$ is a constant.

Proof. Using triangle inequality and Parseval equality, we obtain

$$\begin{aligned}\|f^{m,\delta}(\cdot) - f(\cdot)\| &= \|\widehat{f}^{m,\delta}(\cdot) - \widehat{f}(\cdot)\| \\ &\leq \|\widehat{f}^{m,\delta}(\cdot) - \widehat{f}^{m}(\cdot)\| + \|\widehat{f}^{m}(\cdot) - \widehat{f}(\cdot)\|.\end{aligned}$$

Applying (3.5) and Lemma 4.2, we get

$$\|\widehat{f}^{m,\delta}(\cdot) - \widehat{f}^{m}(\cdot)\| \leq \sqrt{am}\delta \leq C_4 E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, \tag{3.23}$$

where $C_4 = (2(p+2))^{\frac{1}{2}}(\frac{1}{\tau-1})^{\frac{2}{p+2}}$.
For the second part of the right side, we get

$$\begin{aligned}\|\widehat{A}(\widehat{f}^{m}(\cdot) - \widehat{f}(\cdot))\| &= \|(-(1-a\widehat{A}^2)^m)\widehat{g}(\xi)\| \\ &= \|(-(1-a\widehat{A}^2)^m)(\widehat{g}(\xi) - \widehat{g}^{\delta}(\xi))\| + \|(-(1-a\widehat{A}^2)^m)\widehat{g}^{\delta}(\xi)\|.\end{aligned}$$

Combining (1.2) and (3.12), we have

$$\|\widehat{A}(\widehat{f}^{m}(\cdot) - \widehat{f}(\cdot))\| \leq (\tau+1)\delta. \tag{3.24}$$

Using Theorem 2.1 and (2.9), we have

$$\begin{aligned}\|\widehat{f}^{m}(\cdot) - \widehat{f}(\cdot)\|_p &= (\int_{-\infty}^{\infty}(-(1-a\widehat{A}^2)^{2m}\widehat{f}^2(\xi)(\xi^2+k^2)^p)d\xi)^{\frac{1}{2}} \\ &\leq (\int_{-\infty}^{\infty}(\widehat{f}^2(\xi)(\xi^2+k^2)^p)d\xi)^{\frac{1}{2}} \\ &\leq E.\end{aligned}$$

So

$$\|\widehat{f}^{m}(\cdot) - \widehat{f}(\cdot)\| \leq C_1(\tau+1)^{\frac{p}{p+2}}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}. \tag{3.25}$$

Hence

$$\|\widehat{f}^{m,\delta}(\cdot) - \widehat{f}(\cdot)\| \leq (C_1(\tau+1)^{\frac{p}{p+2}} + C_3)E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}. \tag{3.26}$$

The theorem is proved.

4. Numerical implementation and numerical examples

In this section, the numerical results are presented, which verify the validity of the theoretical results of this method.

Example 1. It is easy to see that the function

$$u(x,y) = \begin{cases} (1 - e^{-\sqrt{2}ky}) \sin kx, & y > 0, \\ 0, & y \leq 0, \end{cases} \tag{4.1}$$

and the function

$$f(x) = -2k^2 \sin kx \tag{4.2}$$

are satisfied with the problem (1.1) with exact data

$$g(x) = (1 - e^{-\sqrt{2}k}) \sin kx. \tag{4.3}$$

Suppose that the sequence $g(x_i)_{i=1}^{i=n}$ represents samples from the function $g(x)$ on an equidistant grid, and n is even, then we use the rand function given in Matlab to generate the noisy data,

$$(g^\delta) = g + \varepsilon \; rand(size(g)), \tag{4.4}$$

where

$$g = (g(x_1), \ldots, g(x_n))^T, x_i = (i-1)\triangle x - 5, \tag{4.5}$$

$$\triangle x = \frac{10}{n-1}, \;\; i = 1, \ldots, n. \tag{4.6}$$

The function " $rand(\cdot)$" generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$, and standard deviation $\sigma = 1$. " Rand(size(g))" returns an array of random entries that is the same size as g. The total noise level δ can be measured in the sense of Root Mean Square Error(RMSE) according to

$$\delta = \|g^\delta - g\|_2 = (\frac{1}{n}\sum_{i=1}^{n}(g_i - g_i^\delta)^2)^{\frac{1}{2}}. \tag{4.7}$$

The numerical calculation is described as follows:

Step 1: Take the Fast Fourier Transform (FFT) for the vector $\hat{g}^\delta$.

