AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(9): 9786–9812.
DOI: 10.3934/math.2021569
Received: 08 December 2020
Accepted: 08 June 2021
Published: 29 June 2021

Research article

Well posedness for a singular two dimensional fractional initial boundary value problem with Bessel operator involving boundary integral conditions

Said Mesloub*and Faten Aldosari

Department of Mathematics, King Saud University University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: mesloub@ksu.edu.sa.

Abstract: This paper studies the existence and uniqueness of solutions of a non local initial boundary value problem of a singular two dimensional nonlinear fractional order partial differential equation involving the Caputo fractional derivative by employing the functional analysis. We first establish for the associated linear problem a priori estimate and prove that the range of the operator generated by the considered problem is dense. The technique of obtaining the a priori bound relies on the construction of a suitable multiplicator. From the resulted a priori estimate, we can establish the solvability of the associated linear problem. Then, by applying an iterative process based on the obtained results for the associated linear problem, we establish the existence, uniqueness and continuous dependence of the weak solution of the considered nonlinear problem.

Keywords: fractional differential equation; boundary integral condition; singular initial boundary value problem; well posedness; iterative process; bessel operator
Mathematics Subject Classification: 35D35, 35L20

1. Introduction

Initial boundary value problems with non local and non-classical boundary conditions for integer and fractional order linear and nonlinear evolution partial differential equations, have gained great attention during the last three decades. Especially problems with boundary conditions of integral type (the so called energy specification) which are important from the point of view of their practical application to modelling and investigating various physical phenomena in the context of chemical engineering, thermoelasticity, population dynamics, polymer rheology, aerodynamics, heat conduction processes, plasma physics, underground-water flow, transmission theory, chemical engineering, control theory, fluid flow and many physical and biological processes and systems and so forth, see [19–31]. We should mention that most local phenomena can be examined and modeled in terms of integer order differential equations, while fractional order differential equations model non local phenomena. Accordingly, fractional order partial differential equations describe real world phenomena that cannot be described by classical mathematics literature. This is due to the fact that many models depend on the present and historical states. For integer order case (see for example [1–11] and references therein. For the fractional order case see for example [15–18, 35–37] and references therein. However, the investigation of initial boundary value problems for nonlinear fractional order partial differential equations still needs too much exploration and investigation.

For the proof of the existence and the uniqueness of the solution of the posed problem, we use the energy inequality method based mainly on some a priori estimates and on the density of the range of the operator generated by the considered problem. In the literature, there are few articles using the method of energy inequalities for the proof of existence and uniqueness of fractional initial-boundary value problems in the fractional case (see [12–14, 32–34]). This work, can be considered as a continuation, improvement and generalization of previous works. Many difficulties are encountered while applying the functional analysis method for the posed problem. These difficulties are mainly due to the fact that the considered equation is nonlinear, singular, with fractional order in a two-dimensional space setting, and supplemented with nonlocal conditions.

The outline of this paper is as follows. In section 2, we set the problem and give some preliminaries. In section 3, we pose the associated linear problem and introduce some function spaces used in the sequel. Section 4 is devoted to the uniqueness results for the associated linear problem. The existence of solution of the associated linear problem is considered in section 5. The main results of this paper are given in section 6, it is consecrated to the proof of the existence, uniqueness and continuous dependence of the solution on the data of the nonlinear problem.

2. Problem setting and some preliminaries

In the bounded domain $Q^T = \Omega\times(0,T)$, where $\Omega = (0,a)\times(0,b) \subset \mathbb{R}^2$, we consider the two dimensional singular nonlinear fractional partial differential equation in Caputo sense with Bessel operator

$$\mathcal{L}(\theta) = \partial_t^\delta\theta - \frac{1}{xy}div(xy\nabla\theta) = H(x,y,t,\theta,\theta_x,\theta_y), \tag{2.1}$$

where ∂_t^δ denotes the time fractional Caputo derivative operator of order $\delta\in(1,2]$, the symbols div and ∇, denote respectively the divergence and the gradient operators. θ_x stands for the $x-$derivative of the function θ. Equation (2.1), is supplemented by the initial conditions

$$\ell_1\theta = \theta(x,y,0) = F(x,y)\ ,\ \ell_2\theta = \theta_t(x,y,0) = G(x,y), \tag{2.2}$$

Neumann boundary conditions

$$\theta_x(a,y,t) = 0\ ,\ \theta_y(x,b,t) = 0, \tag{2.3}$$

and the non local weighted boundary integral conditions

$$\int_0^a x\theta(x,y,t)dx = 0\ ,\ \int_0^b y\theta(x,y,t)dx = 0, \tag{2.4}$$

where the functions F, and G are given functions which will be specified later on. We shall assume that the function H is a Lipshitzian function, that is there exists a positive constant λ such that for all $(x, y, t) \in Q^T$

$$\begin{aligned} & H(x,y,t,\theta_1,v_1,w_1) - H(x,y,t,\theta_2,v_2,w_2) \\ \leq \quad & \lambda\left(|\theta_1-\theta_2|+|v_1-v_2|+|w_1-w_2|\right). \end{aligned} \tag{2.5}$$

In equation (2.1), the fractional derivative $\partial_t^\delta \mathcal{E}$ of order $\delta = \beta + 1$, where $0 < \beta < 1$ (see [17]) for a function $\mathcal{E}$ is defined by

$${}^C\partial_t^\delta \mathcal{E}(x,t) = \frac{1}{\Gamma(1-\delta)} \int_0^t \frac{\mathcal{E}_{ss}(x,s)}{(t-s)^\delta} ds, \quad \forall t \in [0,T], \tag{2.6}$$

where $\Gamma(.)$ is the Gamma function.

We begin by giving some important lemmas needed throughout the sequel.

Lemma 2.1 [38]. Let $R(s)$ be nonnegative and absolutely continuous on $[0, T]$, and suppose that for almost all $s \in [0, T]$, R satisfies the inequality

$$\frac{dR}{ds} \leq A_1(s)R(t) + B_1(s), \tag{2.7}$$

where the functions $A_1(s)$ and $B_1(s)$ are summable and nonnegative on $[0, T]$. Then

$$R(s) \leq e^{\int\limits_0^s A_1(t)dt}\left(R(0) + \int\limits_0^s B_1(t)dt\right). \tag{2.8}$$

Lemma 2.2. [14] Let be $M(t)$ a nonnegative absolutely continuous function, such that

$${}^C\partial_t^\lambda M(t) \leq b_1 M(t) + b_2(t), \quad 0 < \lambda < 1, \tag{2.9}$$

for almost all $t \in [0, T]$, where b_1 is a positive constant and $b_2(t)$ is an integrable nonnegative function on $[0, T]$. Then

$$M(t) \leq M(0)E_\lambda(b_1 t^\lambda) + \Gamma(\lambda)E_{\lambda,\lambda}(b_1 t^\lambda)D_t^{-\lambda} b_2(t), \tag{2.10}$$

where

$$E_\lambda(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\lambda} n+1) \text{ and } E_{\lambda,\mu}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\lambda n+\mu)},$$

are the MIttag-Leffler functions, and $D_t^{-\lambda} v(t) = \frac{1}{\Gamma(\lambda)} \int\limits_0^t \frac{v(\tau)}{(t-\tau)^{1-\lambda}} d\tau$ is the Riemann-Liouville integral of order $0 < \lambda < 1$.

Lemma 2.3. [14] For any absolutely continuous function $J(t)$ on the interval $[0, T]$, the following inequality holds

$$J(t)\, {}^C\partial_t^\beta J(t) \geq \frac{1}{2}\, {}^C\partial_t^\beta J^2(t), \quad 0 < \beta < 1 \tag{2.11}$$

3. Statement of the associated linear problem

In this section, we set the associated linear problem and introduce different function spaces needed to investigate this problem. We consider the differential equation

$$\mathcal{L}(\theta) = \partial_t^\delta \theta - \frac{1}{xy} div(xy\nabla\theta) = H(x,y,t), \tag{3.1}$$

supplemented by conditions (2.2)–(2.4). The used method is essentially based on the construction of suitable multipliers for each specific given problem, which provides the a priori estimate from which it is possible to establish the solvability of the posed problem. More precisely, the proofs of uniqueness of the solution is based on an energy inequality and on the density of the range of the operator generated by the abstract formulation of the stated problem.

To investigate the posed problem, we introduce the needed function spaces. We denote by $L_\rho^2(\Omega)$ the Hilbert space of weighted square integrable functions where $\rho = xy$ and with inner product

$$(\mathcal{U}, v)_{L_\rho^2(\Omega)} = (xy\mathcal{U}, v)_{L^2(\Omega)} = \int_\Omega xy\mathcal{U}v dxdy,$$

and with associated norm

$$\|\mathcal{U}\|_{L_\rho^2(\Omega)} = \left(\int_\Omega xy\mathcal{U}^2 dxdy\right)^{\frac{1}{2}}.$$

Let X be Banach space with norm $\|\mathcal{U}\|_X$, and let $\mathcal{U} : (0,T) \to X$ be an abstract function. By $\mathcal{U}(.,.,t)\|_X$ we denote the norm of $\mathcal{U}(.,.,t) \in X$ for fixed t. Let $L^2(0,T;X)$ be the set of all measurable abstract functions $\mathcal{U} : (0,T) \to X$ such that

$$\|\mathcal{U}\|^2_{L^2(0,T;X)} = \int_0^T \|\mathcal{U}(.,.,t)\|_X^2 dt < \infty.$$

If X is a Hilbert space, then $L^2(0,T;X)$ is also a Hilbert space. Let $C(0,T;X)$ be the set of all continuous functions $\mathcal{U} : (0,T) \to X$ such that

$$\|\mathcal{U}\|_{C(0,T;X)} = \max_{t\in[0,T]} \|\mathcal{U}(.,.,t)\|_X < \infty,$$

and denote by $H_\rho^1(\Omega)$ the weighted Sobolev space with norm

$$\|\mathcal{U}\|^2_{H_\rho^1(\Omega)} = \|\mathcal{U}\|^2_{L_\rho^2(\Omega)} + \|\mathcal{U}_x\|^2_{L_\rho^2(\Omega)} + \|\mathcal{U}_y\|^2_{L_\rho^2(\Omega)}.$$

The given problem (3.1), (2.2)–(2.4) can be viewed as the problem of solving the operator equation $A\theta = W = (H, F, G)$, where $A\theta = (\mathcal{L}\theta, \ell_1\theta, \ell_2\theta)$, $\forall\theta \in D(A)$ where A is the operator given by $A = (\mathcal{L}, \ell_1, \ell_2)$ and $D(A)$ is the set of all functions $\theta \in L_\rho^2(Q^T) : \partial_t^\delta\theta, \theta_x, \theta_{xx}, \theta_y, \theta_{yy}, \theta_t \in L_\rho^2(Q^T)$ and θ satisfies conditions (2.2)–(2.4). The operator A acts from B into Y , where B is the Banach space obtained by enclosing $D(A)$ with respect to the finite norm

$$\|\theta\|_B^2 = \sup_{0\le t\le T} \|\theta(x,y,\tau)\|^2_{H^1_\rho(\Omega)} = \|\theta(x,y,\tau)\|^2_{C(0,T.,H^1_\rho(\Omega))}. \tag{3.2}$$

Functions $\theta \in B$ are continuous on $[0,T]$ with values in $L^2_\rho(\Omega)$. Hence the mappings

$$\ell_1\theta \in B \to \ell\theta = \theta(x,y,0) \in L^2_\rho(\Omega),$$

$$\ell_2\theta \in B \to \ell\theta = \theta_t(x,y,0) \in L^2_\rho(\Omega),$$

are defined and continuous on B. And Y is the Hilbert space $L^2(0,T,L^2_\rho(\Omega))\times H^1_\rho(\Omega)\times L^2_\rho(\Omega)$ consisting of vector valued functions $W = (H,F,G)$ for which the norm

$$\|W\|_Y^2 = \|H\|^2_{L^2(0,T;L^2_\rho(\Omega))} + \|F\|^2_{H^1_\rho(\Omega)} + \|G\|^2_{L^2_\rho(\Omega)},$$

is finite. Let $\overline{A}$ be the closure of the operator A with domain of definition $D(\overline{A})$.

