Mathematics

Mini review

Application of fractional differential equation in economic growth model: A systematic review approach

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Abstract: In this paper we review the applications of fractional differential equation in economic growth models. This includes the theories about linear and nonlinear fractional differential equation, including the Fractional Riccati Differential Equation (FRDE) and its applications in economic growth models with memory effect. The method used in this study is by comparing related literatures and evaluate them comprehensively. The results of this study are the chronological order of the applications of the Fractional Differential Equation (FDE) in economic growth models and the development on theories of the FDE solutions, including the FRDE forms of economic growth models. This study also provides a comparative analysis on solutions of linear and nonlinear FDE, and approximate solution of economic growth models involving memory effects using various methods. The main contribution of this research is the chonological development of the theory to find necessary and sufficient conditions to guarantee the existence and uniqueness of the FDE in economic growth and the methods to obtain the solution. Some remarks on how further researches can be done are also presented as a general conclusion.

1. Introduction

The development of mathematics is so fast at present, especially the topic of derivatives and

integrals which initially were only oriented on the natural number order, now it is expanded to the fractional-order, encompasing the rational and real numbers. Although derivatives and integrals with fractional order are believed to have been introduced since 1695, significant developments have only occurred in the early 21st century. This can be seen from the large numbers of scientific papers that examine problems related to derivatives and integrals with fractional order, and their applications in various fields of science, including in the fields of engineering and economics.

To find the solution of a Fractional Differential Equation (FDE), first it needs to show that the FDE indeed has a solution. Thus, it is necessary to analyze the FDE specifically to show the conditions needed to guarantee its existence and uniqueness of the solution given an initial value for the FDE. An FDE is often used to model growth incorporating memory effect. Since many economical processes have memory effect in their nature, the FDE is a suitable concept to model the growth of many economical processes. The memory effect is also a nature in the definition of the fractional derivatives. Hence, calculus fractional serve as a backbone of the effect of memory on the economic growth model.

Fractional calculus is a generalization of classical calculus, in general, calculus here refers to derivatives, integrals, and differential equations. Based on [1], derivatives or integrals with integer-order have local properties (the next state is not influenced by the current and previous state), while fractional derivatives have non-local properties (the next state depends on the current state and all previous states). Thus, FDE has a memory effect, because fractional derivatives or FDE have non-local properties. This is a major advantage of fractional derivatives over classical (integer order) derivatives, where the effect is generally ignored. In addition, this memory effect does not only apply to time variables but can also apply to other variables, such as price. Financial variables such as asset prices or product prices require more long-term memory to estimate price fluctuations in future periods based on fluctuations in previous periods [2].

Before discussing the theory of the effect of memory on the economic growth model, first we discuss the notion of economic growth. Economic growth is the process of changing a country's economic condition towards better conditions for a certain period. Economic growth can also be interpreted as the process of increasing the production capacity of an economy which is manifested in the form of an increase in national income. The existence of economic growth is an indication of the success of economic development in a country. Economic growth shows the increase in the production of goods and services in a region in a certain time interval. In general, the higher the level of economic growth, the faster the process of increasing the output of the region, means that the prospects for regional development are getting better. Hence, technically, the economic development is defined as an increase in output per capita in the long term [3]. When the economic growth is model by an FDE it means it is assumed that there is a memory effect in the economic process being modeled.

In this paper we will review existing literatures of FDE both from mathematical development, such as the techniques developed by the authors to find the solution of the FDE, and from applications sides. Special emphasizing will also be presented on the methods used to prove the existence and uniqueness of the solution. In general the worth of a review paper is the compiling, summarizing, critisizing, and synthesizing the available information on the topic being considered. It is expected that current paper can clarify the state of knowledge and identify needed research in the area of FDE and its applications [4]. The method on how the review is done will be presented in the following section followed by results and their discussion shaping future directtion of the potential research that could be done.

2. Materials and methods

The materials used to conduct the research are composed of two different types, the first one is the conceptual material and the second one is the media in which the concepts are presented, such as conference papers, books, and primary publication in scientific journals.

