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Research article

New methods based $\mathcal{H}$-tensors for identifying the positive definiteness of multivariate homogeneous forms

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Abstract: Positive definite polynomials are important in the field of optimization. $\mathcal{H}$-tensors play an important role in identifying the positive definiteness of an even-order homogeneous multivariate form. In this paper, we propose some new criterion for identifying $\mathcal{H}$-tensor. As applications, we give new conditions for identifying positive definiteness of the even-order homogeneous multivariate form. At last, some numerical examples are provided to illustrate the efficiency and validity of new methods.

Keywords: homogeneous multivariate form; positive definiteness; $\mathcal{H}$-tensors; irreducible; non-zero element chain
Mathematics Subject Classification: 15A18, 15A69, 65F15, 65H17

1. Introduction

Consider the following mth degree homogeneous polynomial of n variables $f(x)$ as

$$f(x) = \sum_{i_1,i_2,\ldots,i_m \in N} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}, \tag{1.1}$$

where $x = (x_1, x_2, \cdots, x_n) \in R^n$. When m is even, $f(x)$ is called positive definite if

$$f(x) > 0, \quad for\ any\ x \in R^n, \quad x \neq 0.$$

The homogeneous polynomial $f(x)$ in (1.1) can be expressed as the tensor product of a symmetric tensor $\mathcal{A}$ with m-order, n-dimension and x^m defined by

$$f(x) \equiv \mathcal{A}x^m = \sum_{i_1,i_2,\ldots,i_m \in N} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}, \tag{1.2}$$

where

$$\mathcal{A} = (a_{i_1 i_2 \cdots i_m}), \quad a_{i_1 i_2 \cdots i_m} \in C(R), \quad i_j = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m,$$

$C(R)$ presents complex (real) number fields. The symmetric tensor $\mathcal{A}$ is called positive definite if $f(x)$ in (1.2) is positive definite [1]. Moreover, a tensor $\mathcal{I} = (\delta_{i_1 i_2 \cdots i_m})$ is called the unit tensor [2], where

$$\delta_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & if\ i_1 = \cdots = i_m, \\ 0, & otherwise. \end{cases}$$

The positive definiteness of tensor has received much attention of researchers' in recent decade [3–5]. Based on the Sturm theorem, the positive definiteness of a multivariate polynomial form can be checked for $n \leq 3$ [6]. For $n > 3$ and $m \geq 4$, it is difficult to determine the positive definiteness of $f(x)$ in (2). Ni et al. [1] provided an eigenvalue method for identifying positive definiteness of a multivariate form. However, all the eigenvalues of the tensor are needed in this method, thus the method is not practical when tensor order or dimension is large.

Recently, based on the criteria of $\mathcal{H}$-tensors, Li et al. [7] provided a practical method for identifying the positive definiteness of an even-order symmetric tensor. $\mathcal{H}$-tensor is a special kind of tensors and an even order symmetric $\mathcal{H}$-tensor with positive diagonal entries is positive definite. Due to this, we may identify the positive definiteness of a tensor via identifying $\mathcal{H}$-tensor. For the latter, with the help of generalized diagonally dominant tensor, various criteria for $\mathcal{H}$-tensors and $\mathcal{M}$-tensors is established [8–16], which only depends on the elements of the tensors and is more effective to determine whether a given tensor is an $\mathcal{H}$-tensor ($\mathcal{M}$-tensor) or not. For example, the following result is given in [16]:

Theorem 1. *Let $\mathcal{A} = (a_{i_1 \cdots i_m})$ be a complex tensor with m-order, n-dimension. If*

$$|a_{ii\cdots i}| > \sum_{\substack{i_2, i_3, \ldots, i_m \in N^{m-1} \setminus N_3^{m-1} \\ \delta_{ii_2 \ldots i_m} = 0}} |a_{ii_2 \cdots i_m}| + \sum_{i_2 i_3 \cdots i_m \in N_3^{m-1}} \max_{j \in \{i_2, i_3, \cdots, i_m\}} \frac{\Lambda_j(\mathcal{A})}{|a_{jj\cdots j}|} |a_{ii_2 \cdots i_m}|, \quad \forall i \in N_1 \cup N_2,$$

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

In this paper, we continue to present new criterions based on $\mathcal{H}$-tensors for identifying positive definiteness of homogeneous polynomial forms. The obtained results extend the corresponding conclusions [16–18]. The validity of our proposed methods are theoretically guaranteed and the numerical experiments show their effciency.

2. Preliminaries

In this section, some notation, definitions and lemmas are given.

Let S be a nonempty subset of $N = \{1, 2, \cdots, n\}$ and let $N \backslash S$ be the complement of S in N. Given an m-order n-dimension complex tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$, we denote

$$\Lambda_i(\mathcal{A}) = \sum_{\substack{i_2, \ldots, i_m \in N \\ \delta_{ii_2 \ldots i_m} = 0}} |a_{ii_2 \cdots i_m}| = \sum_{i_2, \ldots, i_m \in N} |a_{ii_2 \cdots i_m}| - |a_{ii\cdots i}|;$$

$$\begin{aligned}
N_1 &= N_1(\mathcal{A}) = \{i \in N : 0 < |a_{ii\cdots i}| = \Lambda_i(\mathcal{A})\};\\
N_2 &= N_2(\mathcal{A}) = \{i \in N : 0 < |a_{ii\cdots i}| < \Lambda_i(\mathcal{A})\};\\
N_3 &= N_3(\mathcal{A}) = \{i \in N : |a_{ii\cdots i}| > \Lambda_i(\mathcal{A})\};\\
N_0^{m-1} &= N^{m-1} \setminus (N_2^{m-1} \cup N_3^{m-1});\\
q &= \max_{i\in N_2} \frac{\Lambda_i(\mathcal{A}) - |a_{ii\cdots i}|}{\Lambda_i(\mathcal{A})};\\
P_i(\mathcal{A}) &= q\left(\sum_{i_2,\ldots,i_m\in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2,\ldots,i_m\in N_2^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2,\ldots,i_m\in N_3^{m-1}\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}|\right), \qquad \forall i \in N_3;\\
t &= \max_{i\in N_3} \frac{q\left(\sum\limits_{i_2\cdots i_m\in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum\limits_{i_2\cdots i_m\in N_2^{m-1}} \left|a_{ii_2\cdots i_m}\right|\right)}{P_i(\mathcal{A}) - \sum\limits_{\substack{i_2\cdots i_m\in N_3^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} \max\limits_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{P_j(\mathcal{A})}{|a_{jj\cdots j}|} \left|a_{ii_2\cdots i_m}\right|}.
\end{aligned}$$

In this paper, we always assume that neither N_1 or N_2 is empty. Otherwise, we assume that $\mathcal{A}$ satisfies: $a_{ii\cdots i} \neq 0, \Lambda_i(\mathcal{A}) \neq 0, \forall i \in N$.

we may define the following structured tensors extended from matrices.

