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*Research article*

# The generalized inverse eigenvalue problem of Hamiltonian matrices and its approximation

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**Abstract:** Let $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$. A matrix $A \in \mathbb{R}^{2n\times 2n}$ is said to be Hamiltonian if $(AJ)^\top = AJ$. In this paper, we first consider the following generalized inverse eigenvalue problem (GIEP): Given a pair of matrices $(\Lambda, X)$ in the form $\Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_p\} \in \mathbb{C}^{p\times p}$ and $X = [\mathbf{x}_1, \cdots, \mathbf{x}_p] \in \mathbb{C}^{2n\times p}$, where diagonal elements of $\Lambda$ are all distinct with $\mathrm{rank}(X) = p$, and both $\Lambda$ and $X$ are closed under complex conjugation in the sense that $\lambda_{2i} = \bar{\lambda}_{2i-1} \in \mathbb{C}$, $\mathbf{x}_{2i} = \bar{\mathbf{x}}_{2i-1} \in \mathbb{C}^{2n}$ for $i = 1, \cdots, l$, and $\lambda_j \in \mathbb{R}$, $\mathbf{x}_j \in \mathbb{R}^{2n}$ for $j = 2l+1, \cdots, p$. Find Hamiltonian matrices $A$ and $B$ such that $AX\Lambda = BX$. Then, we consider the associated optimal approximation problem (OAP): Given $\tilde{A}, \tilde{B} \in \mathbb{R}^{2n\times 2n}$. Find $(\hat{A}, \hat{B}) \in \mathbb{S}_{\mathbb{E}}$ such that $\|\hat{A} - \tilde{A}\|^2 + \|\hat{B} - \tilde{B}\|^2 = \min_{(A,B)\in\mathbb{S}_{\mathbb{E}}} \left(\|A - \tilde{A}\|^2 + \|B - \tilde{B}\|^2\right)$, where $\mathbb{S}_{\mathbb{E}}$ is the solution set of Problem GIEP. By using the QR-decomposition, we deduce the representation of the general solution of Problem GIEP. Also, we obtain the unique optimal approximation solution $(\hat{A}, \hat{B})$ of Problem OAP.

**Keywords:** generalized inverse eigenvalue problem; Hamiltonian matrix; QR-decomposition; optimal approximation
**Mathematics Subject Classification:** 15A24, 65F18

## 1. Introduction

Throughout this paper, $\mathbb{C}^{m\times n}$, $\mathbb{R}^{m\times n}$, $\mathbb{OR}^{n\times n}$ and $\mathbb{SR}^{n\times n}$ stand for the sets of all $m\times n$ complex matrices, all $m \times n$ real matrices, all $n \times n$ orthogonal matrices and all $n \times n$ real-valued symmetric matrices, respectively. The symbol $A^\top$ and $\mathrm{tr}(A)$ stand for the transpose and the trace of a matrix $A$, respectively. $I_n$ represents the identity matrix of size $n$, and $\mathbb{HR}^{2n\times 2n}$ represents the set of all $2n \times 2n$ Hamiltonian matrices, that is, $\mathbb{HR}^{2n\times 2n} = \left\{A \,\middle|\, (AJ)^\top = AJ,\ A \in \mathbb{R}^{2n\times 2n}\right\}$, where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$.

Hamiltonian matrices are widely applied in Hamiltonian systems of differential equations [1, 2], optimal quadratic linear control [3] and $H_\infty$ optimization [4], etc. For example, Hamiltonian matrix elements from a symmetric wave function are necessary to study the structure of deuterated molecules [5]. Also, the eigenvalue problems for Hamiltonian and skew-Hamiltonian matrices appear frequently in scientific and engineering applications. Such as to compute the conformal parameterization via a constrained energy minimization problem in the field of digital geometry processing [6, 7], quantum mechanical problems with time reversal symmetry [8, 9], the study of closed shell Hartree-Fock wave functions in response theory [10, 11] and total least squares problems with symmetric constraints [12].

Inverse eigenvalue problems emerge from many application areas, and have been studied by many scholars [13–18]. Generalized inverse eigenvalue problems are concerned in structural dynamics [19–21], parameter identification [22] and pole assignment [23], etc. Recently, Zhao and Zhang [24] derived the solvability conditions for the inverse eigenvalue problem of normal skew $J$-Hamiltonian matrices by the Moore-Penrose generalized inverse and the generalized singular value decomposition. Zhang and Yuan [25] solved the generalized inverse eigenvalue problems of Hermitian and $J$-Hamiltonian/skew-Hamiltonian matrices by applying the singular value decomposition and the spectral decomposition. However, the problem of OAP cannot be considered due to the complexity of the expression of the general solution. Very recently, Yuan and Chen [26] solved the inverse eigenvalue problem and the optimal approximation problem for Hamiltonian matrices by using the generalized singular value decomposition. Nevertheless, the generalized inverse eigenvalue problem of Hamiltonian matrices seems rarely to be discussed in the literatures, which motivates us to study such kind of inverse problem and the associated approximation problem. That is, in this paper, we will consider the following generalized inverse eigenvalue problem and the associated optimal approximation problem, which is a generalization of the problems discussed in [26].

**Problem GIEP.** Given a pair of matrices $(\Lambda, X)$ in the form $\Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_p\} \in \mathbb{C}^{p\times p}$, and $X = [\mathbf{x}_1, \cdots, \mathbf{x}_p] \in \mathbb{C}^{2n\times p}$, where diagonal elements of $\Lambda$ are all distinct, $X$ is of full column rank $p$, and both $\Lambda$ and $X$ are closed under complex conjugation in the sense that $\lambda_{2i} = \bar{\lambda}_{2i-1} \in \mathbb{C}$, $\mathbf{x}_{2i} = \bar{\mathbf{x}}_{2i-1} \in \mathbb{C}^{2n}$ for $i = 1, \cdots, l$, and $\lambda_j \in \mathbb{R}$, $\mathbf{x}_j \in \mathbb{R}^{2n}$ for $j = 2l+1, \cdots, p$. Find $A, B \in \mathbb{HR}^{2n\times 2n}$ such that

$$AX\Lambda = BX. \tag{1.1}$$

**Problem OAP.** Given $\tilde{A}, \tilde{B} \in \mathbb{R}^{2n\times 2n}$. Find $(\hat{A}, \hat{B}) \in \mathbb{S}_{\mathbb{E}}$ such that

$$\|\hat{A} - \tilde{A}\|^2 + \|\hat{B} - \tilde{B}\|^2 = \min_{(A,B)\in\mathbb{S}_{\mathbb{E}}} \left(\|A - \tilde{A}\|^2 + \|B - \tilde{B}\|^2\right), \tag{1.2}$$

where $\|\cdot\|$ is the Frobenius norm and $\mathbb{S}_{\mathbb{E}}$ is the solution set of Problem GIEP.

By using the QR-decomposition, the representation of the general solution of Problem GIEP is deduced and the unique optimal approximation solution $(\hat{A}, \hat{B})$ of Problem OAP is obtained. Finally, two numerical examples are presented to illustrate the efficiency of the results.