2.1. Materials

The first type of materials that we consider in this study includes conference papers, books, book chapters, and primary publications in scientific journals. While the second types of material includes mathematical concepts, such as differential equations, and related attributes. What we mean by related attributes are the things like fractional (a mathematical concept) as well as general concepts like its applications in economics and other areas. The attributes that we consider also includes the keywords in this paper, e.g., FRDE, economic growth model, memory effect, etc. For the economic model we emphasize on the following Harror-Domar model

We especially consider the economic growth model of Harrod-Domar [5], i.e.,

$$\frac{dY(t)}{dt} = \frac{1}{v} I(t), \tag{1}$$

where v is a positive constant, called the investment ratio and describes the rate of acceleration, $\frac{1}{v}$ is the marginal productivity from capital (acceleration rate), $I(t)$ is net investment function and $\frac{dY(t)}{dt}$ is the first-order derivative against time t of the function $Y(t)$. Based on [5], it is assumed that the net investment value is a fixed part of the profit proportional to the difference between income, $PY(t)$, and cost, $C(t)$, where cost given by

$$I(t) = m(PY(t) - C(t)), \tag{2}$$

where m is the net investment rate ($0 < m < 1$), that is the profit-sharing used for the net investment. The cost $C(t) = aY(t) + b$ is a linear function, where a is the marginal cost, that is part of the cost that deepends on the value of the output, while b is an independent cost that is part of the cost that does not depend on the value of the output. Eq (2) together with the FDE concept are the central of the review in this paper. The FDE is critically depends on the types of fractional derivatives we used in defining the derivatives. For further discussion, we will use the following definitions.

Definition 2.1 The fractional derivative according to Caputo is defined

$$D^\alpha f(x) = J^{n-\alpha} \left[\frac{d^n}{dx^n} f(x) \right],$$

or usually written in the form

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt,$$

where $n-1 < \alpha < n, n \in N$, where N is a natural number. Thus is not limited to numbers between 0 and 1, but is a rational number or even a real number.

Definition 2.2 The Riemann-Liouville fractional integral of order is defined as

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt,$$

with of any order, a is lower bound, $a < x$, $x > 0$, $f(x)$ is an analytical function and is continuous in the interval $[a, x]$. I^α is a fractional integral of order α with initial condition $a = 0$, thus the symbol of fractional integral of order α with initial condition $a = 0$ of $f(x)$ can be written as $I^\alpha f(x) = J^\alpha f(x) = D^{-\alpha} f(x)$, where D^α is a fractional derivative operator of order α with initial conditions $a = 0$. Meanwhile, $D^{-\alpha}$ is an integral operator which is the inverse of the D^α operator.

2.2. Methods

The research method is done through tracing primary literatures on fractional differential equations with emphasize on its application to economic growth models. We study carefully the development of theories that have been worked on by previous researchers and presented mainly in chronological order to gain relationships among the development and results therein, so that it is hoped that a gap of important issues can be obtained which may direct to new theories needed to be developed and applied in economic growth models with the involvement of memory effect.

This research is conducted in three stages, namely:

1) Review the results that have been achieved in previous researches, encompasing both the applications of the FDR in economic growth model and the mathematical methods used to solve the models. This is necessary to determine and set targets to be achieved in the next research. The study will be conducted through journals, textbooks, or other media.
2) Review current methods of analysis that ensure the existence and uniqueness of linear and nonlinear FDE solutions.
3) Identified the gaps that still open in the area of applications of the FDE in economic growth modelling.

Beside the study of the chronological development of the literatures, we also carried out the bibliometrical analysis. This study is performed to uncover or identify research position in this area using the Publish or Perish application program. The following section summarizes the results of our review.

