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Research article

Planar graphs without $\{4, 6, 8\}$-cycles are 3-choosable

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Abstract: In 2018, Dvořák and Postle introduced DP-coloring and proved that planar graphs without cycles of lengths 4 to 8 are 3-choosable. In this paper, we prove that planar graphs without $\{4, 6, 8\}$-cycles are 3-choosable by using the technique developed in DP-coloring, which also extends the result of Wang and Chen [Sci. China Math., 50 (2007), 1552–1562].

Keywords: planar graph; DP-coloring; 3-choosable
Mathematics Subject Classification: 05C15

1. Introduction

All graphs considered in this paper are simple and finite. Let G be a planar graph, and let $V(G)$, $E(G)$ and $F(G)$ be sets of vertices, edges and faces of G, respectively. Let f be a face in $F(G)$, we write $f = [v_1 v_2 \cdots v_k]$ when the vertices incident with f are in a cyclic order $v_1, v_2, \cdots, v_k$. Let $|C|$ ($|P|$) be the number of edges incident with the cycle C (the path P). A k-path P is a path with $|P| = k$. Let a k-vertex (k^--vertex, k^+-vertex) be a vertex with degree k (at most k, at least k). The notations will be same for cycles and faces. A triangle is a 3-cycle in G. An edge or a vertex is triangular when it is incident with a triangle. Let $Ext(C)$ (or $Int(C)$) be induced by the vertices outside (or inside) of C. Let $\overline{Ext}(C) = C \bigcup Ext(C)$ and $\overline{Int}(C) = C \bigcup Int(C)$. If $Ext(C) \neq \emptyset$ and $Int(C) \neq \emptyset$, then C is separating.

A proper k-coloring of G is a function $f : V(G) \to [k]$ such that $f(u) \neq f(v)$ whenever $uv \in E(G)$. The chromatic number of G is the smallest k such that G is proper k-colorable, denoted by $\chi(G)$. On the 3-colorability of planar graphs, Grötzsch [1] proved that every planar graph without triangles is 3-colorable. In 1976, Steinberg [2] conjectured that planar graphs without $\{4, 5\}$-cycles are 3-colorable. Borodin et al. [3] proved that every planar graph without cycles of lengths 4 to 7 is 3-colorable. Wang and Chen [4] proved that every planar graph without $\{4, 6, 8\}$-cycles is 3-colorable. Choi, Yu and

Zhang [5] proved that planar graphs with girth at least 5 are (3, 4)-colorable. Li and Zhang [6] proved that every planar graph with minimum degree at least 2 and girth at least 8 has an *RE*-*m*-coloring for each integer $m \geq 4$.

List coloring was introduced by Vizing [7] and independently by Erdős, Rubin and Taylor [8]. A list assignment L of G is a mapping that assigns to each v in G a list of available colors $L(v)$. An L-coloring of G is a function $f : V(G) \to \bigcup_{v\in V(G)} L(v)$ such that $f(v) \in L(v)$ for every $v \in V(G)$ and $f(u) \neq f(v)$ whenever $uv \in E(G)$. A graph G is k-choosable if G is L-colorable for each L with $|L(v)| \geq k$. The choice number of G is the smallest k such that G is k-choosable, denoted by $\chi_l(G)$. Thomassen [9] proved that every planar graph without $\{3, 4\}$-cycles is 3-choosable. Dvořák [10] showed that every planar graph with the distance of $\{3, 4\}$-cycles from each other at least 26 is 3-choosable.

The method of identification of vertices can be used for ordinary coloring since all vertices have the same color set. But for list coloring, it is impossible to use the method since different vertices may have different color lists. To overcome this difficulty, Dvořák and Postle [11] introduced DP-coloring. Here we give some definitions.

Definition 1. Let L be a list assignment for a simple graph G. Let $\mathcal{L}(v) = \{v\} \times L(v)$ for each vertex v in G. Let M_{uv} be a matching between $\mathcal{L}(u)$ and $\mathcal{L}(v)$ for each edge uv in G. Let $\mathcal{M} = \bigcup_{uv\in E(G)} M_{uv}$ be a matching assignment for G. For each $v \in V(G)$, if $L(v) = [k]$, then the matching assignment is a k-matching assignment. A cover of G is a graph $\mathcal{G}_{L,\mathcal{M}}$ which satisfies the following two conditions:

(1) The set of vertices of $\mathcal{G}_{L,\mathcal{M}}$ is the disjoint union of $\mathcal{L}(v)$ for all $v \in V(G)$;

(2) The set of edges of $\mathcal{G}_{L,\mathcal{M}}$ is $\mathcal{M}$.

Note that for each v in G, $\mathcal{G}_{L,\mathcal{M}}[\mathcal{L}(v)]$ is an independent set.

Definition 2. Let $\mathcal{G}_{L,\mathcal{M}}$ be a cover of a simple graph G. If $\mathcal{G}_{L,\mathcal{M}}$ has an independent set $\mathcal{I}$ such that for each v in G, $|\mathcal{I} \bigcap \mathcal{L}(v)| = 1$, then G is $\mathcal{M}$-colorable. If G is $\mathcal{M}$-colorable for each k-matching assignment $\mathcal{M}$, then G is DP-k-colorable. The DP-chromatic number of G, denoted by $\chi_{DP}(G)$, is the smallest k such that G is DP-k-colorable.

Let $W = v_1v_2 \cdots v_m$ be a closed walk with length m in G. Let $\mathcal{M}$ be a k-matching assignment on W. If $l_1, l'_1 \in L(v_1)$ and $l_i \in L(v_i)$ for $i = 2, \cdots, m$ such that $(v_i, l_i)(v_{i+1}, l_{i+1})$ is an edge in $M_{v_iv_{i+1}}$ for $i = 1, \cdots, m-1$, and $(v_m, l_m)(v_1, l'_1)$ is an edge in $M_{v_mv_1}$ with $l'_1 \neq l_1$, then $\mathcal{M}$ is inconsistent on W. Otherwise, $\mathcal{M}$ is consistent on W.

Dvořák and Postle [11] showed that planar graphs without cycles of lengths 4 to 8 are 3-choosable and noticed that $\chi_{DP}(G) \leq 3$ if G is a planar graph with girth at least 5. Liu and Li [12] proved that every planar graph without adjacent cycles of length at most 8 is 3-choosable. Bernshteyn et al. [13–17] gave some other results of DP-coloring. Liu et al. [18–20] showed some sufficient conditions for planar graphs to be DP-3-colorable, and [21–23] showed some sufficient conditions for planar graphs to be DP-4-colorable. In this paper, we prove the following result, which extends the results of Dvořák and Postle [11] and Wang and Chen [4].

