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Research article

Integral inequalities for hyperbolic type preinvex functions

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Abstract: In this work, we establish the concept of a new class of non-convex functions, namely hyperbolic type preinvex functions. Secondly few algebraic properties of this class are obtained. Further Hermite-Hadamard type integral inequalities are established for this class. We also derive several new inequalities for the functions for which absolute value of first derivative, with exponent greater or equal to one is hyperbolic type preinvexity. The results are obtained by using both the Hölder's inequality and Hölder-Iscan inequality and compared at the end. Several special cases are discussed as applications of the results.

Keywords: preinvex function; hyperbolic type convexity; hyperbolic type preinvexity; Hermite-Hadamard inequality; Hölder's inequality; Hölder-Iscan integral inequality; power-mean integral inequality
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1. Introduction

The theory of convexity exhibits a significant contribution in the evolution of many branches of science and technology. Over the last few years, several advancements and generalizations of convex sets have been contemplated. One of the remarkable comprehension of convex functions is that of invex functions [1]. Hanson established the invex functions which involves the bifunctional used widely in the study of nonlinear programming involved in mathematical and engineering science. Another comprehension of convex functions is preinvex functions proposed and investigated by Weir and Mond [2]. They related the preinvex function to the formation of sufficient optimality conditions and duality in nonlinear programming. Further Noor [3] demonstrated that a function is preinvex function if it satisfies the Hermite-Hadamard inequality. This result is analogous to the convex functions. For detailed properties, applications and other characteristics of preinvex functions, see [3–7].

Various rectifications of the Hermite-Hadamard inequality for the convex functions and their generalized formations have been obtained by many researchers [3–13, 15, 16]. In [17], Varosanec established the concept of h-convex functions, from which many other classes of convex functions can be obtained as a special case. Another recent concept of convex functions, called the hyperbolic type convex function is introduced by Toplu et.al. [18].

In this work, by using the concept of [18], we introduce the idea of class of hyperbolic type preinvex functions. Also, we establish some Hermite-Hadamard type integral inequalities for this class. The ideas and techniques of this paper may be the starting point for further research in this dynamic field.

2. Preliminaries

Let $X \subset R^n$ be a nonempty set. Let $\phi : X \to R$ be a continuous function and $\zeta : X \times X \to R^n$ be the bifunctional. In the first place, we remind some fundamental ideas and concepts.

Definition 2.1. *[19] A set X_ζ is said to be an invex or ζ-connected set corresponding to the bifunctional ζ, if*

$$\omega + \vartheta\zeta(\upsilon, \omega) \in X_\zeta, \qquad \forall \omega, \upsilon \in X_\zeta, \vartheta \in [0, 1]. \tag{2.1}$$

Definition 2.2. *[7] A function ϕ described on the invex set X_ζ is said to be the preinvex function corresponding to the bifunctional ζ, if*

$$\phi(\omega + \vartheta\zeta(\upsilon, \omega)) \leq (1 - \vartheta)\phi(\omega) + \vartheta\phi(\upsilon), \qquad \forall \omega, \upsilon \in X_\zeta, \vartheta \in [0, 1]. \tag{2.2}$$

The function ϕ is preconcave function if and only if the above inequality is altered in other direction. For $\zeta(\upsilon, \omega) = \upsilon - \omega$, the preinvex function becomes convex function.

Definition 2.3. *[20] A function $\phi : X \to R$ is said to be a convex function, if and only if,*

$$\phi(\vartheta\omega + (1 - \vartheta)\upsilon) \leq (1 - \vartheta)\phi(\omega) + \vartheta\phi(\upsilon), \qquad \forall \omega, \upsilon \in X, \vartheta \in [0, 1]. \tag{2.3}$$

The function ϕ is said to be concave if the inequality reverses. Clearly, every convex function is a preinvex function, but converse may not be true.

Definition 2.4. *[17] Let $h : J \to R$ be a nonnegative function, $h \neq 0$. A nonnegative function $\phi : I \to R$ is said to be h-convex function if*

$$\phi(\vartheta\omega + (1 - \vartheta)\upsilon) \leq h(\vartheta)\phi(\omega) + h(1 - \vartheta)\phi(\upsilon), \qquad \forall \omega, \upsilon \in I, \vartheta \in [0, 1]. \tag{2.4}$$

If we take $h(\vartheta) = \vartheta$ then h-convexity reduces to classical convexity. The h-convex function reduces the h-concave function if the above inequality is reversed.

Definition 2.5. *[14] Let h be a non-negative function $h \neq 0$ defined as $h : [0, 1] \to R$. The positive function ϕ on the invex set X_ζ is said to be h-preinvex with respect to the bifunctional ζ, if*

$$\phi(\omega + \vartheta\zeta(\upsilon, \omega) \leq h(1 - \vartheta)\phi(\omega) + h(\vartheta)\phi(\upsilon)), \qquad \forall \omega, \upsilon \in X_\zeta, \vartheta \in [0, 1]. \tag{2.5}$$

If the above inequality reverses, then the function ϕ is said to be h-preconcave. For any $\vartheta \in [0, 1]$, when $\zeta(\upsilon, \omega) = \upsilon - \omega$ and $h(\vartheta) = \vartheta$ every h-preinvex function reduces to h-convex function.

For the properties and applications, generalization and other aspects of the preinvex functions in variational inequalities and optimization theory, see [21–25] and references therein for more details.