Definition 3. We call a strong solution of problem (3.1), (2.2)-2.4), the solution of the operator equation

$$\overline{A}\theta = W, \quad \forall\theta \in D(\overline{A}).$$

4. Main result of uniqueness of solution for the linear problem

We will establish an a priori estimate for the operator A from which we deduce the uniqueness and continuous dependence of the solution upon the initial conditions (2.2).

Theorem 4.1 For any function $\theta \in D(A)$ we have the a priori estimate

$$\begin{aligned} & \|\theta(x,y,\tau)\|^2_{C(0,T.,H^1_\rho(\Omega))} \\ \le\ & C\left(\|H\|^2_{L^2(0,T;L^2_\rho(\Omega))} + \|F\|^2_{H^1_\rho(\Omega)} + \|G\|^2_{L^2_\rho(\Omega)}\right), \end{aligned} \tag{4.1}$$

where C is a positive constant independent of the function θ given by

$$C = (C_3 e^{C_3 T} T + 1),$$

with

$$\begin{aligned} C_3 &= C_2^2\Gamma(\beta)E_{\beta,\beta}(C_2T^\beta)(\frac{T^\beta}{\beta\Gamma(\beta)}) + C_2, \\ C_2 &= \max\left\{C_1, 1, \frac{T^{1-\beta}}{(1-\beta)\Gamma(1-\beta)}(1+\frac{a^3b^3}{16})\right\}, \\ C_1 &= \max\left\{2, (\frac{a^2b^2}{2} + \frac{a^2}{2} + \frac{b^2}{2})\right\}. \end{aligned}$$

Proof.

Let $\beta + 1 = \delta$, where $0 < \beta \leq 1$, then (3.1) takes the form

$$\partial_t^{\beta+1}\theta - \frac{1}{x}(x\theta_x)_x - \frac{1}{y}(y\theta_y)_y = H(x, y, t). \tag{4.2}$$

Taking the scalar product in $L^2_\rho(\Omega)$ of the partial differential equation (4.2) and the integro-differential operator $M\theta = \theta_t + \Im^2_{xy}(\xi\eta\theta_t)$, where

$$\Im^2_{xy}(\xi\eta\theta_t) = \int\limits_0^x \int\limits_0^y \xi\eta\theta_t(\xi, \eta)d\eta d\xi,$$

then we have

$$\begin{aligned} & (\partial_t^\beta\theta_t, xy\theta_t)_{L^2(\Omega)} - (\theta_x, y\theta_t)_{L^2(\Omega)} - (\theta_{xx}, xy\theta_t)_{L^2(\Omega)} \\ - & (\theta_y, x\theta_t)_{L^2(\Omega)} - (\theta_{yy}, xy\theta_t)_{L^2(\Omega)} + (\partial_t^\beta\theta_t, xy\Im^2_{xy}(\xi\eta\theta_t))_{L^2(\Omega)} \\ - & (\theta_x, y\Im^2_{xy}(\xi\eta\theta_t))_{L^2(\Omega)} - (\theta_{xx}, xy\Im^2_{xy}(\xi\eta\theta_t))_{L^2(\Omega)} \\ - & (\theta_y, x\Im^2_{xy}(\xi\eta\theta_t))_{L^2(\Omega)} - (\theta_{yy}, xy\Im^2_{xy}(\xi\eta\theta_t))_{L^2(\Omega)} \\ = & (H, xy\theta_t)_{L^2(\Omega)} + (H, xy\Im^2_{xy}(\xi\eta\theta_t))_{L^2(\Omega)}. \end{aligned} \tag{4.3}$$

The standard integration by parts of each term in equation (4.3) leads to

$$(\partial_t^\beta\theta_t, xy\theta_t)_{L^2(\Omega)} = (\partial_t^\beta\theta_t, \theta_t)_{L^2_\rho(\Omega)}, \tag{4.4}$$

$$(\partial_t^\beta\theta_t, xy\Im^2_{xy}(\xi\eta\theta_t))_{L^2(\Omega)} = (\partial_t^\beta(\Im_{xy}(\xi\eta\theta_t), \Im_{xy}(\xi\eta\theta_t))_{L^2_\rho(\Omega)}, \tag{4.5}$$

$$-(\theta_x, y\theta_t)_{L^2(\Omega)} = -\int\limits_0^a \int\limits_0^b y\theta_x\theta_t dxdy, \tag{4.6}$$

$$-(\theta_y, x\theta_t)_{L^2(\Omega)} = -\int\limits_0^a \int\limits_0^b x\theta_y\theta_t dxdy, \tag{4.7}$$

$$-(\theta_{xx}, xy\theta_t)_{L^2(\Omega)} = \int\limits_0^a \int\limits_0^b y\theta_x\theta_t dxdy + \frac{\partial}{2\partial t}\left\|\theta_x\right\|^2_{L^2_\rho(\Omega)}, \tag{4.8}$$

$$-(\theta_{yy}, xy\theta_t)_{L^2(\Omega)} = \int\limits_0^a \int\limits_0^b x\theta_y\theta_t dxdy + \frac{\partial}{2\partial t}\left\|\theta_y\right\|^2_{L^2_\rho(\Omega)}, \tag{4.9}$$

$$-(\theta_x, y\Im^2_{xy}(\xi\eta\theta_t))_{L^2(\Omega)} = -\int\limits_0^a \int\limits_0^b y\theta_x\Im^2_{xy}(\xi\eta\theta_t)dxdy, \tag{4.10}$$

$$-(\theta_y, x\Im^2_{xy}(\xi\eta\theta_t))_{L^2(\Omega)} = -\int\limits_0^a \int\limits_0^b x\theta_y\Im^2_{xy}(\xi\eta\theta_t)dxdy, \tag{4.11}$$

$$-(\theta_{xx}, xy\Im^2_{xy}(\xi\eta\theta_t))_{L^2(\Omega)} = \int_0^a\int_0^b \theta_x y\Im^2_{xy}(\xi\eta\theta_t)dxdy + (\theta_x, \Im_x\Im^2_y(\xi\eta\theta_t))_{L^2_\rho(\Omega)}, \tag{4.12}$$

$$-(\theta_{yy}, xy\Im^2_{xy}(\xi\eta\theta_t))_{L^2(\Omega)} = \int_0^a\int_0^b \theta_y x\Im^2_{xy}(\xi\eta\theta_t)dxdy + (\theta_y, \Im_y\Im^2_x(\xi\eta\theta_t))_{L^2_\rho(\Omega)}, \tag{4.13}$$

Substitution of equation (4.4)–(4.13) into equation (4.3) yields

$$\begin{aligned} & (\partial_t^\beta\theta_t, \theta_t)_{L^2_\rho(\Omega)} + (\partial_t^\beta(\Im_{xy}(\xi\eta\theta_t), \Im_{xy}(\xi\eta\theta_t))_{L^2_\rho(\Omega)} \\ + & \frac{\partial}{2\partial t}\|\theta_x\|^2_{L^2_\rho(\Omega)} + \frac{\partial}{2\partial t}\left\|\theta_y\right\|^2_{L^2_\rho(\Omega)} \\ = & \int_\Omega xy\theta_t H dxdy + \int_\Omega xyH\Im^2_{xy}(\xi\eta\theta_t)dxdy \\ - & (\theta_x, \Im_x\Im^2_y(\xi\eta\theta_t))_{L^2_\rho(\Omega)} - (\theta_y, \Im_y\Im^2_x(\xi\eta\theta_t))_{L^2_\rho(\Omega)}. \end{aligned} \tag{4.14}$$

Using Cauchy ϵ- inequality ($AB \leq \frac{\varepsilon}{2}A^2 + \frac{1}{2\varepsilon}B^2$), Poincare' type inequalities ($\|\mathcal{I}_x(\xi\upsilon)\|^2_{L^2(0,1)} \leq \frac{1}{2}\|\upsilon\|^2_{L^2(0,1)}, \left\|\mathcal{I}^2_x(\xi\upsilon)\right\|^2_{L^2(0,1)} \leq \frac{1}{2}\|\mathcal{I}_x(\xi\upsilon)\|^2_{L^2(0,1)}$) [7] and Lemma 2.3, we transform (4.14) to

$$\begin{aligned} & \partial_t^\beta\|\theta_t\|^2_{L^2_\rho(\Omega)} + \partial_t^\beta\left\|\Im_{xy}(\xi\eta\theta_t)\right\|^2_{L^2_\rho(\Omega)} \\ + & \partial/\partial t\|\theta_x\|^2_{L^2_\rho(\Omega)} + \partial/\partial t\left\|\theta_y\right\|^2_{L^2_\rho(\Omega)} \\ \leq & \epsilon_1\|\theta_t\|^2_{L^2_\rho(\Omega)} + (\frac{1}{\epsilon_1} + \epsilon_2)\|H\|^2_{L^2_\rho(\Omega)} \\ + & \epsilon_3\|\theta_x\|^2_{L^2_\rho(\Omega)} + \epsilon_4\left\|\theta_y\right\|^2_{L^2_\rho(\Omega)} \\ & +\left(\frac{a^2b^2}{2\epsilon_2} + \frac{a^2}{2\epsilon_4} + \frac{b^2}{2\epsilon_3}\right)\left\|\Im_{xy}(\xi\eta\theta_t)\right\|^2_{L^2_\rho(\Omega)}. \end{aligned} \tag{4.15}$$

Let $\epsilon_i = 1, \ \ i = 1, 2, 3, 4$, in (4.15), then it follows that

$$\begin{aligned} & \partial_t^\beta\|\theta_t\|^2_{L^2_\rho(\Omega)} + \partial_t^\beta\|\Im_{xy}(\xi\eta\theta_t)\|^2_{L^2(\Omega)} \\ + & \frac{\partial}{\partial t}\|\theta_y\|^2_{L^2_\rho(\Omega)} + \frac{\partial}{\partial t}\|\theta_x\|^2_{L^2_\rho(\Omega)} \\ \leq & C_1\Big(\|\theta_t\|^2_{L^2_\rho(\Omega)} + \|H\|^2_{L^2_\rho(\Omega)} \\ & +\|\theta_x\|^2_{L^2_\rho(\Omega)} + \|\theta_y\|^2_{L^2_\rho(\Omega)} + \|\Im_{xy}(\xi\eta\theta_t)\|^2_{L^2_\rho(\Omega)}\Big), \end{aligned} \tag{4.16}$$

where

$$C_1 = \max\left\{2, (\frac{a^2b^2}{2} + \frac{a^2}{2} + \frac{b^2}{2})\right\}.$$

We infer from (4.16) that

$$D_t^{\beta-1}\|\theta_t\|^2_{L^2_\rho(\Omega)} + D_t^{\beta-1}\|\Im_{xy}(\xi\eta\theta_\tau)\|^2_{L^2(\Omega)} + \|\theta_y\|^2_{L^2_\rho(\Omega)} + \|\theta_x\|^2_{L^2_\rho(\Omega)}$$

$$
\begin{aligned}
\leq\ & C_1\left(\int_0^t \|\theta_\tau\|^2_{L^2_\rho(\Omega)}d\tau + \int_0^t \|H\|^2_{L^2_\rho(\Omega)}d\tau + \int_0^t \|\Im_{xy}(\xi\eta\theta_\tau)\|^2_{L^2_\rho(\Omega)}d\tau\right.\\
& \left.\int_0^t \|\theta_x\|^2_{L^2_\rho(\Omega)}d\tau + \int_0^t \|\theta_y\|^2_{L^2_\rho(\Omega)}d\tau\right) + \|F_x\|^2_{L^2_\rho(\Omega)} + \|F_y\|^2_{L^2_\rho(\Omega)}\\
& + \frac{T^{1-\beta}}{(1-\beta)\Gamma(1-\beta)}\left[\|\Im_{xy}\xi\eta G(\xi,\eta)\|^2_{L^2(\Omega)} + \|G\|^2_{L^2_\rho(\Omega)}\right].
\end{aligned}
\tag{4.17}
$$