Theorem 1. Planar graphs without $\{4, 6, 8\}$-cycles are 3-choosable.

Let $\mathcal{G}$ be a set of planar graphs with no 4-, 6- or 8-cycle. We will prove the following result stronger than Theorem 1.

Theorem 2. Let $G \in \mathcal{G}$, and let $\mathcal{M}$ be a 3-matching assignment for G such that $\mathcal{M}$ is consistent on every closed walk with length 3. Let S be a subset of $V(G)$ such that either $|S| \leq 1$, or S consists of all vertices on the outer face of G, then

(i) G is $\mathcal{M}$-colorable, and

(ii) Let D be the boundary cycle of the outer face of G. If $|D| \leq 12$, then for every $\mathcal{M}$-coloring ϕ_0 of S, G has an $\mathcal{M}$-coloring ϕ whose restriction to S is ϕ_0.

Let $\mathcal{M}$ be a k-matching assignment. If $|E(M_{uv})| = k$, then uv is full in $\mathcal{M}$. If for every $(u, c_1)(v, c_2) \in E(M_{uv})$, $c_1 = c_2$, then uv is straight in $\mathcal{M}$.

Lemma 3. [11] Let $\mathcal{M}$ be a k-matching assignment for a simple graph G. Let J be a subgraph of G. For every cycle C in J, if all edges of C are full and $\mathcal{M}$ is consistent on C, then we obtain a k-matching assignment $\mathcal{M}'$ from $\mathcal{M}$ by renaming $L(u)$ for $u \in J$ such that every edge in J is straight in $\mathcal{M}'$.

2. Proof of Theorem 2

Assume that Theorem 2 fails. If G is a planar graph with girth at least 5, then by Reference [11] it is DP-3-colorable, so G is $\mathcal{M}$-colorable and (i) holds. If G has a 3-cycle C such that its $\mathcal{M}$-coloring cannot be extended either to $Int(C)$ or to $Ext(C)$, then there exists a counterexample to Statement (ii). Let G be a minimal counterexample, such that

$$|V(G)| \text{ is minimized. (a)}$$

Subject to (a),

$$|E(G)| - |E(G[S])| \text{ is minimized. (b)}$$

Subject to (a) and (b),

$$\sum_{uv \in E(G)} |E(M_{uv})| \text{ is maximized. (c)}$$

Lemma 4. If $v \notin S$, then v is a 3^+-vertex.

Proof. Let v be a 2^--vertex in $G - S$. By condition (a), ϕ_0 can be extend to $G - \{v\}$. Since v has at most two neighbors, we can color v such that $(v, \phi(v))(u, \phi(u)) \notin E(M_{uv})$ for each neighbor u of v. Therefore, G is $\mathcal{M}$-colorable, a contradiction.

Lemma 5. Let C be a 12^--cycle in G. If C has a chord e, then $|C| = 10, 11$ or 12 and either e is triangular, or e divides C into a 5-cycle and a 7-cycle when C is a 10-cycle, or e divides C into a 5-cycle and a 9-cycle when C is a 12-cycle, or e divides C into two 7-cycles when C is a 12-cycle.

Furthermore, if C has two chords, then the two chords divide C into two triangles and a 9-cycle when C is an 11-cycle, or the two chords divide C into two triangles and a 10-cycle when C is a 12-cycle. If C has three chords, then C is a 12-cycle and the three chords divide C into three triangles and a 9-cycle.

Proof. Since $G \in \mathcal{G}$, G has no 4-, 6- or 8-cycle. If C is a 9^--cycle, then C has no chord.

If C has one chord, then $|C| = 10, 11$ or 12. If C is a 10-cycle and has a chord e that is not triangular, then e divides C into a 5-cycle and a 7-cycle. If C is an 11-cycle then C has only one triangular chord. If C is a 12-cycle and has a chord e that is not triangular, then e divides C into a 5-cycle and a 9-cycle, or two 7-cycles. If C has two chords e_1 and e_2, then $|C| = 11$ or 12. If C is an 11-cycle, then e_1 and e_2 divide C into two triangles and a 9-cycle. If C is a 12-cycle, then e_1 and e_2 divide C into two triangles and a 10-cycle. If C has three chords, then C is a 12-cycle and the three chords divide C into three triangles and a 9-cycle.

Lemma 6. G is 2-connected.

Proof. First, G is connected by the condition (a),

Next, if v is a cut-vertex of G and $G = G_1 \bigcup G_2$ such that $V(G_1) \bigcap V(G_2) = \{v\}$. If $v \in S$, then by the condition (a), G_1 and G_2 have an $\mathcal{M}$-coloring extending ϕ_0. Thus, G has an $\mathcal{M}$-coloring extending ϕ_0, a contradiction. If $v \notin S$, w.l.o.g., let $S \subset V(G_1)$. Then by the condition (a), G_1 has an $\mathcal{M}$-coloring ϕ_1 extending ϕ_0, and G_2 has an $\mathcal{M}$-coloring ϕ_2 extending $\phi_1(v)$. Thus, ϕ_1 and ϕ_2 together give an extension of ϕ_0 to G, a contradiction.

Lemma 7. G has no separating 12^{-}-cycle.

Proof. Suppose that C is a separating 12^{-}-cycle in G. Then $\overline{Ext}(C)$ has an $\mathcal{M}$-coloring ϕ_1 extending ϕ_0. Then the restriction of ϕ_1 to C can be extended to ϕ_2 of $Int(C)$. Thus, ϕ_1 and ϕ_2 together give a coloring of G that extends ϕ_0, a contradiction.

By Lemma 5 and Lemma 7, all 9^{-}-cycles of G are facial.

Lemma 8. If $|S| \leq 1$, then G is $\mathcal{M}$-colorable. So we still need to consider the case of $S = V(D)$.

Proof. Suppose that $|S| \leq 1$. If $S = \emptyset$, then we can include an arbitrary vertex of G in S, and thus without loss of generality, we can assume that $S = \{v\}$ for some vertex $v \in V(G)$. If v is on a 12^{-}-cycle, then by Lemma 7, v is on a 12^{-}-face f_1 whose boundary cycle is induced in G. Now let G be embedded in a plane such that f_1 is the outer face of the embedding. Let $S_1 = V(f_1)$. We choose an $\mathcal{M}$-coloring ϕ_1 for S_1, then $|E(G)| - |E(G[S_1])| < |E(G)| - |E(G[S])|$ and G has an $\mathcal{M}$-coloring extending ϕ_1 by the condition (b). This implies that G has an $\mathcal{M}$-coloring that extends ϕ_0.