## 2. The solution of Problem GIEP

Define $T_p$ as

$$T_p = \mathrm{diag}\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\ 1 & i\end{bmatrix}, \cdots, \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\ 1 & i\end{bmatrix}, I_{p-2l}\right\} \in \mathbb{C}^{p\times p},$$

where $i = \sqrt{-1}$. It is easy to verify that $T_p$ is a unitary matrix, that is, $\bar{T}_p^\top T_p = I_p$. With this matrix, we have

$$\begin{aligned}\tilde{\Lambda} &= \bar{T}_p^\top \Lambda T_p = \operatorname{diag}\left\{\begin{bmatrix}\alpha_1 & \beta_1\\ -\beta_1 & \alpha_1\end{bmatrix}, \cdots, \begin{bmatrix}\alpha_{2l-1} & \beta_{2l-1}\\ -\beta_{2l-1} & \alpha_{2l-1}\end{bmatrix}, \lambda_{2l+1}, \cdots, \lambda_p\right\}\\ &\triangleq \operatorname{diag}\{\tilde{\Lambda}_1, \cdots, \tilde{\Lambda}_{2l-1}, \lambda_{2l+1}, \cdots, \lambda_p\} \in \mathbb{R}^{p\times p},\end{aligned} \tag{2.1}$$

$$\tilde{X} = XT_p = [\sqrt{2}\mathbf{y}_1, \sqrt{2}\mathbf{z}_1, \cdots, \sqrt{2}\mathbf{y}_{2l-1}, \sqrt{2}\mathbf{z}_{2l-1}, \mathbf{x}_{2l+1}, \cdots, \mathbf{x}_p] \in \mathbb{R}^{2n\times p}, \tag{2.2}$$

where $\alpha_i$ and $\beta_i$ are the real part and imaginary part of the complex number $\lambda_i$, and $\mathbf{y}_i$ and $\mathbf{z}_i$ are, respectively, the real part and imaginary part of the complex vector $\mathbf{x}_i$ for $i = 1, 3, \cdots, 2l-1$. Then, Eq (1.1) can be equivalently written as

$$A\tilde{X}\tilde{\Lambda} = B\tilde{X}, \tag{2.3}$$

clearly, Eq (2.3) is equivalent to

$$AJJ^\top\tilde{X}\tilde{\Lambda} = BJJ^\top\tilde{X}. \tag{2.4}$$

Since rank($X$)= rank($\tilde{X}$)= $p$, the QR-decomposition of $J^\top\tilde{X}$ is of the form

$$J^\top\tilde{X} = Q\begin{bmatrix}R\\0\end{bmatrix} = [Q_1, Q_2]\begin{bmatrix}R\\0\end{bmatrix}, \tag{2.5}$$

where $Q = [Q_1, Q_2] \in \mathbb{OR}^{2n\times 2n}$ with $Q_1 \in \mathbb{R}^{2n\times p}$, and $R \in \mathbb{R}^{p\times p}$ is nonsingular. Partition the parameter matrices $Q^\top AJQ$ and $Q^\top BJQ$ into blocks:

$$Q^\top AJQ = \begin{array}{c}\begin{bmatrix}A_{11} & A_{12}\\ A_{12}^\top & A_{22}\end{bmatrix}\begin{array}{c}p\\2n-p\end{array}\\ \begin{array}{cc}p & 2n-p\end{array}\end{array}, \qquad Q^\top BJQ = \begin{array}{c}\begin{bmatrix}B_{11} & B_{12}\\ B_{12}^\top & B_{22}\end{bmatrix}\begin{array}{c}p\\2n-p\end{array}\\ \begin{array}{cc}p & 2n-p\end{array}\end{array}, \tag{2.6}$$

where $A_{11}$, $A_{22}$, $B_{11}$ and $B_{22}$ are real-valued symmetric matrices. By (2.5) and (2.6), Eq (2.4) is equivalent to

$$\begin{bmatrix}A_{11} & A_{12}\\ A_{12}^\top & A_{22}\end{bmatrix}\begin{bmatrix}R\tilde{\Lambda}\\0\end{bmatrix} = \begin{bmatrix}B_{11} & B_{12}\\ B_{12}^\top & B_{22}\end{bmatrix}\begin{bmatrix}R\\0\end{bmatrix}. \tag{2.7}$$

Then, it follows from Eq (2.7) that

$$A_{11}R\tilde{\Lambda} = B_{11}R, \tag{2.8}$$

$$A_{12}^\top R\tilde{\Lambda} = B_{12}^\top R. \tag{2.9}$$

By Eq (2.8), $B_{11}$ is a symmetric matrix implies that

$$R^\top A_{11}R\tilde{\Lambda} = \tilde{\Lambda}^\top R^\top A_{11}R. \tag{2.10}$$

Write

$$C = R^\top A_{11}R, \tag{2.11}$$

then Eq (2.10) can be written as

$$C\tilde{\Lambda} = \tilde{\Lambda}^\top C, \ \text{ s. t. } C = C^\top. \tag{2.12}$$

By direct calculation, we have

$$C = \operatorname{diag}\left\{\begin{bmatrix}a_1 & b_1\\ b_1 & -a_1\end{bmatrix}, \cdots, \begin{bmatrix}a_{2l-1} & b_{2l-1}\\ b_{2l-1} & -a_{2l-1}\end{bmatrix}, c_{2l+1}, \cdots, c_p\right\} \in \mathbb{R}^{p\times p}, \tag{2.13}$$

where $a_{2i-1}$, $b_{2i-1}$, $i = 1, \cdots, l$, and $c_j$, $j = 2l+1, \cdots, p$, are arbitrary real numbers. Thus

$$A_{11} = R^{-\top} C R^{-1}. \tag{2.14}$$

Combining (2.8) with (2.11), we find that

$$B_{11} = R^{-\top} C \tilde{\Lambda} R^{-1}. \tag{2.15}$$

By Eq (2.9), we have

$$B_{12} = R^{-\top} \tilde{\Lambda}^{\top} R^{\top} A_{12}, \tag{2.16}$$

where $A_{12} \in \mathbb{R}^{p\times(2n-p)}$ is an arbitrary matrix.

Summing up above discussion, we can obtain the following result.

