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Research article

List vertex arboricity of planar graphs without 5-cycles intersecting with 6-cycles

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Abstract: The vertex arboricity $a(G)$ of a graph G is the minimum number of colors required to color the vertices of G such that no cycle is monochromatic. The list vertex arboricity $a_l(G)$ is the list version of this concept. In this paper, we prove that if G is a planar graph without 5-cycles intersecting with 6-cycles, then $a_l(G) \leq 2$.

Keywords: planar graph; list vertex arboricity; intersecting cycles
Mathematics Subject Classification: 05C15

1. Introduction

All graphs considered in this paper are finite, simple and undirected. For a graph G, we use $V(G)$, $E(G)$, $\Delta(G)$, and $\delta(G)$ to denote its vertex set, edge set, maximum degree, and minimum degree, respectively. A *plane* graph is a planar graph with a given planar drawing on the Euclidean plane. If G is a plane graph, let $F(G)$ denote the set of faces in G. We say that two cycles (or faces) are *adjacent* if they share at least one edge. In particular, when they share exactly one edge and two vertices, they are said to be *normally adjacent*. Two cycles (or faces) are *intersecting* if they share at least one vertex.

The *vertex arboricity*, denoted by $a(G)$, of a graph G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a forest. Obviously, $a(G) = 1$ if and only if G itself is a forest. In 1968, Chartrand, Kronk and Wall [2] first introduced the vertex arboricity of a graph and proved that $a(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ for any graph G and $a(G) \leq 3$ for any planar graph G. It is known that there exist infinitely many planar graphs G such that $a(G) = 3$. In 1989, Hakimi and Schmeichel [6] provided a characterization by showing that a plane graph G has $a(G) = 2$ if and only if G^*, the dual of G, contains a connected Eulerian spanning subgraph. In 2008, Raspaud and Wang [10] conjectured that every planar graph G without adjacent 3-cycles has $a(G) \leq 2$. To attack this conjecture, Chen, Raspaud and Wang [3] confirmed a weak version, i.e., every planar graph G without intersecting 3-cycles has $a(G) \leq 2$. Some sufficient conditions for a planar graph G to have $a(G) \leq 2$ have been obtained in [5, 7, 10, 13].

A graph G is said to be *L-forested-colorable* if for any color list $L = \{L(v) \mid v \in V(G)\}$, one can choose a color for each vertex v from its list $L(v)$ so that the subgraph induced by every color class is a forest. The *list vertex arboricity* $a_l(G)$ is the minimum number of integer k such that G is L-forested-colorable with $|L(v)| \geq k$ for each $v \in V(G)$. Obviously, $a(G) \leq a_l(G)$ for any graph G. In 2009, Borodin and Ivanova [1] proved that $a_l(G) \leq 2$ if G is a planar graph without 4-cycles adjacent to 3-cycles. This result has been recently extended by Chen, Huang and Wang [4] to a toroidal graph without 4-cycles adjacent to 3-cycles. In 2020, Wang, Huang and Chen [9] proved that every planar graph G without intersecting 5-cycles has $a_l(G) \leq 2$. The list vertex arboricity of toroidal graphs has also been extensively investigated, see [8, 11, 14].

In this paper, we prove the following result:

Theorem 1. *If G is a planar graph without 5-cycles intersecting with 6-cycles, then $a_l(G) \leq 2$.*

We first introduce a few concepts and terminology used in the paper. Let G be a plane graph. For $x \in V(G) \cup F(G)$, let $d_G(x)$, or simply $d(x)$, denote the degree of x in G. A vertex of degree k (resp., at least k, at most k) is called a k-vertex (resp., k^+-vertex, k^--vertex). Similarly, we can define a k-face, k^+-face and k^--face. For a k-vertex $v \in V(G)$, we usually use $v_0, v_1, \ldots, v_{k-1}$ to denote the neighbors of v in G in a clockwise order, and let f_i denote the incident face of v that contains vv_i, vv_{i+1} as two boundary edges for $i = 0, 1, \ldots, k-1$, where indices are taken modulo k. For a face $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [v_1v_2 \cdots v_n]$ if $v_1, v_2, \ldots, v_n$ are the vertices of $b(f)$ in a clockwise order. Let $V(f) = V(b(f))$. Moreover, the face adjacent to f with $e = v_iv_{i+1}$ as a common boundary edge is simply denoted by $f^{v_iv_{i+1}}$. For $v \in V(G)$, let $F_i(v)$ (or $F_{i^+}(v)$) denote the set of i-faces (or i^+-faces) incident to v. Moreover, let $m_i(v) = |F_i(v)|$ and $m_{i^+}(v) = |F_{i^+}(v)|$. A k-vertex v is called a $(d_0, d_1, \ldots, d_{k-1})$*-vertex* if $d(f_i) = d_i$ for $i = 0, 2, \ldots, k-1$. A cycle C in a plane graph G is called *separating* if both its interior and exterior contain at least one vertex of G.

This paper is organized as follows: In section 2 we give the proof of Theorem 1. Initially, we explore structural properties of a minimal counterexample. Then we use the discharging technique to contradict the existence of such a graph. Finally, in Section 3 we conclude with conjectures.

2. Proof of Theorem 1

We prove Theorem 1 by contradiction. Suppose that G is a minimal counterexample to the Theorem 1, i.e., G is a planar graph satisfying the following conditions:

(i) without 5-cycles intersecting with 6-cycles;

(ii) $a_l(G) > 2$; and

(iii) having as few as possible vertices.

Let L be a list assignment of $V(G)$ such that every $v \in V(G)$ has $|L(v)| = 2$. If G contains a vertex v of degree at most 3, let $H = G - v$. By the minimality of G, H admits an L-forested-coloring π. Based on π, we may color v with a color in $L(v)$ which appears at most once in its neighbors such that π is extended to G, which is a contradiction. Hence suppose that $\delta(G) \geq 4$.