If every cycle incident with v is a 13^{+}-cycle, then G has a 13^{+}-face f_2 incident with v. Let $v_1, v_2 \in V(f_2)$ be adjacent to v. Let $G \bigcup \{v_1v_2\} = G'$, then $f_3 = [v_1v_2v]$ is a 3-face. Let G' be embedded in a plane such that f_3 is the outer face of the embedding. Let $S_2 = \{v_1, v_2, v\}$ and $\mathcal{M}'$ be obtained from $\mathcal{M}$ with $M_{v_1v_2}$ is edgeless. We choose an $\mathcal{M}'$-coloring ϕ_2 for S_2, then $|E(G')| - |E(G'[S_2])| < |E(G)| - |E(G[S])|$ and G' has an $\mathcal{M}'$-coloring extending ϕ_2 by the condition (b). This implies that G has an $\mathcal{M}$-coloring that extends ϕ_0. So we still need to consider the case of $S = V(D)$.

Lemma 9. D has no chord.

Proof. Suppose otherwise that D is divided by a chord into two cycles D_1 and D_2, where $|D_1| \leq |D_2|$. Since $|D| \leq 12$, $|D_1|+|D_2| = |D|+2 \leq 14$. Then we have $3 \leq |D_1| \leq |D_2| \leq 11$. By Lemma 7, both D_1 and D_2 are not separating. This implies that $V(G) = V(D)$ and G has been $\mathcal{M}$-colorable, a contradiction.

P is a dividing path of C if its two end-vertices are in $V(C)$ and all other vertices of P are in $Int(C)$.

Lemma 10. If P is a dividing path of D, then D is divided by P into two cycles D_1 and D_2.

(1) If $|P| = 2$, then one of $\{D_1, D_2\}$ is a 3-cycle;

(2) if $|P| = 3$, then one of $\{D_1, D_2\}$ is a 5-cycle;

(3) if $|P| = 4$, then one of $\{D_1, D_2\}$ is a 5- or 7-cycle;

(4) if $|P| = 5$, then one of $\{D_1, D_2\}$ is a 9^{-}-cycle.

Proof. (1) Let $P = uvw$ and $|D_1|, |D_2| \geq 5$. v has a neighbor v' not in $V(P)$ by Lemma 4. W.l.o.g., let $|D_1| \leq |D_2|$. Since $|D| \leq 12$, $|D_1| + |D_2| = |D| + 2 \times |P| \leq 12 + 2 \times |P| = 16$. Then we have $5 \leq |D_1| < 8$ and $5 \leq |D_2| \leq 11$. Since $|D_1|$ and $|D_2|$ are not separating by Lemma 7, vv' is a chord of D_1 or D_2. Since $5 \leq |D_1| < 8$, D_1 has no chord by Lemma 5. Then vv' is a chord of D_2 and $|D_2| \geq 10$. Since G has no 6-cycle, if $|D_2| = 10$, then $|D_1| = 5$ and vv' splits D_2 into a triangle and a 9-cycle, or a 5-cycle and a 7-cycle. This implies that G contains a 6-cycle or an 8-cycle, a contradiction. If $|D_2| = 11$, then $|D_1| = 5$ and vv' splits D_2 into a triangle and a 10-cycle. This implies that G contains a 6-cycle, a contradiction.

(2) Let $P = uvwx$ and $|D_1|, |D_2| \geq 7$. W.l.o.g., let $|D_1| \leq |D_2|$. Since $|D| \leq 12$, $|D_1| + |D_2| = |D| + 2 \times |P| \leq 12 + 2 \times |P| = 18$. Then we have $7 \leq |D_1| \leq 9$ and $7 \leq |D_2| \leq 11$. If two nonconsecutive vertices of P are connected by an edge e, then a part of P together with e can form a cycle C. Since G has no 4-cycle, $|C| = 3$. W.l.o.g., let $e = uw$ and let e lie inside D_1. Now D is divided by the path uwx into D' and D''. By (1), one of $\{D', D''\}$ is a triangle. By Lemma 7, the triangle is not separating. Since the location of vertex v, D_1 is a 4-cycle, a contradiction. So w (v) has a neighbor w' (v') not in $V(P)$ by Lemma 4.

Since $|D_1|$ and $|D_2|$ are not separating by Lemma 7, vv' and ww' are two chords of D_1 or D_2. Since $7 \leq |D_1| < 9$, D_1 has no chord by Lemma 5. Then vv' and ww' are two chords of D_2. By Lemma 5, $|D_2| = 11$. Since $|D_1| + |D_2| \leq 18$, $|D_1| = 7$ and the two chords split D_2 into two triangles and a 9-cycle. This implies that G has an 8-cycle, a contradiction.

(3) Let $P = vwxyz$ and $|D_1|, |D_2| \geq 9$. W.l.o.g., let $|D_1| \leq |D_2|$. Since $|D| \leq 12$, $|D_1| + |D_2| = |D| + 2 \times |P| \leq 12 + 2 \times |P| = 20$. Then we have $9 \leq |D_1| \leq 10$ and $9 \leq |D_2| \leq 11$. If two nonconsecutive vertices of P are connected by an edge e, then a part of P together with e can form a cycle C. By Lemma 9, D has no chord, so $|C| \neq 5$. Since G has no 4-cycle, $|C| = 3$. W.l.o.g., let $e = wy$ and let e lie inside D_1. Now D is divided by the path $vwyz$ into D' and D''. By (2), one of $\{D', D''\}$ is a 5-cycle. By Lemma 7, the 5-cycle is not separating. Since the location of vertex x, D_1 is a 6-cycle, a contradiction. So y (w, x) has a neighbor y' (w', x') not in $V(P)$ by Lemma 4.

Since $|D_1|$ and $|D_2|$ are not separating by Lemma 7, ww', xx' and yy' are three chords of D_1 or D_2. If $|D_1| = 9$, then $|D_2| \in \{9, 10, 11\}$. By Lemma 5, D_1 has no chord and D_2 cannot have three chords, a contradiction. If $|D_1| = 10$, then $|D_2| = 10$. By Lemma 5, both D_1 and D_2 have at most one chord respectively, a contradiction.