By using a poincare type inequality, (4.17) becomes

$$
\begin{aligned}
& D_t^{\beta-1}\|\theta_t\|^2_{L^2_\rho(\Omega)} + D_t^{\beta-1}\|\Im_{xy}(\xi\eta\theta_\tau)\|^2_{L^2(\Omega)} + \|\theta_y\|^2_{L^2_\rho(\Omega)} + \|\theta_x\|^2_{L^2_\rho(\Omega)}\\
\leq\ & C_2\left(\int_0^t \|\theta_\tau\|^2_{L^2_\rho(\Omega)}d\tau + \int_0^t \|\Im_{xy}(\xi\eta\theta_\tau)\|^2_{L^2_\rho(\Omega)}d\tau\right.\\
& + \int_0^t \|\theta_x\|^2_{L^2_\rho(\Omega)}d\tau + \int_0^t \|\theta_y\|^2_{L^2_\rho(\Omega)}d\tau\\
+\ & \left.\int_0^t \|H\|^2_{L^2_\rho(\Omega)}d\tau + \|F\|^2_{H^1_\rho(\Omega)} + \|G\|^2_{L^2_\rho(\Omega)}\right),
\end{aligned}
\tag{4.18}
$$

where

$$
C_2 = \max\left\{C_1, 1, \frac{T^{1-\beta}}{(1-\beta)\Gamma(1-\beta)}(1+\frac{a^3b^3}{16})\right\}.
$$

By dropping the last two terms from left side of (4.18), and applying Lemma 2.2 by taking

$$
h(t) = \int_0^t \|\theta_\tau\|^2_{L^2_\rho(\Omega)}d\tau + \int_0^t \|\Im_{xy}(\xi\eta\theta_\tau)\|^2_{L^2_\rho(\Omega)}d\tau,
$$

$$
\partial_t^\beta h(t) = D_t^{\beta-1}\|\theta_t\|^2_{L^2_\rho(\Omega)} + D_{t-1}^{\beta}\|\Im_{xy}(\xi\eta\theta_\tau)\|^2_{L^2(\Omega)},
$$

and $h(0) = 0$, we have

$$
\begin{aligned}
h(t)\ \leq\ & C_2\Gamma(\beta)E_{\beta,\beta}(C_1T^\beta)D_t^{-\beta}\left(\int_0^t \|\theta_x\|^2_{L^2_\rho(\Omega)}d\tau\right.\\
+\ & \int_0^t \|\theta_y\|^2_{L^2_\rho(\Omega)}d\tau + \int_0^t \|H\|^2_{L^2_\rho(\Omega)}d\tau\\
+\ & \left.\|F\|^2_{H^1_\rho(\Omega)} + \|G\|^2_{L^2_\rho(\Omega)}\right),
\end{aligned}
\tag{4.19}
$$

Since

$$
D_t^{-\beta}\int_0^t \|J(x,y,\tau)\|^2_{L^2_\rho(\Omega)}d\tau \leq \frac{T^\beta}{\beta\Gamma(\beta)}\int_0^t \|J(x,y,\tau)\|^2_{L^2_\rho(\Omega)}d\tau,
$$

then

$$\begin{aligned} h(t) &\leq C_2\Gamma(\beta)E_{\beta,\beta}(C_2T^\beta)(\frac{T^\beta}{\beta\Gamma(\beta)})\Bigg(\int_0^t \|\theta_x\|^2_{L^2_\rho(\Omega)}d\tau \\ &+ \int_0^t \|\theta_y\|^2_{L^2_\rho(\Omega)}d\tau + \int_0^t \|H\|^2_{L^2_\rho(\Omega)}d\tau \\ &+ \|F\|^2_{H^1_\rho(\Omega)} + \|G\|^2_{L^2_\rho(\Omega)}\Bigg), \end{aligned} \tag{4.20}$$

Hence (4.18) becomes

$$\begin{aligned} & D_t^{\beta-1}\|\theta_t\|^2_{L^2_\rho(\Omega)} + D_t^{\beta-1}\|\Im_{xy}(\xi\eta\theta_\tau)\|^2_{L^2(\Omega)} \\ & +\|\theta_y\|^2_{L^2_\rho(\Omega)} + \|\theta_x\|^2_{L^2_\rho(\Omega)} \\ \leq\ & C_3\Bigg(\int_0^t \|\theta_x\|^2_{L^2_\rho(\Omega)}d\tau + \int_0^t \|\theta_y\|^2_{L^2_\rho(\Omega)}d\tau + \int_0^t \|H\|^2_{L^2_\rho(\Omega)}d\tau \\ & +\|F\|^2_{H^2_\rho(\Omega)} + \|G\|^2_{L^2_\rho(\Omega)}\Bigg), \end{aligned} \tag{4.21}$$

where

$$C_3 = C_2^2\Gamma(\beta)E_{\beta,\beta}(C_2T^\beta)(\frac{T^\beta}{\beta\Gamma(\beta)}) + C_2.$$

Now discard the first two terms from left hand side in (4.21) and use lemma 2.1 with

$$S(t) = \int_0^t \|\theta_y\|^2_{L^2_\rho(\Omega)}d\tau + \int_0^t \|\theta_x\|^2_{L^2_\rho(\Omega)}d\tau,$$

$$dS(t)/dt = \|\theta_y\|^2_{L^2_\rho(\Omega)} + \|\theta_x\|^2_{L^2_\rho(\Omega)},$$
$$S(0) = 0,$$

we see that

$$\begin{aligned} & D_t^{\beta-1}\|\theta_t\|^2_{L^2_\rho(\Omega)} + D_t^{\beta-1}\|\Im_{xy}(\xi\eta\theta_\tau)\|^2_{L^2(\Omega)} \\ + & \|\theta_y\|^2_{L^2_\rho(\Omega)} + \|\theta_x\|^2_{L^2_\rho(\Omega)} \\ \leq & C\Big(\int_0^T \|H\|^2_{L^2_\rho(\Omega)}d\tau + \|F\|^2_{H^2_\rho(\Omega)} + \|G\|^2_{L^2_\rho(\Omega)}\Big), \end{aligned} \tag{4.22}$$

where

$$C = (C_3e^{C_3T}T + 1).$$

Now if we omit the first two terms from the left-hand side of (4.22) and use the fact that (see Lemma 6.3)

$$\|\nabla\theta\|^2_{L^2_\rho(\Omega)} \sim \|\theta\|^2_{H^1_\rho(\Omega)},$$

then we get after passing to the supremum over $(0, T)$

$$\begin{aligned} & \sup_{0\le t\le T} \left\{ \|\theta\|^2_{H^1_\rho(\Omega)} \right\} \\ \le \quad & C\Big(\|H\|^2_{L^2(0,T;L^2_\rho(\Omega))} + \|F\|^2_{H^1_\rho(\Omega)} + \|G\|^2_{L^2_\rho(\Omega)} \Big). \end{aligned} \tag{4.23}$$

Since the only information we have about the range of the operator A, is that $R(A) \subset Y$, we must extend A so that the estimate (4.23) holds for the extension and its range is the hole space Y. To this end, we establish the following proposition.

Proposition 4.2 The operator $A : B \to Y$ admits a closure.

Proof. The proof can be done as in [7]

Let $\overline{A}$ be the closure of the operator A, and $D(\overline{A})$ be its domain. The inequality (4.1) can be extended to strong solutions after passing to limit, that is we have

$$\|\theta\|_B \le C\|\overline{A}\theta\|_Y, \quad \forall\theta \in D(\overline{A}),$$

from which we deduce that $R(\overline{A})$ is closed in Y and that $R(\overline{A}) = \overline{R(A)}$.

Definition 4.3. A solution of the equation

$$\overline{A}\theta = (\mathcal{L}\theta, \ell_1\theta, \ell_2\theta) = (H, F, G)$$

is called a strong solution of problem (3.1), (2.2)-2.4).

5. Solvability of the posed associated linear problem

Theorem 5.1. Problem (3.1), (2.2)-2.4), has a unique strong solution $\theta = \overline{L^{-1}}(H, F, G) = \overline{L^{-1}}(H, F, G)$, that depends continuously on the data, for all $H \in L^2(0, T; L^2_\rho(\Omega))$, $G \in L^2_\rho(\Omega)$ and $F \in H^1_\rho(\Omega)$.

Proof. To prove that problem (3.1), (2.2)-2.4) has unique strong solution for all $W = (H, F, G) \in Y$, it suffices to prove that the range $R(A)$ of the operator A is dense in Y. For this we need to prove the following proposition.

Proposition 5.2. If for some function $g(x, y, t) \in L^2_\rho(Q^T)$ and for all $k(x, y, t) \in D(A)$ satisfies homogeneous initial conditions we have

$$(\mathcal{L}k, g)_{L^2_\rho(Q^T)} = 0, \tag{5.1}$$

then $g \equiv 0$ a.e in Q^T.

Proof. Equation (5.1) implies

$$(\partial_t^\delta k - \frac{1}{x}k_x - k_{xx} - \frac{1}{y}k_y - k_{yy}, g)_{L^2_\rho(Q^T)} = 0, \tag{5.2}$$

Let $P(x, y, t)$ be a function satisfying conditions (2.2)-2.4) and such that P, P_x, P_y, $\Im_t\Im^2_{xy}P$, $\Im_tP_x$, $\Im_tP_y$, $\partial_t^{\beta+1}P$ are all in $L^2_\rho(Q^T)$, then we set

$$k(x, y, t) = \Im^2_t P = \int\limits_0^t \int\limits_0^s P(x, y, z)dzds,$$

and let

$$g(x,y,t) = \Im_t P + \Im_{xy}^2(\xi\eta\Im_t P),$$

then equation (5.2) becomes

$$\begin{aligned}
&(\partial_t^{\beta+1}(\Im_t^2 P) - \frac{1}{x}(\Im_t^2 P_x) - (\Im_t^2 P_{xx}) - \frac{1}{y}(\Im_t^2 P_y) \\
&-(\Im_t^2 P_{yy}), \Im_t P + \Im_{xy}^2(\xi\eta\Im_t P))_{L_\rho^2(Q^T)} \\
= \;& 0,
\end{aligned}$$

that is

$$\begin{aligned}
&(\partial_t^{\beta+1}(\Im_t^2 P), \Im_t P)_{L_\rho^2(\Omega)} + (\partial_t^{\beta+1}(\Im_t^2 P), \Im_{xy}^2(\xi\eta\Im_t P))_{L_\rho^2(\Omega)} \\
- \;& (\frac{1}{x}(\Im_t^2 P_x), \Im_t P)_{L_\rho^2(\Omega)} - (\frac{1}{x}(\Im_t^2 P_x), \Im_{xy}^2(\xi\eta\Im_t P))_{L_\rho^2(\Omega)} \\
- \;& ((\Im_t^2 P_{xx}), \Im_t P)_{L_\rho^2(\Omega)} - ((\Im_t^2 P_{xx}), \Im_{xy}^2(\xi\eta\Im_t P))_{L_\rho^2(\Omega)} \\
- \;& (\frac{1}{y}(\Im_t^2 P_y), \Im_t P)_{L_\rho^2(\Omega)} - (\frac{1}{y}(\Im_t^2 P_y), \Im_{xy}^2(\xi\eta\Im_t P))_{L_\rho^2(\Omega)} \\
- \;& ((\Im_t^2 P_{yy}), \Im_t P)_{L_\rho^2(\Omega)} - ((\Im_t^2 P_{yy}), \Im_{xy}^2(\xi\eta\Im_t P))_{L_\rho^2(\Omega)} \\
= \;& 0.
\end{aligned} \tag{5.3}$$