2.1. Structural properties of G

As given in [9], the following lemma still holds for the current graph G:

Lemma 1. *G contains no a 4-cycle $C = x_1x_2x_3x_4x_1$ with a chord x_1x_3 such that $d_G(x_1) \leq 5$ and $d_G(x_i) = 4$ for $i = 2, 3, 4$, as shown in the configuration C_1 of Figure 1.*

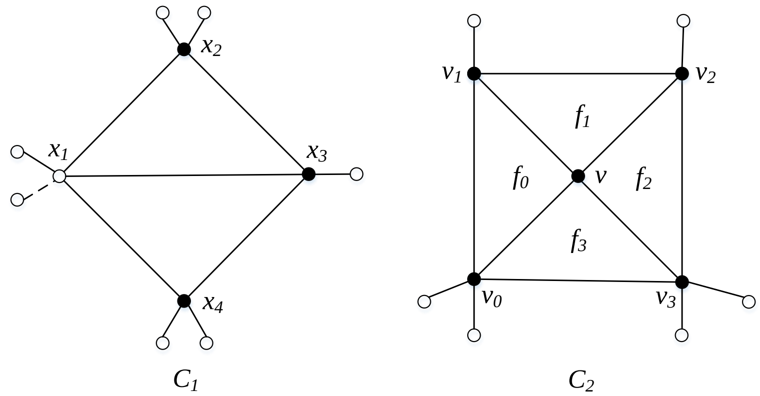

Figure 1. The configurations C_1 and C_2.

For $S \subset V(G)$, let $G[S]$ be a subgraph of G induced on S and let π be an L-forested-coloring of $G[S]$. Note that by definition every color class of π induces a forest in G. We call π a *partial L-forested-coloring* of G. For a vertex $u \in V(G) \setminus S$ and a color $c \in L(u)$, we use $\tau_L(c, u)$ to denote the number of times that c appears in the colored neighbors of u in G. Set

$$\tau_L(u) = \min\{\tau_L(c, u) | c \in L(u)\}$$

Vertex u is said to be *free* with respect to (L, π) if π can be extended into a partial L-forested-coloring of G by coloring u with a color $c \in L(u)$ such that $\tau_L(c, u) = \tau_L(u)$.

Lemma 2. ([12]) *Let $u \in V(G) \setminus S$ be a 4-vertex. If at least one of the following conditions holds, then u is free.*

(1) *At least one of neighbors of u is uncolored*;

(2) *At least three colors appear in the neighbors of u*;

(3) *Some color appears at least three times in the neighbors of u.*

Lemma 3. *G does not contain a internal 4-vertex v incident to four 3-faces such that v_1, v_2 are internal 4-vertices and v_0, v_3 are internal 5-vertices, as shown in the configuration C_2 of Figure 1.*

Proof. Suppose that G contains such a 4-vertex v. By symmetry, we have to discuss the following cases.

Case 1. $v_0v_2 \notin G$.

Let v_0', v_0'' be the neighbors of v_0 other than v, v_1, v_3; v_3', v_3'' be the neighbors of v_3 other that v, v_0, v_2; v_1' be the neighbor of v_1 other than v, v_0, v_2; v_2' be the neighbor of v_2 other than v, v_1, v_3. Consider the graph $H = G - \{v, v_0, v_1, v_2, v_3\}$. By the minimality of G, H has an L-forested-coloring π. Define a sublist L' of colors as follows:

$L'(v_i) = L(v_i) \setminus \{\pi(v_i'), \pi(v_i'')\}$ for $i = 0, 3$.

It is easy to see that $|L'(v_i)| \geq 0$ for $i = 0, 3$. Moreover, if $\pi(v_i') = \pi(v_i'')$, or $\pi(v_i') \neq \pi(v_i'')$ and $L(v_i) \neq \{\pi(v_i'), \pi(v_i'')\}$, then $|L'(v_i)| \geq 1$. Let $S = \{x \in \{v_0, v_3\} \mid |L'(x)| \geq 1\}$. Obviously, $0 \leq |S| \leq 2$. We are going to extend π to G, which leads to a contradiction.

Firstly, we give each v_i a color, $i = 0, 1, 2, 3$. For $i = 1, 2$, since $|L(v_i)| \geq 2$ and v_i has only one neighbor which is colored, there exists a color $a_i \in L(v_i) \setminus \pi(v_i')$ that we use to color v_i with α_i. Suppose that $i \in \{0, 3\}$. If $v_i \in S$, then we color the vertex v_i with a color $b_i \in L'(v_i)$. Otherwise, color v_i with the color $\pi(v_i')$. We have to discuss the following cases.

Case 1.1 $|S| \geq 1$.

By symmetry, assume that $|L'(v_0)| \geq 1$ and $a_0 \in L'(v_0)$. Let $\pi(v_1) = a_1$, $\pi(v_2) = a_2$, and $\pi(v_3) = a_3$.

Case 1.1.1. $a_0 \neq a_3$.

Suppose that $\{a_1, a_2\} \neq \{a_0, a_3\}$. Then v is free by Lemma 2 and π can be extended to G, a contradiction. Otherwise, $\{a_1, a_2\} = \{a_0, a_3\}$. If $a_1 = a_0$ and $a_2 = a_3$, then we recolor v_2 with a color in $L(v_2)\backslash\{a_2\}$ and color v with a color in $L(v)\backslash\{a_0\}$. If $a_1 = a_3$ and $a_2 = a_0$, then we color v with a color in $L(v)$.

Case 1.1.2. $a_0 = a_3$.

If $a_1 \neq a_2$, then v is free by Lemma 2 and π can be extended to G, a contradiction. If $a_1 = a_2 = a_0$, then recolor v_1 with a color in $L(v_1)\backslash\{a_0\}$, and color v with a color in $L(v)\backslash\{a_0\}$. Otherwise, $a_1 = a_2 \neq a_0$, then we recolor v_2 with a color in $L(v_2)\backslash\{a_2\}$ and color v with a color in $L(v)\backslash\{a_0\}$. Now if G does not contain monochromatic cycle, then we get an L-forested-coloring of G, a contradiction. Otherwise, G contains a monochromatic cycle $v_2v_3 \cdots v_2'v_2$, then $|L'(v_3)| = 0$, $L(v_3) = \{\pi(v_3'), \pi(v_3'')\}$ and $\pi(v_3') \neq \pi(v_3'')$. Recoloring v_3 with a color in $L(v_3)\backslash\{a_3\}$, we extend π to G, a contradiction.

Case 1.2. $|S| = 0$.

Without loss of generality, let $\pi(v_i) = a_i$, $i = 0, 1, 2, 3$.

Case 1.2.1. $a_0 \neq a_3$.

Suppose that $\{a_1, a_2\} \neq \{a_0, a_3\}$. Then v is free by Lemma 2 and π can be extended to G, a contradiction. Otherwise, $\{a_1, a_2\} = \{a_0, a_3\}$. If $a_1 = a_0$ and $a_2 = a_3$, then we recolor v_0 with the color $\pi(v_0'')$, and color v with a color in $L(v) \setminus \{a_3\}$. If there is no monochromatic cycle in G, then π is extended to G, a contradiction. Otherwise, if G contains a monochromatic cycle $v_0v_3 \cdots v_0''v_0$, then color v_3 with the color $\pi(v_3'')$. If $a_1 = a_3$ and $a_2 = a_0$, then we color v with a color in $L(v)$.

