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Research article

Embedding theorems for variable exponent fractional Sobolev spaces and an application

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Abstract: In this paper, we mainly discuss the embedding theory of variable exponent fractional Sobolev space $W^{s(\cdot),p(\cdot)}(\Omega)$, and apply this theory to study the $s(x)$-$p(x)$-Laplacian equation:

$$(-\Delta)^{s(\cdot)}_{p(\cdot)}u + V(x)|u|^{p(x)-2}u = f(x,u) + g(x)$$

where $x \in \Omega \subset \mathbb{R}^n$, $(-\Delta)^{s(\cdot)}_{p(\cdot)}$ is $s(x)$-$p(x)$-Laplacian operator with $0 < s(x) < 1 < p(x) < \infty$ and $p(x)s(x) < n$, the nonlinear term $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $V : \mathbb{R}^n \to \mathbb{R}$ is a potential function and $g : \mathbb{R}^n \to \mathbb{R}$ is a perturbation term.

Keywords: variable exponent; fractional Sobolev spaces; embedding; $s(x)$-$p(x)$-Laplacian equation
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1. Introduction

Variable exponent Lebesgue spaces were first studied by Orlicz in 1931 (see [33]). Since the 1990s, variable exponent Lebesgue spaces and variable exponent Sobolev spaces have been used in a variety of fields, the most important of which is the mathematical modeling of electrorheological fluids. In 1997, the variable exponent Lebesgue spaces were applied to the study of image processing: In image reconstruction, the variable exponent interpolation technique can be used to obtain a smoother image. For the theory and applications of variable exponent Lebesgue spaces and variable exponent Sobolev spaces, see [10, 12, 15, 21, 28] and the references therein.

As a part of the theory of variable exponent function spaces, variable exponent fractional Sobolev spacea are also developing vigorously. In [27], Kaufmann et al gave a class of variable exponent fractional Sobolev spaces:

$$W^{s,q(x),p(x,y)}(\Omega) := \left\{ u \in L^{q(x)}(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}} dxdy < \infty \text{ for some } \lambda > 0 \right\}, \tag{1.1}$$

where $s \in (0,1)$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, $q : \bar{\Omega} \to (1,\infty)$ and $p : \bar{\Omega} \times \bar{\Omega} \to (1,\infty)$ are two continuous functions bounded away from 1 and ∞. Assume further that p is symmetric, i.e. $p(x,y) = p(y,x)$.

Afterwards some scholars did further research on theory and applications of this kind of spaces (see [3, 5–7, 13, 25, 32] and the references therein). In [31], we considered the case that the index s is a function $s(x)$, $p(x,y)$ is $\frac{p(x)+p(y)}{2}$, $q(x)$ is $p(x)$, established the so called variable exponent fractional Sobolev spaces $W^{s(\cdot),p(\cdot)}(\Omega)$ and gave some basic properties and an application. In this paper, we will further study basic properties of this kind of spaces, for example: Embedding.

Embedding is always a classical topic in functional analysis, partial differential equations and other fields. The first task of this paper is to give embedding theorems for $W^{s(\cdot),p(\cdot)}(\Omega)$. Related to embedding theorems, we refer to [14, 18, 24, 35] and the references therein.

In recent years, mathematicians have made some achievements in the study of fractional partial differential equations with variable growth. In [7], Bahrouni and Rădulescu extended the classical fractional Laplacian to a class of fractional $p(x,y)$-Laplacian defined as

$$\mathcal{L}u(x) = P.V. \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x - y|^{n+sp(x,y)}} dy,$$

where $\Omega \subset \mathbb{R}^n$, $0 < s < 1$ and $p : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ is continuous satisfing

$$1 < p^- = \min_{(x,y)\in\bar{\Omega}\times\bar{\Omega}} p(x,y) \le p(x,y) \le p^+ = \max_{(x,y)\in\bar{\Omega}\times\bar{\Omega}} p(x,y) < \infty,$$

$$p((x,y) - (z,z)) = p(x,y), \quad (x,y),\ (z,z) \in \Omega \times \Omega.$$

Under certain conditions, they established the existence of solutions to the following problems by means of the Ekeland variational principle:

$$\begin{cases} \mathcal{L}u(x) + |u(x)|^{q(x)-1}u(x) = \lambda|u(x)|^{r(x)-1}u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

In [32] Nguyen further discussed the problem (1.2) to show the existence of the eigenvalues of the following fractional $p(x,y)$-Laplacian operator:

$$\begin{cases} \mathcal{L}u(x) + |u(x)|^{q(x)-2}u(x) = \lambda V(x)|u(x)|^{r(x)-2}u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

In [27], Kaufmann et al considered the existence and uniqueness of the solution of fractional $p(x,y)$-Laplacian equation as follows:

$$\begin{cases} \mathcal{L}u(x) + |u(x)|^{q(x)-2}u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

In [6], comparison and sub-supersolution principles for the fractional $p(x,y)$-Laplacian are given. In [4], Azroul et al studied the existence of nontrivial weak solutions for fractional $p(x,y)$-Kirchhoff type problems. In [3], the existence of eigenvalues of fractional $p(x,y)$-Laplacian is studied by means of Ekeland variational principle. These problems are considered under the condition that the exponent s is constant.

In [34], Xiang et al used the mountain pass theorem and Ekeland variational principle to study the elliptic problems of Laplacian with variable exponent s and constant pc under appropriate assumptions:

$$\begin{cases} (-\Delta)^{s(\cdot)}u + \lambda V(x)u = \alpha|u|^{p(x)-2}u + \beta|u|^{q(x)-2}u, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

where

$$(-\Delta)^{s(\cdot)}u(x) = 2P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s(x,y)}} dy.$$

It is proved that there are at least two different solutions to the above problems. Furthermore, the existence of infinite many solutions for the limit problems is obtained.

In [11], Cheng et al further studied the existence of weak solutions for nonlinear elliptic equations where the exponents s and p are of variable forms, i.e.

$$\begin{aligned} &(-\Delta)^{k(\cdot)}_{\alpha(\cdot)}u + \alpha|u|^{\bar{p}(x)-2}u = f(x)h(u), && x \in \Omega, \\ &u(x) = 0, && x \in \mathbb{R}^n \setminus \Omega. \end{aligned}$$

where the fractional $\alpha(\cdot)$-$k(\cdot)$-Laplacian $(-\Delta)^{k(\cdot)}_{\alpha(\cdot)}$ is defined by

$$(-\Delta)^{k(\cdot)}_{\alpha(\cdot)}u(x) = 2\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} |u(x) - u(y)|^{\alpha(x,y)-2} \frac{u(x) - u(y)}{|x-y|^{n+\alpha(x,y)k(x,y)}} dy, \quad x \in \mathbb{R}^n.$$

As we know that when people studied nonlinear problems of fractional Laplace operators with variable exponents, they mainly focus on the case that the exponent s is constant and p is variable. For the cases that the exponent s is variable and p is constant or both the exponents s and p are variables, there are still few results.

Under the quantum mechanics background, in [29, 30] Laskin expanded the Feynman way integrals from the kind of Braun quantum mechanics way to the kind of Lévy quantum mechanics way, proposed the nonlinear fractional Schrödinger equation. Subsequently, results on the fractional Schrödinger equation gradually appeared

$$(-\Delta)^s u + V(x)u = f(x,u), \quad x \in \Omega$$

where

$$(-\Delta)^s u := P.V. \int_\Omega \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy$$

and f satisfies some conditions, which are stated in details in [17, 22].

As a direct application of embedding theorems for $W^{s(\cdot),p(\cdot)}(\Omega)$, the second task of this paper is to study the existence of multiple solutions for Dirichlet boundary value problem of the $s(x)$-$p(x)$-Laplacian equations in $W^{s(\cdot),p(\cdot)}(\Omega)$:

$$\begin{cases} (-\Delta)^{s(\cdot)}_{p(\cdot)}u + V(x)|u|^{p(x)-2}u = f(x,u) + g(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where $0 < s(x) < 1 < p(x) < \infty$ with $p(x)s(x) < n$, $(-\Delta_{p(\cdot)})^{s(\cdot)}$ is the $s(x)$-$p(x)$-Laplacian operator defined as

$$(-\Delta)^{s(\cdot)}_{p(\cdot)}u(x) := \text{P.V.} \int_\Omega \frac{|u(x) - u(y)|^{\frac{p(x)+p(y)}{2}-2}(u(x) - u(y))}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} dy, \quad x \in \Omega.$$

When $p(x) = 2$ and $s(x) = s$(constant), Eq (1.5) becomes a fractional Laplacian equation

$$(-\varDelta)^s u + V(x)u = f(x,u) + g(x), \quad x \in \Omega.$$

This can be seen as fractional form of the following classic stationary Schrödinger equation

$$-\varDelta u + V(x)u = f(x,u) + g(x), \quad x \in \Omega.$$

Therefore, we think it is meaningful to study problem (1.5), and further, it is very necessary to study the application of $s(x)$-$p(x)$-Laplace equation in $W^{s(\cdot),p(\cdot)}(\Omega)$.