Definition 1. [10] *Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m})$ be an m-order n-dimension complex tensor. $\mathcal{A}$ is called an $\mathcal{H}$-tensor if there is a positive vector $x = (x_1, x_2, \cdots, x_n)^T \in R^n$ such that*

$$|a_{ii\cdots i}|x_i^{m-1} > \sum_{\substack{i_2,\ldots,i_m\in N\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}, \quad \forall i \in N.$$

Definition 2. [2] *An m-order n-dimension complex tensor $\mathcal{A} = (a_{i_1 i_2\cdots i_m})$ is called reducible if there exists a nonempty proper index subset $I \subset N$ such that*

$$a_{i_1 i_2\cdots i_m} = 0, \;\; \forall i_1 \in I, \;\; \forall i_2, \cdots, i_m \notin I.$$

Otherwise, we say $\mathcal{A}$ is irreducible.

Example 1. *Consider the* 4*-order* 4*-dimension tensor $\mathcal{A}$ given*

$$a_{1111} = a_{2222} = a_{3333} = a_{4444} = a_{1444} = a_{2333} = 2,$$

and zero elsewhere. Then $a_{i_1 i_2 i_3 i_4} = 0$ for all $i_1 \in \{1, 4\}$ and for all $i_2, i_3, i_4 \in \{2, 3\}$. From Definition 2*, we have that $\mathcal{A}$ is reducible.*

Definition 3. [12] *Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m})$ be an m-order n-dimension complex tensor, for $i, j \in N (i \neq j)$, if there exist indices $k_1, k_2, \cdots, k_r$ with*

$$\sum_{\substack{i_2,\ldots,i_m\in N\\ \delta_{k_s i_2\ldots i_m}=0, k_{s+1}\in\{i_2,\ldots,i_m\}}} |a_{k_s i_2\cdots i_m}| \neq 0, \qquad s = 0, 1, \ldots, r,$$

where $k_0 = i, k_{r+1} = j$, we call that there is a nonzero elements chain from i to j.

It is shown that for any $\mathcal{H}$-tensor, there exists at least one strictly diagonally dominant row [7]. Further, we have the following conclusion.

Lemma 1. [10] *If $\mathcal{A}$ is a strictly diagonally dominant tensor, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.*

Lemma 2. [7] *Let $\mathcal{A} = (a_{i_1\cdots i_m})$ be a complex tensor with m-order, n-dimension. If there exists a positive diagonal matrix X such that $\mathcal{A}X^{m-1}$ is an $\mathcal{H}$-tensor, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.*

Lemma 3. [7] *Let $\mathcal{A} = (a_{i_1\cdots i_m})$ be a complex tensor with m-order, n-dimension. If $\mathcal{A}$ is irreducible,*

$$|a_{i\cdots i}| \geq \Lambda_i(\mathcal{A}), \quad \forall i \in N,$$

and strictly inequality holds for at least one i, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Lemma 4. [12] *Let $\mathcal{A} = (a_{i_1\cdots i_m})$ be a complex tensor with m-order, n-dimension. If*

- *(i) $|a_{ii\cdots i}| \geq \Lambda_i(\mathcal{A}), \quad \forall i \in N$,*
- *(ii) $N_3 = \{i \in N : |a_{ii\cdots i}| > \Lambda_i(\mathcal{A})\} \neq \emptyset$,*
- *(iii) For any $i \notin N_3$, there exists a nonzero elements chain from i to j such that $j \in N_3$,*

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

3. Criteria for identifying $\mathcal{H}$-tensors

In this section, we give some new criteria for $\mathcal{H}$-tensors.

Theorem 2. *Let $\mathcal{A} = (a_{i_1\cdots i_m})$ be a complex tensor with m-order, n-dimension. If for $i \in N_2$,*

$$\begin{aligned} |a_{ii\cdots i}| \quad > \quad & \frac{\Lambda_i(\mathcal{A})}{\Lambda_i(\mathcal{A}) - |a_{ii\cdots i}|}\left[q\left(\sum_{i_2 i_3\cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2 i_3\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \right) \right. \\ & \left. + \sum_{i_2 i_3\cdots i_m \in N_3^{m-1}} \max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{tP_j(\mathcal{A})}{|a_{jj\cdots j}|} |a_{ii_2\cdots i_m}| \right], \end{aligned} \tag{3.1}$$

and for $i \in N_1$, $|a_{ii\cdots i}| \neq \sum_{\substack{i_2 i_3\cdots i_m \in N_0^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|$, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof. From the definition of q, we know that $0 \leq q < 1$, $q \geq \frac{\Lambda_i(\mathcal{A}) - |a_{ii\cdots i}|}{\Lambda_i(\mathcal{A})} (\forall i \in N_2)$, so for any $i \in N_3$,

$$\begin{aligned} P_i(\mathcal{A}) \quad = \quad & q\left(\sum_{i_2,\ldots,i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2,\ldots,i_m \in N_2^{m-1}} \left|a_{ii_2\cdots i_m}\right| \right. \\ & \left. + \sum_{\substack{i_2,\ldots,i_m \in N_3^{m-1} \\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}| \right) = q\Lambda_i(\mathcal{A}) < q|a_{ii\cdots i}|, \end{aligned}$$