Case 1.2.2. $a_0 = a_3$.

Recolor v_0 with the color $\pi(v_0'')$, and then the proof can be given as in Case 1.2.1.

Case 2. $v_0v_2 \in G$.

Let v_0' be the neighbors of v_0 other than v, v_1, v_2, v_3; v_3', v_3'' be the neighbors of v_3 other that v, v_0, v_2; v_1' be the neighbor of v_1 other than v, v_0, v_2.

Consider the graph $H = G - \{v, v_0, v_1, v_2, v_3\}$. By the minimality of G, H has an L-forested-coloring π. Define a sublist $L'(v_3) = L(v_3) \setminus \{\pi(v_3'), \pi(v_3'')\}$.

Firstly, we give each v_i a color, $i = 0, 1, 2, 3$. For $i = 0, 1$, since $|L(v_i)| \geq 2$ and v_i has only one neighbor which is colored, there exists a color $a_i \in L(v_i) \setminus \{\pi(v_i')\}$ that we use to color v_i with a_i. We color the vertex v_2 with a color $a_2 \in L(v_2) \setminus \{a_0\}$. If $L'(v_3) \geq 1$, then we color the vertex v_3 with a color $a_3 \in L'(v_3)$. Otherwise, color v_3 with the color $\pi(v_3')$.

Case 2.1. $a_0 \neq a_3$.

Suppose that $\{a_1, a_2\} \neq \{a_0, a_3\}$. Then v is free by Lemma 2 and π can be extended to G, a contradiction. Otherwise, $\{a_1, a_2\} = \{a_0, a_3\}$. Thus, $a_0 = a_1$ and $a_2 = a_3$. We recolor v_1 with a color in $L(v_1)\backslash\{a_0\}$ and color v with a color in $L(v)\backslash\{a_3\}$.

Case 2.2. $a_0 = a_3$.

If $a_1 \neq a_2$, then v is free by Lemma 2 and π can be extended to G, a contradiction. If $a_1 = a_2$, then recolor v_1 with a color in $L(v_1)\backslash\{a_2\}$, and color v with a color in $L(v)\backslash\{a_0\}$.

This completes the proof of Lemma 3. □

2.2. The discharging technique

To obtain a contradiction by applying discharging technique, we define a new graph H from G as follows: If there is no separating 3-cycle in G, let $H = G$; otherwise, choose a separating 3-cycle T with the least internal vertices, and let $H = G[V^0(T) \cup V(T)]$, where $V^0(T)$ is the set of internal vertices of T. Let $V^0(H) = V^0(T)$, which are called the set of internal vertices in H. Let f^Δ denote the outer face of H, and let the set of interior faces of H be $F^0(H) = F(H) \setminus \{f^\Delta\}$. For each vertex $v \in V^0(H)$, it holds obviously that $d_H(v) = d_G(v)$. Since G is connected, so is H. In the following, we will contradict the existence of subgraph H and therefore of G, by using the discharging technique. Using Euler's formula $|V(H)| - |E(H)| + |F(H)| = 2$, we can deduce the following identity.

$$\sum_{v \in V(H)} (2d_H(v) - 6) + \sum_{f \in F(H)} (d_H(f) - 6) = -12 \tag{2.1}$$

Define an *initial weight* w on vertices and faces of H by $w(v) = 2d_H(v) - 6$ if $v \in V(H)$ and $w(f) = d_H(f) - 6$ if $f \in F(H)$. It follows from (2.1) that the total sum of weights is equal to -12. In what follows, we will design some discharging rules and redistribute weights accordingly. Once the discharging is finished, a new weight function w' is produced. However, the total sum of weights is kept fixed during the discharging process. Nevertheless, we can show that the sum of $w'(x)$ is at least -11.6 for all $x \in V(H) \cup F(H)$. This leads to the following obvious contradiction

$$-11.6 \leq \sum_{x \in V(H) \cup F(H)} w'(x) = \sum_{x \in V(H) \cup F(H)} w(x) = -12$$

and hence the theorem follows.

For a 3-face $f \in F(G)$, we say that it is *bad* if $V(f)$ contains three internal 4-vertices; *weak* if $V(f)$ contains two internal 4-vertices and one 5^+-vertex, or two internal 4-vertices with $|V(f) \cap V(T)| = 1$; and *good* otherwise.

Observation 1. *Let $f', f'' \in F^0(H)$ with $d_H(f') = 3$ and $d_H(f'') = 4$. If f' is normally adjacent to f'', then $b(f') \cup b(f'')$ contains a 5-cycle.*

Let $C_5 \bowtie C_6$ denote a configuration that a vertex lies in a 5-cycle and a 6-cycle at the same time.

Observation 2. *Let $f', f'' \in F^0(H)$ with $d_H(f') = 3$ and $d_H(f'') = 5$. If f' is normally adjacent to f'', then $b(f') \cup b(f'')$ contains a configuration $C_5 \bowtie C_6$.*

Proof. Suppose that $f'' = [x_1x_2x_3x_4x_5]$ and $f' = [x_1x_2x_6]$. Since f'' is a 5-face, $x_1, x_2, \ldots, x_5$ are mutually distinct. Since f' and f'' are normally adjacent, it follows that $x_6 \notin V(f'')$ and hence a 6-cycle $x_1x_6x_2x_3x_4x_5x_1$ is found. Thus, $b(f') \cup b(f'')$ has a configuration $C_5 \bowtie C_6$ at the vertex x_1. □

Claim 1. *If $v \in V^0(H)$ is a 5^+-vertex, then $m_3(v) \leq \lfloor \frac{3d_H(v)}{4} \rfloor$.*

Proof. Since v is an internal vertex, all the faces incident to v are internal faces. It is easy to check that if v is incident to four consecutive 3-faces, then G will contain a configuration $C_5 \bowtie C_6$. This shows that $m_3(v) \leq \lfloor \frac{3d_H(v)}{4} \rfloor$. □

Claim 2. *If $v \in V^0(H)$ is incident to two 3-faces f_i, f_{i+1}, then $d_H(f_j) \neq 4, 5$ for each $j \in \{i-1, i+2\}$.*

Proof. Without loss of generality, assume that $i = 1$.

• $d_H(f_3) = 4$.