**Theorem 2.1.** *Suppose that $\Lambda = diag\{\lambda_1, \cdots, \lambda_p\} \in \mathbb{C}^{p\times p}$, $X = [\mathbf{x}_1, \cdots, \mathbf{x}_p] \in \mathbb{C}^{2n\times p}$, where diagonal elements of $\Lambda$ are all distinct, $X$ is of full column rank $p$, and both $\Lambda$ and $X$ are closed under complex conjugation. Let the real matrices $\tilde{\Lambda}$ and $\tilde{X}$ be given by* (2.1) *and* (2.2) *and the QR-decomposition of $J^{\top}\tilde{X}$ be given by* (2.5)*. Then the general solution of Problem GIEP can be expressed as*

$$\mathbb{S}_{\mathbb{E}} = \left\{ (A, B) \left| \begin{array}{c} A = Q \begin{bmatrix} R^{-\top} C R^{-1} & A_{12} \\ A_{12}^{\top} & A_{22} \end{bmatrix} Q^{\top} J^{\top}, \\ B = Q \begin{bmatrix} R^{-\top} C \tilde{\Lambda} R^{-1} & R^{-\top} \tilde{\Lambda}^{\top} R^{\top} A_{12} \\ A_{12}^{\top} R \tilde{\Lambda} R^{-1} & B_{22} \end{bmatrix} Q^{\top} J^{\top} \end{array} \right. \right\}, \tag{2.17}$$

*where $A_{12} \in \mathbb{R}^{p\times(2n-p)}$, $A_{22} \in \mathbb{SR}^{(2n-p)\times(2n-p)}$ and $B_{22} \in \mathbb{SR}^{(2n-p)\times(2n-p)}$ are arbitrary matrices, and $C$ is given by* (2.13).

## 3. The solution of Problem OAP

According to (2.17), we know that the solution set $\mathbb{S}_{\mathbb{E}}$ is always nonempty and $\mathbb{S}_{\mathbb{E}}$ is a closed convex subset, which implies that Problem OAP has a unique solution $(\hat{A}, \hat{B}) \in \mathbb{S}_{\mathbb{E}}$ by the optimal approximation theorem (see Ref. [27]). For the given matrices $\tilde{A}, \tilde{B} \in \mathbb{R}^{2n\times 2n}$, write

$$Q^{\top}\tilde{A}JQ = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{matrix} p \\ 2n-p \end{matrix}, \qquad Q^{\top}\tilde{B}JQ = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{bmatrix} \begin{matrix} p \\ 2n-p \end{matrix}, \tag{3.1}$$
$$\qquad\quad p \quad 2n-p \qquad\qquad\qquad\qquad p \quad 2n-p$$

then

$$\begin{aligned} & \|A - \tilde{A}\|^2 + \|B - \tilde{B}\|^2 \\ = \; & \left\| Q \begin{bmatrix} R^{-\top} C R^{-1} & A_{12} \\ A_{12}^{\top} & A_{22} \end{bmatrix} Q^{\top} J^{\top} - \tilde{A} \right\|^2 + \left\| Q \begin{bmatrix} R^{-\top} C \tilde{\Lambda} R^{-1} & R^{-\top} \tilde{\Lambda}^{\top} R^{\top} A_{12} \\ A_{12}^{\top} R \tilde{\Lambda} R^{-1} & B_{22} \end{bmatrix} Q^{\top} J^{\top} - \tilde{B} \right\|^2 \\ = \; & \|R^{-\top} C R^{-1} - \tilde{A}_{11}\|^2 + \|A_{12} - \tilde{A}_{12}\|^2 + \|A_{12}^{\top} - \tilde{A}_{21}\|^2 + \|A_{22} - \tilde{A}_{22}\|^2 \\ + \; & \|R^{-\top} C \tilde{\Lambda} R^{-1} - \tilde{B}_{11}\|^2 + \|R^{-\top} \tilde{\Lambda}^{\top} R^{\top} A_{12} - \tilde{B}_{12}\|^2 + \|A_{12}^{\top} R \tilde{\Lambda} R^{-1} - \tilde{B}_{21}\|^2 + \|B_{22} - \tilde{B}_{22}\|^2. \end{aligned}$$

Therefore, $\|A - \tilde{A}\|^2 + \|B - \tilde{B}\|^2 = \min$ if and only if

$$f(C) = \|R^{-\top} C R^{-1} - \tilde{A}_{11}\|^2 + \|R^{-\top} C \tilde{\Lambda} R^{-1} - \tilde{B}_{11}\|^2 = \min, \tag{3.2}$$

$$\|A_{12}-\tilde{A}_{12}\|^2+\|R^{-\top}\tilde{\Lambda}^\top R^\top A_{12}-\tilde{B}_{12}\|^2+\|A_{12}^\top-\tilde{A}_{21}\|^2+\|A_{12}^\top R\tilde{\Lambda}R^{-1}-\tilde{B}_{21}\|^2=\min, \tag{3.3}$$

$$\|A_{22}-\tilde{A}_{22}\|^2=\min, \tag{3.4}$$

$$\|B_{22}-\tilde{B}_{22}\|^2=\min\,. \tag{3.5}$$

Let

$$R^{-1}=\begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \tag{3.6}$$

where

$$R_1=\begin{bmatrix} R_{1,1} \\ \vdots \\ R_{1,2l-1} \end{bmatrix},\ R_2=\begin{bmatrix} R_{2,2l+1} \\ \vdots \\ R_{2,p} \end{bmatrix},$$

and $R_{1,2i-1}\in\mathbb{R}^{2\times p}$, $R_{2,j}\in\mathbb{R}^{1\times p}$ $(i=1,\cdots,l,\, j=2l+1,\cdots,p)$. Furthermore, let

$$\begin{cases} D_{2i-1}=R_{1,2i-1}^\top F_1 R_{1,2i-1}, \\ D_{2i}=R_{1,2i-1}^\top F_2 R_{1,2i-1}, \\ D_j=R_{2,j}^\top R_{2,j}, \\ E_{2i-1}=R_{1,2i-1}^\top F_1\tilde{\Lambda}_{2i-1}R_{1,2i-1}, \\ E_{2i}=R_{1,2i-1}^\top F_2\tilde{\Lambda}_{2i-1}R_{1,2i-1}, \\ E_j=R_{2,j}^\top\lambda_j R_{2,j}, \\ i=1,\cdots,l,\ j=2l+1,\cdots,p, \end{cases} \tag{3.7}$$

where

$$F_1=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ F_2=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then the relation of (3.2) is equivalent to

$$\begin{aligned} & f(C)=f(a_1,b_1,\cdots,a_{2l-1},b_{2l-1},c_{2l+1},\cdots,c_p) \\ = & \|a_1D_1+b_1D_2+\cdots+a_{2l-1}D_{2l-1}+b_{2l-1}D_{2l}+c_{2l+1}D_{2l+1}+\cdots+c_pD_p-\tilde{A}_{11}\|^2 \\ + & \|a_1E_1+b_1E_2+\cdots+a_{2l-1}E_{2l-1}+b_{2l-1}E_{2l}+c_{2l+1}E_{2l+1}+\cdots+c_pE_p-\tilde{B}_{11}\|^2=\min, \end{aligned}$$