that is

$$q > \frac{P_i(\mathcal{A})}{|a_{ii\cdots i}|}. \tag{3.2}$$

By the definition of $P_i(\mathcal{A})$, we have

$$\frac{q\left(\sum\limits_{i_2i_3\cdots i_m\in N_0^{m-1}}|a_{ii_2\cdots i_m}|+\sum\limits_{i_2i_3\cdots i_m\in N_2^{m-1}}|a_{ii_2\cdots i_m}|\right)}{P_i(\mathcal{A})-\sum\limits_{\substack{i_2i_3\cdots i_m\in N_3^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\max\limits_{j\in\{i_2,i_3,\cdots,i_m\}}\frac{P_j(\mathcal{A})}{|a_{jj\cdots j}|}|a_{ii_2\cdots i_m}|}=\frac{P_i(\mathcal{A})-q\sum\limits_{\substack{i_2\cdots i_m\in N_3^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|}{P_i(\mathcal{A})-\sum\limits_{\substack{i_2i_3\cdots i_m\in N_3^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\max\limits_{j\in\{i_2,i_3,\cdots,i_m\}}\frac{P_j(\mathcal{A})}{|a_{jj\cdots j}|}|a_{ii_2\cdots i_m}|}\leq 1.$$

For any $i \in N_3$, from Inequality (3.2) and $0 \leq t \leq 1$, we conclude that

$$q > \frac{tP_i(\mathcal{A})}{|a_{ii\cdots i}|}, \qquad \forall i \in N_3. \tag{3.3}$$

For any $i \in N_2$, by Inequality (3.1), it holds that

$$\begin{aligned} |a_{ii\cdots i}|\frac{\Lambda_i(A)-|a_{ii\cdots i}|}{\Lambda_i(A)} \;>\; & q\left(\sum_{i_2\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{\substack{i_2\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\left|a_{ii_2\cdots i_m}\right|\right) \\ & +\sum_{i_2\cdots i_m\in N_3^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\frac{tP_j(A)}{\left|a_{jj\cdots j}\right|}\left|a_{ii_2\cdots i_m}\right|. \end{aligned} \tag{3.4}$$

By Inequality (3.3) and Inequality (3.4), there exists a sufficiently small positive number ε such that

$$q > \frac{tP_i(\mathcal{A})}{|a_{ii\cdots i}|} + \varepsilon, \qquad \forall i \in N_3, \tag{3.5}$$

and

$$\begin{aligned} |a_{ii\cdots i}|\frac{\Lambda_i(A)-|a_{ii\cdots i}|}{\Lambda_i(A)} \;>\; & q\left(\sum_{i_2\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{\substack{i_2\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\left|a_{ii_2\cdots i_m}\right|\right) \\ & +\sum_{i_2\cdots i_m\in N_3^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\frac{tP_j(A)}{\left|a_{jj\cdots j}\right|}\left|a_{ii_2\cdots i_m}\right| \\ & +\varepsilon\sum_{i_2\cdots i_m\in N_3^{m-1}}\left|a_{ii_2\cdots i_m}\right|, \qquad \forall i \in N_2, \end{aligned}$$

that is,

$$\varepsilon\sum_{i_2\cdots i_m\in N_3^{m-1}}\left|a_{ii_2\cdots i_m}\right| \;<\; |a_{ii\cdots i}|\frac{\Lambda_i(A)-|a_{ii\cdots i}|}{\Lambda_i(A)}-q\left(\sum_{i_2\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{\substack{i_2\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\left|a_{ii_2\cdots i_m}\right|\right)$$

$$-\sum_{i_2\cdots i_m\in N_3^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\frac{tP_j(A)}{\left|a_{jj\cdots j}\right|}\left|a_{ii_2\cdots i_m}\right|,\qquad \forall i\in N_2. \tag{3.6}$$

By the definition of t, it holds that

$$t\geq\frac{q\left(\sum\limits_{i_2\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum\limits_{i_2\cdots i_m\in N_2^{m-1}}\left|a_{ii_2\cdots i_m}\right|\right)}{P_i(\mathcal{A})-\sum\limits_{\substack{i_2\cdots i_m\in N_3^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\max\limits_{j\in\{i_2,i_3,\cdots,i_m\}}\frac{P_j(\mathcal{A})}{\left|a_{jj\cdots j}\right|}\left|a_{ii_2\cdots i_m}\right|},\qquad \forall i\in N_2,$$

that is

$$q\left(\sum_{i_2\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2\cdots i_m\in N_2^{m-1}}\left|a_{ii_2\cdots i_m}\right|\right)+t\sum_{\substack{i_2\cdots i_m\in N_3^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\frac{P_j(\mathcal{A})}{\left|a_{jj\cdots j}\right|}\left|a_{ii_2\cdots i_m}\right|\leq tP_i(\mathcal{A}),\quad \forall i\in N_2. \tag{3.7}$$

Let the matrix $D=diag(d_1,d_2,\cdots,d_n)$, and denote $\mathcal{B}=\mathcal{A}D^{m-1}=(b_{i_1i_2\cdots i_m})$, where

$$d_i=\begin{cases} q^{\frac{1}{m-1}}, & i\in N_1,\\ \left(\frac{\Lambda_i(\mathcal{A})-|a_{ii\cdots i}|}{\Lambda_i(\mathcal{A})}\right)^{\frac{1}{m-1}}, & i\in N_2,\\ \left(\varepsilon+\frac{tP_i(\mathcal{A})}{|a_{ii\cdots i}|}\right)^{\frac{1}{m-1}}, & i\in N_3.\end{cases}$$