Suppose that $f_3 = [vv_3uv_4]$. If u, v_1, v_2 are mutually distinct, then a 5-cycle $vv_2v_3uv_4v$ and a 6-cycle $vv_1v_2v_3uv_4v$ are found. Thus, $b(f_1) \cup b(f_2) \cup b(f_3)$ has a configuration $C_5 \bowtie C_6$ at the vertex v, a contradiction. If $v_2 = u$, then a separating 3-cycle vv_2v_4v with fewer internal vertices than T appears, contradicting the choice of T. If $v_1 = u$, then a separating 3-cycle vv_1v_3v with fewer internal vertices than T appears, contradicting the choice of T.

• $d_H(f_3) = 5$.

Suppose that $f_3 = [vv_3u_1u_2v_4]$. By Observation 2, f_2, f_3 are not normally adjacent, thus $v_2 = u_1$ or $v_2 = u_2$. If $v_2 = u_1$, then v_3 is a 2-vertex and v is a vertex on T, a contradiction. If $v_2 = u_2$, then a separating 3-cycle vv_2v_4v with fewer internal vertices than T appears, contradicting the choice of T. □

Claim 3. *Let $v \in V^0(H)$ with $d_H(v) \geq 4$. If $d_H(f_i) = d_H(f_{i+1}) = d_H(f_{i+2}) = 3$, then $d_H(f_{i-1}), d_H(f_{i+3}) \geq 7$.*

Proof. Without loss of generality, assume that $i = 1$. By Claim 1, $d_H(f_0) \neq 3$. By Claim 2, $d_H(f_0) \neq 4, 5$. If $d_H(f_0) = 6$, a 5-cycle $vv_1v_2v_3v_4v$ and a 6-cycle f_0 are found. Thus, $b(f_0) \cup b(f_1) \cup b(f_2) \cup b(f_3)$ has a configuration $C_5 \bowtie C_6$ at the vertex v, a contradiction. □

Claim 4. *If $v \in V^0(H)$ is incident to two 3-faces f_i, f_{i+2}, then $d_H(f_{i+1}) \neq 4, 5$.*

Proof. Without loss of generality, assume that $i = 1$.

• $d_H(f_2) = 4$.

Suppose that $f_2 = [vv_2uv_3]$. If u, v_1, v_4 are mutually distinct, then a 5-cycle $vv_1v_2uv_3v$ and a 6-cycle $vv_1v_2uv_3v_4v$ are found. Thus, $b(f_1) \cup b(f_2) \cup b(f_3)$ has a configuration $C_5 \bowtie C_6$ at the vertex v, a contradiction. If $v_1 = u$, then a separating 3-cycle vv_1v_3v with fewer internal vertices than T appears, contradicting the choice of T. If $v_4 = u$, then a separating 3-cycle vv_2v_4v with fewer internal vertices than T appears, contradicting the choice of T.

• $d_H(f_2) = 5$.

Suppose that $f_2 = [vv_2u_1u_2v_3]$. By Observation 2, f_1, f_2 are not normally adjacent and f_2, f_3 are not normally adjacent, thus $v_1 = u_1$ and $v_4 = u_2$. In this case, v_2 and v_3 are both 2-vertices and they are stand on different 3-faces, a contradiction. □

Our discharging rules are defined as follows:

(R0) Let $v \in V(T)$. We carry out the following three subrules (R0.1), (R0.2) and (R0.3):

(R0.1) v sends 0.5 to each incident 4-face, and 0.2 each incident 5-face.

(R0.2) If $d_H(v) \geq 5$, then v sends 2 to each incident internal 3-face. Otherwise, assume that $3 \leq d_H(v) \leq 4$. If $m_3(v) = d_H(v)$, then v sends 1.5 to each incident internal 3-face. If $m_3(v) \leq d_H(v) - 1$, then v sends 2 to each incident internal 3-face.

(R0.3) If $f = [vx_1x_2x_3]$ is a 4-face, then v sends 0.1 to x_2 through the face f.

(R1) Every 4-face gets 0.5 from each of its incident internal vertices.

(R2) Every 5-face gets 0.2 from each of its incident internal vertices.

(R3) Let $v \in V^0(H)$ with $m_3(v) \geq 1$.

(R3.1) Suppose that $d_H(v) = 4$.

• Assume that $m_3(v) = 4$. If v is incident with a bad 3-face and a good 3-face, then v sends 1 to bad 3-face, 0 to good 3-face and 0.5 to each of its incident weak 3-faces. Otherwise, v sends 0.5 to each of its incident 3-faces.

• Assume that $m_3(v) = 3$. If v is incident with a bad 3-face, then v sends 1 to bad 3-face, and 0.5 to each of its other incident 3-faces. If v is incident with at least one good 3-face, then v sends 0.5 to good 3-face, and 0.75 to each of its incident weak 3-faces. Otherwise, v sends 0.5 to each of its incident weak 3-faces.

• If $m_3(v) \leq 2$, then v sends 1 to each of its incident 3-faces.

(R3.2) Suppose that $d_H(v) = 5$.

• If $d_H(f_i) = 3$ for $i = 0, 1, 2$, and $d_H(f_3), d_H(f_4) \geq 7$, then v sends 1.25 to each of f_0, f_1, f_2.

• If $d_H(f_i) = 3$ for $i = 0, 1, 3$, and $d_H(f_2), d_H(f_4) \geq 6$, then v sends 1.25 to each of f_0, f_1 and 1.5 to f_3.

• For other cases, v sends 1.5 to each of its incident 3-faces.

(R3.3) If $d_H(v) \geq 6$, then v sends 1.5 to each of its incident 3-faces.

(R4) Assume that $v \in V^0(H)$ is a $(3, 4, 5, 4)$-vertex with $f_1 = [vv_1x_1v_2]$ and $f_3 = [vv_3x_2v_0]$, as shown in the configuration G_1 of Figure 2. Then each of x_1 and x_2 send 0.1 to v.

(R5) Assume that $v \in V^0(H)$ is an internal $(3, 3, 3, 3)$-vertex and v_0 is an internal $(3, 3, 7^+, 3, 7^+)$-vertex. Let f^*, other than f_3, denote the incident face of the edge v_0v_3. Such configuration G_2 is depicted Figure 2.

(1) If v_1, v_2 are internal 4-vertices and v_3 is an internal 6^+-vertex, then f^* sends 0.25 to f_3.

(2) If v_3 is an internal 4-vertex, then f^* send 0.25 to f_3.