3. Results of literature review

3.1. Review on the techniques to solve an FDE

The existence and uniqueness of the solution of some nonlinear FDE with known initial value can be found in [6]. In this reference, the authors have obtained a unique solution from a nonlinear FDE by the use of a contraction map T and the contraction principle. They proved that a map T has at least one fixed point at *C ([0,1], R)* using the Schauder's fixed-point theorem. The existence and uniqueness of the linear FDE initial value problem can also be found in [7]. In their paper, the existence of linear FDE solution is proven by the help of the Laplace transform on the fractional derivative sequence and the Riemann-Liouville derivative definition, while the uniqueness of linear FDE solution is proven by the use of the linear properties of fractional derivative and Laplace

transform. Gambo et al. [8] studied the fractional Cauchy problem by generalizing the left Caputo fractional derivative in continuous and differentiable function space and proving the existence as well as the uniqueness of a nonlinear FDE solution if it meets Lipschitz's requirements. Further, the existence and singularity of solutions of nonlinear fractional differential equations using various methods, such as fixed point theory, basic theory of inequalities, peano locales and extreme solutions can be found in [9]. Many research on the applications of FDE on the economic growth model generally use linear FDE, while most economic phenomena are nonlinear. So the theory of the FDE needs to be developed into nonlinear FDE forms, including the application of the Fractional Riccati Differential Equation (FRDE) on the economic growth model. One of the most common models is the FRDE with Caputo fractional derivative.

The methodology to find the FRDE solution using the concept of Caputo fractional derivative can be found in [10] and some results for FRDE with incomplete meromorphic functions can be found in [11]. Busawon and Johnson (2005) presented a closed-form analytical solution of FRDE in accordance to homogeneous linear Ordinary Differential Equation (ODE) [12] and then used to obtain analytical solutions of second-order homogeneous linear ODE with FRDE. Several numerical methods have been developed to solve FRDE such as variational iteration method, Laplace-Adomian-Pade method, Adomian decomposition method, Homotopy Perturbation methods, Chebyshev finite difference method, asymptotic decomposition method and Optimal homotopy asymptotic [13–18]. In addition, analytical and exact solutions have attracted the interest of researchers to solve fractional Riccati differential equations [19–21]. Harko et al. [19] presents ten new exact solutions for fractional Riccati differential equations by assuming certain relationships between coefficients in the form of some integral or differential operator. Jaber and Al-Tarawneh (2016) present an exact solution from the fractional Riccati differential equations. The exact solution is obtained using the following stages: 1) reducing it to a second-order linear GDP, 2) converting it into a Bernoulli equation, 3) obtaining the solution by considering an integral condition $R(x)$, where $R(x)$ is the constant part of fractional Riccati differential equations [20]. Further, Khaniyev and Merdan (2016) studied an analytical solution of fractional Riccati differential equations with conformable fractional derivatives [21]. In General, the fractional Riccati differential equations is a nonlinear differential equation with a complex form of equations so that some analytic techniques are difficult to solve, maybe even unsolvable [22].

In finding nonlinear fractional differential equations solutions generaly it can be used several iteration methods such as the Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), Laplace Adomian Decomposition, Differential Transformation, Homotopy Perturbation, and others. Literatures related to the approximate solutions of the fractional differential equation using ADM and alike methods are discussed in the following studies. Initially, Adomian (1988) introduced the decomposition method to find an approximate solution of a Differential Equation [23]. Then Genga et al. [24] studied a piecewise variational iteration method or a modified variational iteration method to solve the fractional Riccati differential equations. Followed by Jafari and Tajadodi (2010) who presented the solution of fractional Riccati differential equations using the He variational iteration method [25]. Further Faraz et al. [26] utilized variational iteration method to solve partial fractional Riccati differential equations. In the following year, Bhakelar and Listdar-Gejji (2012) completed a logistic PDF using the new iterative method, the Adomian decomposition method, and the homotopy perturbation method [27]. Some researchers review the methods, such as Duan et al. [28] who emphasized on the Adomian decomposition method and its modifications, as well as its

application to differential equations. Further development done by Jafari et al. [29] who introduced a modified variational iteration method with Adomian polynomials nonlinear forms to solve the fractional Riccati differential equations. Also a modification in the form of time delay is presented by Mohammedali et al. [30] who give an approximate solution of the Riccati differential equation in the form of a non-homogeneous matrix with a time delay using the variational iteration method. Finally, a combination of the Adomian decomposition method and the Sumudu integral transformation is developed to obtain an approximate solution of the fractional Riccati differential equations in [31].