Definition 2.6. *(Hölder-İşcan integral inequality [26]). Let $r>1$ and $\frac{1}{r}+\frac{1}{s}=1$. Consider ϕ and ψ be the real valued functions defined on $[\omega,\upsilon]$, where $|\phi|^r, |\psi|^s$ are integrable on $[\omega,\upsilon]$, then*

$$\begin{aligned}\int_\omega^\upsilon |\phi(\chi)\psi(\chi)|d\chi \;\leq\; & \frac{1}{\omega-\upsilon}\Big[\Big(\int_\omega^\upsilon(\upsilon-\chi)|\phi(\chi)|^r d\chi\Big)^{\frac{1}{r}}\Big(\int_\omega^\upsilon(\upsilon-\chi)|\psi(\chi)|^s d\chi\Big)^{\frac{1}{s}}\\ &+\Big(\int_\omega^\upsilon(\chi-\omega)|\phi(\chi)|^r d\chi\Big)^{\frac{1}{r}}\Big(\int_\omega^\upsilon(\chi-\omega)|\psi(\chi)|^s d\chi\Big)^{\frac{1}{s}}\Big]. \qquad (2.6)\end{aligned}$$

Definition 2.7. *An improved form of power-mean integral inequality as a distinct form of the Hölder-İşcan inequality is given by [26]. Let ϕ and ψ be two real valued functions described on the interval $[\omega,\upsilon]$. If for any $s\geq 1$, the functions $|\phi|, |\phi||\psi|^s$ are integrable, then*

$$\begin{aligned}\int_\omega^\upsilon |\phi(\chi)\psi(\chi)|d\chi \;\leq\; & \frac{1}{\omega-\upsilon}\Big[\Big(\int_\omega^\upsilon(\upsilon-\chi)|\phi(\chi)|d\chi\Big)^{1-\frac{1}{r}}\Big(\int_\omega^\upsilon(\upsilon-\chi)|\phi(\chi)||\psi(\chi)|^s d\chi\Big)^{\frac{1}{s}}\\ &+\Big(\int_\omega^\upsilon(\chi-\omega)|\phi(\chi)|d\chi\Big)^{1-\frac{1}{s}}\Big(\int_\omega^\upsilon(\chi-\omega)|\phi(\chi)||\psi(\chi)|^s d\chi\Big)^{\frac{1}{s}}\Big]. \qquad (2.7)\end{aligned}$$

3. Main results

In this section, we introduce the concept of hyperbolic type preinvex function. First, we will define a hyperbolic type preinvex function and then give some basic theorems relevant to it.

Definition 3.1. *Let function $\phi: X_\zeta\subset R\to R$, and X_ζ be an invex set regarding the bifunctional ζ, then ϕ is said to be hyperbolic type preinvex function if and only if for each $\omega,\upsilon\in I$ and $\vartheta\in[0,1]$,*

$$\phi(\omega+\vartheta\zeta(\upsilon,\omega))\leq\Big(\frac{sinh1-sinh\vartheta}{sinh1}\Big)\phi(\omega)+\Big(\frac{sinh\vartheta}{sinh1}\Big)\phi(\upsilon). \qquad (3.1)$$

We denote this class by $HP(I)$.
If we choose $\zeta(\upsilon,\omega)=\upsilon-\omega$, then the hyperbolic type preinvex function reduces to hyperbolic type convex function.

Theorem 3.1. *Let $\phi,\psi\in X_\zeta=[\omega,\omega+\zeta(\upsilon,\omega)]\to R$ be hyperbolic type preinvex function. Then for any scalar $\kappa\geq 0, k\phi$ is a hyperbolic type preinvex function and $\phi+\psi$ is also a hyperbolic preinvex function.*

Proof. (i) Let ϕ be a hyperbolic type preinvex function and κ be a non-negative real number. Consider

$$\begin{aligned}\kappa\phi(\omega+(1-\vartheta)\zeta(\upsilon,\omega)) &\leq \kappa\Big[\Big(\frac{sinh\vartheta}{sinh1}\Big)\phi(\omega)+\Big(\frac{sinh1-sinh\vartheta}{sinh1}\Big)\phi(\omega+\zeta(\upsilon,\omega))\Big]\\ &= \Big(\frac{sinh\vartheta}{sinh1}\Big)[\kappa\phi(\omega)]+\Big(\frac{sinh1-sinh\vartheta}{sinh1}\Big)[\kappa\phi(\omega+\zeta(\upsilon,\omega))]\\ &= \Big(\frac{sinh\vartheta}{sinh1}\Big)(\kappa\phi)(\omega)+\Big(\frac{sinh1-sinh\vartheta}{sinh1}\Big)(\kappa\phi)(\omega+\zeta(\upsilon,\omega)),\end{aligned}$$

which completes the proof.
(ii) Let ϕ,ψ be hyperbolic type preinvex functions, then

$$(\phi+\psi)(\omega+(1-\vartheta)\zeta(\upsilon,\omega)) = \phi(\omega+(1-\vartheta)\zeta(\upsilon,\omega))+\psi(\omega+(1-\vartheta)\zeta(\upsilon,\omega))$$

$$
\begin{aligned}
&\leq \Big(\frac{sinh\vartheta}{sinh1}\Big)\phi(\omega) + \Big(\frac{sinh1 - sinh\vartheta}{sinh1}\Big)\phi(\omega + \zeta(\upsilon, \omega)) \\
&+ \Big(\frac{sinh\vartheta}{sinh1}\Big)\psi(\omega) + \Big(\frac{sinh1 - sinh\vartheta}{sinh1}\Big)\psi(\omega + \zeta(\upsilon, \omega)) \\
&= \Big(\frac{sinh\vartheta}{sinh1}\Big)(\phi + \psi)(\omega) \\
&+ \Big(\frac{sinh1 - sinh\vartheta}{sinh1}\Big)(\phi + \psi)(\omega + \zeta(\upsilon, \omega)),
\end{aligned}
$$

which completes the proof. □

Theorem 3.2. *Let $\phi : X_\zeta \to X_\zeta$ be a preinvex function and $\psi : X_\zeta \to R$ be a hyperbolic type preinvex function, then $\psi \circ \phi : J \to R$ is a hyperbolic type preinvex function.*

Proof. For $\omega, \upsilon \in J$ and $\vartheta \in [0, 1]$, consider

$$
\begin{aligned}
\psi \circ \phi(\omega + (1 - \vartheta)\zeta(\upsilon, \omega)) &= \psi(\phi(\omega + (1 - \vartheta)\zeta(\upsilon, \omega)) \\
&\leq \psi(\vartheta\phi(\omega) + (1 - \vartheta)\phi(\upsilon)) \\
&\leq \Big(\frac{sinh\vartheta}{sinh1}\Big)\psi(\phi(\omega)) + \Big(\frac{sinh1 - sinh\vartheta}{sinh1}\Big)\psi(\phi(\upsilon)) \\
&= \Big(\frac{sinh\vartheta}{sinh1}\Big)(\psi \circ \phi)(\omega) + \Big(\frac{sinh1 - sinh\vartheta}{sinh1}\Big)(\psi \circ \phi)(\upsilon),
\end{aligned}
$$

which completes the proof. □

Theorem 3.3. *Let $\phi_\alpha : X_\zeta \subset R \to R$ be any family of hyperbolic type preinvex functions and let $\phi(\omega) = sup_\alpha\phi_\alpha(\omega)$. If I is nonempty such that $I = \{\chi \in X_\zeta : \phi(\chi)<\infty\}$, then I is an invex set regarding the bifunctional ζ and ϕ is a hyperbolic type preinvex function on I.*