For any $i\in N_1$, by $q>\frac{tP_i(\mathcal{A})}{|a_{ii\cdots i}|}(\forall i\in N_3)$, we conclude that

$$\begin{aligned}\Lambda_i(\mathcal{B}) &= q\sum_{\substack{i_2\cdots i_m\in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\left|a_{ii_2\cdots i_m}\right|\\ &\quad+\sum_{i_2\cdots i_m\in N_2^{m-1}}\left|a_{ii_2\cdots i_m}\right|\left(\frac{\Lambda_{i_2}(\mathcal{A})-\left|a_{i_2i_2\cdots i_2}\right|}{\Lambda_{i_2}(\mathcal{A})}\right)^{\frac{1}{m-1}}\cdots\left(\frac{\Lambda_{i_m}(\mathcal{A})-\left|a_{i_mi_m\cdots i_m}\right|}{\Lambda_{i_m}(\mathcal{A})}\right)^{\frac{1}{m-1}}\\ &\quad+\sum_{i_2\cdots i_m\in N_3^{m-1}}\left|a_{ii_2\cdots i_m}\right|\left(\frac{tP_{i_2}(\mathcal{A})}{\left|a_{i_2i_2\cdots i_2}\right|}+\varepsilon\right)^{\frac{1}{m-1}}\cdots\left(\frac{tP_{i_m}(\mathcal{A})}{\left|a_{i_mi_m\cdots i_m}\right|}+\varepsilon\right)^{\frac{1}{m-1}}\\ &\leq q\left(\sum_{\substack{i_2\cdots i_m\in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2\cdots i_m\in N_2^{m-1}}\left|a_{ii_2\cdots i_m}\right|\right)\\ &\quad+\sum_{i_2\cdots i_m\in N_3^{m-1}}\left|a_{ii_2\cdots i_m}\right|\left(\max_{j\in\{i_2,i_3,\cdots,i_m\}}\frac{tP_j(\mathcal{A})}{\left|a_{jj\cdots j}\right|}+\varepsilon\right)\\ &< q\left(\sum_{\substack{i_2\cdots i_m\in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2\cdots i_m\in N_2^{m-1}}\left|a_{ii_2\cdots i_m}\right|\right)+q\sum_{i_2\cdots i_m\in N_3^{m-1}}\left|a_{ii_2\cdots i_m}\right|\end{aligned}$$

$$= \quad q\left|a_{ii\cdots i}\right| = \left|b_{ii\cdots i}\right|.$$

For $\forall i \in N_2$, by Inequality (3.6), then

$$\begin{aligned}
\Lambda_i(\mathcal{B}) &= q \sum_{i_2\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| \\
&+ \sum_{\substack{i_2\cdots i_m \in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right| \left(\frac{\Lambda_{i_2}(\mathcal{A}) - \left|a_{i_2i_2\cdots i_2}\right|}{\Lambda_{i_2}(\mathcal{A})}\right)^{\frac{1}{m-1}} \cdots \left(\frac{\Lambda_{i_m}(\mathcal{A}) - \left|a_{i_mi_m\cdots i_m}\right|}{\Lambda_{i_m}(\mathcal{A})}\right)^{\frac{1}{m-1}} \\
&+ \sum_{i_2\cdots i_m \in N_3^{m-1}} \left|a_{ii_2\cdots i_m}\right| \left(\frac{tP_{i_2}(\mathcal{A})}{\left|a_{i_2i_2\cdots i_2}\right|} + \varepsilon\right)^{\frac{1}{m-1}} \cdots \left(\frac{tP_{i_m}(\mathcal{A})}{\left|a_{i_mi_m\cdots i_m}\right|} + \varepsilon\right)^{\frac{1}{m-1}} \\
&\le q\left(\sum_{i_2\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2\cdots i_m \in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right|\right) \\
&+ \sum_{i_2\cdots i_m \in N_3^{m-1}} \left|a_{ii_2\cdots i_m}\right| \left(\max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{tP_j(\mathcal{A})}{\left|a_{jj\cdots j}\right|} + \varepsilon\right) \\
&= q\left(\sum_{i_2\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2\cdots i_m \in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right|\right) \\
&+ \sum_{i_2\cdots i_m \in N_3^{m-1}} \max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{tP_j(\mathcal{A})}{\left|a_{jj\cdots j}\right|}\left|a_{ii_2\cdots i_m}\right| + \varepsilon \sum_{i_2\cdots i_m \in N_3^{m-1}} \left|a_{ii_2\cdots i_m}\right| \\
&< q\left(\sum_{i_2\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2\cdots i_m \in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right|\right) \\
&+ \sum_{i_2\cdots i_m \in N_3^{m-1}} \max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{tP_j(\mathcal{A})}{\left|a_{jj\cdots j}\right|}\left|a_{ii_2\cdots i_m}\right| \\
&+ \left[\left|a_{ii\cdots i}\right| \frac{\Lambda_i(A) - \left|a_{ii\cdots i}\right|}{\Lambda_i(A)} - q\left(\sum_{i_2\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2\cdots i_m \in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right|\right)\right. \\
&\quad \left. - \sum_{i_2\cdots i_m \in N_3^{m-1}} \max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{tP_j(A)}{\left|a_{jj\cdots j}\right|}\left|a_{ii_2\cdots i_m}\right|\right] \\
&= \left|a_{ii\cdots i}\right| \frac{\Lambda_i(\mathcal{A}) - \left|a_{ii\cdots i}\right|}{\Lambda_i(\mathcal{A})} = \left|b_{ii\cdots i}\right|.
\end{aligned}$$

Finally, for any $i \in N_3$, by Inequality (3.7), thus

$$\begin{aligned}
\Lambda_i(\mathcal{B}) &= q \sum_{i_2\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| \\
&\quad + \sum_{i_2\cdots i_m \in N_2^{m-1}} \left|a_{ii_2\cdots i_m}\right| \left(\frac{\Lambda_{i_2}(\mathcal{A}) - \left|a_{i_2 i_2\cdots i_2}\right|}{\Lambda_{i_2}(\mathcal{A})} \right)^{\frac{1}{m-1}} \cdots \left(\frac{\Lambda_{i_m}(\mathcal{A}) - \left|a_{i_m i_m\cdots i_m}\right|}{\Lambda_{i_m}(\mathcal{A})} \right)^{\frac{1}{m-1}} \\
&\quad + \sum_{\substack{i_2\cdots i_m \in N_3^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right| \left(\frac{tP_{i_2}(\mathcal{A})}{\left|a_{i_2 i_2\cdots i_2}\right|} + \varepsilon \right)^{\frac{1}{m-1}} \cdots \left(\frac{tP_{i_m}(\mathcal{A})}{\left|a_{i_m i_m\cdots i_m}\right|} + \varepsilon \right)^{\frac{1}{m-1}} \\
&\le q \left(\sum_{i_2\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2\cdots i_m \in N_2^{m-1}} \left|a_{ii_2\cdots i_m}\right| \right) \\
&\quad + \sum_{\substack{i_2\cdots i_m \in N_3^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right| \left(\max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{tP_j(\mathcal{A})}{\left|a_{jj\cdots j}\right|} + \varepsilon \right) \\
&= q \left(\sum_{i_2\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2\cdots i_m \in N_2^{m-1}} \left|a_{ii_2\cdots i_m}\right| \right) \\
&\quad + \sum_{\substack{i_2\cdots i_m \in N_3^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{tP_j(\mathcal{A})}{\left|a_{jj\cdots j}\right|} \left|a_{ii_2\cdots i_m}\right| + \varepsilon \sum_{\substack{i_2\cdots i_m \in N_3^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right| \\
&\le tP_i(\mathcal{A}) + \varepsilon \sum_{\substack{i_2\cdots i_m \in N_3^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right| \\
&< tP_i(\mathcal{A}) + \varepsilon |a_{ii\cdots i}| = |b_{ii\cdots i}|.
\end{aligned}$$