Put in mind that the function P verifies the given boundary and initial conditions (2.2)–(2.4), then all terms in (5.3) can be computed as

$$(\partial_t^{\beta+1}(\Im_t^2 P), \Im_t P)_{L_\rho^2(\Omega)} = (\partial_t^{\beta}(\Im_t P), \Im_t P)_{L_\rho^2(\Omega)}, \tag{5.4}$$

$$\begin{aligned}
&(\partial_t^{\beta+1}(\Im_t^2 P), \Im_{xy}^2(\xi\eta\Im_t P))_{L_\rho^2(\Omega)} = (\partial_t^{\beta}(\Im_t P), \Im_{xy}^2(\xi\eta\Im_t P))_{L_\rho^2(\Omega)} \\
= \;& (\partial_t^{\beta}(\Im_{xy}\xi\eta\Im_t P))\Im_{xy}(\xi\eta\Im_t P))_{L^2(\Omega)},
\end{aligned} \tag{5.5}$$

$$-(\frac{1}{x}(\Im_t^2 P_x), \Im_t P)_{L_\rho^2(\Omega)} = -(y\Im_t^2 P_x, \Im_t P)_{L^2(\Omega)}, \tag{5.6}$$

$$-(\frac{1}{y}(\Im_t^2 P_y), \Im_t P)_{L_\rho^2(\Omega)} = -(x\Im_t^2 P_y, \Im_t P)_{L_\rho^2(\Omega)}, \tag{5.7}$$

$$\begin{aligned}
&-(\Im_t^2 P_{xx}, \Im_t P)_{L_\rho^2(\Omega)} = -\int_\Omega xy(\Im_t^2 P_{xx})(\Im_t P)dxdy \\
= \;& (y\Im_t^2 P_x, \Im_t P)_{L^2(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\left\|\Im_t^2 P_x\right\|_{L_\rho^2(\Omega)},
\end{aligned} \tag{5.8}$$

$$\begin{aligned}
&-(\Im_t^2 P_{yy}, \Im_t P)_{L_\rho^2(\Omega)} = -\int_\Omega xy(\Im_t^2 P_{yy})(\Im_t P)dxdy \\
= \;& (x\Im_t^2 P_y, \Im_t P)_{L^2(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\left\|\Im_t^2 P_y\right\|_{L_\rho^2(\Omega)},
\end{aligned} \tag{5.9}$$

$$-\left(\frac{1}{x}(\Im_t^2 P_x), \Im_{xy}^2(\xi\eta\Im_t P)\right)_{L^2_\rho(\Omega)} = -\left(y\Im_t^2 P_x, \Im_{xy}^2(\xi\eta\Im_t P)\right)_{L^2(\Omega)}, \tag{5.10}$$

$$-\left(\frac{1}{y}(\Im_t^2 P_y), \Im_{xy}^2(\xi\eta\Im_t P)\right)_{L^2_\rho(\Omega)} = -\left(x\Im_t^2 P_y, \Im_{xy}^2(\xi\eta\Im_t P)\right)_{L^2(\Omega)}, \tag{5.11}$$

$$\begin{aligned}
&-(\Im_t^2 P_{xx}, \Im_{xy}^2(\xi\eta\Im_t P))_{L^2_\rho(\Omega)} = -\int_\Omega xy(\Im_t^2 P_{xx})\Im_{xy}^2(\xi\eta\Im_t P)dxdy \\
= \;& \left(y\Im_t^2 P_x, \Im_{xy}^2(\xi\eta\Im_t P)\right)_{L^2(\Omega)} + \left(\Im_t^2 P_x, \Im_x\Im_y^2(\xi\eta\Im_t P_x)\right)_{L^2_\rho(\Omega)},
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
&-(\Im_t^2 P_{yy}, \Im_{xy}^2(\xi\eta\Im_t P))_{L^2_\rho(\Omega)} = \int_\Omega xy\Im_t^2 P_{yy}\Im_{xy}^2(\xi\eta\Im_t P)dxdy \\
&+\left(x\Im_t^2 P_y, \Im_{xy}^2(\xi\eta\Im_t P)\right)_{L^2(\Omega)} + \left(\Im_t^2 P_y, \Im_y\Im_x^2(\xi\eta\Im_t P_y)\right)_{L^2_\rho(\Omega)}.
\end{aligned} \tag{5.13}$$

Insertion of equations (5.4)–(5.13) into (5.3), yields

$$\begin{aligned}
&2(\partial_t^\beta(\Im_t P), \Im_t P)_{L^2_\rho(\Omega)} + 2(\partial_t^\beta(\Im_{xy}\xi\eta\Im_t P))\Im_{xy}(\xi\eta\Im_t P))_{L^2(\Omega)} \\
&+\frac{\partial}{\partial t}\left\|\Im_t^2 P_x\right\|_{L^2_\rho(\Omega)} + \frac{\partial}{\partial t}\left\|\Im_t^2 P_y\right\|_{L^2_\rho(\Omega)} \\
= \;& -2\left(\Im_t^2 P_x, \Im_x\Im_y^2(\xi\eta\Im_t P_x)\right)_{L^2_\rho(\Omega)} \\
&-2\left(\Im_t^2 P_y, \Im_y\Im_x^2(\xi\eta\Im_t P_y)\right)_{L^2_\rho(\Omega)}.
\end{aligned} \tag{5.14}$$

Applying Lemma 2.3 and Poincare' type inequality to (5.14), we obtain

$$\begin{aligned}
&\partial_t^\beta\|\Im_t P\|^2_{L^2_\rho(\Omega)} + \partial_t^\beta\|\Im_{xy}(\xi\eta\Im_t P)\|^2_{L^2(\Omega)} \\
&+\frac{\partial}{\partial t}\left\|\Im_t^2 P_x\right\|_{L^2_\rho(\Omega)} + \frac{\partial}{\partial t}\left\|\Im_t^2 P_y\right\|_{L^2_\rho(\Omega)} \\
\leq \;& \left\|\Im_t^2 P_x\right\|_{L^2_\rho(\Omega)} + \left\|\Im_t^2 P_y\right\|_{L^2_\rho(\Omega)} \\
&+\frac{(ab^3 + a^3 b)}{4}\|\Im_{xy}(\xi\eta\Im_t P)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{5.15}$$

We infer from (5.15) that

$$\begin{aligned}
&D_t^{\beta-1}\|\Im_t P\|^2_{L^2_\rho(\Omega)} + D_t^{\beta-1}\|\Im_{xy}(\xi\eta\Im_t P)\|^2_{L^2(\Omega)} \\
+ \;& \|\Im_t^2 P_x\|^2_{L^2_\rho(\Omega)} + \|\Im_t^2 P_y\|^2_{L^2_\rho(\Omega)} \\
\leq \;& C\left(\int_0^t \|\Im_{xy}\xi\eta\Im_\tau P\|^2_{L^2(\Omega)}d\tau + \int_0^t \|\Im_\tau^2 P_x\|^2_{L^2_\rho(\Omega)}d\tau + \int_0^t \|\Im_\tau^2 P_y\|^2_{L^2_\rho(\Omega)}d\tau\right),
\end{aligned} \tag{5.16}$$

where

$$C = \max\Big\{1, \frac{ab^3 + a^3 b}{4}\Big\}.$$

By omitting the first two terms from the left-hand side of (5.16) and applying Lemma 2.1 by letting

$$S(t) = \int_0^t \|\Im_\tau^2 P_x\|^2_{L^2_\rho(\Omega)} d\tau + \int_0^t \|\Im_\tau^2 P_y\|^2_{L^2_\rho(\Omega)} d\tau,$$

$$\frac{dS(t)}{dt} = \|\Im_t^2 P_x\|^2_{L^2_\rho(\Omega)} + \|\Im_t^2 P_y\|^2_{L^2_\rho(\Omega)}$$

$$S(0) = 0,$$

then

$$\begin{aligned} S(t) &= \int_0^t \|\Im_\tau^2 P_x\|^2_{L^2_\rho(\Omega)} d\tau + \int_0^t \|\Im_\tau^2 P_y\|^2_{L^2_\rho(\Omega)} d\tau \\ &\leq Te^{CT} \int_0^t \|\Im_{xy}(\xi\eta P)\|^2_{L^2(\Omega)}. \end{aligned} \tag{5.17}$$

By inserting (5.17) into (5.16), we obtain

$$\begin{aligned} & D_t^{\beta-1}\|\Im_{xy}(\xi\eta\Im_t P)\|^2_{L^2(\Omega)} + D_t^{\beta-1}\|\Im_t P\|^2_{L^2_\rho(\Omega)} \\ + & \|\Im_t^2 P_x\|^2_{L^2_\rho(\Omega)} + \|\Im_t^2 P_y\|^2_{L^2_\rho(\Omega)} \\ \leq & C\Big(1 + Te^{CT}\Big) \int_0^t \|\Im_{xy}\xi\eta\Im_\tau P\|^2_{L^2(Q^T)} d\tau, \end{aligned} \tag{5.18}$$

if we drop the last three terms on the left hand side of (5.18) and apply Lemma 2.2 by taking

$$z(t) = \int_0^t \|\Im_{xy}\xi\eta\Im_\tau P\|^2_{L^2(Q^T)} d\tau,$$

$$\partial_t^\beta z(t) = D_t^{\beta-1}\|\Im_{xy}(\xi\eta\Im_t P)\|^2_{L^2(\Omega)},$$

$$z(0) = 0,$$

it follows that

$$\int_0^t \|\Im_{xy}\xi\eta\Im_\tau P\|^2_{L^2(Q^T)} d\tau \leq 0. \tag{5.19}$$

Consequently, inequality (5.19) implies that g is zero a.e in Q^T.

To complete the proof of Theorem 5.1 , we suppose that for some element $W = (H, F, G) \in R(A)^{\perp}$, we have

$$(\mathcal{L}\theta, H)_{L^2(0,T;L^2_\rho(\Omega))} + (\ell_1\theta, F)_{H^1_\rho(\Omega)} + (\ell_2\theta, G)_{L^2_\rho(\Omega)} = 0. \tag{5.20}$$

We must prove that $W = 0$. If we put $\theta \in D(A)$ satisfying homogeneous conditions into (5.20), we have

$$(\mathcal{L}\theta, H)_{L^2(0,T;L^2_\rho(\Omega))} = 0. \tag{5.21}$$

Applying Proposition 5.2 to (5.21), it follows from that $H = 0$.

Thus (5.20) takes the form

$$(\ell_1\theta, F)_{H^1_\rho(\Omega)} + (\ell_2\theta, G)_{L^2_\rho(\Omega)} = 0. \tag{5.22}$$

But since the range of the operators ℓ_1, ℓ_2 are dense in the spaces $H^1_\rho(\Omega)$, $L^2_\rho(\Omega)$ respectively then relation (5.22) implies $G = F = 0$. Consequently $W = 0$ and Theorem 5.1 follows.

6. The nonlinear problem

This section is consecrated to the proof of the existence, uniqueness and continuous dependence of the solution on the data of the problem (2.1)–(2.4). Let us consider the following auxiliary problem with homogeneous equation

$$\mathcal{L}(U) = \partial_t^{\beta+1} U - \frac{1}{xy} div(xy\nabla U) = 0, \tag{6.1}$$

$$\ell_1 U = U(x, y, 0) = F(x, y)\ ,\ \ell_2 U = U_t(x, y, 0) = G(x, y)\ , \tag{6.2}$$

$$U_x(a, y, t) = 0\ ,\ U_y(x, b, t) = 0\ , \tag{6.3}$$

$$\int_0^a xU(x, y, t)dx = 0\ ,\ \int_0^b yU(x, y, t)dx = 0, \tag{6.4}$$

If θ is a solution of problem (2.1)-(2.4) and U is a solution of problem (6.1)-(6.4), then $w = \theta - U$ satisfies

$$\partial_t^{\beta+1} w - \frac{1}{xy} div(xy\nabla w) = \tilde{H}(x, y, t, w, w_x, w_y), \tag{6.5}$$

$$w(x, y, 0) = 0\ ,\ w_t(x, y, 0) = 0\ , \tag{6.6}$$

$$w_x(a, y, t) = 0\ ,\ w_y(x, b, t) = 0\ , \tag{6.7}$$

$$\int_0^a xw(x, y, t)dx = 0\ ,\ \int_0^b yw(x, y, t)dx = 0, \tag{6.8}$$

where

$$\tilde{H}\left(x, y, t, w, w_x, w_y\right) = H\left(x, y, t, \theta - U, (\theta - U)_x, (\theta - U)_y\right).$$

The function $\tilde{H}$ satisfies the Lipshitz condition

$$\begin{aligned} & \tilde{H}(x, y, t, w_1, w_2, w_3) - \tilde{H}(x, y, t, v_1, v_2, v_3) \\ \leqslant \;& \lambda\left(|w_1 - v_1| + |w_2 - v_2| + |w_3 - v_3|\right), \end{aligned} \tag{6.9}$$

for all $(x, y, t) \in Q^T = (0, a) \times (0, b) \times (0, T)$. According to Theorem (5.1) problem (6.1)–(6.4) has a unique solution depending continuously on $F \in H^1_\rho(\Omega)$ and $G \in L^2_\rho(\Omega)$. It remains to solve problem (6.5)–(6.8). We shall prove that problem (6.5)-(6.8) has a unique weak solution. Suppose that v and w belong to $C^1(Q^T)$ such that $v(x, T) = 0$, $w(x, y, 0) = 0$, $w_t(x, y, 0) = 0$, $\int\limits_0^a xwdx = 0$, $\int\limits_0^b ywdx = 0$, $\int\limits_0^a xvdx = 0$, $\int\limits_0^b yvdx = 0$. For $v \in C^1(Q^T)$, we have

$$\begin{aligned} & (\mathcal{L}w, \mathfrak{I}_{xy}(\xi\eta v))_{L^2(0,T;L^2_\rho(\Omega))} = (\partial_t^\beta w_t, \mathfrak{I}_{xy}(\xi\eta v))_{L^2(0,T;L^2_\rho(\Omega))} \\ - \;& (\frac{1}{x}w_x, \mathfrak{I}_{xy}(\xi\eta v))_{L^2(0,T;L^2_\rho(\Omega))} - (w_{xx}, \mathfrak{I}_{xy}(\xi\eta v))_{L^2(0,T;L^2_\rho(\Omega))} \\ - \;& (\frac{1}{y}w_y, \mathfrak{I}_{xy}(\xi\eta v))_{L^2(0,T;L^2_\rho(\Omega))} - (w_{yy}, \mathfrak{I}_{xy}(\xi\eta v))_{L^2(0,T;L^2_\rho(\Omega))} \\ = \;& (\tilde{H}, \mathfrak{I}_{xy}(\xi\eta v))_{L^2(0,T;L^2_\rho(\Omega))}. \end{aligned} \tag{6.10}$$