(4) Let $P = uvwxyz$ and $|D_1|, |D_2| \geq 10$. W.l.o.g., let $|D_1| \leq |D_2|$. Since $|D| \leq 12$, $|D_1| + |D_2| = |D| + 2 \times |P| \leq 12 + 2 \times |P| = 22$. Then we have $10 \leq |D_1| \leq 11$ and $10 \leq |D_2| \leq 12$. If two nonconsecutive vertices of P are connected by an edge e, then a part of P together with e can form a cycle C. Since G has no 4- or 6-cycle, $|C| = \{3, 5\}$. Let $|C| = 5$. W.l.o.g., let $e = vz$ and let e lie inside D_1. Then D is divided by the path uvz into D' and D''. By (1), one of $\{D', D''\}$ is a triangle. By Lemma 7, the 3-cycle is not separating. Since the location of vertex w, D_1 is a 6-cycle, a contradiction. Let $|C| = 3$. W.l.o.g., let $e = vx$ and let e lie inside D_1. Then D is divided by the path $uvxyz$ into D' and D''. By (3), one of $\{D', D''\}$ is a 5- or 7-cycle. By Lemma 7, the 7^--cycle is not separating. Since the location of vertex w, D_1 is a 6-cycle or an 8-cycle, a contradiction. So x (w, y, v) has a neighbor x' (w', y', v') not in $V(P)$.

Since $|D_1|$ and $|D_2|$ are not separating by Lemma 7, ww', xx', vv' and yy' are four chords of D_1 or D_2. If $|D_1| = 10$, then $|D_2| = \{10, 11, 12\}$. Since D_1 has at most one chord by Lemma 5, $|D_2| = 12$ and D_2 has three chords. Then the three chords split D_2 into three triangles and a 9-cycle, this implies that G has a 4-cycle, a contradiction. If $|D_1| = 11$, then $|D_2| = 11$. By Lemma 5, both D_1 and D_2 have two chords respectively, and the chords split them into two triangles and a 9-cycle respectively. Since G contains no edge that connects two nonconsecutive vertices on P and two chords split D_1 into two triangles and a 9-cycle, vv' and yy' are the two chords of D_1. But ww' and xx' cannot split D_2 into two triangles and a 9-cycle, a contradiction.

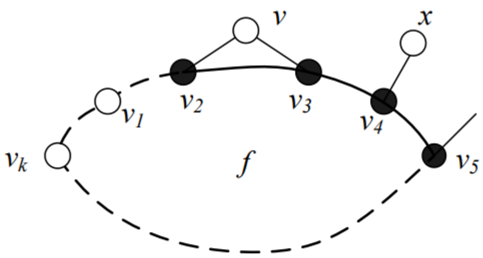
The following Lemma 11 and Lemma 12 about some crucial properties of the 3-matching assignment $\mathcal{M}$ are from [11]. Since the proof processes are similar, they are omitted here.

Lemma 11. [11] If $uv \in E(G)$ and $uv \notin E(D)$, then $|E(M_{uv})| \geq 2$ in the 3-matching assignment $\mathcal{M}$.

Moreover, if uv is not triangular, then uv is full in the 3-matching assignment $\mathcal{M}$.

Lemma 12. [11] Let $T = [uvw]$ be a triangle in G. If $u, w \notin V(D)$ and $d(u) = d(w) = 3$, then all edges on the triangle are full in the 3-matching assignment $\mathcal{M}$.

Let f_0 be the outer face of G, then D is the boundary cycle of f_0. A vertex or an edge is internal when it is not on D. A 3-path P is good if every $v \in V(P)$ is an internal 3-vertex and at least one end-edge of P is triangular as shown in Figure 1.



○ A vertex may have some other neighbors

Figure 1. A good 3-path $v_2v_3v_4v_5$.

Lemma 13. No face of G is incident with a good 3-path.

Proof. Suppose otherwise that G contains a k-face f which is incident with a good 3-path P (see Figure 1). Since G has no 4-, 6- or 8-cycle, $k \geq 9$. Let $f = [v_1v_2\cdots v_k]$ and $P = v_2v_3v_4v_5$. Let v, which is not on P, be the common neighbor of v_2 and v_3. Let x, which is not on P, be the neighbor of v_4. Since G contains no 4-cycle, $x \neq v$ and $x \neq v_1$. We delete $\{v_5, v_4, v_3, v_2\}$ and identify x and v_1. Let G' be the resulting planar graph.

Suppose that two vertices of D are identified, or two vertices of D are connected by an edge. Then P is a dividing 4- or 5-path of D which has $\{x, v_4, v_3, v_2, v_1\}$. Thus G contains a 9^--cycle C formed by a part of D and P' by Lemma 10. Since $|C| \leq 9$, C is facial and neither v nor v_5 lie inside C. Since v_5 is internal, v is on C. Now vv_2 and vv_3 are two chords of C, contrary to that C is facial. For each $uv \in E(G')$, let $M'_{uv} = M_{uv}$ and $\mathcal{M}' = \bigcup_{uv\in E(G')} M'_{uv}$. Then the $\mathcal{M}'$-coloring of D in G' is ϕ_0.

Suppose that an 8^--cycle is created. Then $G - \{v_5\}$ contains a 12^--cycle C that contains $v_1v_2\cdots v_4x$. If v is on C, then G contains a 6^--cycle adjacent to $[vv_2v_3]$, a contradiction. Thus v lies inside C, and C is separating, contradicting Lemma 7. Therefore we create no 8^--cycle. So $G' \in \mathcal{G}$ and $\mathcal{M}'$ is consistent on every closed walk of length 3.

Since $|V(G')| < |V(G)|$, then G' has an $\mathcal{M}'$-coloring ϕ' extending ϕ_0 by condition (a). Since $d(v_2) = d(v_3) = 3$, each edge of $[vv_2v_3]$ is full by Lemma 12. If v_5 is adjacent to x, recall $d(v_4) = d(v_5) = 3$, then each edge of $[xv_4v_5]$ is full by Lemma 12. If v_5 is not adjacent to x, then v_4 is not on a triangle, then by Lemma 11 and Lemma 12, all edges incident with v_2, v_3 and v_4 are full. Since $\mathcal{M}$ is consistent on each closed walk of length 3, we can rename the vertices in $\{v_2, v, v_3, v_4, x\}$ so that each edge in $\{v_1v_2, v_2v, v_2v_3, v_3v, v_3v_4, v_4x\}$ is straight by lemma 3. Now we keep the color of every vertex in G' and give the color of the identified vertex to v_1 and x. We color v_5 and v_4 in the order. Since v_1 and v_4 have different colors, v_2 and v_3 can be colored. Thus, G has an $\mathcal{M}$-coloring extending ϕ_0, a contradiction.

A vertex is bad if it is a triangular internal vertex with degree 3. Otherwise it is good.