Therefore, we obtain that $|b_{ii\cdots i}| > \Lambda_i(\mathcal{B}) (\forall i \in N)$. From Lemma 1, $\mathcal{B}$ is an $\mathcal{H}$-tensor. Further, by Lemma 2, $\mathcal{A}$ is an $\mathcal{H}$-tensor. □

Remark 1. *From Theorem* 2*, we conclude that* $0 \le p < 1$, $0 \le t \le 1$, *and for any* $i \in N_3$,

$$\frac{tP_i(\mathcal{A})}{|a_{ii\cdots i}|} < \frac{\Lambda_i(\mathcal{A})}{|a_{ii\cdots i}|} < 1.$$

Thus, all conditions in Theorem 2 *are weaker than that in Theorem* 1*. Example* 2 *illustrates the superiority of Theorem* 2*.*

Theorem 3. *Let* $\mathcal{A} = (a_{i_1\cdots i_m})$ *be a complex tensor with m-order, n-dimension. If* $\mathcal{A}$ *is irreducible, and for all* $i \in N_2$,

$$|a_{ii\cdots i}| \ge \frac{\Lambda_i(\mathcal{A})}{\Lambda_i(\mathcal{A}) - |a_{ii\cdots i}|} \left[q \left(\sum_{i_2 i_3\cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2 i_3\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \right) \right.$$

$$+\sum_{i_2i_3\cdots i_m\in N_3^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\frac{tP_j(\mathcal{A})}{|a_{jj\cdots j}|}|a_{ii_2\cdots i_m}|\Bigg],\tag{3.8}$$

and at least one strict inequality in (3.8) *holds, then* $\mathcal{A}$ *is an* $\mathcal{H}$*-tensor.*

Proof. Notice that $\mathcal{A}$ is irreducible, this implies that for any $i\in N_3$, $P_i(\mathcal{A})>0$, $t>0$ (Otherwise, $\mathcal{A}$ is reducible).

For any $i\in N_2$, by Inequality (3.8), we obtain

$$\begin{aligned}|a_{ii\cdots i}|\frac{\Lambda_i(\mathcal{A})-|a_{ii\cdots i}|}{\Lambda_i(\mathcal{A})} \geq\ & q\left(\sum_{i_2i_3\cdots i_m\in N_0^{m-1}}|a_{ii_2\cdots i_m}|+\sum_{\substack{i_2i_3\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|\right.\\ &\left.+\sum_{i_2i_3\cdots i_m\in N_3^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\frac{tP_j(\mathcal{A})}{|a_{jj\cdots j}|}|a_{ii_2\cdots i_m}|\right).\end{aligned}\tag{3.9}$$

Let the matrix $D=diag(d_1,d_2,\cdots,d_n)$, denote $\mathcal{B}=\mathcal{A}D^{m-1}=(b_{i_1i_2\cdots i_m})$, where

$$d_i=\begin{cases} q^{\frac{1}{m-1}}, & i\in N_1,\\ \left(\frac{\Lambda_i(\mathcal{A})-|a_{ii\cdots i}|}{\Lambda_i(\mathcal{A})}\right)^{\frac{1}{m-1}}, & i\in N_2,\\ \left(\frac{tP_i(\mathcal{A})}{|a_{ii\cdots i}|}\right)^{\frac{1}{m-1}}, & i\in N_3.\end{cases}$$

For any $i\in N_1$, by $q>\frac{tP_i(\mathcal{A})}{|a_{ii\cdots i}|}(\forall i\in N_3)$, we conclude that

$$\begin{aligned}\Lambda_i(\mathcal{B}) =\ & q\sum_{\substack{i_2\cdots i_m\in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\left|a_{ii_2\cdots i_m}\right|\\ &+\sum_{i_2\cdots i_m\in N_2^{m-1}}\left|a_{ii_2\cdots i_m}\right|\left(\frac{\Lambda_{i_2}(\mathcal{A})-\left|a_{i_2i_2\cdots i_2}\right|}{\Lambda_{i_2}(\mathcal{A})}\right)^{\frac{1}{m-1}}\cdots\left(\frac{\Lambda_{i_m}(\mathcal{A})-\left|a_{i_mi_m\cdots i_m}\right|}{\Lambda_{i_m}(\mathcal{A})}\right)^{\frac{1}{m-1}}\\ &+\sum_{i_2\cdots i_m\in N_3^{m-1}}\left|a_{ii_2\cdots i_m}\right|\left(\frac{tP_{i_2}(\mathcal{A})}{\left|a_{i_2i_2\cdots i_2}\right|}\right)^{\frac{1}{m-1}}\cdots\left(\frac{tP_{i_m}(\mathcal{A})}{\left|a_{i_mi_m\cdots i_m}\right|}\right)^{\frac{1}{m-1}}\\ \leq\ & q\left(\sum_{\substack{i_2\cdots i_m\in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2\cdots i_m\in N_2^{m-1}}\left|a_{ii_2\cdots i_m}\right|\right)\\ &+\sum_{i_2\cdots i_m\in N_3^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\frac{tP_j(\mathcal{A})}{\left|a_{jj\cdots j}\right|}\left|a_{ii_2\cdots i_m}\right|\\ <\ & q\left(\sum_{\substack{i_2\cdots i_m\in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2\cdots i_m\in N_2^{m-1}}\left|a_{ii_2\cdots i_m}\right|\right)+q\sum_{i_2\cdots i_m\in N_3^{m-1}}\left|a_{ii_2\cdots i_m}\right|\end{aligned}$$