3.2. Review on the applications of FDE in economics

One of the important topics of the applications of fractional derivatives and fractional integral is the description of economic phenomena involving the concept of memory effects. The application of fractional derivatives to generalize ideas in the framework of solving economic dynamics problems by involving memory effects can be found in [32]. The authors in this reference suggest that determining the price elasticity of demand for a product and their applications in economics can use fractional derivatives to express the memory effect of the process. This approach has advanced the theory of economic growth.

The theory of economic growth is the theory that explains the phenomenon of socio-economic change. This field now includes more advanced theory, such as the generalization of economic growth models that involve memory effects. In the early stage, the theory used linear fractional differential equation [33] to model the national economic growth, with case study of gross domestic product in Spain. This kind of work is done using derivative and integral models with fractional order [34]. Other case studies who looked at the applications of the fractional derivative are done by several authors, e.g., those who used the fractional derivative of Caputo to simulate gross domestic product growth in China, United States and Italy [35].

The economic growth model was developed by experts regarding the idea of improving socio-economic conditions. In the early development the model was developed by ignoring the effect of memory. However, since in realty there are many socio-economic process that depend on the effect of memory. For example, an acceleration of economic growth usually involves memory effect. Among the authors that considered this effect in their model are [36] in which they developed a model using a discrete-time approach. Other authors [37] discussed the economic processes involving long and short-term memory effects. They modeled the effect by fractional differential equation, where the fractional derivative used is the Grunwald-Letnikov. They found the exact solution using the Fourier transform [37].

The theory of economic growth is grouped into two parts, namely the classical economic growth and the neo-classical economic growth. Adam Smith's classical economic growth theory states that a country's economic growth is determined by two main factors, namely population growth and output growth. Meanwhile, the neo-classical economic growth theory put forward by senior economists named Robert Solow and T. W. Swan focuses on three main factors that affect economic growth, namely capital, labor, and technological development. Based on this, the neo-classical theory believes that an increase in the number of workers can increase per capita income. However, without the development of modern technology, this increase will not be able to provide positive results on national economic growth.

As for some solutions of economic model involving memory effects are investigated by some of

the research. Machado et al. [38] discussed the economic growth model using multidimensional scaling methods and state-space portrait analysis. Paper [39] proved that the local fractional derivative of the differential function is an integer order or zero-order derivative and show the local fractional derivative is the limit of the Caputo fractional derivative. Paper [40] formulated interpretations of the economy by using the concept of T-indicators and derivatives of fractional Caputo which allows describing the economic processes with memory effects. Paper [41] proposed a generalized model of economic growth from the logistic FDE and the Volterra integral equations involving memory effect and crisis effect. The memory effect means that the economic factors and parameters at a given time depend not only on the values at that time but also on the values of the previous time. The mathematical description of the memory effect uses the fractional-order derivative theory, while the crisis effect is a sudden price change in the form of a price explosion which can be represented by a Gaussian function with zero mean and small variance. Paper [42] constructed a mathematical model of the economic process with various types of memory. Luo et al. [43] implemented the calculus of fractional to analyze the economic growth model to simulate the gross domestic product in Spain. Pakhira et al. [44] studied several models Economic Order Quantity (EOQ) depends on the memory effect that has an important role to handle business policy on the inventory system. Paper [45] formulated two dynamic intersection principles with memory effects, namely the principles of changes in the rate of technological growth and changes in dominance. Pakhira et al. [46] presented an inventory model with a memory effect and show the results of both long-term and short-term memory effects on the minimum average total cost and the optimal ordering interval. Paper [47] constructed a mathematical model at economic growth with the memory effect and the delay time distribution that is continuous. This model can be considered a generalization of the standard macroeconomic model. Paper [48] formulated three types of general inventory models that fit the classic inventory model with nonlinear demand levels and then analyzed the behaviors of those models. Tejado et al. [49] presented economic models for countries that are members of the G7 in 1973–2016. Acay et al. [50] investigated several economic problems with the help of non-local fractional operators which include Caputo, Caputo-Fabrizio, Atangana-Baleanu-Caputo (ABC), and development of the Mittag-Leffler kernel. Some literature related to fractional order in economic models can be seen in [51–54].