2. Preliminaries

First we provide some basic concepts and related notations. Suppose that Ω be a Lebesgue measurable subset of $\mathbb{R}^n$ with positive measure. Let $B_k(0), \bar{B}_k(0)$ denote the open and close ball centered at 0 with radius k, respectively. Let $\mathcal{P}(\Omega)$ denote the family of all Lebesgue measurable functions $p : \Omega \to [1, \infty]$ and $\mathcal{S}(\Omega)$ denote the family of all Lebesgue measurable functions $s : \Omega \to (0, 1)$. Denote

$$p^+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x), \quad p^- = \operatorname*{ess\,inf}_{x\in\Omega} p(x),$$
$$s^+ = \operatorname*{ess\,sup}_{x\in\Omega} s(x), \quad s^- = \operatorname*{ess\,inf}_{x\in\Omega} s(x).$$

For a Lebesgue measurable function $u : \Omega \to \mathbb{R}$, define

$$\rho_{p(\cdot),\Omega}(u) = \int_{\Omega\backslash\Omega_\infty} |u(x)|^{p(x)} dx + \|u\|_{L^\infty(\Omega_\infty)},$$

The space $W^{s(\cdot),\infty}(\Omega)$ is defined as the set of functions

$$\left\{ u \in L^\infty(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{s(x)+s(y)}{2}}} \in L^\infty(\Omega \times \Omega) \right\}.$$

When the exponent s is constant, it is the space $W^{s,\infty}(\Omega)$ mentioned in [1, 26]. The norm can be defined as

$$\|u\|_{W^{s(\cdot),\infty}(\Omega)} = \|u\|_{L^\infty(\Omega)} + |u|_{C^{0,s(\cdot)}(\Omega)},$$

where the Hölder semi-norm is defined by

$$|u|_{C^{0,s(\cdot)}(\Omega)} := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^{\frac{s(x)+s(y)}{2}}}.$$

Define

$$\varphi_{s(\cdot),p(\cdot),\Omega}(u) = \int_{\Omega\backslash\Omega_\infty} \int_{\Omega\backslash\Omega_\infty} \frac{|u(x) - u(y)|^{\frac{p(x)+p(y)}{2}}}{|x - y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}} dxdy + \|u\|_{W^{s(\cdot),\infty}(\Omega_\infty)},$$

where $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) := \left\{ u : \exists \lambda > 0, \text{ s.t. } \rho_{p(\cdot),\Omega}(\frac{u}{\lambda}) < \infty \right\}.$$

We define a norm, so called Luxembourg norm, for this space by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\Big\{\lambda > 0 : \rho_{p(\cdot),\Omega}(\frac{u}{\lambda}) < 1\Big\}.$$

The variable exponent fractional Sobolev space $W^{s(\cdot),p(\cdot)}(\Omega)$ is defined by

$$W^{s(\cdot),p(\cdot)}(\Omega) := \Big\{u \in L^{p(\cdot)}(\Omega) : \exists \lambda > 0, \text{ s.t. } \varphi_{s(\cdot),p(\cdot),\Omega}(\frac{u}{\lambda}) < \infty\Big\}.$$

Let

$$[u]_{W^{s(\cdot),p(\cdot)}(\Omega)} = \inf\Big\{\lambda > 0 : \varphi_{s(\cdot),p(\cdot),\Omega}(\frac{u}{\lambda}) < 1\Big\}$$

be the corresponding variable exponent Gagliardo semi-norm. The norm is equipped as

$$\|u\|_{W^{s(\cdot),p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + [u]_{W^{s(\cdot),p(\cdot)}(\Omega)}.$$

It is easy to verify that under this norm this space is a Banach space.

For the sake of convenience, we give some notations. For the variable exponent $p : \Omega\times\Omega \to [1, \infty]$ which is symmetric, i.e. $p(x, y) = p(y, x)$ on $\Omega \times \Omega$, denote

$$\bar{p}^+ = \operatorname*{ess\,sup}_{(x,y)\in\Omega\times\Omega} p(x, y), \quad \bar{p}^- = \operatorname*{ess\,inf}_{(x,y)\in\Omega\times\Omega} p(x, y),$$

$$(\Omega \times \Omega)_\infty = \{(x, y) \in \Omega \times \Omega : p(x, y) = \infty\}.$$

In view of $\rho_{p(\cdot)}$ and $L^{p(\cdot)}(\Omega)$, we can define modular $\bar{\rho}_{p(\cdot,\cdot)}$ and variable exponent Lebesgue spaces $L^{p(\cdot,\cdot)}$ on $\Omega \times \Omega$. The conclusions on $L^{p(\cdot)}(\Omega)$ can be moved to $L^{p(\cdot,\cdot)}(\Omega \times \Omega)$. Here we give another modular and norm in $W^{s(\cdot),p(\cdot)}(\Omega)$. In this case, we only consider the case of $p^+ < \infty$. Modular is defined as:

$$\hat{\rho}_{s(\cdot),p(\cdot),\Omega}(u) = \int_\Omega\int_\Omega \frac{|u(x) - u(y)|^{\frac{p(x)+p(y)}{2}}}{|x - y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}}dxdy + \int_\Omega |u(x)|^{p(x)}dx.$$

According to this modular, we define the norm as:

$$|||u|||_{W^{s(\cdot),p(\cdot)}(\Omega)} = \inf\Big\{\lambda > 0 : \hat{\rho}_{s(\cdot),p(\cdot),\Omega}(\frac{u}{\lambda}) < 1\Big\}.$$

The following conclusions are what we will use later.

Proposition 2.1. *Let $p(\cdot) \in \mathcal{P}(\Omega)$ with $p^+ < \infty$. Then $|||u|||_{W^{s(\cdot),p(\cdot)}(\Omega)}$ is equivalent to $\|u\|_{W^{s(\cdot),p(\cdot)}(\Omega)}$, i.e.*

$$\frac{1}{2}\|u\|_{W^{s(\cdot),p(\cdot)}(\Omega)} \le |||u|||_{W^{s(\cdot),p(\cdot)}(\Omega)} \le \frac{1}{2^{\frac{1}{p^+}}}\|u\|_{W^{s(\cdot),p(\cdot)}(\Omega)}.$$

Proof. By the definition of $\hat{\rho}_{s(\cdot),p(\cdot),\Omega}$, $\rho_{p(\cdot),\Omega}$, $\varphi_{s(\cdot),p(\cdot),\Omega}$, we have

$$\rho_{p(\cdot),\Omega}\Big(\frac{u}{|||u|||_{W^{s(\cdot),p(\cdot)}(\Omega)}}\Big) \le \hat{\rho}_{s(\cdot),p(\cdot),\Omega}\Big(\frac{u}{|||u|||_{W^{s(\cdot),p(\cdot)}(\Omega)}}\Big) \le 1,$$

$$\varphi_{s(\cdot),p(\cdot),\Omega}\Big(\frac{u}{|||u|||_{W^{s(\cdot),p(\cdot)}(\Omega)}}\Big) \le \hat{\rho}_{s(\cdot),p(\cdot),\Omega}\Big(\frac{u}{|||u|||_{W^{s(\cdot),p(\cdot)}(\Omega)}}\Big) \le 1,$$

so

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq |||u|||_{W^{s(\cdot),p(\cdot)}(\Omega)}, \quad [u]_{W^{s(\cdot),p(\cdot)}(\Omega)} \leq |||u|||_{W^{s(\cdot),p(\cdot)}(\Omega)},$$

and further

$$\frac{1}{2}\|u\|_{W^{s(\cdot),p(\cdot)}(\Omega)} \leq |||u|||_{W^{s(\cdot),p(\cdot)}(\Omega)}.$$

On the other hand,

$$\rho_{p(\cdot),\Omega}\Big(\frac{2^{\frac{1}{p^+}}u}{\|u\|_{W^{s(\cdot),p(\cdot)}(\Omega)}}\Big) \leq \rho_{p(\cdot),\Omega}\Big(\frac{2^{\frac{1}{p^+}}u}{\|u\|_{L^{p(\cdot)}(\Omega)}}\Big) \leq \frac{1}{2},$$
$$\varphi_{s(\cdot),p(\cdot),\Omega}\Big(\frac{2^{\frac{1}{p^+}}u}{\|u\|_{W^{s(\cdot),p(\cdot)}(\Omega)}}\Big) \leq \varphi_{s(\cdot),p(\cdot),\Omega}\Big(\frac{2^{\frac{1}{p^+}}u}{[u]_{W^{s(\cdot),p(\cdot)}(\Omega)}}\Big) \leq \frac{1}{2},$$

so by the definition of $|||u|||_{W^{s(\cdot),p(\cdot)}(\Omega)}$,

$$|||u|||_{W^{s(\cdot),p(\cdot)}(\Omega)} \leq \frac{1}{2^{\frac{1}{p^+}}}\|u\|_{W^{s(\cdot),p(\cdot)}(\Omega)}.$$

The equivalence between $|||u|||_{W^{s(\cdot),p(\cdot)}(\Omega)}$ and $\|u\|_{W^{s(\cdot),p(\cdot)}(\Omega)}$ is proved. □

Just like the relationship between norm $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ and module $\rho_{p(\cdot),\Omega}(\cdot)$ in $L^{p(\cdot)}(\Omega)$ space (see [12, 15, 21]), norm $|||u|||_{W^{s(\cdot),p(\cdot)}(\Omega)}$ and module $\hat{\rho}_{s(\cdot),p(\cdot),\Omega}$ have similar results.

Proposition 2.2. *Let Ω be a open set in $\mathbb{R}^n$ and $p(\cdot) \in \mathcal{P}(\Omega)$ with $p^+ < \infty$. Then next statements are correct*
1. $\min\{|||u|||^{p^-}_{W^{s(\cdot),p(\cdot)}(\Omega)}, |||u|||^{p^+}_{W^{s(\cdot),p(\cdot)}(\Omega)}\} \leq \hat{\rho}_{s(\cdot),p(\cdot),\Omega}(u) \leq \max\{|||u|||^{p^-}_{W^{s(\cdot),p(\cdot)}(\Omega)}, |||u|||^{p^+}_{W^{s(\cdot),p(\cdot)}(\Omega)}\}$, *if* $|||u|||_{W^{s(\cdot),p(\cdot)}(\Omega)} < +\infty$.
2. $\min\{\hat{\rho}^{1/p^-}_{s(\cdot),p(\cdot),\Omega}(u), \hat{\rho}^{1/p^+}_{s(\cdot),p(\cdot),\Omega}(u)\} \leq |||u|||_{W^{s(\cdot),p(\cdot)}(\Omega)} \leq \max\{\hat{\rho}^{1/p^-}_{s(\cdot),p(\cdot),\Omega}(u), \hat{\rho}^{1/p^+}_{s(\cdot),p(\cdot),\Omega}(u)\}$, *if* $\hat{\rho}_{s(\cdot),p(\cdot),\Omega}(u) < +\infty$.