Step 2: Compute the vectors

$$\{-\frac{\xi_j^2 + k^2}{1 - e^{-\sqrt{\xi_j^2+k^2}}}[1 - (1 - a(\frac{1 - e^{-\sqrt{\xi_j^2+k^2}}}{\xi_j^2 + k^2})^2)^m]\hat{g}^\delta(\xi_j)\}_{j=-\frac{n}{2}-1}^{\frac{n}{2}}, \tag{4.8}$$

where $\xi_j = 2\pi j$.

Step 3: Take the inverse FFT for the vector (2.12), and obtain (2.13).

When we use the FFT algorithm, the vector $g^\delta(x)$ must be a periodic function. But the vector $g^\delta(x) \in L^2(\mathbb{R})$ is only measurable, it is not a periodic function. Thus we need to modify the algorithm.

The detail of the modify algorithm can be found in [14]. In the other hand, $f(x)$ is unknown, the exact priori bound is not obtained. Thus in our computation, we only give the compute result under the posteriori regularization parameter according to (3.12). In the following numerical experiment, we take $\tau = 1.1$, $n = 100$, and the relative error is given as follows:

$$rerr(f) := \frac{\|f^{m,\delta} - f\|_2}{\|f\|_2}, \tag{4.9}$$

where $\|\cdot\|_2$ is defined by (4.4).

Next we will give two examples to show the effectiveness of this regularization method. In two examples, we first give the unknown source function $f(x)$ and solve the direct problem and obtain $u(x, y)$. Then use $u(x, 1) = g(x)$ and obtain $g^{\delta}(x)$. Last, we solve the inverse problem to obtain the regularization solution $f^{m,\delta}(x)$, and compare the exact solution the regularization solution for different noisy level and k.

Example 2. Consider a piecewise smooth unknown source:

$$f(x) = \begin{cases} 0, & 0 \le -10 \le -5, \\ x + 5, & -5 \le x \le 0, \\ -x + 5, & 0 \le x \le 5, \\ 0, & 5 \le x \le 10. \end{cases} \tag{4.10}$$

Example 3. Consider the following discontinuous unknown source:

$$f(x) = \begin{cases} -1, & 0 \le -10 \le -5, \\ 1, & -5 \le x \le 0, \\ -1, & 0 \le x \le 5, \\ 1, & 5 \le x \le 10. \end{cases} \tag{4.11}$$

Figure 1 shows the comparison of the exact solution $f(x)$ and the Landweber regularization solution $f^{m,\delta}(x)$ in Example 1 for the relative error levels $\varepsilon = 0.001, 0.0001, 0.00001$ with various values $k = 2, 4, 6$. Figure 2 shows the comparison of the exact solution $f(x)$ and the Landweber regularization solution $f^{m,\delta}(x)$ in Example 2 for the relative error levels $\varepsilon = 0.001, 0.0001, 0.00001$ with various values $k = 2, 4, 6$. Figure 3 shows the comparison of the exact solution $f(x)$ and the Landweber regularization solution $f^{m,\delta}(x)$ in Example 3 for the relative error levels $\varepsilon = 0.001, 0.0001, 0.00001$ with various values $k = 2, 4, 6$. From Figures 1–3, we can see the smaller ε and k, the better the fitting effect. Moreover, we can see the numerical effectiveness of Example 1 is better than that of Example 2 and 3, because Example 2 and 3 first needs to solve the direct problem to obtain the data $g(x)$.