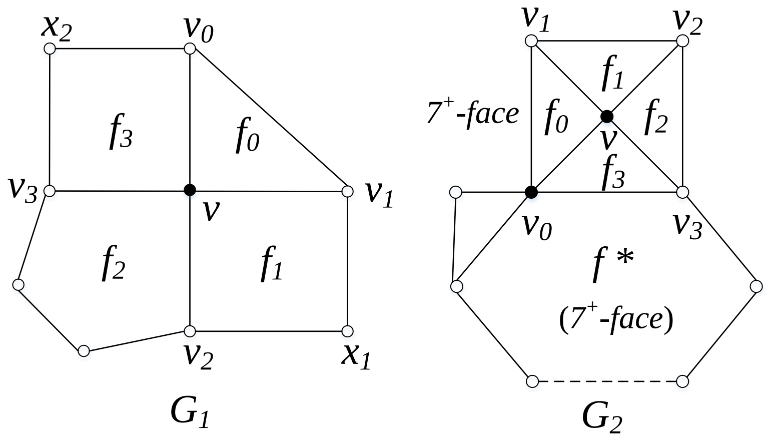

Figure 2. The configurations G_1 and G_2.

If v is a k-vertex incident with m 3-faces, then v is called a k_m*-vertex*. If a k-vertex v sends the weight a to an incident 3-face f, then v is called a k^a*-vertex* for f. A k^a_m*-vertex* for a 3-face f is a k_m-vertex that sends the weight a to f.

For $x, y \in V(H) \cup F(H)$, let $\tau(x \to y)$ denote the amount of weights transferred from x to y according to the rules (R0)-(R5).

Observation 3. *Let v be an internal 4^+-vertex incident to an internal 3-face f. Then*
(1) $\tau(v \to f) \in \{1, 0.75, 0.5, 0\}$ *if* $d_H(v) = 4$;
(2) $\tau(v \to f) \in \{1.25, 1.5\}$ *if* $d_H(v) = 5$;
(3) $\tau(v \to f) = 1.5$ *if* $d_H(v) \geq 6$.

Claim 5. *Suppose that $f = [x_1x_2x_3]$ is an internal 3-face. Let s denote the total number of $4^{0.75}$-vertices, $4^{0.5}$-vertices and 4^0-vertices which are adjacent to f. Then $s \leq 1$.*

Proof. Suppose that $s \geq 2$. By the discharging rules, if x_i is a $4^{0.75}$-vertex, $4^{0.5}$-vertex or 4^0-vertex, then x_i is an internal 4_3-vertex or 4_4-vertex. We consider the following possibilities by symmetry.

• At least one of x_1, x_2 is a 4_3-vertex in $V^0(H)$.

If $f^{x_1x_2}$ and $f^{x_1x_3}$ are 3-faces, then neither x_2 nor x_3 is an internal 4_3-vertex or 4_4-vertex, for otherwise H will contain a separating 3-cycle with fewer internal vertices than T or a configuration $C_5 \bowtie C_6$ at the vertex x_1, a contradiction. The case in which $f^{x_1x_2} = [x_1x_2w]$ and $f^{x_1w} = [x_1wy]$ are 3-faces can be similarly discussed, where y and w are the neighbors of x_1 other than x_2 and x_3.

• Both x_1 and x_2 are internal 4_4-vertices.

In this case, H contains a separating 3-cycle with fewer internal vertices than T or a configuration $C_5 \bowtie C_6$ at the vertex x_1, also a contradiction.

The above discussion implies that $s \leq 1$. □

Lemma 4. $w'(f) \geq 0$ *for each* $f \in F^0(H)$.

Proof. Let $f \in F^0(H)$. Clearly, $d_H(f) \geq 3$.

Case 1. Suppose that $d_H(f) = 3$ and $f = [x_1x_2x_3]$.

Then $w(f) = -3$. Since f is an internal face of H, we can derive that $|V(f) \cap V(T)| \leq 2$.

Case 1.1. $|V(f) \cap V(T)| = 2$.

Then f is a good 3-face. By (R0), f receives 1.5 or 2 from each of its two vertices of T. Hence $w'(f) \geq -3 + 1.5 \times 2 = 0$.

Case 1.2. $|V(f) \cap V(T)| = 1$.

Without loss of generality, let $x_1 \in V(T)$. We distinguish cases depending on whether x_2 and x_3 are internal 4-vertices or internal 5^+-vertices. Assume that both x_2 and x_3 are internal 4-vertices. Then f is a weak 3-face. Rule (R3.1) assures that 4-vertices always send non-zero weight to weak faces. Hence by Observation 3, x_2 and x_3 send either 0.5 or 0.75 or 1 to f. Now, by Claim 5, at most one of x_2 and x_3 is an $4^{0.5}$-vertex or $4^{0.75}$-vertex. Equivalently at least one of them is a 4^1-vertex. Thus, $w'(f) \geq -3 + 1.5 + 1 + 0.5 = 0$ by (R0) and (R3). Assume that x_2 is a internal 4-vertex and x_3 is a internal 5^+-vertex. If x_2 is not a 4^0-vertex, then $w'(f) \geq -3 + 1.5 + 1.25 + 0.5 = 0.25$ by (R0) and (R3). Otherwise, x_2 is a 4^0-vertex. By (R3.1), x_2 is a 4_4-vertex. If $d_H(x_1) = 3$, then $|V(f) \cap V(T)| = 2$, a contradiction. If $d_H(x_1) \geq 4$, then $f^{x_1x_3}$ is a 4^+-face and $m_3(x_1) \leq d_H(x_1) - 1$, otherwise it will produce a 3-vertex in $v^0(H)$, or a separating 3-cycle with fewer internal vertices than T or a configuration $C_5 \bowtie C_6$ at the vertex x_1, a contradiction. Thus f gets 2 from x_1. Therefore, $w'(f) \geq -3 + 2 + 1.25 = 0.25$ by (R0) and (R3). Assume that both x_2 and x_3 are 5^+-vertices, then $w'(f) \geq -3 + 1.5 + 1.25 \times 2 = 1$ by (R0) and (R3).

Case 1.3. $|V(f) \cap V(T)| = 0$.

Let $d_H(x_1) \leq d_H(x_2) \leq d_H(x_3)$.

• Assume that $d_H(x_1) \geq 5$, then f gets at least 1.25 from each of x_1, x_2, x_3 by Observation 3. Thus $w'(f) \geq -3 + 1.25 \times 3 = 0.75$.