3.3. Review on the existence and uniqueness of the FDE solution

Although FDE has been applied to various fields of science, including economics, research on the existence and uniqueness of nonlinear FDE in economic growth models has not been widely researched. FDE involving the Riemann-Liouville fractional differential operator with the order of $0 < \alpha < 1$ is often used in modeling some phenomena of economic growth. This show that it is necessary to find for the existence and uniqueness condition of the FDE solution of this type. In this article, we review the basic theory of existence and uniqueness of the nonlinear FDE solution.

The existence and uniqueness of nonlinear FDE solutions are often obtained using the results of the fixed-point theory, the theory of inequality, local peano, and extreme solutions [9]. The following shows the theorems related to the theory of existence and uniqueness of the FDE solution.

Theorem 3.1. Initial Value Problem [9]

Consider the initial value problem (IVP) for fractional differential equation given by

$$D^q x = f(t, x), \quad x(0) = x_0, \tag{3}$$

where $f \in C([0, T], R), D^q x$ is the fractional derivative of x and q is such that $0 < q < 1$. Since f is assumed continuous, the IVP (3) is equivalent to the following Volterra fractional integral

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, x(s)) ds, 0 \le t \le T \tag{4}$$

that is, every solution of (4) is also a solution of (3) and vice versa. Here and elsewhere Γ denotes the Gamma function.

Theorem 3.2. Strict [9]

Let $v, w \in C([0, T], R)$, $f \in C([0, T] \times R, R)$ and

$$v(t) \le v(0) + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, v(s)) ds, \tag{5}$$

$$w(t) \ge w(0) + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, w(s)) ds, 0 \le t \le T, \tag{6}$$

one of the foregoing inequalities being strict. Suppose further that $f(t, x)$ is nondecreasing in x for each t and

$$v(0) < w(0), \tag{7}$$

then it is obtained

$$v(t) < w(t), \quad 0 \le t \le T. \tag{8}$$

Theorem 3.3. Nonstrict [9]

Assume that the conditions of Theorem 3.2 hold with nonstrict inequality (5) and Eq (6). Suppose further that

$$f(t, x) - f(t, y) \le \frac{L}{1 + t^q} (x - y), \tag{9}$$

whenever $x \ge y$ and $L > 0$. Then, $v(0) < w(0)$ and $L > \Gamma(q + 1)$ implies

$$v(t) \le w(t), \quad 0 \le t \le T. \tag{10}$$

Theorem 3.4. Local Peano Existence [9]

Consider the fractional differential equation $D^q x = f(t, x)$.
Assume $f(t, x)$ is continuous in $[R_0, R]$, where $R_0 = [(T, x): 0 \le t \le a$ and $|x - x_0| \le b]$, and let $|f(t, x)| \le M$, on R_0. Then the IVP for FDE (1) possesses at least one solution $x(t)$ pada $0 \le t \le \alpha$, where

$$t \leq \alpha = min\left(a, \left[\frac{b}{M}\Gamma(q+1)\right]^{\frac{1}{q}}\right), 0 < q < 1,$$

with $M = sup|f(t,x)|, (t,x) \in R_0$. Furthermore, $x(t)$ is the sequence limit $\left(x_{\varepsilon_n}(t)\right)$, uniform on $0 \leq t \leq \alpha$.

Theorem 3.5. Existence [55]

Assume that

$$\mathcal{D} := [0, \mathcal{X}^*] \times \left[y_0^{(0)} - \alpha, y_0^{(0)} + \alpha\right],$$

with some $\mathcal{X}^* > 0$ and some $\alpha > 0$, and let the function $f: \mathcal{D} \to \mathbb{R}$ be continuous. Furthermore, define

$$\mathcal{X} := min\left\{\mathcal{X}^*, \left(\frac{\alpha\Gamma(q+1)}{||f||_\infty}\right)\right\},$$

then, there exists a function $y: [0, \mathcal{X}] \to \mathbb{R}$ solving the IVP for FDE.