By using conditions on w and v , a quick computation of each term on the right and left-hand side of (6.10), gives

$$(\partial_t^\beta w_t, \mathfrak{I}_{xy}(\xi\eta v))_{L^2(0,T;L^2_\rho(\Omega))} = (v, \partial_t^\beta(\mathfrak{I}_{xy}(\xi\eta w_t)))_{L^2(0,T;L^2_\rho(\Omega))}, \tag{6.11}$$

$$-(\frac{1}{x}w_x, \mathfrak{I}_{xy}(\xi\eta v))_{L^2(0,T;L^2_\rho(\Omega))} = (w, \mathfrak{I}_y(\eta v))_{L^2(0,T;L^2_\rho(\Omega))}, \tag{6.12}$$

$$\begin{aligned} -(w_{xx}, \mathfrak{I}_{xy}(\xi\eta v))_{L^2(0,T;L^2_\rho(\Omega))} &= -(w, \mathfrak{I}_y(\eta v))_{L^2(0,T;L^2_\rho(\Omega))} \\ &+ (xw_x, \mathfrak{I}_y(\eta v))_{L^2(0,T;L^2_\rho(\Omega))}, \end{aligned} \tag{6.13}$$

$$\begin{aligned} -(w_{yy}, \mathfrak{I}^2_{xy}(\xi\eta v))_{L^2(0,T;L^2_\rho(\Omega))} &= (yw_y, \mathfrak{I}_x(\xi v))_{L^2(0,T;L^2_\rho(\Omega))} \\ &- (w, \mathfrak{I}_x(\xi v))_{L^2(0,T;L^2_\rho(\Omega))}, \end{aligned} \tag{6.14}$$

$$-(\frac{1}{y}w_y, \mathfrak{I}_{xy}(\xi\eta v))_{L^2(0,T;L^2_\rho(\Omega))} = (w, \mathfrak{I}_x(\xi v))_{L^2(0,T;L^2_\rho(\Omega))}, \tag{6.15}$$

$$(\tilde{H}, \mathfrak{I}_{xy}(\xi\eta v))_{L^2(0,T;L^2_\rho(\Omega))} = (\mathfrak{I}_{xy}(\xi\eta\tilde{H}), v)_{L^2(0,T;L^2_\rho(\Omega))}, \tag{6.16}$$

Insertion of (5.11)-(5.16)into (5.10) yields

$$M(w,v)=(\Im_{xy}(\xi\eta\tilde{H}),v)_{L^2(0,T;L^2_\rho(\Omega))},\tag{6.17}$$

where

$$\begin{aligned}M(w,v) &= (v,\partial_t^\beta(\Im_{xy}(\xi\eta w_t)))_{L^2(0,T;L^2_\rho(\Omega))}+(xw_x,\Im_y(\eta v))_{L^2(0,T;L^2_\rho(\Omega))}\\ &+ (yw_y,\Im_x(\xi v))_{L^2(0,T;L^2_\rho(\Omega))}.\end{aligned}\tag{6.18}$$

Definition 6.1. A function $w\in L^2(0,T;H^1_\rho(\Omega))$ is called weak solution of problem (6.5)-(6.8) if (6.7) and (6.17) hold.

Our main purpose is to construct an iteration sequence $(w^n)_{n\in\mathbf{N}}$ which converges to a certain function $w\in L^2(0,T;H^1_\rho(\Omega))$ which solves problem (6.5)–(6.8). Starting with $w^{(0)}=0$, the sequence $(w^n)_{n\in\mathbf{N}}$ is defined as follows: given the element $w^{(n-1)}$, then for $n=1,2,\ldots$ we solve the iterated problems:

$$\partial_t^\beta w_t^{(n)}-\frac{1}{x}w_x^{(n)}-w_{xx}^{(n)}-\frac{1}{y}w_y^{(n)}-w_{yy}^{(n)}=\tilde{H}(x,y,t,w^{(n)},w_x^{(n)},w_y^{(n)}),\tag{6.19}$$

$$w^{(n)}(x,y,0)=0\ ,\ w_t^{(n)}(x,y,0)=0\ ,\tag{6.20}$$

$$w_x^{(n)}(a,y,t)=0\ ,\ w_y^{(n)}(x,b,t)=0\ ,\tag{6.21}$$

$$\int_0^a xw^{(n)}(x,y,t)dx=0\ ,\ \int_0^b yw^{(n)}(x,y,t)dx=0,\tag{6.22}$$

Theorem (5.1) asserts that for fixed n, each problem (6.19)-(6.22) has unique solution $w^{(n)}(x,y,t)$. If we set $\mathcal{Z}^{(n)}(x,y,t)=w^{(n+1)}(x,y,t)-w^{(n)}(x,y,t)$, then we have the new problem

$$\partial_t^\beta\mathcal{Z}_t^{(n)}-\frac{1}{x}\mathcal{Z}_x^{(n)}-\mathcal{Z}_{xx}^{(n)}-\frac{1}{y}\mathcal{Z}_y^{(n)}-\mathcal{Z}_{yy}^{(n)}=\sigma^{(n-1)}(x,y,t),\tag{6.23}$$

$$\mathcal{Z}^{(n)}(x,y,0)=0\ ,\ \mathcal{Z}_t^{(n)}(x,y,0)=0\ ,\tag{6.24}$$

$$\mathcal{Z}_x^{(n)}(a,y,t)=0\ ,\ \mathcal{Z}_y^{(n)}(x,b,t)=0\ ,\tag{6.25}$$

$$\int_0^a x\mathcal{Z}^{(n)}(x,y,t)dx=0\ ,\ \int_0^b y\mathcal{Z}^{(n)}(x,y,t)dx=0,\tag{6.26}$$

where

$$\sigma^{(n-1)}(x,y,t)=\tilde{H}\left(x,y,t,w^{(n)},w_x^{(n)},w_y^{(n)}\right)-\tilde{H}\left(x,y,t,w^{(n-1)},w_x^{(n-1)},w_y^{(n-1)}\right).$$

Theorem 6.2. Assume that condition (6.9) holds, then for the linearized problem (6.23)–(6.26), we have the a priori estimate

$$\|\mathcal{Z}^{(n)}\|_{L^2(0,T;H^1_\rho(\Omega))}\le K\|\mathcal{Z}^{(n-1)}\|_{L^2(0,T;H^1_\rho(\Omega))},\tag{6.27}$$

where K is positive constant given by

$$K = TC_4\left(\frac{a^2+b^2}{8}+1\right).$$

Proof. Taking the inner product in $L^2(0,\tau;L^2_\rho(\Omega))$,with $0\leq\tau\leq T$ of equation (6.23) and integro-differential operator

$$\mathcal{M}\mathcal{Z} = \mathcal{Z}_t^{(n)} - \Im^2_{xy}(\xi\eta\mathcal{Z}_t^{(n)})$$

we have

$$\begin{aligned}
& (\partial_t^\beta \mathcal{Z}_t^{(n)}, \mathcal{Z}_t^{(n)})_{L^2(0,\tau;L^2_\rho(\Omega))} - (\frac{1}{x}\mathcal{Z}_x^{(n)}, \mathcal{Z}_t^{(n)})_{L^2(0,\tau;L^2_\rho(\Omega))} \\
- \quad & (\mathcal{Z}_{xx}^{(n)}, \mathcal{Z}_t^{(n)})_{L^2(0,\tau;L^2_\rho(\Omega))} - (\frac{1}{y}\mathcal{Z}_y^{(n)}, \mathcal{Z}_t^{(n)})_{L^2(0,\tau;L^2_\rho(\Omega))} \\
- \quad & (\mathcal{Z}_{yy}^{(n)}, \mathcal{Z}_t^{(n)})_{L^2(0,\tau;L^2_\rho(\Omega))} - (\partial_t^\beta \mathcal{Z}_t^{(n)}, \Im^2_{xy}(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L^2_\rho(\Omega))} \\
+ \quad & (\frac{1}{x}\mathcal{Z}_x^{(n)}, \Im^2_{xy}(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L^2_\rho(\Omega))} + (\mathcal{Z}_{xx}^{(n)}, \Im^2_{xy}(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L^2_\rho(\Omega))} \\
+ \quad & (\frac{1}{y}\mathcal{Z}_y^{(n)}, \Im^2_{xy}(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L^2_\rho(\Omega))} + (\mathcal{Z}_{yy}^{(n)}, \Im^2_{xy}(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L^2_\rho(\Omega))} \\
= \quad & (\sigma^{(n-1)}, \mathcal{Z}_t^{(n)})_{L^2(0,\tau;L^2_\rho(\Omega))} - (\sigma^{(n-1)}, \Im^2_{xy}(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L^2_\rho(\Omega))}. \qquad (6.28)
\end{aligned}$$

In the light of conditions (6.25) and (6.26), Cauchy ϵ inequality, Lemma 2.2, Lemma 2.3 and successive integrations by parts of each term of (6.28) leads to

$$\begin{aligned}
& \left(\partial_t^\beta \mathcal{Z}_t^{(n)}, \mathcal{Z}_t^{(n)}\right)_{L^2(0,\tau;L^2_\rho(\Omega))} \geq \frac{1}{2}\int_0^\tau \partial_t^\beta \|\mathcal{Z}_t^{(n)}\|^2_{L^2_\rho(\Omega)} dt, \\
= \quad & \frac{1}{2} D_\tau^{\beta-1} \|\mathcal{Z}_t^{(n)}(x,y,t)\|^2_{L^2_\rho(\Omega)}, \qquad (6.29)
\end{aligned}$$

$$-\left(\frac{1}{x}\mathcal{Z}_x^{(n)}, \mathcal{Z}_t^{(n)}\right)_{L^2(0,\tau;L^2_\rho(\Omega))} = -\left(y\mathcal{Z}_x^{(n)}, \mathcal{Z}_t^{(n)}\right)_{L^2(0,\tau;L^2(\Omega))}, \qquad (6.30)$$

$$-(\frac{1}{y}\mathcal{Z}_y^{(n)}, \mathcal{Z}_t^{(n)})_{L^2(0,\tau;L^2_\rho(\Omega))} = -\left(x\mathcal{Z}_y^{(n)}, \mathcal{Z}_t^{(n)}\right)_{L^2(0,\tau;L^2(\Omega))}, \qquad (6.31)$$

$$\begin{aligned}
- \quad & (\mathcal{Z}_{xx}^{(n)}, \mathcal{Z}_t^{(n)})_{L^2(0,\tau;L^2_\rho(\Omega))} = \left(y\mathcal{Z}_x^{(n)}, \mathcal{Z}_t^{(n)}\right)_{L^2(0,\tau;L^2(\Omega))} \\
+ \quad & \frac{1}{2}\|\mathcal{Z}_x^{(n)}(x,y,\tau)\|^2_{L^2_\rho(\Omega)}, \qquad (6.32)
\end{aligned}$$

$$\begin{aligned}
- \quad & (\mathcal{Z}_{yy}^{(n)}, \mathcal{Z}_t^{(n)})_{L^2(0,\tau;L^2_\rho(\Omega))} = \left(x\mathcal{Z}_y^{(n)}, \mathcal{Z}_t^{(n)}\right)_{L^2(0,\tau;L^2(\Omega))} \\
+ \quad & \frac{1}{2}\|\mathcal{Z}_y^{(n)}(x,y,\tau)\|^2_{L^2_\rho(\Omega)}, \qquad (6.33)
\end{aligned}$$