Proof. Let $\vartheta \in [0, 1]$ and $\omega, \upsilon \in I$, then

$$
\begin{aligned}
\phi(\omega + (1 - \vartheta)\zeta(\upsilon, \omega)) &= sup_\alpha\phi_\alpha(\omega + (1 - \vartheta)\zeta(\upsilon, \omega)) \\
&\leq sup_\alpha\Big[\Big(\frac{sinh\vartheta}{sinh1}\Big)\phi_\alpha(\omega) + \Big(\frac{sinh1 - sinh\vartheta}{sinh1}\Big)\phi_\alpha(\upsilon)\Big] \\
&\leq \Big(\frac{sinh\vartheta}{sinh1}\Big)sup_\alpha\phi_\alpha(\omega) + \Big(\frac{sinh1 - sinh\vartheta}{sinh1}\Big)sup_\alpha\phi_\alpha(\upsilon) \\
&= \Big(\frac{sinh\vartheta}{sinh1}\Big)\phi_\alpha(\omega) + \Big(\frac{sinh1 - sinh\vartheta}{sinh1}\Big)\phi_\alpha(\upsilon) \\
&< \infty. \qquad (3.2)
\end{aligned}
$$

This shows that I is an invex set, as it contains every point between any two of its points, and ϕ is a hyperbolic type preinvex function on I. □

4. Hermite-Hadamard inequalities

Theorem 4.1. *Let $\phi : L = [\omega, \omega + \zeta(\upsilon, \omega)] \to R$ be a hyperbolic type preinvex function. If $\zeta(\upsilon, \omega) > 0$ then*

$$
\phi\Big(\frac{2\omega + \zeta(\upsilon, \omega)}{2}\Big) \leq \frac{1}{\zeta(\upsilon, \omega)} \int_\omega^{\omega+\zeta(\upsilon,\omega)} \phi(z)dz \leq \frac{cosh1 - 1}{sinh1}\phi(\omega) + \frac{e - 1}{esinh1}\phi(\omega + \zeta(\upsilon, \omega)).
$$

Proof. Let ϕ be a hyperbolic type preinvex function, then

$$\begin{aligned}\phi\Big(\frac{2\omega+\zeta(\upsilon,\omega)}{2}\Big) &= \phi\Big(\frac{2\omega+\vartheta\zeta(\upsilon,\omega)+(1-\vartheta)\zeta(\upsilon,\omega)}{2}\Big)\\ &= \phi\Big(\frac{\omega+\vartheta\zeta(\upsilon,\omega)}{2}+\frac{\omega+(1-\vartheta)\zeta(\upsilon,\omega)}{2}\Big)\\ &\le \frac{sinh(1/2)}{sinh1}\phi(\omega+\vartheta\zeta(\upsilon,\omega))+\frac{sinh1-sinh(1/2)}{sinh1}\phi(\omega+(1-\vartheta)\zeta(\upsilon,\omega)).\end{aligned}$$

By taking the integral in last inequality over $\vartheta\in[0,1]$, we have

$$\begin{aligned}\phi\Big(\frac{2\omega+\zeta(\upsilon,\omega)}{2}\Big) &\le \frac{sinh(1/2)}{sinh1}\int_0^1\phi(\omega+\vartheta\zeta(\upsilon,\omega)d\vartheta\\ &+\frac{sinh1-sinh(1/2)}{sinh1}\int_0^1\phi(\omega+(1-\vartheta)\zeta(\upsilon,\omega))d\vartheta\\ &= \frac{sinh(1/2)}{sinh1}\frac{1}{\zeta(\upsilon,\omega)}\int_\omega^{\omega+\zeta(\upsilon,\omega)}\phi(z)dz\\ &+\frac{sinh1-sinh(1/2)}{sinh1}\frac{1}{\zeta(\upsilon,\omega)}\int_\omega^{\omega+\zeta(\upsilon,\omega)}\phi(z)dz\\ &= \frac{1}{\zeta(\upsilon,\omega)}\int_\omega^{\omega+\zeta(\upsilon,\omega)}\phi(z)dz.\end{aligned}$$

By using the property of hyperbolic type preinvex function ϕ, we change the variable as $z=\omega+(1-\vartheta)\zeta(\upsilon,\omega)$, then

$$\begin{aligned}\frac{1}{\zeta(\upsilon,\omega)}\int_\omega^{\omega+\zeta(\upsilon,\omega)}\phi(z)dz &= \int_0^1\phi(\omega+(1-\vartheta)\zeta(\upsilon,\omega))d\vartheta\\ &\le \int_0^1\Big[\frac{sinh\vartheta}{sinh1}\phi(\omega)+\frac{sinh1-sinh\vartheta}{sinh1}\phi(\omega+\zeta(\upsilon,\omega))\Big]d\vartheta\\ &= \frac{\phi(\omega)}{sinh1}\int_0^1 sinh\vartheta d\vartheta+\frac{\phi(\omega+\zeta(\upsilon,\omega))}{sinh1}\int_0^1(sinh1-sinh\vartheta)d\vartheta\\ &= \frac{cosh1-1}{sinh1}\phi(\omega)+\frac{e-1}{esinh1}\phi(\omega+\zeta(\upsilon,\omega)).\end{aligned}$$

This completes the proof of the Theorem. □

5. New inequalities for Hyperbolic type preinvex functions

In the following section, we set up some new approximations that rectify the Hermite-Hadamard inequality for the functions whose first derivative in absolute value, raised to some power equal or more than one, is a hyperbolic type preinvex function. We go prove the following Lemma, which is a generalization of a result in [5].