Proposition 2.3. *([12, 21]) Let $\Omega \subset \mathbb{R}^n$, $p(\cdot) \in \mathcal{P}(\Omega)$ with $p^+ < \infty$ and $u_k, u \in L^{p(\cdot)}(\Omega)$. The following are equivalent:*
1. $\lim_{k\to\infty} \|u_k - u\|_{L^{p(\cdot)}(\Omega)} = 0$,
2. $\lim_{k\to\infty} \rho(u_k - u) = 0$,
3. $u_k \to u$ *in measure and* $\lim_{k\to\infty} \rho(\gamma u_k) = \rho(\gamma u)$ *for some* $\gamma > 0$.

Proposition 2.4. *[31] Let $\Omega \subset \mathbb{R}^n$, $p(\cdot) \in \mathcal{P}(\Omega)$ with $p^+ < \infty$ and $u_k, u \in W^{s(\cdot),p(\cdot)}(\Omega)$. Then $\lim_{k\to\infty} \varphi(u_k - u) = 0$ if and only if $\lim_{k\to\infty} [u_k - u]_{W^{s(\cdot),p(\cdot)}(\Omega)} = 0$.*

Proposition 2.5. *[31] If $|\Omega| < +\infty$ and $p^+ < \infty$, then for $u \in W^{s(\cdot),p(\cdot)}(\Omega)$ and $\{u_k\} \subset W^{s(\cdot),p(\cdot)}(\Omega)$, the following statements are equivalent:*
1. $u_k \xrightarrow{\|\cdot\|} u$.
2. $u_k \xrightarrow{\rho} u$ *and* $u_k \xrightarrow{\varphi} u$.
3. $u_k \to u$ *in measure and* $\rho(\gamma u_k) \to \rho(\gamma u)$, $\varphi(\delta u_k) \to \varphi(\delta u)$ *for some* $\gamma, \delta > 0$.

Proposition 2.6. *Suppose that $\Omega \subset \mathbb{R}^n$, $s(\cdot) \in \mathcal{S}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $p^+ < \infty$ and $0 < s^- \leq s(x) \leq s^+ < 1$. Then $C_0^\infty(\Omega) \subset W^{s(\cdot),p(\cdot)}(\Omega)$.*

Proof. Let $u \in C_0^\infty(\Omega)$ with supp$u \subset \Omega$, we already know $u \in L^{p(\cdot)}(\Omega)$. Now we prove:

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy < \infty.$$

Suppose that supp$u \subset B_r(0) \cap \Omega$, then

$$\begin{aligned}
&\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy \\
&= \int_{B_r(0)\cap\Omega} \int_\Omega \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy + \int_{\Omega\setminus B_r(0)} \int_{B_r(0)\cap\Omega} \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy \\
&\le 2\int_{B_r(0)} \int_\Omega \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy \\
&\le 2\int_{B_r(0)} \int_{B_{2r}(0)} \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy + 2\int_{B_r(0)} \int_{\Omega\setminus B_{2r}(0)} \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy \\
&= 2I_1 + 2I_2.
\end{aligned}$$

Now we estimate I_1 and I_2. Since $u \in C_0^\infty(\Omega)$, we have

$$u(x) - u(y) = \nabla u(\theta x + (1-\theta)y)\cdot(x-y)$$

for $x \in B_r(0), y \in B_{2r}(0), 0 < \theta < 1$. So

$$\begin{aligned}
I_1 &= \int_{B_r(0)} \int_{B_{2r}(0)} \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy \\
&= \int_{B_{2r}(0)} \int_{B_r(0)} \frac{|\nabla u(\theta x + (1-\theta)y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{(s(x)-1)p(x)+(s(y)-1)p(y)}{2}}}dxdy \\
&\le \int_{B_{2r}(0)} \int_{B_{2r}(0)} \frac{\|u\|_{C^1(\Omega)}^{p^+} + \|u\|_{C^1(\Omega)}^{p^-}}{|x-y|^{n+\frac{(s(x)-1)p(x)+(s(y)-1)p(y)}{2}}}dxdy \\
&\le C\int_{B_{\frac{1}{2}}(0)} \left(\int_{B_{\frac{1}{2}}(0)} \frac{1}{|z|^{n-(1-s^+)p^-}}dz\right)dx,
\end{aligned}$$

where constant C depends on $\|u\|_{C^1(\Omega)}$, r, p^- and p^+. Since $n-(1-s^+)p^- < n$, we know that $\int_{B_{\frac{1}{2}}(0)} \frac{1}{|z|^{n-(1-s+)p^-}}dz$ is finite and further I_1 is also finite.

Next

$$\begin{aligned}
I_2 &= \int_{B_r(0)} \int_{\Omega\setminus B_{2r}(0)} \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy \\
&= \int_{B_r(0)} \int_{\Omega\setminus B_{2r}(0)} \frac{|u(x)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy
\end{aligned}$$

$$\leq \int_{B_r(0)} \int_{\mathbb{R}^n \setminus B_{2r}(0)} \frac{M^{p^-} + M^{p^+}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} dxdy$$
$$\leq C \int_{B_1(0)} \Big(\int_{\mathbb{R}^n \setminus B_2(0)} \frac{1}{|z|^{n+s^-p^-}} dz \Big) dx$$

where $M = \max_{x \in \text{supp} u} |u(x)|$ and constant C depends on M, r, p^- and p^+. Since $n + s^-p^- > n$, we have $\int_{\mathbb{R}^n \setminus B_2(0)} \frac{1}{|z|^{n+s^-p^-}} dz$ is finite and further I_2 is also finite.

Based on the discussion above, we arrive at the conclusion. □

In view of Proposition 2.6, it is reasonable to define $W_0^{s(\cdot),p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{s(\cdot),p(\cdot)}(\Omega)$. According to Remark 3.2 on the trace theorem of in [13], we know that under the condition $s^-p^- > 1$, the trace of a function in $W_0^{s(\cdot),p(\cdot)}(\Omega)$ can be guaranteed to be zero.

Next, we list the theorems will use.

Theorem 2.1. *([12, 15]) Give $r(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$. Define $p(\cdot) \in \mathcal{P}(\Omega)$ by*

$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}.$$

Then there exists a constant C such that for all $u \in L^{q(\cdot)}(\Omega)$ and $v \in L^{r(\cdot)}(\Omega)$, $uv \in L^{p(\cdot)}(\Omega)$ and

$$||uv||_{L^{p(\cdot)}(\Omega)} \leq C||u||_{L^{q(\cdot)}(\Omega)}||v||_{L^{r(\cdot)}(\Omega)}.$$

At the end of this section, we consider the $s(x)$-$p(x)$-Laplacian operator $(-\Delta)^{s(\cdot)}_{p(\cdot)}$ on $W_0^{s(\cdot),p(\cdot)}(\Omega)$. Here, we denote by $(W_0^{s(\cdot),p(\cdot)}(\Omega))'$ the space dual to $W_0^{s(\cdot),p(\cdot)}(\Omega)$, and by $\langle \cdot, \cdot \rangle$ denote the scalar product on the pair $[(W_0^{s(\cdot),p(\cdot)}(\Omega))', W_0^{s(\cdot),p(\cdot)}(\Omega)]$.

The operator $(-\Delta)^{s(\cdot)}_{p(\cdot)}$ can be thought of as a mapping from $W_0^{s(\cdot),p(\cdot)}(\Omega)$ into $(W_0^{s(\cdot),p(\cdot)}(\Omega))'$ by

$$\langle (-\Delta)^{s(\cdot)}_{p(\cdot)} u, v \rangle := \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} dxdy \tag{2.1}$$

for $u, v \in W_0^{s(\cdot),p(\cdot)}(\Omega)$ and this definition makes sense. Indeed, we can use Theorem 2.1 to get the desired result very easily.

3. Embedding theorems for $W^{s(\cdot),p(\cdot)}(\Omega)$

Theorem 3.1. *Let Ω be a bounded open set in $\mathbb{R}^n$ and $p \in \mathcal{P}(\Omega)$, $p^+ < \infty$. $s_1, s_2 \in \mathcal{S}(\Omega)$ and $s_2(x) \geq s_1(x)$ a.e. on Ω, then there exists a positive constant $C = C(p, s_1, s_2, \Omega)$ such that, for any $u \in W^{s_2(\cdot),p(\cdot)}(\Omega)$, we have*

$$||u||_{W^{s_1(\cdot),p(\cdot)}(\Omega)} \leq C||u||_{W^{s_2(\cdot),p(\cdot)}(\Omega)},$$

i.e. the space $W^{s_2(\cdot),p(\cdot)}(\Omega)$ is continuously embedded in $W^{s_1(\cdot),p(\cdot)}(\Omega)$.

Proof. For convenience, let $[u]_{W^{s_2(\cdot),p(\cdot)}(\Omega)} = 1$ and

$$C = \sup_{(x,y) \in \Omega \times \Omega} |x-y|^{\frac{p(x)(s_2(x)-s_1(x))+p(y)(s_2(y)-s_1(y))}{p(x)+p(y)}}$$

then

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{C^{\frac{p(x)+p(y)}{2}}|x-y|^{n+\frac{p(x)s_1(x)+p(y)s_1(y)}{2}}}dxdy\\
=&\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{p(x)s_2(x)+p(y)s_2(y)}{2}}}\cdot\frac{|x-y|^{\frac{p(x)(s_2(x)-s_1(x))+p(y)(s_2(y)-s_1(y))}{2}}}{C^{\frac{p(x)+p(y)}{2}}}dxdy\\
\le&\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{p(x)s_2(x)+p(y)s_2(y)}{2}}}\\
\le&1,
\end{aligned}$$

therefore

$$[u]_{W^{s_1(\cdot),p(\cdot)}(\Omega)}\le C[u]_{W^{s_2(\cdot),p(\cdot)}(\Omega)}$$

and further

$$\|u\|_{W^{s_1(\cdot),p(\cdot)}(\Omega)}\le C\|u\|_{W^{s_2(\cdot),p(\cdot)}(\Omega)}.$$

□

Theorem 3.2. *Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain. $p,\ s$ are continuous on $\bar\Omega$ with $1>s(x)\ge s^->0$ and $p(x)\ge 1,\ s(x)p(x)<n$ for $x\in\bar\Omega$. Assume that $q:\bar\Omega\to[1,\infty)$ is a continuous function with*

$$q(x)<p^*(x):=\frac{np(x)}{n-s(x)p(x)}$$

for $x\in\bar\Omega$, then there exists a constant $C=C(n,s,p,q,\Omega)$ such that for every $u\in W^{s(\cdot),p(\cdot)}(\Omega)$, there holds

$$\|u\|_{L^{q(\cdot)}(\Omega)}\le C\|u\|_{W^{s(\cdot),p(\cdot)}(\Omega)},$$

i.e. the space $W^{s(\cdot),p(\cdot)}(\Omega)$ is continuously embedded in $L^{q(\cdot)}(\Omega)$. Moreover, this embedding is compact.