Lemma 14. No face of G has five bad vertices which are consecutive. Furthermore, if a 9^+-face f of G contains a path $v_0v_1\cdots v_5$ and the vertices $v_1,\cdots,v_4$ are bad, then each edge in $\{v_0v_1,\ v_2v_3,\ v_4v_5\}$ is triangular.

Proof. Since G has no 4-, 6- or 8-cycle, f has no bad vertex when $d(f)\le 7$. Let $d(f)\ge 9$. Suppose that f contains five bad vertices in a path $v_1v_2\cdots v_5$. Since v_3 is triangular and G contains no 4-cycle, we can assume that v_2v_3 is triangular by symmetry. Since $d(v_3)=3$, v_3v_4 is not triangular, and thus the triangle which is incident with v_4 contains v_4v_5. Then $v_2v_3v_4v_5$ is a good 3-path, a contradiction.

Similarly, if a 9^+-face f of G contains a path $v_0v_1\cdots v_5$ and the vertices $v_1,\cdots,v_4$ are bad, then either each edge in $\{v_1v_2, v_3v_4\}$ or each edge in $\{v_0v_1, v_2v_3, v_4v_5\}$ is triangular. In the former case, the path $v_1v_2v_3v_4$ is a good 3-path, contrary to Lemma 13. Therefore, only the latter case remains. Then each edge in $\{v_0v_1, v_2v_3, v_4v_5\}$ is triangular.

Lemma 15. G has no 5-face containing five internal 3-vertices.

Proof. Suppose otherwise that a 5-face f in G contains five internal 3-vertices. Let $f=[v_1v_2\cdots v_5]$, and let u_i be the neighbor of v_i not on f for $1\le i\le 5$. Since G has no 4- or 6-cycle, the vertices in $\{u_1,u_2,\cdots,u_5\}$ are pairwise distinct. Since G has no 6- or 8-cycle, each vertex in $V(f)$ is on two 7^+-faces. By Lemma 11, each edge in $\{u_2v_2, v_2v_1, v_1u_1, v_1v_5, v_5u_5\}$ is full. We can rename the vertices in $\{v_2, v_1, u_1, v_5, u_5\}$ so that each edge in $\{u_2v_2, v_2v_1, v_1u_1, v_1v_5, v_5u_5\}$ is straight by Lemma 3. We delete $\{v_1, v_2,\cdots, v_5\}$ and add an edge between u_2 and u_5. Let G' be the resulting planar graph.

Assume $u_5, u_2\in V(D)$. Then the path $u_5v_5v_1v_2u_2$ is a dividing 4-path of D, and it together with a part of D form a 5- or 7-cycle C in G by Lemma 10. Since C is a 7^--cycle, C is facial. According to the location of vertices u_1 and u_4, C cannot be facial, a contradiction. For each $uv\in E(G')\bigcap E(G)$, let $M'_{uv}=M_{uv}$ and let $M'_{u_2u_5}$ be a full and straight matching. Let $\mathcal{M}'=\bigcup_{uv\in E(G')}M'_{uv}$. Then the $\mathcal{M}'$-coloring of D in G' is ϕ_0.

Suppose that an 8^--cycle is created. Then $G-\{v_4,v_3\}$ contains an 11^--cycle C which contains $u_2v_2v_1v_5u_5$. Since $|C|\le 11$, no vertex in $\{u_1,v_3,v_4\}$ lies inside C by Lemma 7. Then u_1 is on C and C is divided by u_1v_1 into two cycles C_1 and C_2 and $|C_1|+|C_2|=|C|+2\le 13$. Recall each vertex in $V(f)$ is on two 7^+-faces, $|C_1|+|C_2|\ge 14$, a contradiction. Therefore no 8^--cycle is created. So $G'\in\mathcal{G}$ and $\mathcal{M}'$ is consistent on every closed walk of length 3.

Since $|V(G')|<|V(G)|$, then G' has an $\mathcal{M}'$-coloring ϕ' extending ϕ_0 by condition (a). Now we keep the color of every vertex in G'. Since $M'_{u_2u_5}$ is full and straight, u_2 and u_5 have different colors. So at least one color of $\{u_5,u_2\}$ is different from the color of u_1. W.l.o.g., let the color of u_1 be different from u_2. Since each edge in $\{u_2v_2, v_2v_1, v_1u_1, v_1v_5, v_5u_5\}$ is straight, we give v_2 the color of u_1 and color v_3, v_4, v_5 and v_1 in the order. Thus, G has an $\mathcal{M}$-coloring extending ϕ_0, a contradiction.

Lemma 16. G has no 7-face containing seven internal 3-vertices.

Proof. Suppose otherwise that a 7-face f in G contains seven internal 3-vertices. Let $f=[v_1v_2\cdots v_7]$, and let u_i be the neighbor of v_i not on f for $1\le i\le 7$. Since G has no 4-, 6- or 8-cycle, the vertices in $\{u_1,u_2,\cdots,u_7\}$ are pairwise distinct. By Lemma 11, each edge in $\{u_1v_1, v_1v_7, v_7u_7, v_7v_6, v_6u_6, v_6v_5, v_5u_5\}$ is full. We can rename the vertices in $\{v_1, v_7, u_7, v_6, u_6, v_5, u_5\}$ so that each edge in $\{u_1v_1, v_1v_7, v_7u_7, v_7v_6, v_6u_6, v_6v_5, v_5u_5\}$ is straight by Lemma 3. We delete $\{v_1, v_2,\cdots, v_7\}$ and add an edge between u_1 and u_5. Let G' be the resulting planar graph.

Suppose $\{u_1,u_5\}\in V(D)$. Then the path $u_1v_1v_7v_6v_5u_5$ is a dividing 5-path of D, and it together with a part of D form a 9^--cycle C in G by Lemma 10. Since C is a 9^--cycle, C is facial. Since v_2 is internal,

u_6, $u_7 \in V(C)$. Then v_6u_6 and v_7u_7 are two chords of C, contrary to that C is facial. For each $uv \in E(G') \bigcap E(G)$, let $M'_{uv} = M_{uv}$ and let $M'_{u_1u_5}$ be a full and straight matching. Let $\mathcal{M}' = \bigcup_{uv \in E(G')} M'_{uv}$. Then the $\mathcal{M}'$-coloring of D in G' is ϕ_0.