$$= q\,|a_{ii\cdots i}| = |b_{ii\cdots i}|\,.$$

For any $i \in N_2$, by Inequality (3.9), it holds that

$$\begin{aligned}
\Lambda_i(\mathcal{B}) &= q \sum_{i_2\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| \\
&\quad + \sum_{\substack{i_2\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right| \left(\frac{\Lambda_{i_2}(\mathcal{A}) - \left|a_{i_2i_2\cdots i_2}\right|}{\Lambda_{i_2}(\mathcal{A})}\right)^{\frac{1}{m-1}} \cdots \left(\frac{\Lambda_{i_m}(\mathcal{A}) - \left|a_{i_mi_m\cdots i_m}\right|}{\Lambda_{i_m}(\mathcal{A})}\right)^{\frac{1}{m-1}} \\
&\quad + \sum_{i_2\cdots i_m \in N_3^{m-1}} \left|a_{ii_2\cdots i_m}\right| \left(\frac{tP_{i_2}(\mathcal{A})}{\left|a_{i_2i_2\cdots i_2}\right|}\right)^{\frac{1}{m-1}} \cdots \left(\frac{tP_{i_m}(\mathcal{A})}{\left|a_{i_mi_m\cdots i_m}\right|}\right)^{\frac{1}{m-1}} \\
&\leq q \left(\sum_{i_2\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right| \right) \\
&\quad + \sum_{i_2\cdots i_m \in N_3^{m-1}} \max_{j \in \{i_2, i_3, \cdots, i_m\}} \frac{tP_j(\mathcal{A})}{\left|a_{jj\cdots j}\right|} \left|a_{ii_2\cdots i_m}\right| \\
&\leq |a_{ii\cdots i}| \frac{\Lambda_i(\mathcal{A}) - |a_{ii\cdots i}|}{\Lambda_i(\mathcal{A})} = |b_{ii\cdots i}|\,.
\end{aligned}$$

Next, for any $i \in N_3$, by Inequality (3.7), then

$$\begin{aligned}
\Lambda_i(\mathcal{B}) &= q \sum_{i_2\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| \\
&\quad + \sum_{i_2\cdots i_m \in N_2^{m-1}} \left|a_{ii_2\cdots i_m}\right| \left(\frac{\Lambda_{i_2}(\mathcal{A}) - \left|a_{i_2i_2\cdots i_2}\right|}{\Lambda_{i_2}(\mathcal{A})}\right)^{\frac{1}{m-1}} \cdots \left(\frac{\Lambda_{i_m}(\mathcal{A}) - \left|a_{i_mi_m\cdots i_m}\right|}{\Lambda_{i_m}(\mathcal{A})}\right)^{\frac{1}{m-1}} \\
&\quad + \sum_{\substack{i_2\cdots i_m \in N_3^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right| \left(\frac{tP_{i_2}(\mathcal{A})}{\left|a_{i_2i_2\cdots i_2}\right|}\right)^{\frac{1}{m-1}} \cdots \left(\frac{tP_{i_m}(\mathcal{A})}{\left|a_{i_mi_m\cdots i_m}\right|}\right)^{\frac{1}{m-1}} \\
&\leq q \left(\sum_{i_2\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2\cdots i_m \in N_2^{m-1}} \left|a_{ii_2\cdots i_m}\right| \right) \\
&\quad + \sum_{\substack{i_2\cdots i_m \in N_3^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j \in \{i_2, i_3, \cdots, i_m\}} \frac{tP_j(\mathcal{A})}{\left|a_{jj\cdots j}\right|} \left|a_{ii_2\cdots i_m}\right| \\
&\leq tP_i(\mathcal{A}) = \frac{tP_i(\mathcal{A})}{|a_{ii\cdots i}|} \times |a_{ii\cdots i}| = |b_{ii\cdots i}|\,.
\end{aligned}$$

Therefore, $|b_{ii\cdots i}| \geq \Lambda_i(\mathcal{B})$ ($\forall i \in N$), and for all $\forall i \in N_2$, at least one strict inequality in (10) holds, that is, there exists an $i_0 \in N_2$ such that $|b_{i_0i_0\cdots i_0}| > \Lambda_{i_0}(\mathcal{B})$.

On the other hand, since $\mathcal{A}$ is irreducible and so is $\mathcal{B}$. Then, by Lemma 3, we have that $\mathcal{B}$ is an $\mathcal{H}$-tensor. By Lemma 2, $\mathcal{A}$ is also an $\mathcal{H}$-tensor. □

Let

$$K(\mathcal{A}) = \left\{ i \in N_2 : |a_{ii\cdots i}| > \frac{\Lambda_i(\mathcal{A})}{\Lambda_i(\mathcal{A}) - |a_{ii\cdots i}|} \left[q \left(\sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2 i_3 \cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \right) + \sum_{i_2 i_3 \cdots i_m \in N_3^{m-1}} \max_{j \in \{i_2, i_3, \cdots, i_m\}} \frac{tP_j(\mathcal{A})}{|a_{jj\cdots j}|} |a_{ii_2\cdots i_m}| \right] \right\}.$$