• Assume that $d_H(x_1) = 4, d_H(x_2) \geq 5$, then f gets at least 1.25 from each of x_2, x_3 by (R3.2) and (R3.3). If x_1 is not a 4^0-vertex, then f gets at least 0.5 from x_1 by (R3.1). Hence, $w'(f) \geq -3 + 0.5 + 1.25 \times 2 = 0$. Otherwise, x_1 is a 4^0-vertex, then x_1 is a internal 4_4-vertex and x_1 is incident with a bad 3-face by (R3.1). By Lemma 1 and Lemma 3, x_2 or x_3 is a internal 6^+-vertex, say x_2, and f gets 1.5 from x_2 by (R3.3). If x_3 is a internal 6^+-vertex, or a internal 5-vertex with $m_3(x_3) = 2$, then f gets 1.5 from x_3 by (R3.2) and (R3.3). Thus, $w'(f) \geq -3 + 1.5 \times 2 = 0$. Otherwise, x_3 is a internal 5-vertex with $m_3(x_3) = 3$. If x_3 is a internal $(3, 3, 3, 6^+, 6^+)$-vertex, a $(3, 3, 6, 3, 6^+)$-vertex, or a $(3, 3, 6^+, 3, 6)$-vertex, it will contain a configuration $C_5 \bowtie C_6$ at the vertex x_3, a contradiction. If x_3 is a internal $(3, 3, 7^+, 3, 7^+)$-vertex, it is the construction G_2 and f gets 0.25 from the 7^+-face $f^{x_2x_3}$ by (R5). Thus, $w'(f) \geq -3 + 1.5 + 1.25 + 0.25 = 0$ by (R3.2) and (R3.3).

• Assume that $d_H(x_1) = d_H(x_2) = 4$ and $d_H(x_3) \geq 5$. By definition, f is a weak 3-face and f gets at least 0.5 from each of x_1, x_2 by (R3.1). By Claim 5, at least one of x_1 and x_2 is a 4^1-vertex. Without loss of generality, assume that x_2 is a 4^1-vertex. If x_1 is a 4^1-vertex or a $4^{0.75}$-vertex, then $w'(f) \geq -3 + 1 + 0.75 + 1.25 = 0$ by (R3). Otherwise, x_1 is a $4^{0.5}$-vertex. If x_3 is a internal 6^+-vertex or a $5^{1.5}$-vertex, then $w'(f) \geq -3 + 1 + 1.5 + 0.5 = 0$ by (R3). Therefore, x_3 is a internal 5-vertex with $m_3(x_3) = 3$. By Lemma 1 and the definition of good 3-face, we can derive that x_1 is incident with a good 3-face. If x_1 is a internal 4_3-vertex, then f gets 0.75 from x_1 by (R3.1), a contradiction. Otherwise, x_1 is a internal 4_4-vertex. If x_3 is a $(3, 3, 3, 6^+, 6^+)$-vertex, $(3, 3, 6, 3, 6^+)$-vertex or $(3, 3, 6^+, 3, 6)$-vertex, it will contain a configuration $C_5 \bowtie C_6$ at the vertex x_3, which is a contradiction. If x_3 is a $(3, 3, 7^+, 3, 7^+)$-vertex, it is the construction G_2 and f gets 0.25 from the 7^+-face $f^{x_2x_3}$ by (R5). Thus, $w'(f) \geq -3 + 1 + 0.5 + 1.25 + 0.25 = 0$ by (R3).

• Assume that $d_H(x_1) = d_H(x_2) = d_H(x_3) = 4$.

By definition, f is a bad 3-face and f gets 1 from each of x_1, x_2, x_3 by (R3). Thus, $w'(f) \geq -3 + 1 \times 3 = 0$.

Case 2. $4 \leq d_H(f) \leq 6$.

(1) If $d_H(f) = 4$, then $w'(f) = -2 + 0.5 \times 4 = 0$ by (R0) and (R1).

(2) If $d_H(f) = 5$, then $w'(f) = -1 + 0.2 \times 5 = 0$ by (R0) and (R2).

(3) If $d_H(f) = 6$, then $w'(f) = 0$.

Next, if $f = [v_1v_2v_3 \cdots v_n]$ is a 7^+-face, v_iv_{i+1} is in the configuration G_2, $[v_iv_{i+1}x]$ is a 3-face and x is a 4_4-vertex, then we say that f is adjacent to a configuration G_2 by v_iv_{i+1}. Let t denote the number of the configuration G_2 which f is adjacent to.

Claim 6. *Suppose that f is a 7^+-face in H, then $t \leq \lfloor \frac{2d_H(f)}{3} \rfloor$.*

Proof. Let $f = [v_1v_2v_3 \cdots v_n]$ be a 7^+-face in H.

If f is adjacent to a configuration G_2 by v_iv_{i+1}, then v_i or v_{i+1} is a internal $(3, 3, 7^+, 3, 7^+)$-vertex. By symmetry, let v_i be a internal $(3, 3, 7^+, 3, 7^+)$-vertex. If f is adjacent to another G_2 by edge $v_{i-1}v_i$, then v_i would be a $(3, 3, 3, 3, 7^+)$-vertex, contradicting the definition of G_2. Hence, if f is adjacent to a G_2 by edge v_iv_{i+1} at least one of $v_{i-1}v_i$ or $v_{i+1}v_{i+2}$ is not adjacent to G_2. It means that the three consecutive edges on f are adjacent to at most two G_2 configurations. Consequently, $t \leq \lfloor \frac{2d_H(f)}{3} \rfloor$. □

Case 3. $d_H(f) \geq 7$.

Suppose that $d_H(f)$=7. By Claim 6, $t \leq \lfloor \frac{2d_H(f)}{3} \rfloor = 4$, then $w'(f) \geq 7 - 6 - 0.25 \times 4 = 0$ by (R5). Suppose that $d_H(f) \geq 8$, then $w'(f) \geq d_H(f) - 6 - 0.25 \times \lfloor \frac{2d_H(f)}{3} \rfloor \geq \frac{5}{6}d_H(f) - 6 \geq 0$ by (R5).

This completes the proof of Lemma 4. □

Lemma 5. $w'_H(v) \geq 0$ *for each* $v \in V^0(H)$.

Proof. Since v is a vertex of G and $\delta(G) \geq 4$, it follows that $d_H(v) = d_G(v) \geq 4$. We have six cases to consider.

Case 1. $d_H(v) = 4$.

Case 1.1. Suppose that v is x_1 (or x_2) in the configuration G_1.

In the configuration G_1, we can derive that $m_5^+(v) \geq 2$ and $m_3(v) \leq 1$, otherwise it will produce a separating 3-cycle with fewer internal vertices than T or a configuration $C_5 \bowtie C_6$, a contradiction. Hence, $w'(v) \geq 2 - 0.5 - 0.1 - 0.2 \times 2 - 1 = 0$.