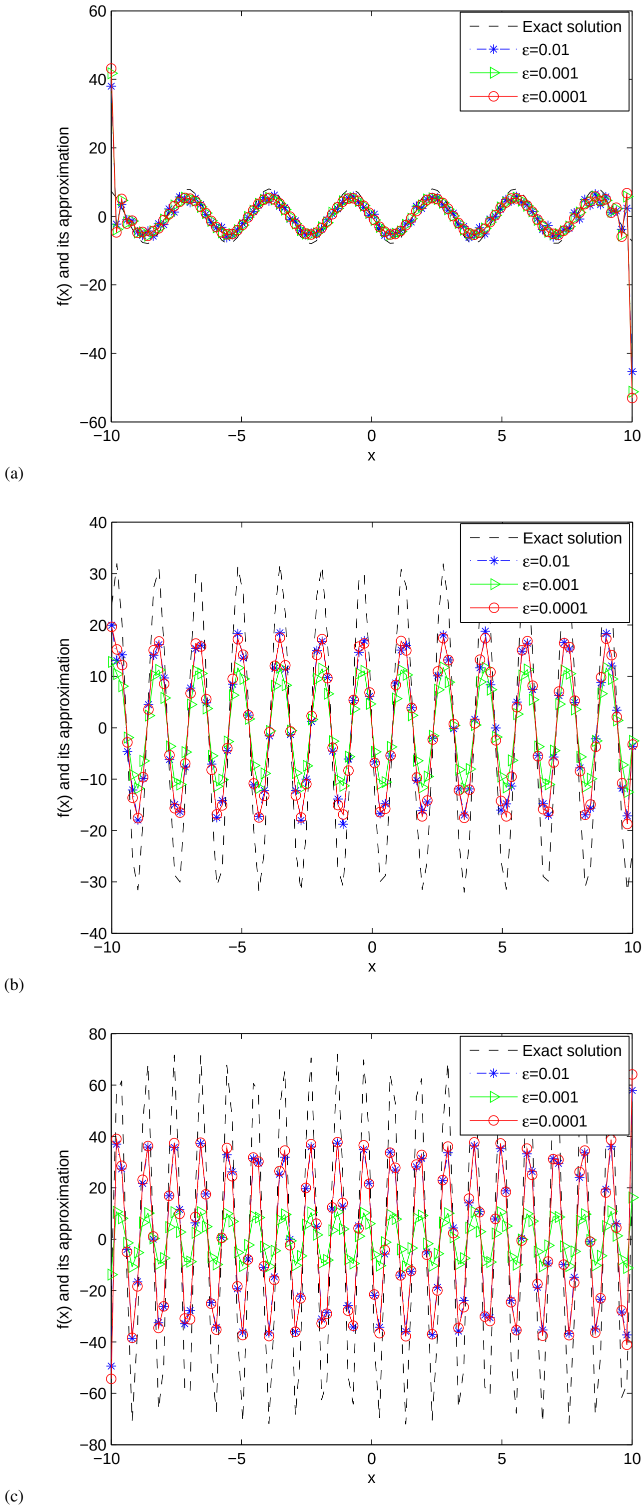
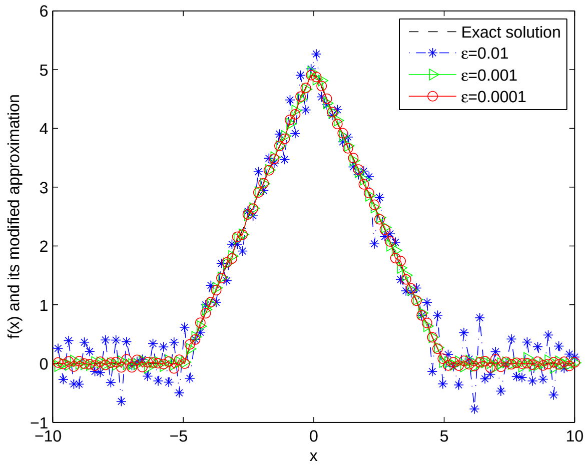
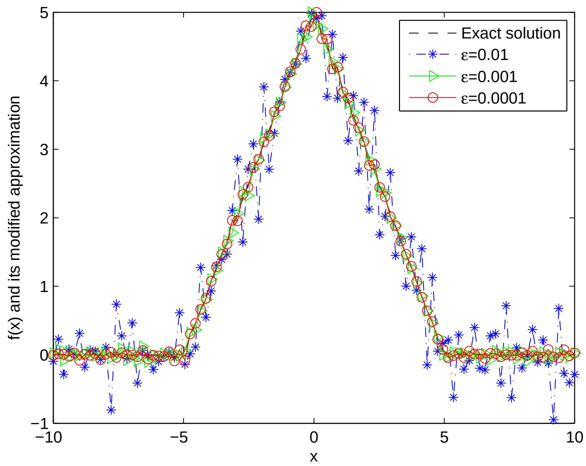