The embedding theorem given in [11] (the space involved is $X^{k(\cdot),\alpha(\cdot)}$), the exponent $\alpha(\cdot)$ is restricted by the exponent $p_1(\cdot)$ in the space $L^{p_1(\cdot)}$ under the condition: $\alpha(z,s)<p_1(z)$ for $(z,s)\in\bar\Omega\times\bar\Omega$, but the conclusion of our theorem does not require such a requirement. In addition, in the statement of the embedding theorem in this paper, the case that the variable exponent p and q are equal to 1 is considered, which is not mentioned in references [8, 11].

In order to prove this embedding theorem, we will use embedding theorem for constant exponent fractional Sobolev space. In order to make the proof more clear, we list this theorem here.

Theorem 3.3. *[16] (Embedding theorem for constant exponent fractional Sobolev space) Let $s\in(0,1)$ and $p\in[1,+\infty)$ be constants and satisfy $sp<n$. Denote $p^*=\frac{np}{n-sp}$. Let $\Omega\subset\mathbb{R}^n$ be an extension domain for $W^{s,p}(\Omega)$. Then there exists a positive constant $C=C(n,p,s,\Omega)$ such that for any $u\in W^{s,p}(\Omega)$, we have*

$$\|u\|_{L^q(\Omega)}\le C\|u\|_{W^{s,p}(\Omega)}$$

for any $q\in[p,p^*]$. i.e. the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q\in[p,p^*]$.

If in addition Ω is bounded, then the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q\in[1,p^*]$. Moreover, this embedding is compact for $q\in[1,p^*)$.

With these preparations, we will now prove the Theorem 3.2.

Proof. Since $p,\ s,\ q$ are continuous on $\bar{\Omega}$ and Ω is bounded, there exists a positive constant ξ such that

$$\frac{np(x)}{n - s(x)p(x)} - q(x) \geq \xi > 0 \tag{3.1}$$

for every $x \in \bar{\Omega}$.

In view of the continuity of p and (3.1), we can find a constant $\varepsilon = \varepsilon(n, p, q, s, \Omega)$ and a fnite family of disjoint Lipschitz sets O_i such that

$$\Omega = \bigcup_{i=1}^{N} O_i$$

and

$$\sup_{(x,y)\in O_i\times O_i} |p(x) - p(y)| < \varepsilon, \quad \sup_{(x,y)\in O_i\times O_i} |s(x) - s(y)| < \varepsilon$$

such that

$$\frac{np(y)}{n - s(z)p(y)} - q(x) \geq \frac{\xi}{2}$$

for every $x, y, z \in O_i$.

We can choose constant p_i and t_i, with $p_i = \inf_{y\in O_i} p(y),\ 0 < t_i < s_i := \inf_{y\in O_i} s(y)$, such that

$$p_i^* = \frac{np_i}{n - t_i p_i} \geq \frac{\xi}{3} + q(x) \tag{3.2}$$

for each $x \in O_i$.

By Theoremn 3.3, there exists a constant $C = C(n, \varepsilon, t_i, p_i, O_i)$, such that

$$||u||_{L^{p_i^*}(O_i)} \leq C(||u||_{L^{p_i}(O_i)} + [u]_{W^{t_i,p_i}(O_i)}) \tag{3.3}$$

Now, we prove the following inequalities.

(a) There exists a constant c_1 such that

$$\sum_{i=1}^{N} ||u||_{L^{p_i^*}(O_i)} \geq c_1 ||u||_{L^{q(\cdot)}(\Omega)}.$$

(b) There exists a constant c_2 such that

$$c_2[u]_{W^{\bar{s}(\cdot),p(\cdot)}(\Omega)} \geq \sum_{i=1}^{N} [u]_{W^{t_i,p_i}(O_i)}.$$

where $\bar{s}(x) := s_i \chi_{O_i}(x),\ x \in \Omega$.

(c) There exists a constant c_3 such that

$$\sum_{i=1}^{N} ||u||_{L^{p_i}(O_i)} \leq c_3 ||u||_{L^{p(\cdot)}(\Omega)}.$$

If the above three inequalities hold, a conclusion can be drawn by combining (3.3) and Theorem 3.1 as the following:

$$\begin{aligned}\|u\|_{L^{q(\cdot)}(\Omega)} &\leq C\sum_{i=1}^{N}\|u\|_{L^{p_i^*}(O_i)}\\ &\leq C\sum_{i=1}^{N}(\|u\|_{L^{p_i}(O_i)}+[u]_{W^{t_i,p_i}(O_i)})\\ &\leq C(\|u\|_{L^{p(\cdot)}(\Omega)}+[u]_{W^{\bar{s}(\cdot),p(\cdot)}(\Omega)})\\ &= C\|u\|_{W^{\bar{s}(\cdot),p(\cdot)}(\Omega)}\\ &\leq C\|u\|_{W^{s(\cdot),p(\cdot)}(\Omega)}.\end{aligned} \tag{3.4}$$

First prove (a). We have

$$|u(x)| = \sum_{i=1}^{N}|u(x)|\chi_{O_i}$$

i.e.

$$\|u\|_{L^{q(\cdot)}(\Omega)} \leq \sum_{i=1}^{N}\|u\|_{L^{q(\cdot)}(O_i)}$$

Since for each i, $p_i^* > q(x)$ for $x \in O_i$, these exists α_i such that

$$\frac{1}{q(x)} = \frac{1}{p_i^*} + \frac{1}{\alpha_i(x)}.$$

According to Theorem 2.1, we have

$$\begin{aligned}\|u\|_{L^{q(\cdot)}(O_i)} &\leq C\|u\|_{L^{p_i^*}(O_i)}\|1\|_{L^{\alpha_i(\cdot)}(O_i)}\\ &= C\|u\|_{L^{p_i^*}(O_i)}\end{aligned}$$

In this way, (a) is proved.

Next prove (b). Set

$$F_i(x,y) := \frac{|u(x)-u(y)|}{|x-y|^{s_i}}$$

then

$$\begin{aligned}[u]_{W^{t_i,p_i}(O_i)} &= \Big(\int_{O_i}\int_{O_i}\frac{|u(x)-u(y)|^{p_i}}{|x-y|^{n+t_ip_i+s_ip_i-s_ip_i}}dxdy\Big)^{\frac{1}{p_i}}\\ &= \Big(\int_{O_i}\int_{O_i}\Big(\frac{|u(x)-u(y)|}{|x-y|^{s_i}}\Big)^{p_i}\frac{1}{|x-y|^{n+(t_i-s_i)p_i}}dxdy\Big)^{\frac{1}{p_i}}\\ &= \|F_i\|_{L^{p_i}(O_i\times O_i)}\\ &\leq C\|F_i\|_{L^{\frac{p(x)+p(y)}{2}}(\mu,O_i\times O_i)}\|1\|_{L^{\beta_i(x,y)}(\mu,O_i\times O_i)}\\ &\leq C\|F_i\|_{L^{\frac{p(x)+p(y)}{2}}(\mu,O_i\times O_i)}\end{aligned}$$

where

$$\frac{1}{p_i} = \frac{1}{\frac{p(x)+p(y)}{2}} + \frac{1}{\beta_i(x,y)}$$

and

$$d\mu(x,y) = \frac{dxdy}{|x-y|^{n+(t_i-s_i)p_i}}$$

is a measure on $O_i \times O_i$.

Set $\lambda = [u]_{W^{s_i,p(\cdot)}(O_i)}$ and $k = \max_i \left\{ \sup_{(x,y)\in O_i\times O_i} \{|x-y|^{\frac{2p_i(s_i-t_i)}{p(x)+p(y)}}\} \right\}$. We have

$$\begin{aligned}
&\int_{O_i}\int_{O_i} \left(\frac{|u(x)-u(y)|}{k\lambda|x-y|^{s_i}}\right)^{\frac{p(x)+p(y)}{2}} \frac{1}{|x-y|^{n+(t_i-s_i)p_i}} dxdy \\
=&\int_{O_i}\int_{O_i} \frac{|x-y|^{(s_i-t_i)p_i}}{k^{\frac{p(x)+p(y)}{2}}} \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{\lambda^{\frac{p(x)+p(y)}{2}}|x-y|^{n+\frac{s_i(p(x)+p(y))}{2}}} dxdy \\
<&\int_{O_i}\int_{O_i} \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{\lambda^{\frac{p(x)+p(y)}{2}}|x-y|^{n+\frac{s_i(p(x)+p(y))}{2}}} dxdy \\
\le&1
\end{aligned}$$

Therefore

$$\begin{aligned}
||F_i||_{L^{\frac{p(x)+p(y)}{2}}(\mu, O_i\times O_i)} &\le k[u]_{W^{s_i,p(\cdot)}(O_i)} \\
&\le k[u]_{W^{\bar{s}(\cdot),p(\cdot)}(\Omega)}
\end{aligned}$$

and further

$$[u]_{W^{t_i,p_i}(O_i)} \le C[u]_{W^{\bar{s}(\cdot),p(\cdot)}(\Omega)}$$

In this way, (b) is proved.

By the same way to prove (a), we can prove (c).