Lemma 5.1. *Let X_ζ be an open invex subset in R regarding the bifunctional $\zeta(.,.)$, and $\omega,\upsilon\in X_\zeta$ with $\zeta(\omega,\upsilon)>0$. If $\phi: I\to R$ be a differentiable function such that $\phi'\in L$, then the following inequality* *holds*

$$\frac{\phi(\omega)+\phi(\omega+\zeta(\upsilon,\omega))}{2}-\frac{1}{\zeta(\upsilon,\omega)}\int_{\omega}^{\omega+\zeta(\upsilon,\omega)}\phi(\chi)d\chi$$
$$=\frac{\zeta(\upsilon,\omega)}{2}\int_0^1(1-2\vartheta)\phi'(\omega+(1-\vartheta)\zeta(\upsilon,\omega))d\vartheta. \tag{5.1}$$

Proof. Consider

$$\int_0^1(1-2\vartheta)\phi(\omega+(1-\vartheta)\zeta(\upsilon,\omega))$$
$$=\frac{\phi'(\omega+(1-\vartheta)\zeta(\upsilon,\omega))}{-\zeta(\upsilon,\omega)}(1-2\vartheta)|_0^1-\int_0^1\frac{2\phi(\omega+(1-\vartheta)\zeta(\upsilon,\omega))}{\zeta(\upsilon,\omega)}d\vartheta$$
$$=\frac{\phi(\omega)+\phi(\omega+\zeta(\upsilon,\omega))}{\zeta(\upsilon,\omega)}-\frac{2}{\zeta(\upsilon,\omega)^2}\int_{\omega}^{\omega+\zeta(\upsilon,\omega)}\phi(\chi)d\chi,$$

after rearranging the terms, the required result is obtained. □

Theorem 5.1. *Let A be an open invex set in R with respect to* $\zeta(.,.)$. *Let* $\phi : A \to R$ *be a differential function on A and* $\omega,\upsilon\in A$ *with* $\zeta(\omega,\upsilon)>0$. *If* $|\phi'|\in L$, *then for* $\vartheta\in[0,1]$,

$$\left|\frac{\phi(\omega)+\phi(\omega+\zeta(\upsilon,\omega))}{2}-\frac{1}{\zeta(\upsilon,\omega)}\int_{\omega}^{\omega+\zeta(\upsilon,\omega)}\phi(\chi)d\chi\right|$$
$$\leq\frac{\zeta(\upsilon,\omega)}{2(sinh1)}\Big[\phi'(\omega)(cosh1-2sinh1+4sinh(1/2)-1)$$
$$+\phi'(\omega+\zeta(\upsilon,\omega))(\frac{5}{2}sinh1-cosh1-4sinh(1/2))+1\Big]. \tag{5.2}$$

Proof. Suppose that $\omega,\upsilon\in A$, which is an open invex set with respect to ζ. Consider Lemma (5.1) and using the hyperbolic type preinvexity of ϕ,

$$\left|\frac{\phi(\omega)+\phi(\omega+\zeta(\upsilon,\omega))}{2}-\frac{1}{\zeta(\upsilon,\omega)}\int_{\omega}^{\omega+\zeta(\upsilon,\omega)}\phi(\chi)d\chi\right|$$
$$\leq\left|\frac{\zeta(\upsilon,\omega)}{2}\int_0^1(1-2\vartheta)\phi'(\omega+(1-\vartheta)\zeta(\upsilon,\omega))d\vartheta\right|$$
$$\leq\frac{\zeta(\upsilon,\omega)}{2}\int_0^1|1-2\vartheta|\Big[\frac{sinh\vartheta}{sinh1}|\phi'(\omega)|+\frac{sinh1-sinh\vartheta}{sinh1}|\phi'(\upsilon)|\Big]d\vartheta$$
$$\leq\frac{\zeta(\upsilon,\omega)}{2}\Big(\frac{|\phi'(\omega)|}{sinh1}\int_0^1|1-2\vartheta|sinh\vartheta d\vartheta+\frac{|\phi'(\upsilon|)}{sinh1}\int_0^1|1-2\vartheta|(sinh1-sinh\vartheta)d\vartheta\Big)$$
$$=\frac{\zeta(\upsilon,\omega)}{2sinh1}\Big(|\phi'(\omega)|K_1+|\phi'(\upsilon)|K_2\Big),$$

where
$K_1=\int_0^1|1-2\vartheta|sinh\vartheta d\vartheta=cosh1-2sinh1+4sinh(1/2)-1,$

$K_2=\int_0^1|1-2\vartheta|(sinh1-sinh\vartheta)=\frac{5}{2}sinh1-cosh1-4sinh(1/2))+1.$ □

This completes the proof of the Theorem.

Theorem 5.2. *Let $\phi : A \subseteq R \to R$ be a differentiable function on A and $\phi' \in L$. If for some $s > 1$, such that $1/r + 1/s = 1$ the function $|\phi'|^s$ is a hyperbolic type preinvex function on A then*

$$\left|\frac{\phi(\omega) + \phi(\omega + \zeta(\upsilon, \omega))}{2} - \frac{1}{\zeta(\upsilon, \omega)} \int_{\omega}^{\omega+\zeta(\upsilon,\omega)} \phi(\chi) d\chi\right| \leq \frac{\zeta(\upsilon, \omega)}{2}\left(\frac{1}{r+1}\right)^{1/r}\left[\frac{|\phi'(\omega)|^s}{sinh1}(cosh1 - 1) + \frac{|\phi'(\upsilon)|^s}{sinh1}(sinh1 - cosh1 + 1)\right]^{1/s}, \tag{5.3}$$

for all $\upsilon, \vartheta \in A$ with $\zeta(\upsilon, \omega) > 0$.