Case 1.2. Suppose that v is not x_1 (or x_2) in the construction G_1.

• Assume that $m_3(v) = 0$, then $w'(v) \geq 2 - 0.5 \times 4 = 0$ by (R1) and (R2).

• Assume that $m_3(v) = 1$. If $m_4(v) + m_5(v) \leq 2$, then $w'(v) \geq 2 - 1 - 0.5 \times 2 = 0$ by (R1), (R2) and (R3.1). Suppose that $m_4(v) + m_5(v) = 3$. If $m_5(v) \geq 2$, then $w'(v) \geq 2 - 1 - 0.5 - 0.2 \times 2 = 0.1$ by (R1), (R2) and (R3.1). Suppose that $m_5(v) = 1$ and $m_4(v) = 2$. If v is a $(4, 3, 5, 4)$-vertex, then it will produce a separating 3-cycle with fewer internal vertices than T or a configuration $C_5 \bowtie C_6$ at the vertex v, a contradiction. If v is a $(4, 3, 4, 5)$-vertex, then it is the configuration G_1. Thus, $w'(v) \geq 2 - 1 - 0.5 \times 2 - 0.2 + 0.1 \times 2 = 0$ by (R1), (R2), (R3.1) and (R4). If $m_4(v) = 3$, then it will produce a separating 3-cycle or a configuration $C_5 \bowtie C_6$, a contradiction.

• Assume that $m_3(v) = 2$. If $d_H(f_0) = d_H(f_1) = 3$, then $d_H(f_2) \geq 6$ and $d_H(f_3) \geq 6$ by Claim 2. Thus, $w'(v) \geq 2 - 1 \times 2 = 0$ by (R3.1). If $d_H(f_0) = d_H(f_2) = 3$, then $d_H(f_1) \geq 6$ and $d_H(f_3) \geq 6$ by Claim 4. Thus, $w'(v) \geq 2 - 1 \times 2 = 0$ by (R3.1).

• Assume that $m_3(v) = 3$. Suppose that $d_H(f_0) = d_H(f_1) = d_H(f_2) = 3$, then $d_H(f_3) \geq 7$ by Claim 3. By Lemma 1, at least one of v_0, v_1, v_2, v_3 is an internal 5^+-vertex or belongs to $V(T)$. If v_0 or v_3 is an internal 5^+-vertex or belongs to $V(T)$, say v_0, then one of v_1, v_2, v_3 is an internal 5^+-vertex or belongs to $V(T)$ by Lemma 1 and v is incident with at most one bad 3-face. If v_1 or v_2 is an internal 5^+-vertex or belongs to $V(T)$, then v is incident with at most one bad 3-face. Thus, $w'(v) \geq 2 - 1 - 0.5 \times 2 = 0$, or $w'(v) \geq 2 - 0.5 - 0.75 \times 2 = 0$, or $w'(v) \geq 2 - 0.5 \times 3 = 0.5$ by (R3.1).

• Assume that $m_3(v) = 4$. If v is not incident with any bad 3-face, then $w'(v) \geq 2 - 0.5 \times 4 = 0$ (R3.1). Otherwise, v is incident with bad 3-faces. By Lemma 1, at least two of v_0, v_1, v_2, v_3 are internal 5^+-vertices or belong to $V(T)$, it means that v is incident with one bad 3-face, one good 3-face and two weak 3-faces. Thus, $w'(v) \geq 2 - 1 - 0.5 \times 2 = 0$ by (R3.1).

Case 2. $d_H(v) = 5$.

Note that $w(v) = 4$. By Claim 1, $m_3(v) \leq \lfloor \frac{3}{4} d_H(v) \rfloor = 3$. If $m_{6^+}(v) \geq 3$, then $w'(v) \geq 4 - 1.5 \times 2 = 0$. Assume that $m_{6^+}(v) \leq 2$. If $m_3(v) = 0$, then $w'(v) \geq 4 - 0.5 \times 5 - 0.1 \times 5 = 1$ by (R1)-(R4). If $m_3(v) = 1$, then $w'(v) \geq 4 - 1.5 - 0.5 \times 4 - 0.1 \times 4 = 0.1$ by (R1)-(R4). Suppose that $m_3(v) = 2$. If $d_H(f_0) = d_H(f_1) = 3$, then $d_H(f_2) \geq 6$ and $d_H(f_4) \geq 6$ by Claim 2. Thus, $w'(v) \geq 4 - 1.5 \times 2 - 0.5 - 0.1 = 0.4$ by (R1)-(R4). If $d_H(f_0) = d_H(f_2) = 3$, then $d_H(f_1) \geq 6$ by Claim 4. If $d_H(f_3) = d_H(f_4) = 4$, it will produce a separating 3-cycle with fewer internal vertices than T or a configuration $C_5 \bowtie C_6$ at the vertex v, a contradiction. Thus $w'(v) \geq 4 - 1.5 \times 2 - 0.5 - 0.1 - 0.2 = 0.2$ by (R1)-(R4).

Suppose that $m_3(v) = 3$. If $d_H(f_0) = d_H(f_1) = d_H(f_2) = 3$, then $d(f_3) \geq 7$ and $d(f_4) \geq 7$ by Claim 3. Thus $w'(v) \geq 4 - 1.25 \times 3 = 0.25$ by (R3.2). If $d_H(f_0) = d_H(f_1) = d_H(f_3) = 3$, then $d_H(f_2) \geq 6$ and $d_H(f_4) \geq 6$ by Claim 2. Thus, $w'(v) \geq 4 - 1.25 \times 2 - 1.5 = 0$ by (R3.2).

Case 3. $d_H(v) = 6$.

Note that $w(v) = 6$. By Claim 1, $m_3(v) \leq \lfloor \frac{3}{4} d_H(v) \rfloor = 4$. If $m_{6^+}(v) \geq 2$, then $w'(v) \geq 6 - 1.5 \times 4 = 0$ by (R3.3).