Finally, prove the compactness of this embedding. Let $\{u_k\}$ be a sequence in $W^{s(\cdot),p(\cdot)}(\Omega)$ with $||u_k||_{W^{s(\cdot),p(\cdot)}(\Omega)} \le M$. According to (3.4), for any i, $||u_k||_{W^{t_i,p_i}(O_i)} \le M$. By Theorem 3.3 and (3.2), $\{u_k\}$ has a subsequence $\{u_k^1\}$ such that $\{u_k^1|_{O_1}\}$ converges in $L^{p_1^*-\frac{\xi}{3}}(O_1)$ to some $u^1 \in L^{p_1^*-\frac{\xi}{3}}(O_1)$. Similarly, $\{u_k^1\}$ has a subsequence $\{u_k^2\}$ such that $\{u_k^2|_{O_2}\}$ converges in $L^{p_2^*-\frac{\xi}{3}}(O_2)$ to some $u^2 \in L^{p_2^*-\frac{\xi}{3}}(O_2)$. And so on, $\{u_k^{N-1}\}$ has a subsequence $\{u_k^N\}$ such that $\{u_k^N|_{O_N}\}$ converges in $L^{p_N^*-\frac{\xi}{3}}(O_N)$ to some $u^N \in L^{p_N^*-\frac{\xi}{3}}(O_N)$. Set

$$u(x) = \sum_{i=1}^{N} u^i(x)\chi_{O_i},$$

then

$$||u_k^N - u||_{L^{q(\cdot)}(\Omega)} \le C\sum_{i=1}^{N} ||u_k^N|_{O_i} - u^i||_{L^{p_i^*-\frac{\xi}{3}}(O_i)} \to 0 \text{ as } k\to\infty. \tag{3.5}$$

Now the proof is finished. □

Remark.

1. We can reduce the condition that q is continuous in the Theorem 3.2 to $\operatorname{ess\,inf}(p^*-q) > 0$;
2. Theorem 3.2 remains true if we replace $W^{s(\cdot),p(\cdot)}(\Omega)$ by $W_0^{s(\cdot),p(\cdot)}(\Omega)$.

4. An application

For problem (1.5), we make the following assumptions.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and

(PQS) $p, q, s \in C(\bar{\Omega})$, $0 < s(x) < 1$, $s(x)p(x) < n$, $1 < s^- p^- < p(x) \leq p^+ < q^- \leq q(x) < p^*(x) := \dfrac{np(x)}{n - s(x)p(x)}$ for all $x \in \bar{\Omega}$,

(F) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exist constant $a_1 > 0$, $r > 0$, $\mu > p^+$ such that

(F1) $|f(x,t)| \leq a_1(1 + |t|^{q(x)-1})$ for a.e. $x \in \Omega$ and for each $t \in \mathbb{R}$,

(F2) $0 < \mu F(x,t) \leq f(x,t)t$ for a.e. $x \in \Omega$ and for each t, $|t| \geq r$, where

$$F(x,t) = \int_0^t f(x,\tau)d\tau \quad \text{for a.e. } x \in \Omega \ \text{ and for each } t \in \mathbb{R},$$

(F3) $f(x,t) = o(|t|^{p(x)-1})$ as $t \to 0$, uniformly for $x \in \Omega$.

(V) $V \in C(\overline{\Omega})$ and $V_0 := \min\limits_{x \in \overline{\Omega}} V(x) > 0$,

(G) $g \in L^{p'(\cdot)}(\Omega)$, where $p'(\cdot)$ defined by equality $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$ for all $x \in \bar{\Omega}$.

Definition 4.1. *We say that $u \in W_0^{s(\cdot),p(\cdot)}(\Omega)$ is a weak solution of problem* (1.5) *if for all $v \in W_0^{s(\cdot),p(\cdot)}(\Omega)$ we have*

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{\frac{p(x)+p(y)}{2}-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} dxdy$$
$$+ \int_\Omega V(x)|u(x)|^{p(x)-2}u(x)v(x)dx = \int_\Omega f(x,u)v(x)dx + \int_\Omega g(x)v(x)dx.$$

Theorem 4.1. *Let (PQS), (F), (F1)–(F3) and (V) hold and suppose that $0 \not\equiv g \in L^{p'(\cdot)}(\Omega)$. Then there exists a constant $\delta_0 > 0$ such that problem* (1.5) *admits at least two nontrivial solutions in $W_0^{s(\cdot),p(\cdot)}(\Omega)$ provided that $\|g\|_{L^{p'(\cdot)}(\Omega)} \leq \delta_0$.*

Corresponding to the problem (1.2), consider the energy functional $I : W_0^{s(\cdot),p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = J(u) - H(u) - G(u),$$

where

$$J(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{\frac{p(x)+p(y)}{2}}}{\frac{p(x)+p(y)}{2}|x - y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} dxdy + \int_\Omega \frac{V(x)}{p(x)}|u(x)|^{p(x)}dx,$$
$$H(u) = \int_\Omega F(x, u(x))dx,$$
$$G(u) = \int_\Omega g(x)u(x)dx.$$

We know that a critical point of I is a weak solution to the problem (1.2). To prove Theorem 4.1, we give some lemmas.

Lemma 4.1. *Suppose that (V) hold. Then $J \in C^1(W_0^{s(\cdot),p(\cdot)}(\Omega))$ and*

$$\langle J'(u), v\rangle = \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy + \int_\Omega V(x)|u(x)|^{p(x)-2}u(x)v(x)dx \tag{4.1}$$

for all $u, v \in W_0^{s(\cdot),p(\cdot)}(\Omega)$. Moreover, J is weakly lower semi-continuous on $W_0^{s(\cdot),p(\cdot)}(\Omega)$.

Proof. We can easily verify the Gâteaux differentiability of J on $W_0^{s(\cdot),p(\cdot)}(\Omega)$ and (4.1) holds for all $u, v \in W_0^{s(\cdot),p(\cdot)}(\Omega)$.

Now prove $J \in C^1(W_0^{s(\cdot),p(\cdot)}(\Omega))$. For any $\{u_n\} \subset W_0^{s(\cdot),p(\cdot)}(\Omega)$ and $u_n \to u$ in $W_0^{s(\cdot),p(\cdot)}(\Omega)$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\int_\Omega\int_\Omega\Big(\frac{|u_n(x)-u_n(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} - \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}\Big)dxdy = 0. \tag{4.2}$$

Without loss of generality, we further assume that

$$u_n \to u \text{ a.e. in } \Omega \text{ as } n \to \infty.$$

By (4.2),

$$\left\{\frac{|u_n(x)-u_n(y)|^{\frac{p(x)+p(y)}{2}-2}(u_n(x)-u_n(y))}{|x-y|^{(n+\frac{s(x)p(x)+s(y)p(y)}{2})(\frac{p(x)+p(y)-2}{p(x)+p(y)})}}\right\}_n$$

is bounded in $L^{\frac{p(x)+p(y)}{p(x)+p(y)-2}}(\Omega)$ and by Brezis-Lieb Lemma in [23] we have

$$\lim_{n\to\infty}\int_\Omega\int_\Omega\Big(\frac{|u_n(x)-u_n(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} - \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}\Big)dxdy = 0.$$

Similarly,

$$\lim_{n\to\infty}\int_\Omega V(x)\Big||u_n(x)|^{p(x)-2}u_n(x) - |u(x)|^{p(x)-2}u(x)\Big|^{\frac{p(x)+p(y)}{p(x)+p(y)-2}}dx = 0.$$

By Hölder inequality,

$$\|J'(u_n)-J'(u)\|_{(W_0^{s(\cdot),p(\cdot)}(\Omega))'} = \sup_{\substack{v\in W_0^{s(\cdot),p(\cdot)}(\Omega)\\ \|v\|_{W_0^{s(\cdot),p(\cdot)}(\Omega)}=1}} |\langle J'(u_n)-J'(u), v\rangle| \to 0$$

as $n \to \infty$. Hence $J \in C^1(W_0^{s(\cdot),p(\cdot)}(\Omega))$.

Next we prove J is weakly lower semi-continuous on $W_0^{s(\cdot),p(\cdot)}(\Omega)$. Let $\{u_n\} \subset W_0^{s(\cdot),p(\cdot)}(\Omega)$ and $u_n \rightharpoonup u$ weakly in $W_0^{s(\cdot),p(\cdot)}(\Omega)$ as $n \to \infty$. Notice that for $w, v \in W_0^{s(\cdot),p(\cdot)}(\Omega)$,

$$\begin{aligned}
J(\frac{w+v}{2}) &= \int_\Omega\int_\Omega \frac{|\frac{w(x)+v(x)}{2} - \frac{w(y)+v(y)}{2}|^{\frac{p(x)+p(y)}{2}}}{\frac{p(x)+p(y)}{2}|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy + \int_\Omega \frac{V(x)}{p(x)}\Big|\frac{w(x)+v(x)}{2}\Big|^{p(x)}dx \\
&\le \frac{1}{2}\Big(\int_\Omega\int_\Omega \frac{|w(x)-w(y)|^{\frac{p(x)+p(y)}{2}}}{\frac{p(x)+p(y)}{2}|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy + \int_\Omega \frac{V(x)}{p(x)}|w(x)|^{p(x)}dx\Big) \\
&\quad + \frac{1}{2}\Big(\int_\Omega\int_\Omega \frac{|v(x)-v(y)|^{\frac{p(x)+p(y)}{2}}}{\frac{p(x)+p(y)}{2}|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy + \int_\Omega \frac{V(x)}{p(x)}|v(x)|^{p(x)}dx\Big) \\
&= \frac{1}{2}J(w) + \frac{1}{2}J(v).
\end{aligned}$$

Thus J is a convex functional on $W_0^{s(\cdot),p(\cdot)}(\Omega)$.

Because $J \in C^1(W_0^{s(\cdot),p(\cdot)}(\Omega))$, $J'(u)$ is subgradient of J at point $u \in W_0^{s(\cdot),p(\cdot)}(\Omega)$ and by the definition of a subgradient we have

$$J(u_n) - J(u) \ge \langle J'(u), u_n - u\rangle.$$

Letting $n \to \infty$, we have

$$J(u) \le \liminf_{n\to\infty} J(u_n),$$

i.e. J is weakly lower semi-continuous. □

Lemma 4.2. *Suppose that (F1) and (F3) hold. Then $H \in C^1(W_0^{s(\cdot),p(\cdot)}(\Omega))$ and*

$$\langle H'(u), v\rangle = \int_\Omega f(x, u(x))v(x)dx \tag{4.3}$$

for all $u, v \in W_0^{s(\cdot),p(\cdot)}(\Omega)$. Moreover H is weakly continuous on $W_0^{s(\cdot),p(\cdot)}(\Omega)$.