Proof. Considering the Lemma (5.1) and applying the Hölder's integral inequality,

$$\begin{aligned}
&\left|\frac{\phi(\omega) + \phi(\omega + \zeta(\upsilon, \omega))}{2} - \frac{1}{\zeta(\upsilon, \omega)} \int_{\omega}^{\omega+\zeta(\upsilon,\omega)} \phi(\chi) d\chi\right| \\
\leq& \left|\frac{\zeta(\upsilon, \omega)}{2} \int_0^1 (1 - 2\vartheta)\phi'(\omega + (1 - \vartheta)\zeta(\upsilon, \omega)) d\vartheta\right| \\
\leq& \frac{\zeta(\upsilon, \omega)}{2} \int_0^1 |1 - 2\vartheta||\phi'(\omega + (1 - \vartheta)\zeta(\upsilon, \omega))| d\vartheta \\
\leq& \frac{\zeta(\upsilon, \omega)}{2}\left(\int_0^1 |1 - 2\vartheta|^r d\vartheta\right)^{1/r}\left(\int_0^1 |\phi'(\omega + (1 - \vartheta)\zeta(\upsilon, \omega))|^s d\vartheta\right)^{1/s} \\
\leq& \frac{\zeta(\upsilon, \omega)}{2}\left(\frac{1}{2(r+1)}\right)^{1/r}\left[\int_0^1 \left(\frac{sinh\vartheta}{sinh1}|\phi'(\omega)|^s + \frac{sinh1 - sinh\vartheta}{sinh1}|\phi'(\upsilon)|^s\right) d\vartheta\right]^{1/s} \\
=& \frac{\zeta(\upsilon, \omega)}{2}\left(\frac{1}{2(r+1)}\right)^{1/r}\left[\frac{|\phi'(\omega)|^s}{sinh1}(cosh1 - 1) + \frac{|\phi'(\upsilon)|^s}{sinh1}(sinh1 - cosh1 + 1)\right]^{1/s},
\end{aligned}$$

which completes the proof. □

Theorem 5.3. *Let $\phi : A \subseteq R \to R$ be a differentiable function on A and $\phi' \in L$. If for some $s \geq 1$, the function $|\phi'|^s$ is a hyperbolic type preinvex function on A then for $\vartheta \in [0, 1]$,*

$$\begin{aligned}
&\left|\frac{\phi(\omega) + \phi(\omega + \zeta(\upsilon, \omega))}{2} - \frac{1}{\zeta(\upsilon, \omega)} \int_{\omega}^{\omega+\zeta(\upsilon,\omega)} \phi(\chi) d\chi\right| \leq \frac{\zeta(\upsilon, \omega)}{2}\left(\frac{1}{2}\right)^{1-1/s} \\
&\times\left[\frac{|\phi'(\omega)|^s}{sinh1}\left(\frac{2(e-1)}{\sqrt{e}} - \frac{e^2 - 3}{2e} - 1\right) + \frac{|\phi'(\upsilon)|^s}{sinh1}\left(\frac{3e^2 - 7}{4e} + \frac{2}{\sqrt{2}} - 2\sqrt{e} + 1\right)\right]^{1/s},
\end{aligned} \tag{5.4}$$

for all $\upsilon, \vartheta \in A$ with $\zeta(\upsilon, \omega) > 0$.

Proof. We assume that $s>1$. Consider (5.1), using Hölder's integral inequality and the property of $|\phi'|^s$ being a hyperbolic type preinvex function, we have

$$\left|\frac{\phi(\omega) + \phi(\omega + \zeta(\upsilon, \omega))}{2} - \frac{1}{\zeta(\upsilon, \omega)} \int_{\omega}^{\omega+\zeta(\upsilon,\omega)} \phi(\chi) d\chi\right|$$

$$\leq\left|\frac{\zeta(\upsilon,\omega)}{2}\int_0^1(1-2\vartheta)\phi'(\omega+(1-\vartheta)\zeta(\upsilon,\omega))d\vartheta\right|$$
$$\leq\frac{\zeta(\upsilon,\omega)}{2}\int_0^1|1-2\vartheta||\phi'(\omega+(1-\vartheta)\zeta(\upsilon,\omega))|d\vartheta$$
$$\leq\frac{\zeta(\upsilon,\omega)}{2}\Big(\int_0^1|1-2\vartheta|d\vartheta\Big)^{1-1/s}\Big(\int_0^1|1-2\vartheta||\phi'(\omega+(1-\vartheta)\zeta(\upsilon,\omega))|^s d\vartheta\Big)^{1/s}$$
$$=\frac{\zeta(\upsilon,\omega)}{2}\Big(\frac{1}{2}\Big)^{1-1/s}\Big[\int_0^1|1-2\vartheta|\Big(\frac{sinh\vartheta}{sinh1}|\phi'(\omega)|^s+\frac{sinh1-sinh\vartheta}{sinh1}|\phi'(\upsilon)|^s\Big)d\vartheta\Big]^{1/s}$$
$$=\frac{\zeta(\upsilon,\omega)}{2}\Big(\frac{1}{2}\Big)^{1-1/s}\Big[\frac{|\phi'(\omega)|^s}{sinh1}\int_0^1|1-2\vartheta|sinh\vartheta d\vartheta+\frac{|\phi'(\upsilon)|^s}{sinh1}\int_0^1|1-2\vartheta|(sinh1-sinh\vartheta)d\vartheta\Big]^{1/s}$$
$$=\frac{\zeta(\upsilon,\omega)}{2sinh1}\Big(\frac{1}{2}\Big)^{1-1/s}\Big[|\phi'(\omega)|^sJ_1+|\phi'(\upsilon)|^sJ_2\Big]^{1/s},$$

where

$$J_1=\int_0^1|1-2\vartheta|sinh\vartheta d\vartheta=\frac{2(e-1)}{\sqrt{e}}-\frac{e^2-3}{2e}-1$$
$$J_2=\int_0^1|1-2\vartheta|(sinh1-sinh\vartheta)d\vartheta=\frac{3e^2-7}{4e}+\frac{2}{\sqrt{2}}-2\sqrt{e}+1.$$ □

For $s=1$, we use the estimates from the Theorem (5.1), which also follows stepwise the above approximations. This completes the proof.

Corollary 5.1. *Under the assumption of above Theorem with $s=1$, we obtain the result of Theorem (5.1).*

In next Theorem we will use the Hölder-Işcan integral inequality to prove our results. The new results obtained shows a better approximation than (5.3).