that is,

$$
\begin{aligned}
& f(a_1, b_1, \cdots, a_{2l-1}, b_{2l-1}, c_{2l+1}, \cdots, c_p) \\
= \; & \mathrm{tr}[(a_1 D_1^\top + b_1 D_2^\top + \cdots + a_{2l-1} D_{2l-1}^\top + b_{2l-1} D_{2l}^\top + c_{2l+1} D_{2l+1}^\top + \cdots + c_p D_p^\top - \tilde{A}_{11}^\top) \\
& (a_1 D_1 + b_1 D_2 + \cdots + a_{2l-1} D_{2l-1} + b_{2l-1} D_{2l} + c_{2l+1} D_{2l+1} + \cdots + c_p D_p - \tilde{A}_{11}) \\
+ \; & (a_1 E_1^\top + b_1 E_2^\top + \cdots + a_{2l-1} E_{2l-1}^\top + b_{2l-1} E_{2l}^\top + c_{2l+1} E_{2l+1}^\top + \cdots + c_p E_p^\top - \tilde{B}_{11}^\top) \\
& (a_1 E_1 + b_1 E_2 + \cdots + a_{2l-1} E_{2l-1} + b_{2l-1} E_{2l} + c_{2l+1} E_{2l+1} + \cdots + c_p E_p - \tilde{B}_{11})] \\
= \; & a_1^2 g_{1,1} + 2a_1 b_1 g_{1,2} + \cdots + 2a_1 a_{2l-1} g_{1,2l-1} + 2a_1 b_{2l-1} g_{1,2l} + 2a_1 c_{2l+1} g_{1,2l+1} + \cdots \\
+ \; & 2a_1 c_p g_{1,p} - 2a_1 h_1 \\
+ \; & b_1^2 g_{2,2} + \cdots + 2a_{2l-1} b_1 g_{2,2l-1} + 2b_1 b_{2l-1} g_{2,2l} + 2b_1 c_{2l+1} g_{2,2l+1} + \cdots + 2b_1 c_p g_{2,p} \\
- \; & 2b_1 h_2 \\
+ \; & \cdots, \cdots \\
+ \; & a_{2l-1}^2 g_{2l-1,2l-1} + 2a_{2l-1} b_{2l-1} g_{2l-1,2l} + 2a_{2l-1} c_{2l+1} g_{2l-1,2l+1} + \cdots + 2a_{2l-1} c_p g_{2l-1,p} \\
- \; & 2a_{2l-1} h_{2l-1} \\
+ \; & b_{2l-1}^2 g_{2l,2l} + 2b_{2l-1} c_{2l+1} g_{2l,2l+1} + \cdots + 2b_{2l-1} c_p g_{2l,p} - 2b_{2l-1} h_{2l} \\
+ \; & c_{2l+1}^2 g_{2l+1,2l+1} + \cdots + 2c_{2l+1} c_p g_{2l+1,p} - 2c_{2l+1} h_{2l+1} \\
+ \; & \cdots, \cdots \\
+ \; & c_p^2 g_{p,p} - 2c_p h_p + e,
\end{aligned}
$$

where $g_{m,n} = \mathrm{tr}(D_m^\top D_n) + \mathrm{tr}(E_m^\top E_n)$, $h_m = \mathrm{tr}(D_m^\top \tilde{A}_{11}) + \mathrm{tr}(E_m^\top \tilde{B}_{11})$, $e = \mathrm{tr}(\tilde{A}_{11}^\top \tilde{A}_{11}) + \mathrm{tr}(\tilde{B}_{11}^\top \tilde{B}_{11})$, $m, n = 1, \cdots, p$.
Consequently,

$$
\begin{aligned}
\frac{\partial f(a_1,b_1,\cdots,a_{2l-1},b_{2l-1},c_{2l+1},\cdots,c_p)}{\partial a_1} &= 2a_1 g_{1,1} + 2b_1 g_{1,2} + \cdots + 2a_{2l-1} g_{1,2l-1} + 2b_{2l-1} g_{1,2l} + 2c_{2l+1} g_{1,2l+1} \\
&\quad + \cdots + 2c_p g_{1,p} - 2h_1, \\
\frac{\partial f(a_1,b_1,\cdots,a_{2l-1},b_{2l-1},c_{2l+1},\cdots,c_p)}{\partial b_1} &= 2a_1 g_{2,1} + 2b_1 g_{2,2} + \cdots + 2a_{2l-1} g_{2,2l-1} + 2b_{2l-1} g_{2,2l} + 2c_{2l+1} g_{2,2l+1} \\
&\quad + \cdots + 2c_p g_{2,p} - 2h_2, \\
\cdots, \cdots, & \\
\frac{\partial f(a_1,b_1,\cdots,a_{2l-1},b_{2l-1},c_{2l+1},\cdots,c_p)}{\partial a_{2l-1}} &= 2a_1 g_{2l-1,1} + 2b_1 g_{2l-1,2} + \cdots + 2a_{2l-1} g_{2l-1,2l-1} + 2b_{2l-1} g_{2l-1,2l} \\
&\quad + 2c_{2l+1} g_{2l-1,2l+1} + \cdots + 2c_p g_{2l-1,p} - 2h_{2l-1}, \\
\frac{\partial f(a_1,b_1,\cdots,a_{2l-1},b_{2l-1},c_{2l+1},\cdots,c_p)}{\partial b_{2l-1}} &= 2a_1 g_{2l,1} + 2b_1 g_{2l,2} + \cdots + 2a_{2l-1} g_{2l,2l-1} + 2b_{2l-1} g_{2l,2l} + 2c_{2l+1} g_{2l,2l+1} \\
&\quad + \cdots + 2c_p g_{2l,p} - 2h_{2l}, \\
\frac{\partial f(a_1,b_1,\cdots,a_{2l-1},b_{2l-1},c_{2l+1},\cdots,c_p)}{\partial c_{2l+1}} &= 2a_1 g_{2l+1,1} + 2b_1 g_{2l+1,2} + \cdots + 2a_{2l-1} g_{2l+1,2l-1} + 2b_{2l-1} g_{2l+1,2l} \\
&\quad + 2c_{2l+1} g_{2l+1,2l+1} + \cdots + 2c_p g_{2l+1,p} - 2h_{2l+1}, \\
\cdots, \cdots, & \\
\frac{\partial f(a_1,b_1,\cdots,a_{2l-1},b_{2l-1},c_{2l+1},\cdots,c_p)}{\partial c_p} &= 2a_1 g_{p,1} + 2b_1 g_{p,2} + \cdots + 2a_{2l-1} g_{p,2l-1} + 2b_{2l-1} g_{p,2l} + 2c_{2l+1} g_{p,2l+1} \\
&\quad + \cdots + 2c_p g_{p,p} - 2h_p.
\end{aligned}
$$

Clearly, $f(a_1, b_1, \cdots, a_{2l-1}, b_{2l-1}, c_{2l+1}, \cdots, c_p) = \min$ if and only if

$$
\frac{\partial f(a_1, b_1, \cdots, a_{2l-1}, b_{2l-1}, c_{2l+1}, \cdots, c_p)}{\partial a_1} = 0, \cdots, \frac{\partial f(a_1, b_1, \cdots, a_{2l-1}, b_{2l-1}, c_{2l+1}, \cdots, c_p)}{\partial c_p} = 0.
$$