Suppose that an 8^{-}-cycle is created. Then $G - \{v_4, v_3, v_2\}$ contains a 12^{-}-cycle C which contains $u_1v_1v_7v_6v_5u_5$. Since $|C| \leq 12$, none of v_2, v_3 and v_4 lie inside C by Lemma 7. It follows that both u_6 and u_7 are on C, then v_6u_6 and v_7u_7 are two chords of C. If $|C| \leq 10$, then C has at most one chord by Lemma 5, a contradiction. If $|C| = 11$ and C has two chords, then the chords split C into two triangles and a 9-cycle by Lemma 5, contrary to that the vertices in $\{u_1, u_2, \cdots, u_7\}$ are pairwise distinct. If $|C| = 12$ and C has two chords, then the chords split C into two triangles and a 10-cycle by Lemma 5, contrary to that the vertices in $\{u_1, u_2, \cdots, u_7\}$ are pairwise distinct. Therefore we create no 8^{-}-cycle. So $G' \in \mathcal{G}$ and $\mathcal{M}'$ is consistent on every closed walk of length 3.

Since $|V(G')| < |V(G)|$, then G' has an $\mathcal{M}'$-coloring ϕ' extending ϕ_0 by condition (a). Now we keep the color of every vertex in G'. Since $M'_{u_1u_5}$ is straight, u_1 and u_5 have different colors. Since each edge in $\{u_1v_1, v_1v_7, v_7u_7, v_7v_6, v_6u_6, v_6v_5, v_5u_5\}$ is straight, if u_7 and u_1 have different colors, then color v_7 same as u_1 and color v_6, v_5, v_4, v_3, v_2, v_1 in the order. If u_7 and u_1 have the same color, then u_7 and u_5 have different colors. If u_6 and u_5 have different colors, then color v_6 same as the color of u_5 and color v_7, v_1, v_2, v_3, v_4, v_5 in the order. If u_6 and u_5 have the same color, then u_7 and u_6 have different colors, color v_7 same as u_6 and color v_1, v_2, v_3, v_4, v_5, v_6 in the order. Thus, G has an $\mathcal{M}$-coloring extending ϕ_0, a contradiction.

A 9-face is light if it has two internal 4-vertices and seven bad vertices. By Lemma 14, a light 9-face is adjacent to five 3-faces. So it is isomorphic to the graph in Figure 2.

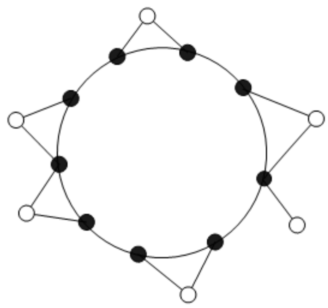


Figure 2. A light 9-face.

Let the initial charge $\mu(x) = d(x) - 4$ for $x \in V(G) \bigcup F(G) - \{f_0\}$, and $\mu(f_0) = d(f_0) + 4$.

The rules:

(R1): Let f be a 3-face. If v is on f, then f receives $\frac{1}{3}$ from v.

(R2): Let v be a 3-vertex not on D and let f be a face that contains v.

① If $d(f) = 5$, then f sends $\frac{1}{4}$ to v.

② Suppose $d(f) \geq 7$. If v is a bad vertex, then f sends $\frac{2}{3}$ to v. If one of the other two faces that contain v is a 5-face, then f sends $\frac{3}{8}$ to v. If all faces that contain v are 7^{+}-faces, then f sends $\frac{1}{3}$ to v.

(R3): Let v be a 4-vertex not on D.

① If v is on two triangles, then v receives $\frac{1}{3}$ from the other two faces that contain v.

② Suppose that v is on one triangle f. If v is not on a light 9-face, then v receives $\frac{1}{6}$ from the two incident 9^+-faces adjacent to f, respectively. If v is on a light 9-face f_1 adjacent to f, then v receives $\frac{1}{6}$ from each incident face except f and f_1. If v is on two light 9-faces adjacent to f, then v receives $\frac{1}{3}$ from the face incident with v which is not adjacent to f.

(R4): The outer face f_0 sends $\frac{4}{3}$ to each vertex on D.

(R5): Let v be a vertex on D and be incident with a 5^+-face $f \neq f_0$.

① If v is a 2-vertex, then v receives $\frac{2}{3}$ from f.

② Suppose that v is a 3-vertex. If v is on a 3-face, then v receives $\frac{1}{12}$ from f. If v is not on a 3-face, then f receives $\frac{1}{12}$ from v.

③ If $d(v) \geq 4$, then f receives $\frac{1}{3}$ from v.

After discharging, let $\mu^*(x)$ be the final charge.

Lemma 17. $\mu^*(f) \geq 0$ for each $f \in F(G)$.

Proof. First suppose that all vertices on f are internal. By Lemma 4, $d(v) \geq 3$ for each $v \in V(f)$. Since G has no 4-, 6- or 8-cycle, if $d(f) = 5$, then f cannot be adjacent to a 3- or 5-face. If $d(f) = 7$, then f cannot be adjacent to a 3-face.

Let $d(f) = 3$. Then $\mu^*(f) = \mu(f) + \frac{1}{3} \times 3 = 0$ by (R1).

Let $d(f) = 5$. Since f has at most four 3-vertices by Lemma 15, $\mu^*(f) = \mu(f) - \frac{1}{4} \times 4 = 0$ by (R2) ①.

Let $d(f) = 7$. Since f is adjacent to no 3-face, by (R2) ② and (R3) ②, f sends at most $\frac{3}{8}$ to each $v \in V(f)$. Therefore, $\mu^*(f) \geq \mu(f) - 7 \times \frac{3}{8} > 0$.

Let $d(f) \geq 9$. Let $D(f)$ be the set of bad vertices of f. Let uvw be a 2-path on f and $A(f) = \{v : v$ is good, both u and w are bad$\}$, $B(f) = \{v : v$ is good, u is bad, and w is good$\}$, $C(f) = \{v : v$ is good, both u and w are good$\}$. Then $V(f) = D(f) \bigcup C(f) \bigcup B(f) \bigcup A(f)$ and $D(f)$, $A(f)$, $B(f)$ and $C(f)$ are pairwise disjoint. Let $n_1 = |A(f)|$, $n_2 = |B(f)|$, $n_3 = |C(f)|$ and $n_4 = |D(f)|$. If $v \in D(f)$, then f sends $\frac{2}{3}$ to v by (R2) ②. If $v \in A(f)$. Since v is good, if v is a 3-vertex, then v cannot be triangular. Since u and w are bad and G has no 6-cycle, v is incident with no 5-face. By (R2) ②, f sends $\frac{1}{3}$ to v. If v is a 4-vertex, then f sends at most $\frac{1}{3}$ to v by (R3) ① ②. So if $v \in A(f)$, v receives at most $\frac{1}{3}$ from f. If $v \in B(f) \bigcup C(f)$, then f sends at most $\frac{3}{8}$ to v by (R2) ② and (R3) ① ②. Therefore,

$$\begin{aligned} \mu^*(f) &\geq d(f) - 4 - n_1 \times \frac{1}{3} - (n_2 + n_3) \times \frac{3}{8} - (d(f) - n_1 - n_2 - n_3) \times \frac{2}{3} \\ &= \frac{1}{3} \times d(f) + \frac{1}{3} \times n_1 + \frac{7}{24} \times (n_2 + n_3) - 4 \end{aligned} \tag{2.1}$$

Clearly, if $n_2 \neq 0$, then n_2 is even. If $n_2 = 0$, then $n_3 = d(f)$ or $n_3 = 0$.