Theorem 3.6. Uniqueness [55]

Assume that

$$\mathcal{D} := [0, \mathcal{X}^*] \times \left[y_0^{(0)} - \alpha, y_0^{(0)} + \alpha\right],$$

with some $\mathcal{X}^* > 0$ and some $\alpha > 0$. Furthermore, let the function $f: \mathcal{D} \to \mathbb{R}$ be bounded pada $\mathcal{D}$ and fulfill a Lipschitz condition concerning the second variable; i.e.,

$$|f(x,y) - f(x,z)| \leq L|y - z|, \tag{11}$$

with some constant $L > 0$ independent of x, y, and z. Then, denoting $\mathcal{X}$ as in Theorem 1 there exists at most one function $y: [0, \mathcal{X}] \to \mathbb{R}$ solving the IVP for FDE.

The theorem on the existence and uniqueness of the FDE solution is very similar to the classical theorem in the case of differential equations [45]. The solution is equivalent to the nonlinear Volterra integral equation of the second kind. Based on the Ascoli-Arzela theorem and Schauder's fixed point theorem, the theorem on the existence and uniqueness of the FDE solution is obtained uses a unique fixed-point as an aid.

3.4. Fractional Riccati Differential Equation (FRDE)

The economic growth model with the involvement of the effects of memory loss, is given by the differential equation with fractional order $\alpha > 0$ of Eq (1) that describes the relationship between net

investment and the value of marginal output. By substituting Eq (2) into (1), we obtain

$$(D_{0+}^{\alpha}Y)(t) = \frac{1}{v}\{m(PY(t) - C(t))\} \tag{12}$$

Furthermore, if the production cost is linear, i.e., *C(t) = aY(t) + b*, then Eq (3) becomes:

$$(D_{0+}^{\alpha}Y)(t) = \frac{m}{v}(P - a)Y(t) - \frac{mb}{v} \tag{13}$$

So we have a linear FDE of order *α > 0* representing the economic growth model that involves memory effect. We have seen in the review results that this kind of equations has received much attention is literatures. Meanwhile if the production cost is quadratic, then the equation will end up to the first order Riccati differential equation. The first order Riccati differential equation is a special form of the nonlinear differential equation, that is

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x), \tag{14}$$

with *P(x) ≠ 0* and *P(x), Q(x), R(x) is* a function of *x*. Meanwhile, the Fractional Riccati Differential Equation is given by

$$D_*^{\alpha}y(x) = A(x) + B(x)y(x) + C(x)y^2(x), x \in \mathbb{R}, 0 < \alpha \le 1, t > 0, \tag{15}$$

with the initial conditions

$$y^{(k)}(0) = d_k, k = 0, 1, 2, \dots, n - 1,$$

where α is the order of the fractional derivative, n is an integer, $A(x), B(x),$ and $C(x)$ are a real, and d_k is a constant.

The economic growth model can be developed by assuming the cost function *C(t)* in the form of the quadratic, cube, even root form. Furthermore, for the quadratic form of the cost function, it means that if the output value is greater, the cost will be quadratically bigger. If the cost function is quadratic in the form form *C(t) = aY²(t) + kY(t) + b*, where *a, k* is the marginal cost that is part of the cost that depends on the output value. As for *b* is an independent cost, that is, the part of the cost that does not depend on the output value, then Eq (4) becomes:

$$(D_0^{\alpha} + Y)(t) = -\frac{ma}{v}Y^2(t) + \left[\frac{m(P-k)}{v}\right]Y(t) - \frac{mb}{v} \tag{16}$$

The meaning of the variables in Eq (16) is as follows:
$(D_0^{\alpha} + Y)(t)$ = fractional derivative of order α from *Y(t)* against *t*;
D_0^{α} = fractional derivative operator of order α against *t* with *t > 0*;
t = time;
Y(t) = output value / number of products produced during time *t* process production.