Therefore,

$$\begin{aligned}
&a_1g_{1,1}+b_1g_{1,2}+\cdots+a_{2l-1}g_{1,2l-1}+b_{2l-1}g_{1,2l}+c_{2l+1}g_{1,2l+1}+\cdots+c_pg_{1,p}=h_1,\\
&a_1g_{2,1}+b_1g_{2,2}+\cdots+a_{2l-1}g_{2,2l-1}+b_{2l-1}g_{2,2l}+c_{2l+1}g_{2,2l+1}+\cdots+c_pg_{2,p}=h_2,\\
&\cdots,\cdots,\\
&a_1g_{2l-1,1}+b_1g_{2l-1,2}+\cdots+a_{2l-1}g_{2l-1,2l-1}+b_{2l-1}g_{2l-1,2l}+c_{2l+1}g_{2l-1,2l+1}+\cdots+c_pg_{2l-1,p}=h_{2l-1},\\
&a_1g_{2l,1}+b_1g_{2l,2}+\cdots+a_{2l-1}g_{2l,2l-1}+b_{2l-1}g_{2l,2l}+c_{2l+1}g_{2l,2l+1}+\cdots+c_pg_{2l,p}=h_{2l},\\
&a_1g_{2l+1,1}+b_1g_{2l+1,2}+\cdots+a_{2l-1}g_{2l+1,2l-1}+b_{2l-1}g_{2l+1,2l}+c_{2l+1}g_{2l+1,2l+1}+\cdots+c_pg_{2l+1,p}=h_{2l+1},\\
&\cdots,\cdots,\\
&a_1g_{p,1}+b_1g_{p,2}+\cdots+a_{2l-1}g_{p,2l-1}+b_{2l-1}g_{p,2l}+c_{2l+1}g_{p,2l+1}+\cdots+c_pg_{p,p}=h_p.
\end{aligned}\tag{3.8}$$

If let

$$G=\begin{bmatrix}
g_{1,1} & g_{1,2} & \cdots & g_{1,2l-1} & g_{1,2l} & g_{1,2l+1} & \cdots & g_{1,p}\\
g_{2,1} & g_{2,2} & \cdots & g_{2,2l-1} & g_{2,2l} & g_{2,2l+1} & \cdots & g_{2,p}\\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\
g_{2l-1,1} & g_{2l-1,2} & \cdots & g_{2l-1,2l-1} & g_{2l-1,2l} & g_{2l-1,2l+1} & \cdots & g_{2l-1,p}\\
g_{2l,1} & g_{2l,2} & \cdots & g_{2l,2l-1} & g_{2l,2l} & g_{2l,2l+1} & \cdots & g_{2l,p}\\
g_{2l+1,1} & g_{2l+1,2} & \cdots & g_{2l+1,2l-1} & g_{2l+1,2l} & g_{2l+1,2l+1} & \cdots & g_{2l+1,p}\\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\
g_{p,1} & g_{p,2} & \cdots & g_{p,2l-1} & g_{p,2l} & g_{p,2l+1} & \cdots & g_{p,p}
\end{bmatrix},$$

$$T=\begin{bmatrix} a_1\\ b_1\\ \vdots\\ a_{2l-1}\\ b_{2l-1}\\ c_{2l+1}\\ \vdots\\ c_p\end{bmatrix},\ H=\begin{bmatrix} h_1\\ h_2\\ \vdots\\ h_{2l-1}\\ h_{2l}\\ h_{2l+1}\\ \vdots\\ h_p\end{bmatrix},$$

where $G$ is symmetric matrix. Then Eq (3.8) is equivalent to

$$GT=H,\tag{3.9}$$

and the solution of Eq (3.9) is

$$T=G^{-1}H.\tag{3.10}$$

Substituting (3.10) into (2.13), we can obtain $C$ explicitly. Similarly, Eq (3.3) is equivalent to

$$\begin{aligned}
f(A_{12}) &= tr[(A_{12}^\top-\tilde{A}_{12}^\top)(A_{12}-\tilde{A}_{12})]+tr[(A_{12}^\top P-\tilde{B}_{12}^\top)(P^\top A_{12}-\tilde{B}_{12})]\\
&\quad+tr[(A_{12}-\tilde{A}_{21}^\top)(A_{12}^\top-\tilde{A}_{21})]+tr[(P^\top A_{12}-\tilde{B}_{21}^\top)(A_{12}^\top P-\tilde{B}_{21})].
\end{aligned}$$

Thus,

$$\frac{\partial f(A_{12})}{\partial A_{12}}=2A_{12}-2\tilde{A}_{12}+2PP^\top A_{12}-2P\tilde{B}_{12}+2A_{12}-2\tilde{A}_{21}^\top+2PP^\top A_{12}-2P\tilde{B}_{21}^\top,$$

setting $\frac{\partial f(A_{12})}{\partial A_{12}} = 0$, we obtain

$$A_{12} = \frac{1}{2}(I_p + PP^\top)^{-1}(\tilde{A}_{12} + P\tilde{B}_{12} + \tilde{A}_{21}^\top + P\tilde{B}_{21}^\top), \tag{3.11}$$

where $P = R\tilde{\Lambda}R^{-1}$. $A_{22}$, $B_{22}$ are symmetric matrices implies that the relations of (3.4) and (3.5) are equivalent to

$$\|A_{22} - \tilde{A}_{22}\|^2 = \left\|A_{22} - \frac{1}{2}(\tilde{A}_{22} + \tilde{A}_{22}^\top)\right\|^2 + \left\|\frac{1}{2}(\tilde{A}_{22} - \tilde{A}_{22}^\top)\right\|^2,$$

$$\|B_{22} - \tilde{B}_{22}\|^2 = \left\|B_{22} - \frac{1}{2}(\tilde{B}_{22} + \tilde{B}_{22}^\top)\right\|^2 + \left\|\frac{1}{2}(\tilde{B}_{22} - \tilde{B}_{22}^\top)\right\|^2,$$

therefore, we have

$$A_{22} = \frac{1}{2}(\tilde{A}_{22} + \tilde{A}_{22}^\top), \ B_{22} = \frac{1}{2}(\tilde{B}_{22} + \tilde{B}_{22}^\top). \tag{3.12}$$

**Theorem 3.1.** *Given $\tilde{A}, \tilde{B} \in \mathbb{R}^{2n\times 2n}$, then the Problem OAP has a unique solution and the unique solution of Problem OAP is*

$$\hat{A} = Q\begin{bmatrix} R^{-T}CR^{-1} & A_{12} \\ A_{12}^\top & A_{22} \end{bmatrix} Q^\top J^\top, \ \hat{B} = Q\begin{bmatrix} R^{-T}C\tilde{\Lambda}R^{-1} & R^{-T}\tilde{\Lambda}^\top R^\top A_{12} \\ A_{12}^\top R\tilde{\Lambda}R^{-1} & B_{22} \end{bmatrix} Q^\top J^\top, \tag{3.13}$$

*where $A_{12}$ and $A_{22}$, $B_{22}$ are given by* (3.11) *and* (3.12), *and $a_1, b_1, \cdots, a_{2l-1}, b_{2l-1}, c_{2l+1}, \cdots, c_p$ are given by* (3.10)*, respectively.*

## 4. Numerical examples

According to Theorems 2.1 and 3.1, we have the following algorithm for solving Problem OAP.