Theorem 4. *Let $\mathcal{A} = (a_{i_1\cdots i_m})$ be a complex tensor with m-order, n-dimension. For any $i \in N_2$,*

$$|a_{ii\cdots i}| \geq \frac{\Lambda_i(\mathcal{A})}{\Lambda_i(\mathcal{A}) - |a_{ii\cdots i}|} \left[q \left(\sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2 i_3 \cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \right) + \sum_{i_2 i_3 \cdots i_m \in N_3^{m-1}} \max_{j \in \{i_2, i_3, \cdots, i_m\}} \frac{tP_j(\mathcal{A})}{|a_{jj\cdots j}|} |a_{ii_2\cdots i_m}| \right],$$

and if for any $i \in N \backslash K(\mathcal{A}) \neq \emptyset$, there exists a nonzero elements chain from i to j such that $j \in K(\mathcal{A}) \neq \emptyset$, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof. Let the matrix $D = diag(d_1, d_2, \cdots, d_n)$, and denote $\mathcal{B} = \mathcal{A}D^{m-1} = (b_{i_1 i_2\cdots i_m})$, where

$$d_i = \begin{cases} q^{\frac{1}{m-1}}, & i \in N_1, \\ \left(\frac{\Lambda_i(\mathcal{A}) - |a_{ii\cdots i}|}{\Lambda_i(\mathcal{A})}\right)^{\frac{1}{m-1}}, & i \in N_2, \\ \left(\frac{tP_i(\mathcal{A})}{|a_{ii\cdots i}|}\right)^{\frac{1}{m-1}}, & i \in N_3. \end{cases}$$

A similar argument to that of Theorem 2, we can prove that $|b_{ii\cdots i}| \geq \Lambda_i(\mathcal{B})(\forall i \in N)$, and there exists at least an $i \in N_2$ such that $|b_{ii\cdots i}| > \Lambda_i(\mathcal{B})$.

On the other hand, if $|b_{ii\cdots i}| = \Lambda_i(\mathcal{B})$, then $i \in N \setminus K(\mathcal{A})$, by the assumption, we know that there exists a nonzero elements chain of $\mathcal{A}$ from i to j, such that $j \in K(\mathcal{A})$. Hence, there exists a nonzero elements chain of $\mathcal{B}$ from i to j, such that j satisfying $|b_{jj\cdots j}| > \Lambda_j(\mathcal{B})$.

Based on above analysis, we get that $\mathcal{B}$ satisfies the conditions of Lemma 4, so $\mathcal{B}$ is an $\mathcal{H}$-tensor. By Lemma 2, $\mathcal{A}$ is an $\mathcal{H}$-tensor. □

Example 2. *Consider the* 3*-order* 3*-dimension tensor $\mathcal{A} = (a_{ijk})$ defined as follows:*

$$\mathcal{A} = [A(1,:,:), A(2,:,:), A(3,:,:)],$$

$$A(1,:,:) = \begin{pmatrix} 12 & 1 & 0 \\ 1 & 6 & 0 \\ 1 & 0 & 15 \end{pmatrix}, A(2,:,:) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A(3,:,:) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{pmatrix}.$$

Obviously,

$$|a_{111}| = 12,\ \ \Lambda_1(\mathcal{A}) = 24,\ \ |a_{222}| = 6,\ \ \Lambda_2(\mathcal{A}) = 3,\ \ |a_{333}| = 16,\ \ \Lambda_3(\mathcal{A}) = 2.$$

so $N_1 = \emptyset, N_2 = \{1\}, N_3 = \{2, 3\}$. *By calculations, we have*

$$q_{i=1} = \frac{24-12}{24} = \frac{1}{2} = q,$$

$$P_2(\mathcal{A}) = \frac{1}{2}(1+1+1) = \frac{3}{2}, \quad P_3(\mathcal{A}) = \frac{1}{2}(0+1+1) = 1,$$

$$\frac{P_2(\mathcal{A})}{|a_{222}|} = \frac{\frac{3}{2}}{6} = \frac{1}{4}, \quad \frac{P_3(\mathcal{A})}{|a_{333}|} = \frac{1}{16},$$

$$t_{i=2} = \frac{\frac{1}{2}(1+1)}{\frac{3}{2} - \frac{1}{4}\times 1} = \frac{4}{5}, \quad t_{i=3} = \frac{\frac{1}{2}(0+1)}{1 - \frac{1}{4}\times 1} = \frac{2}{3}, \quad t = \frac{4}{5}.$$

When $i = 1$, *we get*

$$\begin{aligned} & \frac{\Lambda_1(\mathcal{A})}{\Lambda_1(\mathcal{A}) - |a_{111}|}\left[q\left(\sum_{i_2i_3 \in N_0^2} |a_{1i_2i_3}| + \sum_{\substack{i_2i_3 \in N_2^2 \\ \delta_{1i_2i_3}=0}} |a_{1i_2i_3}| \right) + \sum_{i_2i_3 \in N_3^2} \max_{j\in\{i_2,i_3\}} \frac{tP_j(\mathcal{A})}{|a_{jjj}|}|a_{1i_2i_3}| \right] \\ = & \frac{24}{24-12}\left[\frac{1}{2}(3+0) + \frac{4}{5}\times\frac{1}{4}\times 21\right] = \frac{57}{5} < 12 = |a_{111}|, \end{aligned}$$

so $\mathcal{A}$ *satisfies the conditions of Theorem* 2, *then* $\mathcal{A}$ *is an* $\mathcal{H}$*-tensor. However,*

$$\sum_{\substack{i_2i_3 \in N^2 \setminus N_3^2 \\ \delta_{1i_2i_3}=0}} \left|a_{1i_2i_3}\right| + \sum_{i_2i_3 \in N_3^2} \max_{j\in\{i_2,i_3\}} \frac{\Lambda_j(A)}{\left|a_{jjj}\right|} \left|a_{1i_2i_3}\right| = 3 + \frac{1}{2}\times 21 = \frac{27}{2} > 12 = |a_{111}|,$$

so $\mathcal{A}$ *does not satisfy the conditions of Theorem* 1.

4. An application: the positive definiteness of homogeneous polynomial forms

Based on the criteria of $\mathcal{H}$-tensors in Section 3, we present some criteria for identifying the positive definiteness of an even-order real symmetric tensor. First, we recall the following lemma.

Lemma 5. [7] *Let* $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ *be an even-order real symmetric tensor with m-order, n-dimension, and* $a_{k\cdots k} > 0$ *for all* $k \in N$. *If* $\mathcal{A}$ *is an* $\mathcal{H}$*-tensor, then* $\mathcal{A}$ *is positive definite.*

From Theorems 2 – 4 and Lemma 5, we obtain easily the following result.