Unlike the linear fractional model of economis growth, which has received much attention is literatures, the literature on economic growth models taking form as FRDE is still regarded rare. This is a gap in the literature of FRDE used in economic growth models. Figures 1–3 show this gap visually reflecting most (less) heavily discussed among related concepts in economic growth models

using fractional derivative in the modeling process and analysis. The figures are generated by the VOSviewer application program as a results of bibliometrical analysis using the Publish or Perish application program.
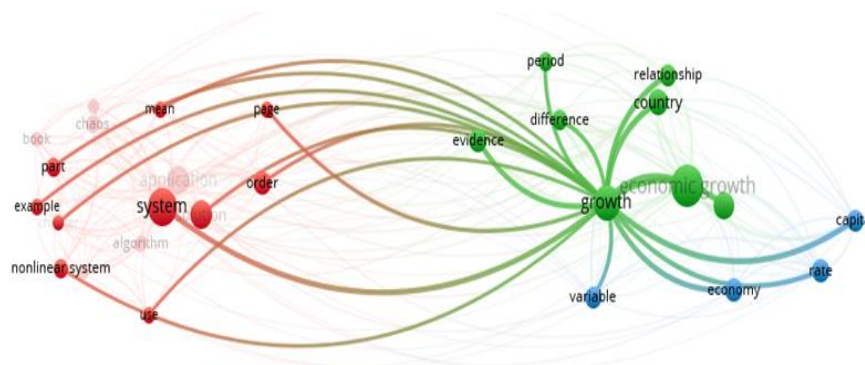

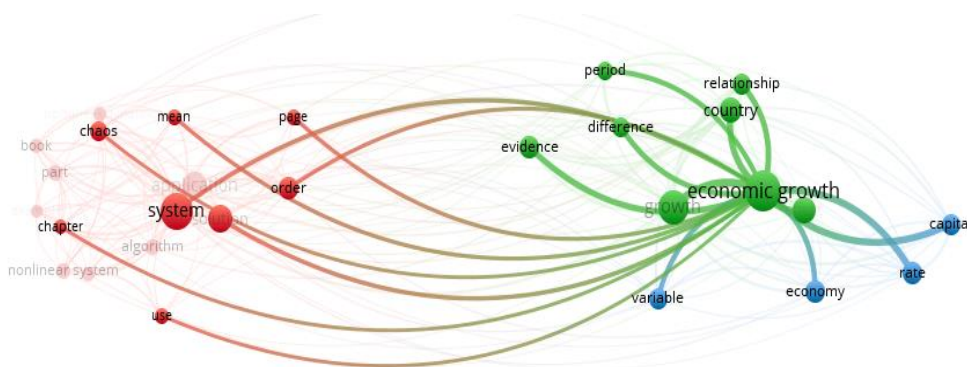**Figure 1.** Visualization map of related concepts for the keywords nonlinear FDE and growth.

Figure 2. Visualization map of related concepts for the keywords nonlinear FDE and economic growth.
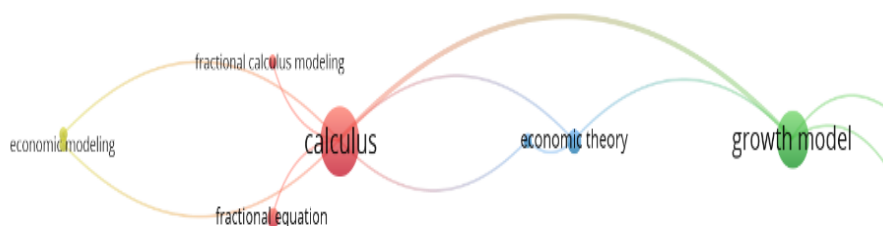

Figure 3. Visualization map of related concepts for the keywords fractional calculus and economic growth.

4. Conclusions

In this paper we have reviewed literatures on fractional derivatives, linear and nonlinear FDEs,

together with their applications on natural economic growth having the effects of memory in the models. Furthermore, we have presented the review results on the solution of the economic growth models in the form of FRDE. The results show that unlike the linear fractional model of economic growth, which has received much attention is literatures, the literature on economic growth models taking form as FRDE is still regarded rare. This is a gap in the literature of FRDE used in economic growth models since many economic processes have memory effect in their nature. This finding can be used as a base to formulate future research in the area of applications of fractional differential equation in economic growth modeling.

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Conflict of interest

All the authors declare no conflict of interest.

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