Theorem 5.4. *Let $\phi: A\subseteq R\to R$ be a differentiable function on A and $\phi'\in L$. If for some $s>1$, such that $1/r+1/s=1$ the function $|\phi'|^s$ is a hyperbolic type preinvex function on A then*

$$\left|\frac{\phi(\omega)+\phi(\omega+\zeta(\upsilon,\omega))}{2}-\frac{1}{\zeta(\upsilon,\omega)}\int_\omega^{\omega+\zeta(\upsilon,\omega)}\phi(\chi)d\chi\right|$$
$$\leq\frac{\zeta(\upsilon,\omega)}{2}\Big(\frac{1}{2(r+1)}\Big)^{1/r}\Big[\Big(\frac{|\phi'(\omega)|^s}{sinh1}\Big(\frac{1}{e}\Big)+\frac{|\phi'(\upsilon)|^s}{sinh1}\Big(\frac{e^2-5}{4e}\Big)\Big)^{1/s}$$
$$+\Big(\frac{|\phi'(\omega)|^s}{sinh1}\Big(\frac{e^2-2e-1}{2e}\Big)+\frac{|\phi'(\upsilon)|^s}{sinh1}\Big(\frac{-e^2-2e+1}{4e}\Big)\Big)^{1/s}\Big],\tag{5.5}$$

for all $\upsilon,\vartheta\in A$ with $\zeta(\upsilon,\omega)>0$.

Proof. Consider Lemma (5.1) and applying Hölder-Iscan integral inequality

$$\left|\frac{\phi(\omega)+\phi(\omega+\zeta(\upsilon,\omega))}{2}-\frac{1}{\zeta(\upsilon,\omega)}\int_\omega^{\omega+\zeta(\upsilon,\omega)}\phi(\chi)d\chi\right|$$

$$
\begin{aligned}
&\leq \frac{\zeta(\upsilon,\omega)}{2}\int_0^1 |1-2\vartheta||\phi'(\omega+(1-\vartheta)\zeta(\upsilon,\omega))|d\vartheta \\
&\leq \frac{\zeta(\upsilon,\omega)}{2}\Bigg[\bigg(\int_0^1 \vartheta|1-2\vartheta|^r d\vartheta\bigg)^{1/r}\bigg(\int_0^1 \vartheta|\phi'(\omega+(1-\vartheta)\zeta(\upsilon,\omega))|^s d\vartheta\bigg)^{1/s} \\
&+\bigg(\int_0^1 (1-\vartheta)|1-2\vartheta|^r d\vartheta\bigg)^{1/r}\bigg(\int_0^1 (1-\vartheta)|\phi'(\omega+(1-\vartheta)\zeta(\upsilon,\omega))|^s d\vartheta\bigg)^{1/s}\Bigg] \\
&\leq \frac{\zeta(\upsilon,\omega)}{2}\Bigg[\bigg(\int_0^1 \vartheta|1-2\vartheta|^r d\vartheta\bigg)^{1/r}\bigg(\int_0^1 \vartheta\Big(\frac{sinh\vartheta}{sinh1}|\phi'(\omega)|^s+\frac{sinh1-sinh\vartheta}{sinh1}|\phi'(\upsilon)|^s\Big)d\vartheta\bigg)^{1/s} \\
&+\bigg(\int_0^1 (1-\vartheta)|1-2\vartheta|^r d\vartheta\bigg)^{1/r} \\
&\times\bigg(\int_0^1 (1-\vartheta)\Big(\frac{sinh\vartheta}{sinh1}|\phi'(\omega)|^s+\frac{sinh1-sinh\vartheta}{sinh1}|\phi'(\upsilon)|^s\Big)d\vartheta\bigg)^{1/s}\Bigg] \\
&\leq \frac{\zeta(\upsilon,\omega)}{2}\Bigg[\bigg(\int_0^1 \vartheta|1-2\vartheta|^r d\vartheta\bigg)^{1/r} \\
&\times\bigg(\frac{|\phi'(\omega)|^s}{sinh1}\int_0^1 \vartheta sinh\vartheta d\vartheta+\frac{|\phi'(\upsilon)|^s}{sinh1}\int_0^1 \vartheta(sinh1-sinh\vartheta)d\vartheta\bigg)^{1/s}
\end{aligned}
$$

$$
\begin{aligned}
&+\bigg(\int_0^1 (1-\vartheta)|1-2\vartheta|^r d\vartheta\bigg)^{1/r} \\
&\times\bigg(\frac{|\phi'(\omega)|^s}{sinh1}\int_0^1 (1-\vartheta)sinh\vartheta d\vartheta+\frac{|\phi'(\upsilon)|^s}{sinh1}\int_0^1 (1-\vartheta)(sinh1-sinh\vartheta)d\vartheta\bigg)^{1/s}\Bigg] \\
&= \frac{\zeta(\upsilon,\omega)}{2sinh1}\Big(\frac{1}{2(p+1)}\Big)^{1/p}\Big[\Big(|\phi'(\omega)|^s I_1+|\phi'(\upsilon)|^s I_2\Big)^{1/s}+\Big(|\phi'(\omega)|^s I_3+|\phi'(\upsilon)|^s I_4\Big)^{1/s}\Big], \qquad (5.6)
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_0^1 \vartheta|1-2\vartheta|^r d\vartheta = \int_0^1 (1-\vartheta)|1-2\vartheta|^r d\vartheta = 1/2(r+1) \\
&I_1 = \int_0^1 \vartheta sinh\vartheta d\vartheta = 1/e \\
&I_2 = \int_0^1 \vartheta(sinh1-sinh\vartheta)d\vartheta = (e^2-5)/(4e) \\
&I_3 = \int_0^1 (1-\vartheta)sinh\vartheta d\vartheta = (e^2-2e-1)/(2e) \\
&I_4 = \int_0^1 (1-\vartheta)(sinh1-sinh\vartheta)d\vartheta = (-e^2+4e+1)/4e.
\end{aligned}
$$