Proof. We can easily verify Gâteaux differentiability of H on $W_0^{s(\cdot),p(\cdot)}(\Omega)$ and (4.3) holds for all $u, v \in W_0^{s(\cdot),p(\cdot)}(\Omega)$.

Now consider $H \in C^1(W_0^{s(\cdot),p(\cdot)}(\Omega))$. For any $\{u_n\} \subset W_0^{s(\cdot),p(\cdot)}(\Omega)$ and $u_n \to u$ in $W_0^{s(\cdot),p(\cdot)}(\Omega)$ as $n \to \infty$. By Theorem 3.2,

$$u_n \to u \ \text{ in } \ L^{q(\cdot)}(\Omega) \text{ as } n \to \infty.$$

By (F1) and Theorem 1.16 in [21], from $u \in L^{q(\cdot)}(\Omega)$ we have $f(x,u) \in L^{q'(\cdot)}(\Omega)$. Since $u_n \to u$ in $L^{q(\cdot)}(\Omega)$, by [20] we get

$$f(x, u_n) \to f(x, u) \text{ in } L^{q'(\cdot)}(\Omega).$$

Let $v \in W_0^{s(\cdot),p(\cdot)}(\Omega)$ with $\|v\|_{W_0^{s(\cdot),p(\cdot)}(\Omega)} = 1$. By Therefore 3.2, $v \in L^{q(\cdot)}(\Omega)$ and further by Hölder inequality,

$$\begin{aligned}
|\langle H'(u_n), v - H'(u), v\rangle| &\le \int_\Omega |f(x, u_n(x)) - f(x, u(x))||v(x)|dx \\
&\le C\|f(x, u_n) - f(x, u)\|_{L^{q'(\cdot)}(\Omega)}\|v\|_{L^{q(\cdot)}(\Omega)} \\
&\le C\|f(x, u_n) - f(x, u)\|_{L^{q'(\cdot)}(\Omega)},
\end{aligned} \tag{4.4}$$

so

$$\|H'(u_n), v - H'(u)\|_{(W_0^{s(\cdot),p(\cdot)}(\Omega))'} \le C\|f(x,u_n) - f(x,u)\|_{L^{q'(\cdot)}(\Omega)} \to 0$$

as $n \to \infty$. Therefore $H \in C^1(W_0^{s(\cdot),p(\cdot)}(\Omega))$.

At last we prove that H is weakly continuous on $W_0^{s(\cdot),p(\cdot)}(\Omega)$. Let $u_n \rightharpoonup u$ weakly in $W_0^{s(\cdot),p(\cdot)}(\Omega)$. By Theorem 3.2, we have $u_n \to u$ in $L^{q(\cdot)}(\Omega)$. Then similar to [9] we can get the conclusion. □

Lemma 4.3. *$G \in C^1(W_0^{s(\cdot),p(\cdot)}(\Omega))$ and*

$$\langle G'(u), v\rangle = \int_\Omega g(x)v(x)dx \tag{4.5}$$

for all $u, v \in W_0^{s(\cdot),p(\cdot)}(\Omega)$. Moreover G is weakly continuous on $W_0^{s(\cdot),p(\cdot)}(\Omega)$.

Proof. We can easily prove that $G \in C^1(W_0^{s(\cdot),p(\cdot)}(\Omega))$ and (4.5).

Let $u_n \rightharpoonup u$ weakly in $W_0^{s(\cdot),p(\cdot)}(\Omega)$. By Theorem (3.2), we have $u_n \to u$ in $L^{q(\cdot)}(\Omega)$. By Hölder inequality,

$$\begin{aligned}|G(u_n) - G(u)| &\le \int_\Omega |g(x)(u_n(x) - u(x)|dx \\ &\le C\|g\|_{L^{q'(\cdot)}(\Omega)}\|u_n - u\|_{L^{q(\cdot)}(\Omega)} \\ &\to 0,\end{aligned}$$

as $n \to \infty$. Thus G is weakly continuous on $W_0^{s(\cdot),p(\cdot)}(\Omega)$. □

By Lemmas (4.1)–(4.3), we get the following conclusion.

Lemma 4.4. *Suppose that (F1)–(F3) and (V) hold, then $I \in C^1(W_0^{s(\cdot),p(\cdot)}(\Omega))$ and I is weakly lower semi-continuous on $W_0^{s(\cdot),p(\cdot)}(\Omega)$.*

Lemma 4.5. *Suppose that (F1), (F3) and (V) hold. Then there exist constants $0 < \rho_0 < 1$, $\alpha_0, \delta_0 > 0$ such that $I(u) \ge \alpha_0$ for all $u \in W_0^{s(\cdot),p(\cdot)}(\Omega)$ with $\|u\|_{W_0^{s(\cdot),p(\cdot)}(\Omega)} = \rho_0$ and all $g \in L^{p'(\cdot)}(\Omega)$ with $\|g\|_{L^{p'(\cdot)}(\Omega)} \le \delta_0$.*

Proof. By (F1) and (F3), we can get

$$\begin{aligned}|F(x,t)| &\le |t|^{p(x)} + \frac{1}{q(x)}(a_1 + \frac{a_1}{\delta^{q(x)-1}})|t|^{q(x)} \\ &\le |t|^{p(x)} + \frac{1}{q^-}(a_1 + \frac{a_1}{\delta^{q^+-1}})|t|^{q(x)}\end{aligned}$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

By Hölder inequalities, Proposition 2.1 and Theorem 3.2, in the case that $\|u\|_{W_0^{s(\cdot),p(\cdot)}(\Omega)}$ is small enough, we have

$$\begin{aligned}I(u) \ge{}& \frac{\min\{1, V_0\}}{p^+}|||u|||^{p^+}_{W_0^{s(\cdot),p(\cdot)}(\Omega)} - \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} - \frac{1}{q^-}(a_1 + \frac{a_1}{\delta^{q^+-1}})\|u\|^{q^-}_{L^{q(\cdot)}(\Omega)} \\ &- C\|g\|_{L^{p'(\cdot)}(\Omega)}\|u\|_{L^{p(\cdot)}(\Omega)} \\ \ge{}& \|u\|_{W_0^{s(\cdot),p(\cdot)}(\Omega)}\Big(\frac{\min\{1, V_0\}}{2^{p^+}p^+}\|u\|^{p^+-1}_{W_0^{s(\cdot),p(\cdot)}(\Omega)} - \|u\|^{p^--1}_{W_0^{s(\cdot),p(\cdot)}(\Omega)} \\ &- \frac{1}{q^-}(a_1 + \frac{a_1}{\delta^{q^+-1}})C_q\|u\|^{q^--1}_{W_0^{s(\cdot),p(\cdot)}(\Omega)} - C_p\|g\|_{L^{p'(\cdot)}(\Omega)}\Big).\end{aligned}$$

For all $t \in \mathbb{R}$, let

$$\eta(t) = \frac{\min\{1, V_0\}}{2^{p^+} p^+}|t|^{p^+-1} - |t|^{p^--1} - \frac{1}{q^-}(a_1 + \frac{a_1}{\delta^{q^+-1}})C_q|t|^{q^--1},$$

then there exists $\rho_0 > 0$ such that $\max_{t\in\mathbb{R}} \eta(t) = \eta(\rho_0) > 0$. Taking $\delta_0 := \frac{\eta(\rho_0)}{2C_p}$, we have $I(u) \geq \alpha_0 = \rho_0\eta(\rho_0)/2 > 0$ for all u in $W_0^{s(\cdot),p(\cdot)}(\Omega)$ with $\|u\|_{W_0^{s(\cdot),p(\cdot)}(\Omega)} = \rho_0$ and for all $g \in L^{p'(\cdot)}(\Omega)$ with $\|g\|_{L^{p'(\cdot)}(\Omega)} \leq \delta_0$. □

Lemma 4.6. *Suppose that (F1)–(F3), (V) hold, then there exists a function* $v \in C_0^\infty(\Omega)$ *such that* $I(v) < 0$ *and* $\|v\|_{W_0^{s(\cdot),p(\cdot)}(\Omega)} > \rho_0$, *where* $\rho_0 > 0$ *is the one in Lemma 4.5.*

Proof. From condition (F2), we have

$$F(x,t) \geq a|t|^\mu - a_1|t|^{p(x)} \text{ all } (x,t) \in \Omega \times \mathbb{R}, \tag{4.6}$$

where $a,\ a_1$ are constants. Thus by (4.6) and (F2), for $u \in C_0^\infty(\Omega)$ with $\|u\|_{W_0^{s(\cdot),p(\cdot)}(\Omega)} = 1$, we have as $t \to +\infty$

$$\begin{aligned}
I(tu) &= \int_\Omega\int_\Omega \frac{|tu(x) - tu(y)|^{\frac{p(x)+p(y)}{2}}}{\frac{p(x)+p(y)}{2}|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy \\
&\quad + \int_\Omega \frac{V(x)}{p(x)}|tu(x)|^{p(x)}dx - \int_\Omega F(x, tu(x))dx - t\int_\Omega g(x)u(x)dx \\
&\leq \frac{t^{p^+}}{p^-}[u]^{p^+}_{W_0^{s(\cdot),p(\cdot)}(\Omega)} + \frac{V_1 t^{p^+}}{p^-}\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} - at^\mu\|u\|^\mu_{L^\mu(\Omega)} + a_1 t^{p^+}\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} - t\int_\Omega g(x)u(x)dx \\
&\leq \left(\frac{1+V_1}{p^-} + a_1\right)t^{p^+}\|u\|^{p^+}_{W_0^{s(\cdot),p(\cdot)}(\Omega)} - at^\mu\|u\|^\mu_{L^\mu(\Omega)} + a_1 - t\int_\Omega g(x)u(x)dx \\
&\to -\infty,
\end{aligned} \tag{4.7}$$

where $V_1 = \sup_{x\in\bar{\Omega}} V(x)$. We conclude the lemma by taking $v = t_0 u$ with $t_0 > 0$ large enough. □