$$
\begin{aligned}
& - \quad (\partial_t^\beta \mathcal{Z}_t^{(n)}, \mathfrak{I}_{xy}^2(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L_\rho^2(\Omega))} \\
& \geq \quad \frac{1}{2}\int_0^\tau \partial_t^\beta \left\|\mathfrak{I}_{xy}(\xi\eta\mathcal{Z}_t^{(n)})\right\|_{L_\rho^2(\Omega)}^2 dt \\
& = \quad \frac{1}{2}D_\tau^{\beta-1}\|\mathfrak{I}_{xy}\xi\eta\mathcal{Z}_t^{(n)}(x,y,t)\|_{L_\rho^2(\Omega)}^2, \qquad (6.34)
\end{aligned}
$$

$$
(\frac{1}{x}\mathcal{Z}_x^{(n)}, \mathfrak{I}_{xy}^2(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L_\rho^2(\Omega))} = \left(y\mathcal{Z}_x^{(n)}, \mathfrak{I}_{xy}^2(\xi\eta\mathcal{Z}_t^{(n)})\right)_{L^2(0,\tau;L^2(\Omega))}, \qquad (6.35)
$$

$$
(\frac{1}{y}Z_y^{(n)}, \mathfrak{I}_{xy}^2(\xi\eta Z_t^{(n)}))_{L^2(0,\tau;L_\rho^2(\Omega))} = \left(x\mathcal{Z}_y^{(n)}, \mathfrak{I}_{xy}^2(\xi\eta\mathcal{Z}_t^{(n)})\right)_{L^2(0,\tau;L^2(\Omega))}, \qquad (6.36)
$$

$$
\begin{aligned}
& (\mathcal{Z}_{xx}^{(n)}, \mathfrak{I}_{xy}^2(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L_\rho^2(\Omega))} \\
= \; & -\left(y\mathcal{Z}_x^{(n)}, \mathfrak{I}_{xy}^2(\xi\eta\mathcal{Z}_t^{(n)})\right)_{L^2(0,\tau;L^2(\Omega))} \\
& -(\mathcal{Z}_x^{(n)}, \mathfrak{I}_x\mathfrak{I}_y^2(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L_\rho^2(\Omega))}, \qquad (6.37)
\end{aligned}
$$

$$
\begin{aligned}
& (Z_{yy}^{(n)}, \mathfrak{I}_{xy}^2(\xi\eta Z_t^{(n)}))_{L^2(0,\tau;L_\rho^2(\Omega))} \\
= \; & -\left(x\mathcal{Z}_y^{(n)}, \mathfrak{I}_{xy}^2(\xi\eta\mathcal{Z}_t^{(n)})\right)_{L^2(0,\tau;L^2(\Omega))} \\
& -(\mathcal{Z}_y^{(n)}, \mathfrak{I}_y\mathfrak{I}_x^2(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L_\rho^2(\Omega))}, \qquad (6.38)
\end{aligned}
$$

Combination of (6.28)–(6.38) yields

$$
\begin{aligned}
& D_\tau^{\beta-1}\|\mathcal{Z}_t^{(n)}(x,y,t)\|_{L_\rho^2(\Omega)}^2 + D_\tau^{\beta-1}\|\mathfrak{I}_{xy}(\xi\eta\mathcal{Z}_t^{(n)})(x,y,t)\|_{L_\rho^2(\Omega)}^2 \\
& +\|\mathcal{Z}_x^{(n)}(x,y,\tau)\|_{L_\rho^2(\Omega)}^2 + \|\mathcal{Z}_y^{(n)}(x,y,\tau)\|_{L_\rho^2(\Omega)}^2 \\
= \; & 2(\mathcal{Z}_x^{(n)}, \mathfrak{I}_x\mathfrak{I}_y^2(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L_\rho^2(\Omega))} + 2(\mathcal{Z}_y^{(n)}, \mathfrak{I}_y\mathfrak{I}_x^2(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L_\rho^2(\Omega))} \\
& +2(\sigma^{(n-1)}, \mathcal{Z}_t^{(n)})_{L^2(0,\tau;L_\rho^2(\Omega))} + 2(\sigma^{(n-1)}, \mathfrak{I}_{xy}^2(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L_\rho^2(\Omega))}. \qquad (6.39)
\end{aligned}
$$

Estimation of the right-hand side of (6.39) gives

$$
\begin{aligned}
& 2(\sigma^{(n-1)}, \mathcal{Z}_t^{(n)})_{L^2(0,\tau;L_\rho^2(\Omega))} \\
\leq \; & \|\mathcal{Z}_t^{(n)}\|_{L^2(0,\tau;L_\rho^2(\Omega))}^2 + \|\sigma^{(n-1)}\|_{L^2(0,\tau;L_\rho^2(\Omega))}^2 \\
\leq \; & \|\mathcal{Z}_t^{(n)}\|_{L^2(0,\tau;L_\rho^2(\Omega))}^2 + \frac{3}{2}\lambda^2\left(\|\mathcal{Z}^{(n-1)}\|_{L^2(0,\tau;L_\rho^2(\Omega))}^2\right. \\
& \left.+\|\mathcal{Z}_x^{(n-1)}\|_{L^2(0,\tau;L_\rho^2(\Omega))}^2 + \|\mathcal{Z}_y^{(n-1)}\|_{L^2(0,\tau;L_\rho^2(\Omega))}^2\right), \qquad (6.40)
\end{aligned}
$$

$$
\begin{aligned}
& 2(\sigma^{(n-1)}, \mathfrak{I}_{xy}^2(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L_\rho^2(\Omega))} \\
\leq \; & \frac{a^2b^2}{4}\|\mathfrak{I}_{xy}(\xi\eta\mathcal{Z}_t^{(n)})\|_{L^2(0,\tau;L_\rho^2(\Omega))}^2 + \frac{3}{2}\lambda^2\left(\|\mathcal{Z}^{(n-1)}\|_{L^2(0,\tau;L_\rho^2(\Omega))}^2\right.
\end{aligned}
$$

$$+\|\mathcal{Z}_x^{(n-1)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + \|\mathcal{Z}_y^{(n-1)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))}\Big), \tag{6.41}$$

$$\begin{aligned} & 2(\mathcal{Z}_x^{(n)}, \Im_x \Im_y^2(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L^2_\rho(\Omega))} \\ \leq\ & \|\mathcal{Z}_x^{(n)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + b^2\|\Im_{xy}(\xi\eta\mathcal{Z}_t^{(n)})\|^2_{L^2(0,\tau;L^2_\rho(\Omega))}, \end{aligned} \tag{6.42}$$

$$\begin{aligned} & 2(\mathcal{Z}_y^{(n)}, \Im_y \Im_x^2(\xi\eta\mathcal{Z}_t^{(n)}))_{L^2(0,\tau;L^2_\rho(\Omega))} \\ \leq\ & \|\mathcal{Z}_y^{(n)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + a^2\|\Im_{xy}(\xi\eta\mathcal{Z}_t^{(n)})\|^2_{L^2(0,\tau;L^2_\rho(\Omega))}. \end{aligned} \tag{6.43}$$

Upon substitution of (6.40)–(6.43) into (6.39), we obtain

$$\begin{aligned} & D_\tau^{\beta-1}\|Z_t^{(n)}(x,y,t)\|^2_{L^2_\rho(\Omega)} + D_\tau^{\beta-1}\|\Im_{xy}(\xi\eta Z_t^{(n)})(x,y,t)\|^2_{L^2_\rho(\Omega)} \\ +\ & \|\mathcal{Z}_x^{(n)}(x,y,\tau)\|^2_{L^2_\rho(\Omega)} + \|\mathcal{Z}_y^{(n)}(x,y,\tau)\|^2_{L^2_\rho(\Omega)} \\ \leq\ & C_1\Big(\|\mathcal{Z}_t^{(n)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + \|\mathcal{Z}^{(n-1)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} \\ +\ & \|\mathcal{Z}_x^{(n-1)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + \|\mathcal{Z}_y^{(n-1)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} \\ & +\|\mathcal{Z}_x^{(n)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + \|\mathcal{Z}_y^{(n)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} \\ & +\|\Im_{xy}(\xi\eta\mathcal{Z}_t^{(n)})\|^2_{L^2(0,\tau;L^2_\rho(\Omega))}\Big), \end{aligned} \tag{6.44}$$

where

$$C_1 = \max\{1, \frac{3}{2}\lambda^2, \frac{a^2b^2}{4}, a^2, b^2\}.$$

Now by discarding the first two terms from left hand side of (6.44) and using Lemma 2.1 with

$$\mathcal{P}(\tau) = \|\mathcal{Z}_x^{(n)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + \|\mathcal{Z}_y^{(n)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))},\ \mathcal{P}(0) = 0,$$

we obtain

$$\begin{aligned} & \|\mathcal{Z}_x^{(n)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + \|\mathcal{Z}_y^{(n)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} \\ \leq\ & C_2\Big(\|\mathcal{Z}^{(n-1)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + \|\mathcal{Z}_x^{(n-1)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} \\ +\ & \|\mathcal{Z}_y^{(n-1)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + \|\mathcal{Z}_t^{(n)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} \\ & +\|\Im_{xy}(\xi\eta\mathcal{Z}_t^{(n)})\|^2_{L^2(0,\tau;L^2_\rho(\Omega))}\Big), \end{aligned} \tag{6.45}$$

where $C_2 = C_1 T e^{C_1 T}$. Hence, inequality (6.44) becomes

$$\begin{aligned} & D_\tau^{\beta-1}\|Z_t^{(n)}(x,y,t)\|^2_{L^2_\rho(\Omega)} + D_\tau^{\beta-1}\|\Im_{xy}(\xi\eta Z_t^{(n)})(x,y,t)\|^2_{L^2(\Omega)} \\ & +\|\mathcal{Z}_x^{(n)}(x,y,\tau)\|^2_{L^2_\rho(\Omega)} + \|\mathcal{Z}_y^{(n)}(x,y,\tau)\|^2_{L^2_\rho(\Omega)} \\ \leq\ & C_3\Big(\|\mathcal{Z}^{(n-1)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + \|\mathcal{Z}_x^{(n-1)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} \end{aligned}$$

$$
\begin{aligned}
&+\|\mathcal{Z}_y^{(n-1)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + \|\mathcal{Z}_t^{(n)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))}\\
&+\|\mathfrak{I}_{xy}(\xi\eta\mathcal{Z}_t^{(n)})\|^2_{L^2(0,\tau;L^2_\rho(\Omega))}\Big),
\end{aligned}
\tag{6.46}
$$

where $C_3 = C_1(1 + C_2)$.