This completes the proof of the Theorem. □

Remark 5.1. *The results in inequality* (5.5) *are better then the results in inequality* (5.3)*. Using the concavity of function* $f : [0, \infty) \to R, f(x) = x^{\alpha}, 0<\alpha \leq 1$*, then*

$$\begin{aligned}
&\frac{\zeta(\upsilon,\omega)}{2}\left(\frac{1}{2(r+1)}\right)^{1/r}\Bigg[\left(\frac{|\phi'(\omega)|^s}{sinh1}\Big(\frac{1}{e}\Big)+\frac{|\phi'(\upsilon)|^s}{sinh1}\Big(\frac{e^2-5}{4e}\Big)\right)^{1/s}\\
&+\left(\frac{|\phi'(\omega)|^s}{sinh1}\Big(\frac{e^2-2e-1}{2e}\Big)+\frac{|\phi'(\upsilon)|^s}{sinh1}\Big(\frac{-e^2-2e+1}{4e}\Big)\right)^{1/s}\Bigg]\\
&\leq 2\frac{\zeta(\upsilon,\omega)}{2}\left(\frac{1}{2(r+1)}\right)^{1/r}\left(\frac{1}{2}\frac{|\phi'(\omega)|^s}{sinh1}\frac{(e-1)^2}{2e}+\frac{|\phi'(\upsilon)|^s}{sinh1}\frac{e-1}{e}\right)^{1/s}\\
&=\frac{\zeta(\upsilon,\omega)}{2}\left(\frac{1}{r+1}\right)^{1/r}\left[\frac{|\phi'(\omega)|^s}{sinh1}(cosh1-1)+\frac{|\phi'(\upsilon)|^s}{sinh1}(sinh1-cosh1+1)\right]^{1/s},
\end{aligned}$$

whereas

$$cosh1-1=\frac{(e-1)^2}{2e}$$
$$sinh1-cosh1+1=\frac{e-1}{e}.$$

Theorem 5.5. *Let* $\phi : A \subseteq R \to R$ *be a differentiable function on A and* $\phi' \in L$*. If for some* $s \geq 1$*, the function* $|\phi'|^s$ *is a hyperbolic type preinvex function on A then*

$$\begin{aligned}
&\left|\frac{\phi(\omega)+\phi(\omega+\zeta(\upsilon,\omega))}{2}-\frac{1}{\zeta(\upsilon,\omega)}\int_{\omega}^{\omega+\zeta(\upsilon,\omega)}\phi(\chi)d\chi\right|\\
&\leq\frac{\zeta(\upsilon,\omega)}{2}\left(\frac{1}{2}\right)^{2-2/s}\\
&\times\Bigg[\left(\frac{|\phi'(\omega)|^s}{sinh1}\Big(\frac{3}{\sqrt{e}}-\frac{3e}{2}-\frac{5}{2e}+5\sqrt{e}-5\Big)+\frac{|\phi'(\upsilon)|^s}{sinh1}\Big(\frac{19}{8e}+\frac{13e}{8}-5\sqrt{e}-\frac{3}{\sqrt{e}}+5\Big)\right)^{1/s}\\
&+\left(\frac{|\phi'(\omega)|^s}{sinh1}\Big(\frac{4}{e}+e-\frac{5}{\sqrt{e}}-3\sqrt{e}+4\Big)+\frac{|\phi'(\upsilon)|^s}{sinh1}\Big(\frac{5}{\sqrt{e}}-\frac{7e}{8}+\frac{33}{8e}+3\sqrt{e}-4\Big)\right)^{1/s}\Bigg] \qquad (5.7)
\end{aligned}$$

for all $\upsilon, \vartheta \in A$ *with* $\zeta(\upsilon, \omega) > 0$.

Proof. Let $|\phi'|^s$ be a hyperbolic type preinvex function and take $s > 1$. Consider Lemma (5.1) and using the power-mean integral inequality, we obtain

$$\begin{aligned}
&\left|\frac{\phi(\omega)+\phi(\omega+\zeta(\upsilon,\omega))}{2}-\frac{1}{\zeta(\upsilon,\omega)}\int_{\omega}^{\omega+\zeta(\upsilon,\omega)}\phi(\chi)d\chi\right|\\
&\leq\frac{\zeta(\upsilon,\omega)}{2}\int_0^1|(1-2\vartheta)||\phi'(\omega+(1-\vartheta)\zeta(\upsilon,\omega))|d\vartheta\\
&\leq\frac{\zeta(\upsilon,\omega)}{2}\Bigg[\left(\int_0^1(1-\vartheta)|(1-2\vartheta)|d\vartheta\right)^{1-1/s}\left(\int_0^1(1-\vartheta)|(1-2\vartheta)||\phi'(\omega+(1-\vartheta)\zeta(\upsilon,\omega))|^s d\vartheta\right)^{1/s}
\end{aligned}$$

$$+\left(\int_0^1 \vartheta|(1-2\vartheta)|d\vartheta\right)^{1-1/s}\left(\int_0^1 \vartheta|(1-2\vartheta)||\phi'(\omega+(1-\vartheta)\zeta(\upsilon,\omega))|^s d\vartheta\right)^{1/s}\Bigg]$$

$$\leq \frac{\zeta(\upsilon,\omega)}{2}\left(\int_0^1 (1-\vartheta)|(1-2\vartheta)|d\vartheta\right)^{1-1/s}$$

$$\times\left(\int_0^1 (1-\vartheta)|(1-2\vartheta)|\Big[\frac{sinh\vartheta}{sinh1}|\phi'(\omega)|^s+\frac{sinh1-sinh\vartheta}{sinh1}|\phi'(\upsilon)|^s\Big]\right)^{1/s}$$

$$+\frac{\zeta(\upsilon,\omega)}{2}\left(\int_0^1 \vartheta|(1-2\vartheta)|d\vartheta\right)^{1-1/s}$$

$$\times\left(\int_0^1 \vartheta|(1-2\vartheta)|\Big[\frac{sinh\vartheta}{sinh1}|\phi'(\omega)|^s+\frac{sinh1-sinh\vartheta}{sinh1}|\phi'(\upsilon)|^s\Big]d\vartheta\right)^{1/s}$$

$$=\frac{\zeta(\upsilon,\omega)}{2}\Big(\frac{1}{4}\Big)^{1-1/s}\left(\frac{|\phi'(\omega)|^s}{sinh1}\int_0^1 (1-\vartheta)|(1-2\vartheta)|sinh\vartheta d\vartheta\right.$$