**Algorithm 4.1.**

1). Input $\Lambda$, $X$, $J$, $\tilde{A}$, $\tilde{B}$.

2). Compute real-valued matrices $\tilde{\Lambda}$, $\tilde{X}$ by (2.1) and (2.2), respectively.

3). Compute the QR-decomposition of the matrix $J^\top \tilde{X}$ by (2.5).

4). Compute $\tilde{A}_{ij}$, $\tilde{B}_{ij}$ by (3.1), $i, j = 1, 2$.

5). Compute $R^{-1}$ by (3.6) to form $R_1, R_2$.

6). Compute $D_{2i-1}, D_{2i}, D_j, E_{2i-1}, E_{2i}$ and $E_j$ ($i = 1, \cdots, l, j = 2l+1, \cdots, p$) by (3.7).

7). Compute $g_{m,n} = \mathrm{tr}(D_m^\top D_n) + \mathrm{tr}(E_m^\top E_n)$ and $h_m = \mathrm{tr}(D_m^\top \tilde{A}_{11}) + \mathrm{tr}(E_m^\top \tilde{B}_{11})$ $(m, n = 1, \cdots, p)$.

8). Compute $a_1, b_1, \cdots, a_{2l-1}, b_{2l-1}, c_{2l+1}, \cdots, c_p$ by (3.10).

9). Compute $A_{12}$ by (3.11).

10). Compute $A_{22}$ and $B_{22}$ by (3.12).

11). Compute $\hat{A}$ and $\hat{B}$ by (3.13).

**Remark 4.1.** After statistics, we find that the amount of computations required by Algorithm 1 is about $p^5 + p^4 + \frac{19}{3}p^3 + 14np^2 + 64n^3$ flops.

**Example 4.1.** Let $n = 5,\ p = 5$, and the matrices $\Lambda,\ X,\ \tilde{A}$ and $\tilde{B}$ be given by

$$\Lambda = \mathrm{diag}\{-0.2218 + 2.0231i,\ -0.2218 - 2.0231i,\ -0.1617 + 0.5721i,\ -0.1617 - 0.5721i,\ 2.7670\},$$

$$X = \begin{bmatrix} -0.6377 - 0.1444i & -0.6377 + 0.1444i & -0.0405 + 0.3341i & -0.0405 - 0.3341i & -1.0000 \\ 0.2678 - 0.0983i & 0.2678 + 0.0983i & -0.2615 - 0.3973i & -0.2615 + 0.3973i & 0.1151 \\ 0.4260 + 0.5740i & 0.4260 - 0.5740i & 0.1348 - 0.2946i & 0.1348 + 0.2946i & 0.0278 \\ -0.2032 - 0.0489i & -0.2032 + 0.0489i & -0.5238 + 0.4762i & -0.5238 - 0.4762i & 0.0357 \\ 0.2111 - 0.1510i & 0.2111 + 0.1510i & 0.7982 - 0.1668i & 0.7982 + 0.1668i & 0.6793 \\ -0.5233 + 0.1151i & -0.5233 - 0.1151i & -0.3033 - 0.4132i & -0.3033 + 0.4132i & -0.1091 \\ 0.4820 - 0.2541i & 0.4820 + 0.2541i & 0.1398 - 0.0715i & 0.1398 + 0.0715i & 0.8550 \\ 0.3183 - 0.3431i & 0.3183 + 0.3431i & -0.2716 - 0.3411i & -0.2716 + 0.3411i & 0.6436 \\ 0.1376 + 0.3007i & 0.1376 - 0.3007i & -0.1391 + 0.0981i & -0.1391 - 0.0981i & 0.0883 \\ -0.0672 + 0.0189i & -0.0672 - 0.0189i & 0.3143 + 0.1462i & 0.3143 - 0.1462i & -0.2828 \end{bmatrix},$$

$$\tilde{A} = \begin{bmatrix} 1.8351 & 3.0635 & 9.3900 & 1.9476 & 9.7975 & 1.1742 & 7.3033 & 6.2406 & 2.6187 & 9.0372 \\ 3.6848 & 5.0851 & 8.7594 & 2.2592 & 4.3887 & 2.9668 & 4.8861 & 6.7914 & 3.3536 & 8.9092 \\ 6.2562 & 5.1077 & 5.5016 & 1.7071 & 1.1112 & 3.1878 & 5.7853 & 3.9552 & 6.7973 & 3.3416 \\ 7.8023 & 8.1763 & 6.2248 & 2.2766 & 2.5806 & 4.2417 & 2.3728 & 3.6744 & 1.3655 & 6.9875 \\ 0.8113 & 7.9483 & 5.8704 & 4.3570 & 4.0872 & 5.0786 & 4.5885 & 9.8798 & 7.2123 & 1.9781 \\ 9.2939 & 6.4432 & 2.0774 & 3.1110 & 5.9490 & 0.8552 & 9.6309 & 0.3774 & 1.0676 & 0.3054 \\ 7.7571 & 3.7861 & 3.0125 & 9.2338 & 2.6221 & 2.6248 & 5.4681 & 8.8517 & 6.5376 & 7.4407 \\ 4.8679 & 8.1158 & 4.7092 & 4.3021 & 6.0284 & 8.0101 & 5.2114 & 9.1329 & 4.9417 & 5.0002 \\ 4.3586 & 5.3283 & 2.3049 & 1.8482 & 7.1122 & 0.2922 & 2.3159 & 7.9618 & 7.7905 & 4.7992 \\ 4.4678 & 3.5073 & 8.4431 & 9.0488 & 2.2175 & 9.2885 & 4.8890 & 0.9871 & 7.1504 & 9.0472 \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} 6.0987 & 1.6793 & 0.9673 & 4.5380 & 3.9926 & 1.0622 & 4.2284 & 6.6653 & 3.6892 & 1.2061 \\ 6.1767 & 9.7868 & 8.1815 & 4.3239 & 5.2688 & 3.7241 & 5.4787 & 1.7813 & 4.6073 & 5.8951 \\ 8.5944 & 7.1269 & 8.1755 & 8.2531 & 4.1680 & 1.9812 & 9.4274 & 1.2801 & 9.8164 & 2.2619 \\ 8.0549 & 5.0047 & 7.2244 & 0.8347 & 6.5686 & 4.8969 & 4.1774 & 9.9908 & 1.5640 & 3.8462 \\ 5.7672 & 4.7109 & 1.4987 & 1.3317 & 6.2797 & 3.3949 & 9.8305 & 1.7112 & 8.5552 & 5.8299 \\ 1.8292 & 0.5962 & 6.5961 & 1.7339 & 2.9198 & 9.5163 & 3.0145 & 0.3260 & 6.4476 & 2.5181 \\ 2.3993 & 6.8197 & 5.1859 & 3.9094 & 4.3165 & 9.2033 & 7.0110 & 5.6120 & 3.7627 & 2.9044 \\ 8.8651 & 0.4243 & 9.7297 & 8.3138 & 0.1549 & 0.5268 & 6.6634 & 8.8187 & 1.9092 & 6.1709 \\ 0.2867 & 0.7145 & 6.4899 & 8.0336 & 9.8406 & 7.3786 & 5.3913 & 6.6918 & 4.2825 & 2.6528 \\ 4.8990 & 5.2165 & 8.0033 & 0.6047 & 1.6717 & 2.6912 & 6.9811 & 1.9043 & 4.8202 & 8.2438 \end{bmatrix}.$$