We now need to eliminate the last two terms on the right-hand side of (6.46) by using Lemma 2.2 and setting

$$
\begin{aligned}
\mathcal{K}(\tau) &= \|\mathcal{Z}_t^{(n)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + \|\mathfrak{I}_{xy}(\xi\eta\mathcal{Z}_t^{(n)})\|^2_{L^2(0,\tau;L^2_\rho(\Omega))},\\
\partial_\tau^\beta\mathcal{K}(t) &= D_\tau^{\beta-1}\|Z_t^{(n)}(x,y,t)\|^2_{L^2_\rho(\Omega)} + D_\tau^{\beta-1}\|\mathfrak{I}_{xy}(\xi\eta Z_t^{(n)})(x,y,t)\|^2_{L^2(\Omega)}\\
\mathcal{K}(0) &= 0,
\end{aligned}
$$

then we have

$$
\begin{aligned}
&\|\mathcal{Z}_t^{(n)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + \|\mathfrak{I}_{xy}(\xi\eta\mathcal{Z}_t^{(n)})\|^2_{L^2(0,\tau;L^2_\rho(\Omega))}\\
\leq\ &\mathcal{K}(0)E_\beta(C_3\tau^\beta) + \Gamma(\beta)E_{\beta,\beta}(C_3\tau^\beta)D_\tau^{-\beta-1}\Big(\|\mathcal{Z}^{(n-1)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))}\\
&+\|\mathcal{Z}_x^{(n-1)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + \|\mathcal{Z}_y^{(n-1)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))}\Big).
\end{aligned}
\tag{6.47}
$$

Inequality (6.47) implies

$$
\begin{aligned}
&\|\mathcal{Z}_t^{(n)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + \|\mathfrak{I}_{xy}(\xi\eta\mathcal{Z}_t^{(n)})\|^2_{L^2(0,\tau;L^2_\rho(\Omega))}\\
\leq\ &\Gamma(\beta)E_{\beta,\beta}(C_3\tau^\beta)\Big(D_\tau^{-\beta-1}\|\mathcal{Z}^{(n-1)}\|^2_{L^2_\rho(\Omega)}\\
&+D_\tau^{-\beta-1}\|\mathcal{Z}_x^{(n-1)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))} + D_\tau^{-\beta-1}\|\mathcal{Z}_y^{(n-1)}\|^2_{L^2(0,\tau;L^2_\rho(\Omega))}\Big).
\end{aligned}
\tag{6.48}
$$

It is obvious that

$$
D_\tau^{-\beta-1}\|\mathcal{Z}^{(n-1)}\|^2_{L^2_\rho(\Omega)} \leq \frac{T^\beta}{\Gamma(\beta+1)}\int_0^\tau \|\mathcal{Z}^{(n-1)}\|^2_{L^2_\rho(\Omega)}d\tau,
\tag{6.49}
$$

If we discard the first two terms on the left-hand side of (6.46) and combine with (6.48) and (6.49), we obtain

$$
\|\nabla\mathcal{Z}^{(n)}(x,y,\tau)\|^2_{L^2_\rho(\Omega)} \leq C_4\|\mathcal{Z}^{(n-1)}\|^2_{L^2(0,T;H^1_\rho(\Omega))},
\tag{6.50}
$$

where

$$
C_4 = C_3\left(1 + \Gamma(\beta)E_{\beta,\beta}(C_3T^\beta)\frac{T^\beta}{\Gamma(\beta+1)}\right).
\tag{6.51}
$$

Lemma 6.3 For $\theta \in H^1_\rho(\Omega)$ satisfying $\int_0^a x\theta dx = 0$, $\int_0^b y\theta dy = 0$, we have

$$
\|\theta\|^2_{L^2_\rho(\Omega)} \leq \frac{a^2+b^2}{8}\|\nabla\theta\|^2_{L^2_\rho(\Omega)}.
\tag{6.52}
$$

Now by using the above lemma and equivalence of norms

$$
\|\nabla\theta\|_{L^2_\rho(\Omega)} \sim \|\theta\|_{H^1_\rho(\Omega)},
\tag{6.53}
$$

which comes from

$$\|\nabla\theta\|^2_{L^2_\rho(\Omega)} \le \|\theta\|^2_{H^1_\rho(\Omega)} \le \left(\frac{a^2+b^2}{8}+1\right)\|\nabla\theta\|^2_{L^2_\rho(\Omega)},$$

we infer from (6.50) and (6.53) that

$$\|\mathcal{Z}^{(n)}\|^2_{L^2(0,T;H^1_\rho(\Omega))} \le K\|\mathcal{Z}^{(n-1)}\|^2_{L^2(0,T;H^1_\rho(\Omega))}, \tag{6.54}$$

where

$$K = TC_4\left(\frac{a^2+b^2}{8}+1\right). \tag{6.55}$$

From the criteria of convergence of series, we see from (6.54) that the series $\sum_{n=0}^{\infty}\mathcal{Z}^{(n)}$ convergence if $K < 1$. Since $\mathcal{Z}^{(n)}(x,y,t) = w^{(n+1)}(x,y,t) - w^{(n)}(x,y,t)$, then it follows that the sequence $(w^{(n)})_{n\in N}$ defined by

$$\begin{aligned} w^{(n)}(x,y,t) &= \sum_{k=0}^{n-1}(w^{(n+1)}(x,y,t) - w^{(k)}(x,y,t)) + w^{(0)}(x,y,t),\\ &= \sum_{k=0}^{n-1}\mathcal{Z}^{(k)} + w^{(0)}(x,y,t), \quad n = 1,2,3,...... \end{aligned} \tag{6.56}$$

converges to $w \in L^2(0,T;H^1_\rho(\Omega))$. Now to prove that this limit function w is a solution of problem under consideration (6.5)–(6.8), we should show that w satisfies (6.7) and (6.17) as mentioned in Definition 6.1. For problem (6.19)–(6.22), we have

$$M(w^{(n)},v) = \left(v, \Im_{xy}(\xi\eta\tilde{H}(x,y,t,w^{(n-1)},\frac{\partial w^{(n-1)}}{\partial\xi},\frac{\partial w^{(n-1)}}{\partial\eta})\right)_{L^2(0,T;L^2_\rho(\Omega))}. \tag{6.57}$$

From (6.57), we have

$$\begin{aligned} &M(w^{(n)}-w,v) + M(w,v)\\ =\ & (v, \Im_{xy}\xi\eta\tilde{H}(\xi,\eta,t,w^{(n-1)},\frac{\partial w^{(n-1)}}{\partial\xi},\frac{\partial w^{(n-1)}}{\partial\eta})\\ -\ & \Im_{xy}\xi\eta\tilde{H}(\xi,\eta,t,w,\frac{\partial w}{\partial\xi},\frac{\partial w}{\partial\eta}))_{L^2(0,T;L^2_\rho(\Omega))}\\ +\ & (v, \Im_{xy}\xi\eta\tilde{H}(\xi,\eta,t,w,\frac{\partial w}{\partial\xi},\frac{\partial w}{\partial\eta})_{L^2(0,T;L^2_\rho(\Omega))}, \end{aligned} \tag{6.58}$$

From the partial differential equation (6.19), we have

$$\begin{aligned} &\left(v, \partial_t^{\beta+1}\Im_{xy}\xi\eta\left(w^{(n)}-w\right)\right)_{L^2(0,T;L^2\rho(\Omega))}\\ &-\left(v, \Im_{xy}\eta\left(w^{(n)}_\xi - w_\xi\right)\right)_{L^2(0,T;L^2\rho(\Omega))}\\ &-\left(v, \Im_{xy}\xi\left(w^{(n)}_\eta - w_\eta\right)\right)_{L^2(0,T;L^2\rho(\Omega))}\\ &-\left(v, \Im_{xy}\xi\eta\left(w^{(n)}_{\xi\xi} - w_{\xi\xi}\right)\right)_{L^2(0,T;L^2\rho(\Omega))} \end{aligned}$$

$$
\begin{aligned}
& -\left(v, \Im_{xy}\xi\eta\left(w^{(n)}_{\eta\eta} - w_{\eta\eta}\right)\right)_{L^2(0,T;L^2\rho(\Omega))} \\
= \;& M(w^{(n)} - w, v). && (6.59)
\end{aligned}
$$

Inner products in (6.59) can be evaluated by using conditions on functions v and w, and this leads to

$$
\begin{aligned}
& \left(\Im_{xy}(\xi\eta\partial_T^{\beta+1}v), (w^{(n)} - w)\right)_{L^2(0,T;L^2\rho(\Omega))} \\
& +\left(\Im_y\eta v, \Im_x(w^{(n)}_\xi - w_\xi\right)_{L^2(0,T;L^2\rho(\Omega))} \\
& +\left(\Im_x\xi v, \Im_y(w^{(n)}_\eta - w_\eta\right)_{L^2(0,T;L^2\rho(\Omega))} \\
& +\left(\Im_y\eta v, x(w^{(n)}_x - w_x)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\
- \;& \left(\Im_y\eta v, (w^{(n)} - w)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\
& +\left(\Im_x\xi v, y(w^{(n)}_y - w_y)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\
& -\left(\Im_x\xi v, (w^{(n)} - w)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\
= \;& M(w^{(n)} - w, v), && (6.60)
\end{aligned}
$$

We now estimate terms on the left-hand side of (6.60) to see that

$$
\begin{aligned}
& \left(\Im_{xy}(\xi\eta\partial_T^{\beta+1}v), (w^{(n)} - w)\right)_{L^2(0,T;L^2\rho(\Omega))} \\
\leq \;& \|w^{(n)} - w\|_{L^2(0,T;L^2_\rho(\Omega))} \times \|\Im_{xy}(\xi\eta\partial_T^{\beta+1}v)\|_{L^2(0,T;L^2_\rho(\Omega))} \\
\leq \;& \frac{a^2b^2}{4}\|w^{(n)} - w\|_{L^2(0,T;L^2_\rho(\Omega))} \times \|\partial_T^{\beta+1}v\|_{L^2(0,T;L^2_\rho(\Omega))}, && (6.61)
\end{aligned}
$$

$$
\begin{aligned}
& \left(\Im_y\eta v, \Im_x(w^{(n)}_\xi - w_\xi)\right)_{L^2(0,T;L^2\rho(\Omega))} \\
\leq \;& \|\Im_x(w^{(n)}_\xi - w_\xi)\|_{L^2(0,T;L^2_\rho(\Omega))} \times \|\Im_y\eta v)\|_{L^2(0,T;L^2_\rho(\Omega))} \\
\leq \;& \frac{a^2b^2}{4}\|v\|_{L^2(0,T;L^2_\rho(\Omega))} \times \|w^{(n)} - w\|_{L^2(0,T;H^1_\rho(\Omega))}, && (6.62)
\end{aligned}
$$

$$
\begin{aligned}
& \left(\Im_x\xi v, \Im_y(w^{(n)}_\eta - w_\eta\right)_{L^2(0,T;L^2\rho(\Omega))} \\
\leq \;& \|\Im_y(w^{(n)}_\eta - w_\eta)\|_{L^2(0,T;L^2_\rho(\Omega))} \times \|\Im_x\xi v)\|_{L^2(0,T;L^2_\rho(\Omega))} \\
\leq \;& \frac{a^2b^2}{4}\|v\|_{L^2(0,T;L^2_\rho(\Omega))} \times \|w^{(n)} - w\|_{L^2(0,T;H^1_\rho(\Omega))}, && (6.63)
\end{aligned}
$$

$$
\begin{aligned}
& \left(\Im_y(\eta v), x(w^{(n)}_x - w_x)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\
\leq \;& a\|\Im_y(\eta v)\|_{L^2(0,T;L^2_\rho(\Omega))} \times \|w^{(n)}_x - w_x\|_{L^2(0,T;L^2_\rho(\Omega))}
\end{aligned}
$$

$$\leq \quad \frac{ab^2}{2}\|v\|_{L^2(0,T;L^2_\rho(\Omega))} \times \|w^{(n)} - w\|_{L^2(0,T;H^1_\rho(\Omega))}, \tag{6.64}$$

$$\begin{aligned} & -\left(\Im_y \eta v, (w^{(n)} - w)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ \leq \quad & \|\Im_y(\eta v)\|_{L^2(0,T;L^2_\rho(\Omega))} \times \|w^{(n)} - w\|_{L^2(0,T;L^2_\rho(\Omega))} \\ \leq \quad & \frac{b^2}{2}\|v\|_{L^2(0,T;L^2_\rho(\Omega))} \times \|w^{(n)} - w\|_{L^2(0,T;H^1_\rho(\Omega))}, \end{aligned} \tag{6.65}$$

$$\begin{aligned} & \left(\Im_x \xi v, y(w_y^{(n)} - w_y)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ \leq \quad & b\|\Im_x(\xi v)\|_{L^2(0,T;L^2_\rho(\Omega))} \times \|w_y^{(n)} - w_y\|_{L^2(0,T;L^2_\rho(\Omega))} \\ \leq \quad & \frac{a^2 b}{2}\|v\|_{L^2(0,T;L^2_\rho(\Omega))} \times \|w^{(n)} - w\|_{L^2(0,T;H^1_\rho(\Omega))}, \end{aligned} \tag{6.66}$$

$$\begin{aligned} & -\left(\Im_x \xi v, (w^{(n)} - w)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ \leq \quad & \|\Im_x(\xi v)\|_{L^2(0,T;L^2_\rho(\Omega))} \times \|w^{(n)} - w\|_{L^2(0,T;L^2_\rho(\Omega))} \\ \leq \quad & \frac{a^2}{2}\|v\|_{L^2(0,T;L^2_\rho(\Omega))} \times \|w^{(n)} - w\|_{L^2(0,T;H^1_\rho(\Omega))}. \end{aligned} \tag{6.67}$$