Lemma 4.7. *Suppose that (F1)–(F3), (V) hold. Then there exists a function* $w \in W_0^{s(\cdot),p(\cdot)}(\Omega)$ *such that* $I(w) < 0$ *and* $\|w\|_{W_0^{s(\cdot),p(\cdot)}(\Omega)} < \rho_0$, *where* $\rho_0 > 0$ *is the one in Lemma 4.5.*

Proof. The proof is similar to that of Lemma 4.6 with minor changes in the proof of inequality (4.7). Let $t \in (0,1)$ be small enough, then inequality (4.7) becomes

$$I(tu) \leq \left(\frac{1+V_1}{p^-} + a_1\right)t^{p^-}\|u\|^{p^-}_{W_0^{s(\cdot),p(\cdot)}(\Omega)} - at^\mu\|u\|^\mu_{L^\mu(\Omega)} - t\int_\Omega g(x)u(x)dx. \tag{4.8}$$

In order to ensure that the right side of inequality (4.8) is less than zero, we just have to make $\int_\Omega g(x)u(x)dx > 0$. Since $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$ and $|g|^{p'(\cdot)-2}g \in L^{p(\cdot)}(\Omega)$, there exists $g_{n_0} > 0$ such that $g_{n_0} \in C_0^\infty(\Omega)$ and

$$\|g_{n_0} - |g|^{p'(\cdot)-2}g\|_{L^{p(\cdot)}(\Omega)} \leq \frac{1}{8\|g\|_{L^{p'(\cdot)}(\Omega)}}\int_\Omega |g(x)|^{p'(x)}dx.$$

So

$$\int_\Omega g_{n_0}(x)g(x)dx \geq -4\|g_{n_0} - |g|^{p'(\cdot)-2}g\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{p'(\cdot)}(\Omega)} + \int_\Omega |g(x)|^{p'(x)}dx > 0.$$

Take $u = g_{n_0} \in W_0^{s(\cdot),p(\cdot)}(\Omega)$ and $\theta = \min\left\{1, \frac{\rho_0}{\|g_{n_0}\|_{W_0^{s(\cdot),p(\cdot)}(\Omega)}}\right\}$ and choose $t_0 \in (0, \theta)$ such that $I(t_0 u) < 0$. Let $w = t_0 u$, then w is the one we expect. □

Definition 4.2. *[2] Let X be Banach space. I is a functional on X. We say that I satisfies PS condition in X, if any PS sequence $\{u_n\}_n \subset X$, i.e. $\{I(u_n)\}_n$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, admits a strongly convergent subsequence in X.*

Lemma 4.8. *Let (F1)–(F3) and (V) hold, then I satisfies the PS condition.*

Proof. Let $\{u_n\}$ be a PS sequence in $W_0^{s(\cdot),p(\cdot)}(\Omega)$. Then there exists $C > 0$ such that $|\langle I'(u_n), u_n\rangle| \leq C\|u_n\|_{W_0^{s(\cdot),p(\cdot)}(\Omega)}$ and $|I(u_n)| \leq C$. Thus by (F2), Proposition 2.2 and Theorem 3.2, we get

$$\begin{aligned}
& C + C\|u_n\|_{W_0^{s(\cdot),p(\cdot)}(\Omega)} \\
&\geq I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \\
&\geq \frac{1}{2}(\frac{1}{p^+} - \frac{1}{\mu}) \min\{1, V_0\} \min\{\|u_n\|^{p^+}_{W_0^{s(\cdot),p(\cdot)}(\Omega)}, \|u_n\|^{p^-}_{W_0^{s(\cdot),p(\cdot)}(\Omega)}\} \\
&\quad - \frac{1}{\mu}\int_\Omega \mu F(x, u_n(x)) - f(x, u_n(x))u_n(x)dx - C_p(1 - \frac{1}{\mu})\|g\|_{L^{p'(\cdot)}(\Omega)}\|u_n\|_{W_0^{s(\cdot),p(\cdot)}(\Omega)} \\
&\geq \frac{1}{2}(\frac{1}{p^+} - \frac{1}{\mu}) \min\{1, V_0\} \min\{\|u_n\|^{p^+}_{W_0^{s(\cdot),p(\cdot)}(\Omega)}, \|u_n\|^{p^-}_{W_0^{s(\cdot),p(\cdot)}(\Omega)}\} \\
&\quad - C_p(1 - \frac{1}{\mu})\|g\|_{L^{p'(\cdot)}(\Omega)}\|u_n\|_{W_0^{s(\cdot),p(\cdot)}(\Omega)}.
\end{aligned}$$

Hence $\{u_n\}$ is bounded in $W_0^{s(\cdot),p(\cdot)}(\Omega)$. By Theorem 3.2, take a subsequence if necessary, then we get

$$\begin{aligned}
& u_n \rightharpoonup u \ \text{ in } W_0^{s(\cdot),p(\cdot)}(\Omega), \\
& u_n \to u \ \text{ a.e. in } \Omega, \\
& u_n \to u \ \text{ in } L^{q(\cdot)}(\Omega).
\end{aligned} \tag{4.9}$$

Now we want to prove that $\{u_n\}$ converges to u in $W_0^{s(\cdot),p(\cdot)}(\Omega)$. For $\psi \in W_0^{s(\cdot),p(\cdot)}(\Omega)$, define a linear functional B_ψ on $W_0^{s(\cdot),p(\cdot)}(\Omega)$ as

$$B_\psi(v) = \int_\Omega\int_\Omega \frac{|\psi(x) - \psi(y)|^{\frac{p(x)+p(y)}{2}-2}(\psi(x) - \psi(y))(v(x) - v(y))}{|x - y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy.$$

By Hölder inequality,

$$|B_\psi(v)| \leq \max\{\|\psi\|^{p^+-1}_{W_0^{s(\cdot),p(\cdot)}(\Omega)}, \ \|\psi\|^{p^--1}_{W_0^{s(\cdot),p(\cdot)}(\Omega)}\}\|v\|_{W_0^{s(\cdot),p(\cdot)}(\Omega)},$$

hence B_ψ is continuous.

By (F1) and (F3), there exists a constant $C > 0$ such that

$$|f(x, t)| \leq |t|^{p(x)-1} + C|t|^{q(x)-1}$$

for all $x \in \Omega$ and $t \in \mathbb{R}$. By Hölder inequality,

$$\begin{aligned}
&\int_\Omega |(f(x,u_n) - f(x,u))(u_n - u)|dx \\
&\le \int_\Omega (|u_n|^{p(x)-1} + |u|^{p(x)-1} + C(|u_n|^{q(x)-1} + |u|^{q(x)-1})|u_n - u|dx \\
&\le (\|u_n\|^{p^+-1}_{L^{p(\cdot)}(\Omega)} + \|u_n\|^{p^--1}_{L^{p(\cdot)}(\Omega)} + \|u\|^{p^+-1}_{L^{p(\cdot)}(\Omega)} + \|u\|^{p^--1}_{L^{p(\cdot)}(\Omega)})\|u_n - u\|_{L^{p(\cdot)}(\Omega)} \\
&\quad + C(\|u_n\|^{q^+-1}_{L^{q(\cdot)}(\Omega)} + \|u_n\|^{q^--1}_{L^{q(\cdot)}(\Omega)} + \|u\|^{q^+-1}_{L^{q(\cdot)}(\Omega)} + \|u\|^{q^--1}_{L^{q(\cdot)}(\Omega)})\|u_n - u\|_{L^{q(\cdot)}(\Omega)},
\end{aligned}$$

then

$$\lim_{n\to\infty} \int_\Omega |(f(x,u_n) - f(x,u))(u_n - u)|dx = 0. \tag{4.10}$$

The fact that I satisfies PS condition in $W_0^{s(\cdot),p(\cdot)}(\Omega)$ and (4.9) imply

$$\lim_{n\to\infty} \langle I'(u_n) - I'(u), u_n - u\rangle = 0, \tag{4.11}$$

so by (4.9)–(4.11),

$$\begin{aligned}
o(1) &= \langle I'(u_n) - I'(u), u_n - u\rangle \\
&= B_{u_n}(u_n - u) - B_u(u_n - u) + \int_\Omega V(x)(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)dx \\
&\quad - \int_\Omega (f(x,u_n) - f(x,u))(u_n - u)dx \\
&= B_{u_n}(u_n - u) - B_u(u_n - u) + \int_\Omega V(x)(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)dx + o(1)
\end{aligned}$$

i.e.

$$B_{u_n}(u_n - u) - B_u(u_n - u) + \int_\Omega V(x)(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)dx \to 0$$

as $n \to \infty$. By Simon Inequality, we can get

$$\begin{aligned}
&B_{u_n}(u_n - u) - B_u(u_n - u) \ge 0, \\
&\int_\Omega V(x)(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)dx \ge 0,
\end{aligned}$$

and further

$$\begin{aligned}
&\lim_{n\to\infty} (B_{u_n}(u_n - u) - B_u(u_n - u)) = 0, \\
&\lim_{n\to\infty} \int_\Omega (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)dx = 0.
\end{aligned} \tag{4.12}$$

Next we apply Simon inequality again to prove $u_n \to u$ in $W_0^{s(\cdot),p(\cdot)}(\Omega)$ as $n \to \infty$. Let $\Omega_1 = \{x \in \Omega : p(x) \ge 2\}$ and $\Omega_2 = \{x \in \Omega : p(x) < 2\}$, then

$$\begin{aligned}
\rho_{p(\cdot),\Omega}(u_n - u) &= \int_{\Omega_1} |u_n - u|^{p(x)}dx + \int_{\Omega_2} |u_n - u|^{p(x)}dx \\
&= Z_1 + Z_2.
\end{aligned}$$