Theorem 5. *Let* $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ *be an even-order real symmetric tensor with m-order, n-dimension, and* $a_{ii\cdots i} > 0$ *for all* $i \in N$. *If one of the following holds:*

- (i) $\mathcal{A}$ *satisfies all the conditions of Theorem* 2*,*
- (ii) $\mathcal{A}$ *satisfies all the conditions of Theorem* 3*,*

- (*iii*) $\mathcal{A}$ *satisfies all the conditions of Theorem* 4*,*

then $\mathcal{A}$ *is positive definite.*

Example 3. *Let*

$$f(x) \;=\; \mathcal{A}x^4 = 16x_1^4 + 20x_2^4 + 30x_3^4 + 33x_4^4 - 8x_1^3x_4 + 12x_1^2x_2x_3 - 12x_2x_3^2x_4 - 24x_1x_2x_3x_4$$

be a 4*th-degree homogeneous polynomial. We can get the* 4*-order* 4*-dimension real symmetric tensor* $\mathcal{A} = (a_{i_1i_2i_3i_4})$*, where*

$$\begin{aligned}
&a_{1111} = 16,\;\; a_{2222} = 20,\;\; a_{3333} = 30,\;\; a_{4444} = 33,\\
&a_{1114} = a_{1141} = a_{1411} = a_{4111} = -2,\\
&a_{1123} = a_{1132} = a_{1213} = a_{1312} = a_{1231} = a_{1321} = 1,\\
&a_{2113} = a_{2131} = a_{2311} = a_{3112} = a_{3121} = a_{3211} = 1,\\
&a_{2334} = a_{2343} = a_{2433} = a_{4233} = a_{4323} = a_{4332} = -1,\\
&a_{3234} = a_{3243} = a_{3324} = a_{3342} = a_{3423} = a_{3432} = -1,\\
&a_{1234} = a_{1243} = a_{1324} = a_{1342} = a_{1423} = a_{1432} = -1,\\
&a_{2134} = a_{2143} = a_{2314} = a_{2341} = a_{2413} = a_{2431} = -1,\\
&a_{3124} = a_{3142} = a_{3214} = a_{3241} = a_{3412} = a_{3421} = -1,\\
&a_{4123} = a_{4132} = a_{4213} = a_{4231} = a_{4312} = a_{4321} = -1,
\end{aligned}$$

and zero elsewhere. By calculations, we have

$$a_{1111} = 16 < 18 = \Lambda_1(\mathcal{A}),$$

and

$$a_{4444}\,(a_{1111} - \Lambda_1(\mathcal{A}) + |a_{1444}|) = -66 < 0 = \Lambda_4(\mathcal{A})|a_{1444}|.$$

Then $\mathcal{A}$ *is not strictly diagonally dominate as defined in [17] or quasidoubly strictly diagonally dominant as defined in [18]. Hence, we cannot use Theorem 3 in [17] and Theorem 4 in [18] to identify the positive definiteness of* $\mathcal{A}$*. However, it can be verified that* $\mathcal{A}$ *satisfies all the conditions of Theorem* 2*.*

$$\Lambda_1(\mathcal{A}) = 18,\;\; \Lambda_2(\mathcal{A}) = 12,\;\; \Lambda_3(\mathcal{A}) = 15,\;\; \Lambda_4(\mathcal{A}) = 11,$$

so $N_1 = \emptyset, N_2 = \{1\}, N_3 = \{2, 3, 4\}$*. By calculations, we have*

$$q_{i=1} = \frac{18-16}{18} = \frac{1}{9} = q,$$

$$P_2(\mathcal{A}) = \frac{1}{9}(9+0+3) = \frac{4}{3},\qquad P_3(\mathcal{A}) = \frac{1}{9}(9+0+6) = \frac{5}{3},\qquad P_4(\mathcal{A}) = \frac{1}{9}(6+2+3) = \frac{11}{9},$$

$$\frac{P_2(\mathcal{A})}{|a_{2222}|} = \frac{\frac{4}{3}}{20} = \frac{1}{15},\qquad \frac{P_3(\mathcal{A})}{|a_{3333}|} = \frac{\frac{5}{3}}{30} = \frac{1}{18},\qquad \frac{P_4(\mathcal{A})}{|a_{4444}|} = \frac{\frac{11}{9}}{33} = \frac{1}{27},$$

$$t_{i=2} = \frac{\frac{1}{9}(9+0)}{\frac{4}{3} - \frac{1}{15}\times 3} = \frac{15}{17},\qquad t_{i=3} = \frac{\frac{1}{9}(9+1)}{\frac{5}{3} - \frac{1}{15}\times 6} = \frac{15}{19},$$

$$t_{i=4} = \frac{\frac{1}{9}(6+2)}{\frac{11}{9} - \frac{1}{15} \times 3} = \frac{10}{13}, \qquad t = \frac{15}{17}.$$

When $i = 1$, we get

$$\frac{\Lambda_1(\mathcal{A})}{\Lambda_1(\mathcal{A}) - |a_{1111}|}\left[q\left(\sum_{i_2i_3i_4\in N_0^3}|a_{1i_2i_3i_4}| + \sum_{\substack{i_2i_3i_4\in N_2^3\\ \delta_{1i_2i_3i_4}=0}}|a_{1i_2i_3i_4}|\right) + \sum_{i_2i_3i_4\in N_3^3}\max_{j\in\{i_2,i_3,i_4\}}\frac{tP_j(\mathcal{A})}{|a_{jjjj}|}|a_{1i_2i_3i_4}|\right]$$
$$= \frac{18}{18-16}\left[\frac{1}{9}(12+0) + \frac{15}{17} \times \frac{1}{15} \times 6\right] = \frac{258}{17} < 16 = |a_{1111}|.$$

Therefore, from Theorem 5, we have that $\mathcal{A}$ is positive definite, that is, $f(x)$ is positive definite.

5. Conclusions

In this paper, we given some inequalities to identify whether a tensor is an $\mathcal{H}$-tensor, which was also used to identify the positive definiteness of an even degree homogeneous polynomial $f(x) \equiv \mathcal{A}x^m$. These inequalities were expressed in terms of the elements of $\mathcal{A}$, so they can be checked easily.

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Conflict of interest

The authors declare that they have no competing interests.

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