If we combine equality (6.60) and inequalities (6.61)–(6.67), we obtain

$$\begin{aligned} & M(w^{(n)} - w, v) \\ \leq \quad & C_5\|v\|_{L^2(0,T;L^2_\rho(\Omega))}\Big(\|w^{(n)} - w\|_{L^2(0,T;H^1_\rho(\Omega))} \\ & + \left\|\partial_T^{\beta+1} v\right\|_{L^2(0,T;L^2_\rho(\Omega))} \|w^{(n)} - w\|_{L^2(0,T;L^2_\rho(\Omega))}\Big) \end{aligned} \tag{6.68}$$

where

$$C_5 = \max\left\{\frac{a^2b^2}{4}, \frac{a^2b}{2}, \frac{ab^2}{2}, \frac{a^2}{2}, \frac{b^2}{2}\right\}$$

On the other side we have

$$\begin{aligned} & \Bigg(v, \Im_{xy}\xi\eta\tilde{H}(\xi,\eta,t,w^{(n-1)}, \frac{\partial w^{(n-1)}}{\partial \xi}, \frac{\partial w^{(n-1)}}{\partial \eta}) \\ & -\Im_{xy}\xi\eta\tilde{H}(\xi,\eta,t,w,\frac{\partial w}{\partial \xi},\frac{\partial w}{\partial \eta}))\Bigg)_{L^2(0,T;L^2(\Omega))} \\ \leq \quad & \frac{(ab)^{\frac{3}{2}}\lambda}{8}\|v\|_{L^2(0,T;L^2_\rho(\Omega))} \times \|w^{(n-1)} - w\|_{L^2(0,T;H^1_\rho(\Omega))}. \end{aligned} \tag{6.69}$$

Taking into account (6.68) and (6.69) and passing to the limit in (6.58) as $n \to \infty$ to obtain

$$M(w, v) = \left(v, \Im_{xy}\xi\eta\tilde{H}(\xi,\eta,t,w,\frac{\partial w}{\partial \xi},\frac{\partial w}{\partial \eta})\right)_{L^2(0,T;L^2_\rho(\Omega))}. \tag{6.70}$$

Hence (6.17) holds. Now to conclude that problem (6.5)-(6.8) has weak solution, we show that (6.7) holds. Since $w \in L^2(0,T;H^1_\rho(\Omega))$, then $\int\limits_0^t \frac{\partial w}{\partial x}(x,y,s)ds$, $\int\limits_0^t \frac{\partial w}{\partial y}(x,y,s)ds \in C(Q^T)$ and we conclude that $\frac{\partial w}{\partial x}(a,y,t) = 0$, $\frac{\partial w}{\partial y}(x,b,t) = 0$, *a.e* □

Thus we have proved the following

Theorem 6.4. Suppose that condition (6.9) holds and that $K < 1$, then problem (6.5)–(6.8) has a weak solution belonging to $L^2(0,T;H^1_\rho(\Omega))$.

It remains to prove that problem (6.5)-(6.8) admits a unique solution.

Theorem 6.5 Assume that condition (6.9) holds, then problem (6.5)-(6.8) admits a unique solution.

Proof. Suppose that $w_1, w_2 \in L^2(0,T;H^1_\rho(\Omega))$ are two solutions of (6.5)-(6.8), then $V = w_1 - w_2 \in L^2(0,T;H^1_\rho(\Omega))$ and satisfies

$$\partial_t^\beta V_t - \frac{1}{x}V_x - V_{xx} - \frac{1}{y}V_y - V_{yy} = \sigma(x,y,t), \tag{6.71}$$

$$V(x,y,0) = 0 \ , \ V_t(x,y,0) = 0 \ , \tag{6.72}$$

$$V_x(a,y,t) = 0 \ , \ V_y(x,b,t) = 0 \ , \tag{6.73}$$

$$\int\limits_0^a xV(x,y,t)dx = 0 \ , \ \int\limits_0^b yV(x,y,t)dx = 0, \tag{6.74}$$

where

$$\sigma(x,y,t) = \tilde{H}\left(x,y,t,w_1,\frac{\partial w_1}{\partial x},\frac{\partial w_1}{\partial y}\right) - \tilde{H}\left(x,y,t,w_2,\frac{\partial w_2}{\partial x},\frac{\partial w_2}{\partial y}\right). \tag{6.75}$$

Taking the inner product in $L^2(0,T;L^2_\rho(\Omega))$ of (6.71) and the integro-differential operator

$$MV = V_t - \Im^2_{xy}\xi\eta V_t \tag{6.76}$$

and following the same procedure done in establishing the proof of Theorem 6.2, we have

$$\|V\|_{L^2(0,T;H^1_\rho(\Omega))} \leq K\|V\|_{L^2(0,T;H^1_\rho(\Omega))}, \tag{6.77}$$

where

$$K = TC_4\left(\frac{a^2+b^2}{8}+1\right). \tag{6.78}$$

Since $K < 1$, it follows from (6.77) that

$$(1-K)\|V\|_{L^2(0,T;H^1_\rho(\Omega))} = 0. \tag{6.79}$$

Consequently (6.79) implies that $V = w_1 - w_2 = 0$ and hence $w_1 = w_2 \in L^2(0,T;H^1_\rho(\Omega))$.

7. Conclusions

Here we studied a non local mixed problem for a two dimensional singular nonlinear fractional order equation in the Caputo sense. We prove the existence, uniqueness and continuous dependence of a strong solution of the posed problem. We first establish for the associated linear problem a priori estimate and prove that the range of the operator generated by the considered problem is dense. The technique of deriving the a priori estimate is based on constructing a suitable multiplier. From the resulted energy estimate, it is possible to establish the solvability of the linear problem. Then, by applying an iterative process based on the obtained results for the linear problem, we establish the existence, uniqueness and continuous dependence of the weak solution of the nonlinear problem. The main contribution is that we applied and developed the a priori estimate method for a two dimensional singular nonlinear fractional order partial differential equation with Bessel operator that have never been treated in the literature of integer and fractional differential equations.

Acknowledgement

The authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for its funding this Research group No. (RG-117).

Conflict of interest

The authors declare no conflict of interest.

References

1. J. R. Cannon, J. Van der Hoek, The existence and the continuous dependence for the solution of the heat equation subject to the specification of energy, *Boll. Uni. Math. Ital. Suppl.,* **1** (1981), 253–282.
2. J. R. Cannon, The solution of heat equation subject to the specification of energy, *Quart. Appl. Math.*, **21** (1963), 155–160.
3. J. R. Canon, S. P. Esteva, J. Van der Hoek, A Galerkin procedure for the diffusion equation subject to the specification of mass, *SIAM Numer. Anal.*, **24** (1987), 499–515.
4. V. I. Korzyuk, V. Dainyak, A weak solution of a Dirichlet type problem for a third order nonclassical linear differential equation, *Differentsial'nye Uravneniya*, **28** (1992), 1056–1066.
5. S. Mesloub, A. Bouziani, Mixed problem with integral conditions for a certain class of hyperbolic equations, *J. Appl. Math.*, **1** (2001), 107–116.
6. S. Mesloub, N. Lekrine, On a nonlocal hyperbolic mixed problem, *Acta Sci. Math.*, **70** (2004), 65–75.
7. S. Mesloub, A nonlinear nonlocal mixed problem for a second order parabolic equation, *J. Math. Anal. Appl.*, **316** (2006), 189–209.
8. S. Mesloub, On a singular two dimensional nonlinear evolution equation with non local conditions, *Nonlinear Anal.*, **68** (2008), 2594–2607.

9. L. S. Pulkina, A nonlocal problem with integral conditions for hyperbolic equations, *Electron. J. Diff. Eqns.*, **45** (1999), 1–6.
10. L. S. Pulkina, On solvability in L^2 of nonlocal problem with integral conditions for a hyperbolic equation, *Differents. Uravn.*, **2** (2000).
11. P. Shi, Weak solution to an evolution problem with a nonlocal constraint, *Siam. J. Math. Anal.*, **24** (1993), 46–58.
12. S. Mesloub, F. Aldosari, Even higher order fractional initial boundary value problem with nonlocal constraints of purely integral type, *Symmetry*, **11** (2019), 305.
13. S. Mesloub, Existence and uniqueness results for a fractional two-times evolution problem with constraints of purely integral type, *Math. Methods Appl. Sci.*, **39** (2016), 1558–1567.
14. A. A. A. Alikhanov, Priori estimates for solutions of boundary value problems for fractional order equations, *Partial Differential Equations*, **46** (2010), 660–666.
15. A. Kilbas, H. Srivastava, J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.
16. F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wavephenomena, *Chaos Solitons Fractals*, **7** (1996), 1461–1477.
17. I. Podlubny, Fractional DifferentialEquations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, vol.198. Elsevier, Amsterdam, 1998.
18. A. Giusti, F. Mainardi, A dynamic viscoelastic analogy for fluid filled elastic tubes, *Meccanica*, **51** (2016), 2321–2330.
19. S. Ladaci, J. J. Loiseau, A. Charef, Fractional order adaptive high-gain controllers for a class of linear systems, *Commun. Nonlinear Sci.*, **13** (2008), 707–714.
20. B. Ahmad, J. J. Nieto, Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions, *Boundary Value Probl.*, **2009** (2009), 11.
21. V. Gafiychuk, B. Datsko, V. Meleshko, Mathematical modeling of time fractional reactiondiffusion systems, *J. Comput. Appl. Math.*, **220** (2008), 215–225.
22. J. Allison, N. Kosmatov, Multi-point boundary value problems of fractional order, *Commun. Appl. Anal.*, **12** (2008), 451–458.
23. R. W. Ibrahim, M. Darus, Subordination and superordination for univalent solutions for fractional differential equations, *J. Math. Anal. Appl.*, **345** (2008), 871–879.
24. K. V. Chukbar, The stochastic transfer and fractional derivatives, *Zh. Eksp. Teor. Fiz.*, **108** (1995), 1875–1884.
25. V. M. Goloviznin, V. P. Kisilev, I. A. Korotkin, Yu. I. Yurkov, Direct problems of nonclassical radionuclide transfer in geological formations, *Izv. Ross. Akad. Nauk, Energ.*, **4** (2004), 121–130.
26. S. Mesloub, On a mixed nonlinear one point boundary value problem for an integrodifferential equation, *Boundary Value Probl.*, **2008** (2008), 8, Article ID 814947.
27. L. Magin Richard, Fractional calculus in bioengineering, Begell House Redding, 2006.

28. A. Boucherif, Second-order boundary value problems with integral boundary conditions, *Nonlinear Anal-Theor.*, **70** (2009), 364–371.

29. Y. K. Chang, J. J. Nieto, W. S. Li, Controllability of semilinear differential systems with nonlocal initial conditions in Banach spaces, to appear in Journal of Optimization Theory and Applications, **142** (2009), 267–273.

30. A. Bachir, J. J. Nieto, Existence Results for Nonlinear Boundary Value Problems of Fractional Integrodifferential Equations with Integral Boundary Conditions, Boundary Value Problems volume 2009, Article number: 708576 (2009).

31. A. Bachir, A. Alsaedi, B. S. Alghamdi, Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions, *Nonlinear Anal.: Real World Appl.*, **9** (2008), 1727–1740.

32. L. Kasmi, A. Guerfi, S. Mesloub, Existence of solution for 2-D time-fractional differential equations with a boundary integral condition, *Adv. Differ. Equ.*, **2019** (2019), 511.

33. A. Akilandeeswari, K. Balachandran, N. Annapoorani, Solvability of hyperbolic fractional partial differential equations, *J. Appl. Anal. Comput.*, **7** (2017), 1570–1585.

34. S. Mesloub, I. Bachar, On a nonlocal 1-d initial value problem for a singular fractional-order parabolic equation with Bessel operator, *Adv. Differ. Equ.*, **2019** (2019), 254.

35. X. Liu, L. Liu, Y. Wu, Existence of positive solutions for a singular nonlinear fractional differential equation with integral boundary conditions involving fractional derivatives, *Boundary Value Probl.*, **2018** (2018), 24.

36. H. Li, L. Liu, Y. Wu, Positive solutions for singular nonlinear fractional differential equation with integral boundary conditions, *Boundary Value Probl.*, **2015** (2015), 232.

37. A. Bashir, M. M. Matar, R. P. Agarwal, Existence results for fractional differential equations of arbitrary order with nonlocal integral boundary conditions, *Boundary Value Probl.*, **2015** (2015), 220.

38. O. L. Ladyzhenskaya, The boundary value problems of mathematical physics, Springer-Verlag, New York, 1985.