Suppose $d(f) = 9$. If $n_1 \geq 3$, then by inequality (2.1), $\mu^*(f) \geq 0$. So we need to consider the following cases.

Case 1: $n_1 \leq 2$, $n_2 = 0$ and $n_3 = 0$. Then $n_1 = 2$ by Lemma 13. Let $f = [v_1 \cdots v_9]$ and every edge in $\{v_1v_2, v_3v_4, v_4v_5, v_6v_7, v_8v_9\}$ be triangular, then $A(f) = \{v_4, v_8\}$ and $D(f) = V(f) - A(f)$. If one vertex in $\{v_4, v_8\}$ is a 5^+-vertex, then the 5^+-vertex receives nothing from f by the rules. Therefore, $\mu^*(f) \geq \mu(f) - 7 \times \frac{2}{3} - \frac{1}{3} = 0$. If $d(v_4) = d(v_8) = 4$, then f is a light 9-face and sends no charge to v_8 by (R3) ②. Therefore, $\mu^*(f) \geq \mu(f) - \frac{2}{3} \times 7 - \frac{1}{3} = 0$.

Case 2: $n_1 = 1$, $n_2 = 2$ and $n_3 = 0$. Then $A(f) \bigcup B(f)$ divides $D(f)$ into 4+2 (four consecutive bad vertices and two other consecutive bad vertices) or 3+3 (three consecutive bad vertices and three other consecutive bad vertices) on f by Lemma 13.

In the former case 4+2, let $f = [v_1 \cdots v_9]$, $A(f) = \{v_1\}$ and $B(f) = \{v_4, v_5\}$. By Lemma 14, $d(v_1), d(v_5) \geq 4$. If $d(v_1) \geq 5$, then f sends no charge to v_1 by the rules. Therefore, $\mu^*(f) \geq \mu(f) - 6 \times \frac{2}{3} - 2 \times \frac{3}{8} > 0$. Now let $d(v_1) = 4$. By Lemma 14, v_1v_9 is triangular. If v_1 is on one 3-face, then f sends at most $\frac{1}{6}$ to v_1 by (R3) ②. Therefore, $\mu^*(f) \geq \mu(f) - 6 \times \frac{2}{3} - \frac{1}{6} - 2 \times \frac{3}{8} > 0$. If v_1 is on two 3-faces, then $d(v_4), d(v_5) \geq 4$. Then by (R3) ①, f sends at most $\frac{1}{3}$ to v_1. By (R3) ① ②, f sends at most $\frac{1}{3}$ to v_4 and v_5, respectively. Therefore, $\mu^*(f) \geq d(f) - 4 - 6 \times \frac{2}{3} - 3 \times \frac{1}{3} = 0$.

In the latter case 3+3, let $f = [v_1 \cdots v_9]$ and $A(f) = \{v_1\}$, then $B(f) = \{v_5, v_6\}$ and $D(f) = \{v_2, v_3, v_4, v_7, v_8, v_9\}$. By Lemma 13, at least one vertex in $\{v_5, v_6\}$ should be a 4^+-vertex. If one vertex in $\boldsymbol{\{v_5, v_6\}}$is a 5^+-vertex, then the 5^+-vertex receives nothing from f by the rules. Therefore, $\mu^*(f) \geq \mu(f) - \frac{2}{3} \times 6 - \frac{1}{3} - \frac{3}{8} > 0$. So $d(v_5), d(v_6) \leq 4$. If $d(v_5) = d(v_6) = 4$, then f sends at most $\frac{1}{3}$ to v_5 and v_6 by (R3) ① ②, respectively. Therefore, $\mu^*(f) \geq \mu(f) - 6 \times \frac{2}{3} - 3 \times \frac{1}{3} = 0$. If one vertex in $\boldsymbol{\{v_5, v_6\}}$is a 3-vertex, then v_5v_6 cannot be on a 3-face since $\{v_5, v_6\} \in B(f)$. W.l.o.g., let $d(v_5) = 3$ and $d(v_6) = 4$. Since v_5 is good and all vertices in $\{v_2, v_3, v_4\}$ are bad, every edge in $\{v_1v_2, v_3v_4\}$ is triangular. Recall $A(f) = \{v_1\}$, $d(v_1) \neq 3$. If $d(v_1) \geq 5$, then f sends no charge to v_1 by the rules. Therefore, $\mu^*(f) \geq \mu(f) - \frac{2}{3} \times 6 - \frac{1}{3} - \frac{3}{8} > 0$. If $d(v_1) = 4$ and v_1 is on precise one 3-face, recall v_1v_2 is triangular, then f sends at most $\frac{1}{6}$ to v_1 by (R3) ②. Therefore, $\mu^*(f) \geq \mu(f) - 6 \times \frac{2}{3} - \frac{1}{6} - \frac{1}{3} - \frac{3}{8} > 0$. If v_1 is on two 3-face, then v_6v_7 is not triangular and v_6 is on at most one 3-face. If v_6 is not triangular, then v receives nothing by (R3) ① ②. Therefore, $\mu^*(f) \geq \mu(f) - \frac{2}{3} \times 6 - \frac{1}{3} - \frac{3}{8} > 0$. If v_6 is on a 3-face T, then T is not adjacent to f. By (R3) ②, f sends at most $\frac{1}{3}$ to v_6. Since G has no 6-cycle, both v_4v_5 and v_5v_6 are not on a 5-face. Then f sends at most $\frac{1}{3}$ to v_5 by (R2) ②. Therefore, $\mu^*(f) \geq \mu(f) - \frac{2}{3} \times 6 - \frac{1}{3} \times 3 = 0$.

Case 3: $n_1 = 0$, $n_2 = 2$ and $n_3 \leq 2$. Since $n_2 + n_3 \leq 4$, f has at least five consecutive bad vertices, contradicting Lemma 14.