$$\left.+\frac{|\phi'(\upsilon)|^s}{sinh1}\int_0^1 (1-\vartheta)|(1-2\vartheta)|(sinh1-sinh\vartheta)d\vartheta\right)^{1/s}$$

$$+\frac{\zeta(\upsilon,\omega)}{2}\Big(\frac{1}{4}\Big)^{1-1/s}\left(\frac{|\phi'(\omega)|^s}{sinh1}\int_0^1 \vartheta|(1-2\vartheta)|sinh\vartheta d\vartheta\right.$$

$$\left.+\frac{|\phi'(\upsilon)|^s}{sinh1}\int_0^1 \vartheta|(1-2\vartheta)|(sinh1-sinh\vartheta)d\vartheta\right)^{1/s}$$

$$=\frac{\zeta(\upsilon,\omega)}{2}\Big(\frac{1}{2}\Big)^{2-2/s}\left[\left(\frac{|\phi'(\omega)|^s}{sinh1}K_1+\frac{|\phi'(\upsilon)|^s}{sinh1}K_2\right)^{1/s}+\left(\frac{|\phi'(\omega)|^s}{sinh1}K_3+\frac{|\phi'(\upsilon)|^s}{sinh1}K_4\right)^{1/s}\right], \tag{5.8}$$

where

$\int_0^1(1-\vartheta)|(1-2\vartheta)|d\vartheta = \int_0^1 \vartheta|(1-2\vartheta)|d\vartheta = 1/4$

$K_1 = \int_0^1(1-\vartheta)|(1-2\vartheta)|sinh\vartheta d\vartheta = \frac{3}{\sqrt{e}} - \frac{3e}{2} - \frac{5}{2e} + 5\sqrt{e} - 5$

$K_2 = \int_0^1(1-\vartheta)|(1-2\vartheta)|(sinh1 - sinh\vartheta)d\vartheta = \frac{19}{8e} + \frac{13e}{8} - 5\sqrt{e} - \frac{3}{\sqrt{e}} + 5$

$K_3 = \int_0^1 \vartheta|(1-2\vartheta)|sinh\vartheta d\vartheta = \frac{4}{e} + e - \frac{5}{\sqrt{e}} - 3\sqrt{e} + 4$

$K_4 = \int_0^1 \vartheta|(1-2\vartheta)|(sinh1 - sinh\vartheta)d\vartheta = \frac{5}{\sqrt{e}} - \frac{7e}{8} + \frac{33}{8e} + 3\sqrt{e} - 4.$

For $s = 1$, we use the approximations from the Theorem (5.3), which also follows above estimates step by step. Hence it completes the proof. □

Remark 5.2. *The inequality* (5.7) *gives better results than the inequality* (5.4). *Consider* (5.7) *and using the concavity of the function* $f : [0,\infty) \to R, f(x) = x^{\alpha}, 0<\alpha \leq 1$*, then*

$$\frac{\zeta(\upsilon,\omega)}{2}\left(\frac{1}{2}\right)^{2-2/s}\left[\left(\frac{|\phi'(\omega)|^s}{sinh1}\Big(\frac{3}{\sqrt{e}}-\frac{3e}{2}-\frac{5}{2e}+5\sqrt{e}-5\Big)\right.\right.$$

$$+\frac{|\phi'(\upsilon)|^s}{sinh1}\Big(\frac{19}{8e}+\frac{13e}{8}-5\sqrt{e}-\frac{3}{\sqrt{e}}+5\Big)\Big)^{1/s}$$
$$+\Big(\frac{|\phi'(\omega)|^s}{sinh1}\Big(\frac{4}{e}+e-\frac{5}{\sqrt{e}}-3\sqrt{e}+4\Big)+\frac{|\phi'(\upsilon)|^s}{sinh1}\Big(\frac{5}{\sqrt{e}}-\frac{7e}{8}+\frac{33}{8e}+3\sqrt{e}-4\Big)\Big)^{1/s}\Bigg]$$
$$\leq 2\frac{\zeta(\upsilon,\omega)}{2}\Big(\frac{1}{2}\Big)^{2-2/s}\Bigg[\frac{1}{2}\Big(\frac{|\phi'(\omega)|^s}{sinh1}\Big(\frac{3}{\sqrt{e}}-\frac{3e}{2}-\frac{5}{2e}+5\sqrt{e}-5\Big)$$
$$+\frac{|\phi'(\upsilon)|^s}{sinh1}\Big(\frac{19}{8e}+\frac{13e}{8}-5\sqrt{e}-\frac{3}{\sqrt{e}}+5\Big)\Big)$$

$$+\frac{1}{2}\Big(\frac{|\phi'(\omega)|^s}{sinh1}\Big(\frac{4}{e}+e-\frac{5}{\sqrt{e}}-3\sqrt{e}+4\Big)+\frac{|\phi'(\upsilon)|^s}{sinh1}\Big(\frac{5}{\sqrt{e}}-\frac{7e}{8}+\frac{33}{8e}+3\sqrt{e}-4\Big)\Big)\Bigg]^{1/s}$$
$$=\frac{\zeta(\upsilon,\omega)}{2}\Big(\frac{1}{2}\Big)^{1-1/s}$$
$$\times\Bigg[\frac{|\phi'(\omega)|^s}{sinh1}\Big(\frac{2(e-1)}{\sqrt{e}}-\frac{e^2-3}{2e}-1\Big)+\frac{|\phi'(\upsilon)|^s}{sinh1}\Big(\frac{3e^2-7}{4e}+\frac{2}{\sqrt{2}}-2\sqrt{e}+1\Big)\Bigg]^{1/s},$$

which completes the proof.

6. Conclusions

In this work, we have introduced the concept of hyperbolic type preinvex functions. For this function few elementary algebraic properties are being satisfied. The Hermite-Hadamard inequality for this function has been introduced. Several inequalities for the differentiable hyperbolic type preinvex functions are obtained. To establish these inequalities the Holder inequality and Holder-Iscan inequalities are being used. It is shown that the results obtained by applying the Holder-Iscan inequality provide better estimate than Holder's inequality. The presented results are more generalized and quite exciting for the further development of this topic.

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