Consider Z_1 and Z_2. First

$$Z_1 \leq C \int_\Omega (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)dx \to 0.$$

By (4.9) and Theorem 1.3 in [21], there exists $K > 0$ such that $\rho_{p(\cdot),\Omega}(u_n) + \rho_{p(\cdot),\Omega}(u) \leq K$. By Hölder inequality

$$\begin{aligned}
Z_2 &\leq C \int_\Omega [(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)]^{\frac{p(x)}{2}} (|u_n|^{p(x)} + |u|^{p(x)})^{\frac{2-p(x)}{2}} dx \\
&\leq C\Big[\Big(\int_\Omega (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)dx\Big)^{\frac{p^+}{2}} \\
&\quad + \Big(\int_\Omega (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)dx\Big)^{\frac{p^-}{2}}\Big] \\
&\quad \times [(\rho_{p(\cdot),\Omega}(u_n) + \rho_{p(\cdot),\Omega}(u))^{\frac{2-p^+}{2}} + (\rho_{p(\cdot),\Omega}(u_n) + \rho_{p(\cdot),\Omega}(u))^{\frac{2-p^-}{2}}] \\
&\leq C(K^{\frac{2-p^+}{2}} + K^{\frac{2-p^-}{2}})\Big[\Big(\int_\Omega (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)dx\Big)^{\frac{p^+}{2}} \\
&\quad + \Big(\int_\Omega (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)dx\Big)^{\frac{p^-}{2}}\Big] \\
&\to 0
\end{aligned}$$

as $n \to \infty$. So $\rho_{p(\cdot),\Omega}(u_n - u) \to 0$ and further by Proposition (2.3),

$$\|u_n - u\|_{L^{p(\cdot)}(\Omega)} \to 0 \tag{4.13}$$

as $n \to \infty$.

On the other hand. Let

$$(\Omega \times \Omega)_1 = \{(x, y) \in \Omega \times \Omega : p(x) + p(y) \geq 4\},$$

$$(\Omega \times \Omega)_2 = \{(x, y) \in \Omega \times \Omega : p(x) + p(y) < 4\},$$

then

$$\begin{aligned}
\varphi_{s(\cdot),p(\cdot),\Omega}(u_n - u) &= \iint_{(\Omega\times\Omega)_1} \frac{|u_n(x) - u_n(y) - u(x) + u(y)|^{\frac{p(x)+p(y)}{2}}}{|x - y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}} dxdy \\
&\quad + \iint_{(\Omega\times\Omega)_2} \frac{|u_n(x) - u_n(y) - u(x) + u(y)|^{\frac{p(x)+p(y)}{2}}}{|x - y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}} dxdy \\
&= \Phi_1 + \Phi_2.
\end{aligned}$$

We investigate Φ_1 and Φ_2. First

$$\begin{aligned}
\Phi_1 &= \iint_{(\Omega\times\Omega)_1} \frac{|u_n(x)-u_n(y)-u(x)+u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}} dxdy \\
&\le C\iint_{(\Omega\times\Omega)_1} \frac{|u_n(x)-u_n(y)|^{\frac{p(x)+p(y)}{2}-2}(u_n(x)-u_n(y)) - |u(x)-u(y)|^{\frac{p(x)+p(y)}{2}-2}(u(x)-u(y))}{|x-y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}} \\
&\quad \times (u_n(x)-u_n(y)-u(x)+u(y))dxdy \\
&\le C(B_{u_n}(u_n-u)-B_u(u_n-u)) \\
&\to 0
\end{aligned}$$

as $n\to\infty$. By Hölder inequality,

$$\begin{aligned}
\Phi_2 &= \iint_{(\Omega\times\Omega)_2} \frac{|u_n(x)-u_n(y)-u(x)+u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}} dxdy \\
&\le C\iint_{(\Omega\times\Omega)_2} \Big[\frac{|u_n(x)-u_n(y)|^{\frac{p(x)+p(y)}{2}-2}(u_n(x)-u_n(y)) - |u(x)-u(y)|^{\frac{p(x)+p(y)}{2}-2}(u(x)-u(y))}{|x-y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}} \\
&\quad \times (u_n(x)-u_n(y)-u(x)+u(y))\Big]^{\frac{p(x)+p(y)}{4}} \\
&\quad \times \Big(\frac{|u_n(x)-u_n(y)|^{\frac{p(x)+p(y)}{2}} + |u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}}\Big)^{\frac{4-p(x)-p(y)}{4}} dxdy \\
&\le C[(B_{u_n}(u_n-u)-B_u(u_n-u))^{\frac{p^+}{2}} + (B_{u_n}(u_n-u)-B_u(u_n-u))^{\frac{p^-}{2}}] \\
&\quad \times [(\varphi_{s(\cdot),p(\cdot),\Omega}(u_n)+\varphi_{s(\cdot),p(\cdot),\Omega}(u))^{\frac{2-p^+}{2}} + (\varphi_{s(\cdot),p(\cdot),\Omega}(u_n)+\varphi_{s(\cdot),p(\cdot),\Omega}(u))^{\frac{2-p^-}{2}}].
\end{aligned}$$

By (4.9) and Proposition 2.3 in [31], there exists $M>0$ such that $\varphi_{s(\cdot),p(\cdot),\Omega}(u_n)+\varphi_{s(\cdot),p(\cdot),\Omega}(u)\le M$, then

$$\begin{aligned}
\Phi_2 &\le C(M^{\frac{2-p^+}{2}} + M^{\frac{2-p^-}{2}})[(B_{u_n}(u_n-u)-B_u(u_n-u))^{\frac{p^+}{2}} + (B_{u_n}(u_n-u)-B_u(u_n-u))^{\frac{p^-}{2}}] \\
&\to 0
\end{aligned}$$

as $n\to\infty$. So $\varphi_{s(\cdot),p(\cdot),\Omega}(u_n-u)\to 0$ and further by Proposition (2.4),

$$[u_n-u]_{W^{s(\cdot),p(\cdot)}(\Omega)} \to 0 \tag{4.14}$$

as $n\to\infty$. By (4.13) and (4.14), we have $\|u_n-u\|_{W^{s(\cdot),p(\cdot)}(\Omega)}\to 0$ as $n\to\infty$. Therefore I satisfies PS condition. □

In the proof of Theorem 4.1, we will apply Mountain Pass Theorem and Ekeland variational principle. In order to make the proof more clear, we first state the two theorems:

Theorem 4.2. *[2] (Mountain Pass Theorem) Let X be a Banach space. $f\in C^1(X,\mathbb{R})$ satisfies the following conditions*

(1) $f(0)=0$ and there exists a constant $\rho>0$ such that $f|_{\partial B_\rho(0)}\ge\alpha>0$;

(2) there exists $x_0 \in X \setminus \bar{B}_\rho(0)$ such that $f(x_0) \le 0$. Let

$$\Gamma = \{g \in C([0,1], X) : g(0) = 0,\ g(1) = x_0\},$$

$$C = \inf_{g\in\Gamma} \max_{t\in[0,1]} f(g(t)),$$

then $C \ge \alpha$. If f satisfies PS conditions, then C is a critical value of f.

Theorem 4.3. *[19] (Ekeland Variational Principle) Let (X, d) be a complete metric space. $f : X \to \mathbb{R} \cup \{+\infty\}$ is bounded from below and lower semi-continuous. If for any $\varepsilon > 0, \delta > 0$ there exists $u = u(\varepsilon, \delta) \in X$ such that*

$$f(u) \le \inf_{x\in X} f(x) + \varepsilon,$$

then there exists some point $v = v(\varepsilon, \delta) \in X$ satisfies

$$f(v) \le f(u),$$

$$d(u, v) \le \delta,$$

$$f(v) < f(x) + \frac{\varepsilon}{\delta} d(v, x), \quad \text{for all } x \ne v.$$

Proof of Theorem 4.1. By Lemma 4.5, Lemma 4.6 and Lemma 4.8, I has mountain pass structure. By Mountain Pass Theorem, there exists a critical value $C_1 \ge \alpha_0 > 0$ and a corresponding critical point $u_1 \in W^{s(\cdot),p(\cdot)}(\Omega)$ such that $I(u_1) = C_1$, where α_0 is the one in Lemma 4.5.

On the other hand, by Lemma 4.7, we have

$$C_2 = \inf\{I(u) : u \in \bar{B}_{\rho_0}\} < 0.$$

Since I is lower semi-continuous, by Ekeland variational principle and Lemma 4.5, there exists a sequence $\{u_n\} \subset B_{\rho_0}$ such that

$$C_2 \le I(u_n) \le C_2 + \frac{1}{n} \text{ and } I(v) \ge I(u_n) - \frac{1}{n}\|v - u_n\|_{W^{s(\cdot),p(\cdot)}(\Omega)}$$

for all $v \in B_{\rho_0}$. Then we can infer that $\{u_n\}$ is a PS sequence. By Lemma 4.5 and Lemma 4.8, there exists a critical point $u_2 \in B_{\rho_0}$ such that $I(u_2) = C_2 < 0$ and $u_1 \ne u_2 \ne 0$. □

5. Conclusions

We obtain embedding theorems for variable exponent fractional Sobolev space $W^{s(\cdot),p(\cdot)}(\Omega)$: In the case that Ω is a bounded open set, if $s_2(x) \ge s_1(x)$, space $W^{s_2(\cdot),p(\cdot)}(\Omega)$ can be continuously embedded into $W^{s_1(\cdot),p(\cdot)}(\Omega)$. In the case that Ω is a Lipschitz bounded domain, if $s(x)p(x) < n$, for continuous function q with $1 < q(x) < p^*(x)$, $W^{s(\cdot),p(\cdot)}(\Omega)$ can not only be continuously embedded, but also be compactly embedded into $L^{q(\cdot)}(\Omega)$. As an application of the embedding theorems, we obtain that the problem (1.5) of $s(x)$-$p(x)$-Laplacian equations has at least two nontrivial weak solutions when the nonlinear function f satisfies conditions (F1)–(F3), the potential function V satisfies condition (V), the exponen p, q, s satisfies condition (PQS) and g satisfies condition (G).

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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