Suppose $d(f) \geq 10$. If $2 \times n_1 + n_2 \geq 4$, then $\mu^*(f) \geq 0$ by inequality (2.1). Now, we assume either $n_1 \leq 1$ and $n_2 = 0$, or $n_1 = 0$ and $n_2 = 2$. If $n_1 \leq 1$ and $n_2 = 0$, then $n_3 = 0$ and f contains at least nine consecutive bad vertices, contradicting Lemma 14. If $n_1 = 0$ and $n_2 = 2$, then $n_4 \leq 4$ by Lemma 14. So $n_3 \geq 4$ and $\mu^*(f) \geq 0$ by inequality (2.1).

Next assume f and f_0 have some common vertices. If $f = f_0$, then $\mu^*(f_0) \geq d(f_0)+4-\frac{4}{3}\times d(f_0) \geq 0$ by R5 since $d(f_0) \leq 12$. Now, let $f \neq f_0$. If $d(f) \leq 7$, then the vertices in $V(f) \bigcap V(f_0)$ are consecutive one by one by Lemma 5, Lemma 7 and Lemma 10. Furthermore, there is at most one 2-vertex in $V(f)$ when f is a 5-face, and at most two 2-vertices in $V(f)$ when f is a 7-face.

Let $d(f) = 3$. By (R1), $\mu^*(f) = \mu(f) + \frac{1}{3} \times 3 = 0$.

Let $d(f) = 5$. If there is no 2-vertex in $V(f)$, then by (R2) ①, f sends at most $\frac{1}{4}$ to each incident internal vertex. Therefore, $\mu^*(f) \geq \mu(f) - \frac{1}{4} \times 4 = 0$. If there is one 2-vertex in $V(f)$, then f contains two 3^+-vertices of f_0. By (R5), both of the 3^+-vertices send at least $\frac{1}{12}$ to f. Therefore, $\mu^*(f) \geq \mu(f) - \frac{2}{3} - 2 \times \frac{1}{4} + 2 \times \frac{1}{12} = 0$.

Let $d(f) = 7$. Since f cannot be adjacent to a triangle, by (R2) ② and (R3) ②, f sends at most $\frac{3}{8}$ to each incident internal vertex. If f and f_0 has a common 4^+-vertex v, then f receives $\frac{1}{3}$ from v by (R5) ③. Therefore, $\mu^*(f) \geq \mu(f) - 2\times\frac{2}{3} - 3\times\frac{3}{8} + \frac{1}{3} > 0$. If f and f_0 has no common 4^+-vertex, then f and f_0 have at least two common 3-vertices u and v. Since $d(f) = 7$, both u and v are not triangular, then f receives $2 \times \frac{1}{12}$ from them by (R5) ②. Since f cannot be adjacent to a triangle, by (R2) ② and (R3) ②, f sends at most $\frac{3}{8}$ to each incident internal vertex. Therefore, $\mu^*(f) \geq \mu(f) - 2 \times \frac{2}{3} - 3 \times \frac{3}{8} + 2 \times \frac{1}{12} > 0$.

Let $d(f) \geq 9$. If f and f_0 have a common 4^+-vertex v, then by (R5) ③, v sends $\frac{1}{3}$ to f. Therefore, $\mu^*(f) \geq \mu(f) - (d(f) - 1) \times \frac{2}{3} + \frac{1}{3} = \frac{1}{3}(d(f) - 9) \geq 0$. If f and f_0 has no common 4^+-vertex, then f and f_0

have at least two common 3-vertices. By (R5) ②, f sends at most $\frac{1}{12}$ to the two 3-vertices, respectively. Therefore, $\mu^*(f) \geq \mu(f) - \frac{2}{3} \times (d(f) - 2) - 2 \times \frac{1}{12} = \frac{1}{6}(2d(f) - 17) > 0$.

Lemma 18. $\mu^*(v) \geq 0$ for each $v \in V(G)$.

Proof. If v internal, then by Lemma 4, $d(v) \geq 3$.

Let $d(v) = 3$. The lengths of the faces that contain v is $\{7^+, 7^+, 7^+\}$, $\{3, 9^+, 9^+\}$ or $\{5, 7^+, 7^+\}$ since G is in $\mathcal{G}$. By (R1) and (R2) ① ② $\mu^*(v) \geq 0$.

Let $d(v) = 4$. the charge that v receives is equal to the charge that v sends out by (R1) and (R3) ① ②. Therefore, $\mu^*(v) = \mu(v) = 0$.

If $d(v) \geq 5$, then v sends $\frac{1}{3}$ to each incident 3-face by (R1). Therefore, $\mu^*(v) \geq \mu(v) - \frac{d(v)}{2} \times \frac{1}{3} = \frac{5}{6} \times d(v) - 4 > 0$.

If v is not internal, then $d(v) \geq 2$.

If $d(v) = 2$, then v is not triangular by Lemma 9. By (R4) and (R5) ①, $\mu^*(v) \geq \mu(v) + \frac{4}{3} + \frac{2}{3} = 0$.

Let $d(v) = 3$. If v is triangular, then $\mu^*(v) \geq \mu(v) + \frac{4}{3} - \frac{1}{3} + \frac{1}{12} > 0$ by (R4) and (R5) ②. If v is not triangular, then $\mu^*(v) \geq \mu(v) + \frac{4}{3} - \frac{1}{12} > 0$ by (R4) and (R5) ②.

Let $d(v) \geq 4$. By (R4), f_0 sends $\frac{4}{3}$ to v. By (R5) ③ and (R1), v sends $\frac{1}{3}$ to each other incident face than f_0. Therefore, $\mu^*(v) \geq \mu(v) + \frac{4}{3} - \frac{1}{3} \times (d(v) - 1) = \frac{2}{3} \times d(v) - \frac{7}{3} > 0$.

Proof of Theorem 2. Clearly, $V(G) \neq V(D)$. Let x_0 be a 3^+-vertex on D. From Lemma 18, $\mu^*(x_0) > 0$. Therefore, $\sum_{x \in V \bigcup F} \mu^*(x) > 0$ from Lemmas 17 and 18, contradicting Euler's Formula. So Theorem 2 is true.

3. Conclusions

It is well known that the problem of deciding whether a planar graph is 3-colorable is NP-complete. This provides motivation for finding some sufficient conditions for 3-coloring of planar graphs. DP-coloring is a stronger version of list coloring.

In this paper, we prove that planar graphs without $\{4, 6, 8\}$-cycles are 3-choosable by using the technique developed in DP-coloring. We like to conclude this paper by raising the following conjecture:

Conjecture 1 Planar graphs without $\{4, 6, 8\}$-cycles are DP-3-colorable.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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