Suppose that $m_{6^+}(v) \leq 1$. If $m_3(v) \leq 2$, then $w'(v) \geq 6 - 1.5 \times 2 - 0.5 \times 4 - 0.1 \times 4 = 0.6$ by (R1)-(R4). Assume that $m_3(v) = 3$. If $d_H(f_0) = d_H(f_1) = d_H(f_2) = 3$, then $d_H(f_3) \geq 7$ and $d_H(f_5) \geq 7$ by Claim 3, a contradiction. If $d_H(f_0) = d_H(f_1) = d_H(f_3) = 3$, then $d_H(f_2) \geq 6$ and $d_H(f_5) \geq 6$ by Claim 2, a contradiction. If $d_H(f_0) = d_H(f_2) = d_H(f_4) = 3$, then $d_H(f_1) \geq 6$, $d_H(f_3) \geq 6$ and $d_H(f_5) \geq 6$ by Claim 4. Assume that $m_3(v) = 4$. If $d_H(f_0) = d_H(f_1) = d_H(f_2) = d_H(f_4) = 3$, then $d_H(f_3) \geq 7$ and $d_H(f_5) \geq 7$ by Claim 3, a contradiction. If $d_H(f_0) = d_H(f_1) = d_H(f_3) = d_H(f_4) = 3$, then $d_H(f_2) \geq 6$ and $d_H(f_5) \geq 6$ by Claim 2, a contradiction.

Case 4. $d_H(v) = 7$.

Note that $w(v) = 8$. By Claim 1, $m_3(v) \leq \lfloor \frac{3}{4} d_H(v) \rfloor = 5$. If $m_{6^+}(v) \geq 2$, then $w'(v) \geq 8 - 1.5 \times 5 = 0.5$ by (R3.3). Suppose that $m_{6^+}(v) \leq 1$. If $m_3(v) \leq 4$, then $w'(v) \geq 8 - 1.5 \times 4 - 0.5 \times 3 - 0.1 \times 3 = 0.2$ by (R1)-(R4). Assume that $m_3(v) = 5$. If $d_H(f_0) = d_H(f_1) = d_H(f_2) = 3$, then $d_H(f_3) \geq 7$ and $d_H(f_6) \geq 7$ by Claim 3, a contradiction. If $d_H(f_0) = d_H(f_1) = 3$, then $d_H(f_2) \geq 6$ and $d_H(f_6) \geq 6$ by Claim 2, a contradiction.

Case 5. $d_H(v) = 8$.

Note that $w(v) = 10$. By Claim 1, $m_3(v) \leq \lfloor \frac{3}{4} d_H(v) \rfloor = 6$. If $m_{6^+}(v) \geq 2$, then $w'(v) \geq 10 - 1.5 \times 6 = 1$ by (R3.3). Suppose that $m_{6^+}(v) \leq 1$. If $m_3(v) \leq 5$, then $w'(v) \geq 10 - 1.5 \times 5 - 0.5 \times 3 - 0.1 \times 3 = 0.7$ by (R1)-(R4). Assume that $m_3(v) = 6$. If $d_H(f_0) = d_H(f_1) = d_H(f_2) = 3$, then $d_H(f_3) \geq 7$ and $d_H(f_7) \geq 7$ by Claim 3, a contradiction. If $d_H(f_0) = d_H(f_1) = 3$, then $d_H(f_2) \geq 6$ and $d_H(f_7) \geq 6$ by Claim 2, a contradiction.

Case 6. $d_H(v) \geq 9$.

We have the following estimation:

$$\begin{aligned} w'(v) &\geq 2d_H(v) - 6 - 1.5m_3(v) - 0.5(d_H(v) - m_3(v)) - 0.1(d_H(v) - m_3(v)) \\ &\geq 1.4d_H(v) - 0.9m_3(v) - 6 \\ &\geq 1.4d_H(v) - 0.9 \times \lfloor \frac{3d_H(v)}{4} \rfloor - 6 \\ &\geq \frac{29d_H(v)}{40} - 6 > 0 \end{aligned}$$

This completes the proof of Lemma 5. □

Lemma 6. $w'(f^{\Delta}) + \Sigma_{x \in V(T)} w'(x) \geq -11.6$.

Proof. By (R0), $w'(f^{\Delta}) \geq 3 - 6 = -3$. Let $v \in V(T)$. First assume that $d_H(v) = 2$. Note that v is not be incident with any 3-face or 4-face for otherwise we can find another separating 3-cycle with fewer internal vertices than T, which is impossible. Consequently, $w'(v) \geq 2 \times 2 - 6 - 0.2 = -2.2$ by (R0). Next assume that $d_H(v) = 3$. If $m_3(v) = 3$, then $w'(v) \geq 2 \times 3 - 6 - 1.5 \times 2 = -3$ by (R0). Otherwise, $w'(v) \geq 2 \times 3 - 6 - 2 - 0.6 = -2.6$ by (R0). Now assume that $d_H(v) = 4$. If $m_3(v) = 4$, then $w'(v) \geq 2 \times 4 - 6 - 1.5 \times 3 = -2.5$ by (R0). Otherwise, $w'(v) \geq 2 \times 4 - 6 - 2 \times 2 - 0.6 = -2.6$ by (R0). Finally assume that $d_H(v) \geq 5$. It follows that $m_3(v) \leq d_H(v) - 1$, otherwise a configuration $C_5 \bowtie C_6$ at the vertex v will be found. Thus, $w'(v) \geq 2 \times d_H(v) - 6 - 2 \times (d_H(v) - 2) - 0.6 = -2.6$ by (R0).

If f^{Δ} is incident with a vertex that is not a 3_3-vertex, then $w'(f^{\Delta}) + \Sigma_{x \in V(T)} w'(x) \geq -3 - 3 - 3 - 2.6 = -11.6$. Otherwise there is an internal 3-vertex, which is impossible.

Recall that the sum of the initial weights of vertices and faces of H equals -12. From Lemma 4-6, the corresponding sum of the new weights satisfies $\Sigma_{x \in V^0(H) \cup F^0(H)} w'(x) + w'(f^{\Delta}) + \Sigma_{x \in V(T)} w'(x) \geq -11.6$. Since the total weight of H has been preserved during the discharging process, this is a contradiction to our initial assumptions that G, the minimal counterexample to Theorem 1, exists. This completes the proof of Theorem 1. □

3. Conjectures

In this paper, we show that the list vertex arboricity of every planar graph without a 5-cycle intersecting with a 6-cycle is at most 2. This implies directly that if G is a planar graph without 5-cycles or without 6-cycles, then $a_l(G) \leq 2$. We like to conclude this paper by raising the following conjectures:

Conjecture 1. *Every planar graph G without intersecting 4-cycles has $a_l(G) \leq 2$.*

Conjecture 2. *If G is a planar graph without a vertex lying on all cycles of length from 3 to 7, then $a_l(G) \leq 2$.*

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Conflict of interest

The authors declare that they have no competing interests.

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