By applying Algorithm 4.1, we can obtain the unique solution $(\hat{A}, \hat{B})$ of Problem OAP as follows:

$$\hat{A} = \begin{bmatrix} 3.0180 & -0.4225 & -0.1555 & 0.8402 & 0.9125 & 4.0071 & 3.8840 & 3.2121 & 4.0178 & 6.8173 \\ -0.3558 & -1.7726 & -0.3149 & 2.4300 & -2.4421 & 3.8840 & 1.9734 & 5.0988 & 2.4334 & 6.2185 \\ 1.8766 & -3.0189 & -0.9503 & -1.1732 & -1.8790 & 3.2121 & 5.0988 & 2.6896 & 6.6750 & 5.4058 \\ 1.9510 & -0.1294 & -0.2763 & -0.7668 & -1.9008 & 4.0178 & 2.4334 & 6.6750 & 2.9195 & 5.2316 \\ 3.0211 & 0.2273 & -0.6075 & -0.9217 & -2.7818 & 6.8173 & 6.2185 & 5.4058 & 5.2316 & 3.9665 \\ 6.3731 & 6.9288 & 5.5322 & 4.4397 & 5.4356 & -3.0180 & 0.3558 & -1.8766 & -1.9510 & -3.0211 \\ 6.9288 & 4.5231 & 6.2946 & 7.2049 & 3.6851 & 0.4225 & 1.7726 & 3.0189 & 0.1294 & -0.2273 \\ 5.5322 & 6.2946 & 4.8453 & 3.7706 & 4.4618 & 0.1555 & 0.3149 & 0.9503 & 0.2763 & 0.6075 \\ 4.4397 & 7.2049 & 3.7706 & 2.8249 & 6.6239 & -0.8402 & -2.4300 & 1.1732 & 0.7668 & 0.9217 \\ 5.4356 & 3.6851 & 4.4618 & 6.6239 & 2.8468 & -0.9125 & 2.4421 & 1.8790 & 1.9008 & 2.7818 \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} 0.6709 & -1.2381 & -0.7557 & -1.1218 & -1.9835 & -1.1000 & 3.7817 & 3.8263 & 0.6376 & 3.5484 \\ 1.3451 & 0.2456 & -0.0257 & -0.2862 & 0.2915 & 3.7817 & 5.3559 & 5.3224 & 3.6947 & 8.6240 \\ 3.0095 & 1.2171 & -1.0933 & 0.9515 & 2.1292 & 3.8263 & 5.3224 & 1.2147 & 6.8013 & 2.4324 \\ 2.0994 & -1.8754 & 3.1477 & -0.1903 & 0.6793 & 0.6376 & 3.6947 & 6.8013 & 3.3691 & 7.2508 \\ 1.4047 & 0.3032 & -1.6547 & -2.5697 & 1.2202 & 3.5484 & 8.6240 & 2.4324 & 7.2508 & 6.0558 \\ 2.7999 & 2.6538 & 5.9368 & 4.0530 & 1.3808 & -0.6709 & -1.3451 & -3.0095 & -2.0994 & -1.4047 \\ 2.6538 & 5.3390 & 2.3523 & 2.6326 & 4.1208 & 1.2381 & -0.2456 & -1.2171 & 1.8754 & -0.3032 \\ 5.9368 & 2.3523 & 8.4193 & 6.9480 & 4.5598 & 0.7557 & 0.0257 & 1.0933 & -3.1477 & 1.6547 \\ 4.0530 & 2.6326 & 6.9480 & 8.0838 & 5.2373 & 1.1218 & 0.2862 & -0.9515 & 0.1903 & 2.5697 \\ 1.3808 & 4.1208 & 4.5598 & 5.2373 & 3.4028 & 1.9835 & -0.2915 & -2.1292 & -0.6793 & -1.2202 \end{bmatrix},$$

and

$$\|\hat{A}X\Lambda - \hat{B}X\| = 2.0806 \times 10^{-14},$$

which implies that $\hat{A}X\Lambda = \hat{B}X$ reproduces the desired eigenvalues and eigenvectors.

**Example 4.2.** We consider an inverse problem for the spectral conformal parameterization (see Refs. [6,7]). Let $n = 5,\ p = 4$, and the matrices $\Lambda,\ X,\ \tilde{B}$ and $\tilde{L}_C$ be given by

$$\Lambda = \mathrm{diag}\{-0.0822,\ -0.0250,\ 0,\ 0.0757\} \triangleq \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\},$$

$$X = \begin{bmatrix} 0.5760 & 0.2684 & 0.0000 & 0.1340 \\ -0.1510 & -0.0673 & 0.0000 & -1.0000 \\ -0.3890 & -0.0610 & 0.0000 & -0.1246 \\ -0.0359 & -0.1401 & 0.0000 & 0.9907 \\ 0.3409 & 1.0000 & 1.0000 & 0.0366 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.5672 & 0.0665 & 0.0000 & 0.3072 \\ 1.0000 & -0.1288 & 0.0000 & -0.1395 \\ -0.4328 & 0.0623 & 0.0000 & -0.1677 \\ 0.0650 & 0.2282 & 0.2000 & -0.1508 \end{bmatrix} \triangleq \mathrm{diag}\{f_1,\ f_2,\ f_3,\ f_4\},$$

$$
\tilde{B}=\begin{bmatrix}
0.7000 & 0.5483 & 0.5369 & 0.6919 & 0.6032 & 0 & -0.8083 & 0.0306 & -0.7715 & 0.0234\\
0.5483 & 1.6617 & 1.3425 & 0.6037 & 1.4860 & 0.8083 & 0 & 0.4237 & -0.5357 & -0.9292\\
0.5369 & 1.3425 & 1.5075 & 0.9112 & 1.0372 & -0.0306 & -0.4237 & 0 & 0.4304 & 0.0488\\
0.6919 & 0.6037 & 0.9112 & 1.5583 & 0.9459 & 0.7715 & 0.5357 & -0.4304 & 0 & -0.0255\\
0.6032 & 1.4860 & 1.0372 & 0.9459 & 0.6742 & -0.0234 & 0.9292 & -0.0488 & 0.0255 & 0\\
0 & 0.8083 & -0.0306 & 0.7715 & -0.0234 & 0.7000 & 0.5483 & 0.5369 & 0.6919 & 0.6032\\
-0.8083 & 0 & -0.4237 & 0.5357 & 0.9292 & 0.5483 & 1.6617 & 1.3425 & 0.6037 & 1.4860\\
0.0306 & 0.4237 & 0 & -0.4304 & -0.0488 & 0.5369 & 1.3425 & 1.5075 & 0.9112 & 1.0372\\
-0.7715 & -0.5357 & 0.4304 & 0 & 0.0255 & 0.6919 & 0.6037 & 0.9112 & 1.5583 & 0.9459\\
0.0234 & -0.9292 & 0.0488 & -0.0255 & 0 & 0.6032 & 1.4860 & 1.0372 & 0.9459 & 0.6742
\end{bmatrix},
$$