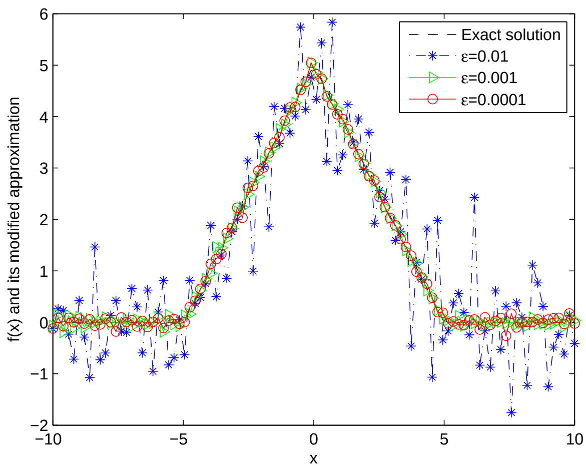
Figure 1. The comparison of the numerical effects between the exact solution and its computed approximations for $\varepsilon = 0.01, 0.001, 0.0001$ with example 1. (a) $k = 2$. (b) $k = 4$. (c) $k = 6$.



(a)

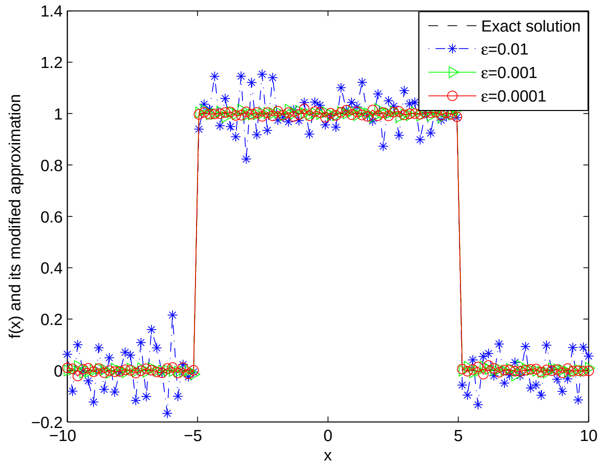


(b)

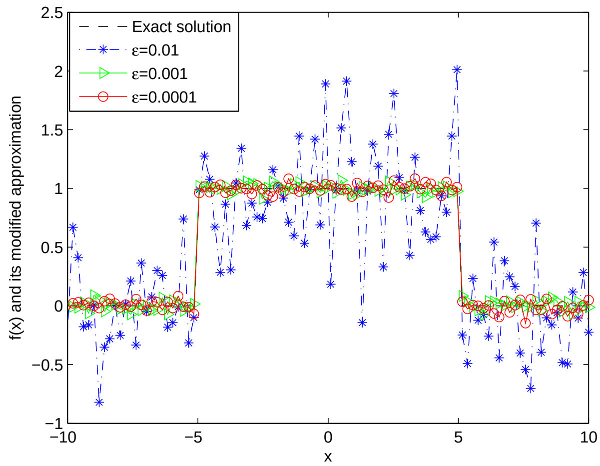


(c)

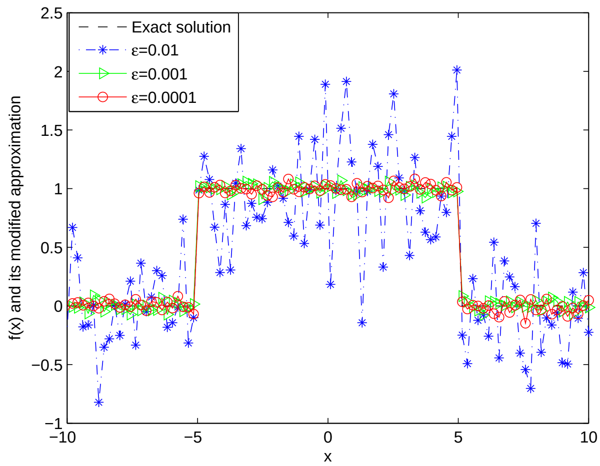
Figure 2. The Comparison of the numerical effects between the exact solution and its computed approximations for $\varepsilon = 0.01, 0.001, 0.0001$ with example 2. (a) $k = 2$. (b) $k = 4$. (c) $k = 6$.



(a)



(b)



(c)

Figure 3. The comparison of the numerical effects between the exact solution and its computed approximations for $\varepsilon = 0.01, 0.001, 0.0001$ with example 3. (a) $k = 2$. (b) $k = 4$. (c) $k = 6$.

5. Conclusions

In this paper, we use the Landweber regularization method to reconstruct the unknown source for the modified Helmholtz equation. The error estimates between the exact solution and the regularization solution are obtained under the priori regularization parameter choice and the posteriori regularization parameter choice. Meanwhile, different kind numerical examples are used to verify the efficiency and accuracy of this method.

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Conflict of interest

The authors declare that they have no competing interests.

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