$$
\tilde{L}_C=\begin{bmatrix}
1.7000 & 2.5483 & 3.5369 & 4.6919 & -9.3968 & 0 & -1.6166 & 0.0612 & -1.5430 & 0.0468\\
2.5483 & 4.6617 & 5.3425 & -9.3963 & 2.4860 & 1.6166 & 0 & 0.8474 & -1.0714 & -1.8584\\
3.5369 & 5.3425 & -8.4925 & 1.9112 & 3.0372 & -0.0612 & -0.8474 & 0 & 0.8608 & 0.0976\\
4.6919 & -9.3963 & 1.9112 & 3.5583 & 3.9459 & 1.5430 & 1.0714 & -0.8608 & 0 & -0.0510\\
-9.3968 & 2.4860 & 3.0372 & 3.9459 & 4.6742 & -0.0468 & 1.8584 & -0.0976 & 0.0510 & 0\\
0 & 1.6166 & -0.0612 & 1.5430 & -0.0468 & 1.7000 & 2.5483 & 3.5369 & 4.6919 & -9.3968\\
-1.6166 & 0 & -0.8474 & 1.0714 & 1.8584 & 2.5483 & 4.6617 & 5.3425 & -9.3963 & 2.4860\\
0.0612 & 0.8474 & 0 & -0.8608 & -0.0976 & 3.5369 & 5.3425 & -8.4925 & 1.9112 & 3.0372\\
-1.5430 & -1.0714 & 0.8608 & 0 & 0.0510 & 4.6919 & -9.3963 & 1.9112 & 3.5583 & 3.9459\\
0.0468 & -1.8584 & 0.0976 & -0.0510 & 0 & -9.3968 & 2.4860 & 3.0372 & 3.9459 & 4.6742
\end{bmatrix}.
$$

By calculating, we can obtain the unique solution $(\hat{B},\hat{L}_C)$ of Problem OAP as follows:

$$
\hat{B}=\begin{bmatrix}
1.0866 & -0.0847 & -0.0520 & 0.0501 & 0.1093 & 0.0000 & -0.0330 & 0.0833 & -0.0503 & 0.0134\\
-0.1591 & 1.7380 & -0.2522 & -0.3467 & -0.2414 & 0.0330 & -0.0000 & -0.0191 & 0.0087 & 0.0234\\
-0.1345 & -0.2315 & 1.6597 & -0.2656 & -0.2222 & -0.0833 & 0.0191 & 0.0000 & 0.0206 & -0.0230\\
-0.0049 & -0.3583 & -0.2945 & 1.6496 & -0.3472 & 0.0503 & -0.0087 & -0.0206 & -0.0000 & -0.1716\\
0.2724 & -0.0519 & -0.1132 & -0.1159 & 0.8731 & -0.0134 & -0.0234 & 0.0230 & 0.1716 & 0.0000\\
0.0000 & 0.0864 & -0.2014 & 0.1221 & -0.0239 & 1.0866 & -0.1591 & -0.1345 & -0.0049 & 0.2724\\
-0.0864 & -0.0000 & 0.0415 & 0.0244 & 0.1288 & -0.0847 & 1.7380 & -0.2315 & -0.3583 & -0.0519\\
0.2014 & -0.0415 & 0.0000 & -0.1586 & -0.1553 & -0.0520 & -0.2522 & 1.6597 & -0.2945 & -0.1132\\
-0.1221 & -0.0244 & 0.1586 & -0.0000 & -0.0000 & 0.0501 & -0.3467 & -0.2656 & 1.6496 & -0.1159\\
0.0239 & -0.1288 & 0.1553 & 0.0000 & 0.0000 & 0.1093 & -0.2414 & -0.2222 & -0.3472 & 0.8731
\end{bmatrix},
$$

$$
\hat{L}_C=\begin{bmatrix}
3.0749 & 3.0182 & 2.9855 & 3.3985 & -0.0221 & -0.0000 & -0.5788 & 0.0130 & 1.0166 & 0.1107\\
0.8672 & 0.7497 & 1.1429 & 0.2011 & 0.4437 & 0.5788 & 0.0000 & -0.0656 & -0.3528 & -2.2185\\
0.7590 & 0.8360 & 0.5193 & 0.6574 & 0.1300 & -0.0130 & 0.0656 & 0.0000 & -0.0313 & -0.6502\\
0.1065 & 0.3474 & -0.4752 & 0.5081 & -0.2263 & -1.0166 & 0.3528 & 0.0313 & 0.0000 & 1.1314\\
-0.1139 & 0.6946 & -0.6294 & -0.1520 & 0.0000 & -0.1107 & 2.2185 & 0.6502 & -1.1314 & 0.0000\\
0.0000 & -0.3435 & 0.7467 & -0.4158 & 0.0228 & 3.0749 & 0.8672 & 0.7590 & 0.1065 & -0.1139\\
0.3435 & 0.0000 & 0.9259 & 0.1743 & -0.1389 & 3.0182 & 0.7497 & 0.8360 & 0.3474 & 0.6946\\
-0.7467 & -0.9259 & -0.0000 & -1.3216 & 0.1259 & 2.9855 & 1.1429 & 0.5193 & -0.4752 & -0.6294\\
0.4158 & -0.1743 & 1.3216 & -0.0000 & 0.0304 & 3.3985 & 0.2011 & 0.6574 & 0.5081 & -0.1520\\
-0.0228 & 0.1389 & -0.1259 & -0.0304 & -0.0000 & -0.0221 & 0.4437 & 0.1300 & -0.2263 & 0.0000
\end{bmatrix}.
$$

Furthermore, we can obtain the following numerical results:

**Table 4.1.** Residuals of the eigenpairs $(\lambda_i, f_i)$.

| $(\lambda_i, f_i)$ | $(\lambda_1, f_1)$ | $(\lambda_2, f_2)$ | $(\lambda_3, f_3)$ | $(\lambda_4, f_4)$ |
|---|---|---|---|---|
| $\|\lambda_i \hat{B} f_i - \hat{L}_C f_i\|$ | $1.1762 \times 10^{-15}$ | $1.1322 \times 10^{-15}$ | $1.0372 \times 10^{-15}$ | $9.7141 \times 10^{-16}$ |

Therefore, the new model $\hat{B}X\Lambda = \hat{L}_C X$ reproduces the desired eigenvalues and eigenvectors.

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## Conflict of interest

The authors declare no conflict of interest.

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